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On a Model for Soft Landing with State-Dependent Delay

Hans-Otto Walther

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Automatic soft landing is modeled by a differential equation with state-dependent delay. It is shown that in the model soft landing occurs for an open set of initial data, which is determined by means of a smooth invariant manifold.

KEY WORDS: State-dependent delay; soft landing; functional differential equations, local invariant manifolds.

AMS SUBJECT CLASSIFICATION: 34 K 05; 34 K 19; 70 Q 05; 93 C 20; 93 C 85.

1. INTRODUCTION

This paper deals with a simple model for automatic soft landing. What we have in mind is an object moving along a line and approaching a particular spot without collision. The control mechanism involves a nonconstant time lag and can be written as a differential equation with state-dependent delay. Initial value problems for such equations are not covered by the established theory of retarded functional differential equations [3,11], and some basic theory for equations with state-dependent delay, from well-posedness to linearization and local invariant manifolds at stationary points, has been developed only recently [17,12,22,21]. The present paper is a case study which shows how to use the new framework. Local invariant manifolds, notably a submanifold of the stable manifold, will be essential in this study.

Mathematisches Institut, Universität Giessen, Arndtstr. 2, 35392 Giessen, Germany. E-mail: Hans-Otto.Walther@math.uni-giessen.de

In order to achieve soft landing, control by echo is used in the following way. The object emits signals which are reflected by the desired landing place and then, after reflection, sensed by the object. The travel time of the signals is used to compute a position. This computed position may be different from the true position of the object, due to its motion during the signal travel time. Depending on the computed position the object regulates its acceleration.

Details of the model are as follows. Let $u \in \mathbb{R}$ denote the position, assume the desired landing place is at $u=0$, and let c denote the constant speed of the signal. Let t denote time, and consider an approach from above, i. e., with $u=u(t)\geq 0$. The equation relating the (true) position and the signal running time $s=s(t)$ is

$$c\,s = u(t-s) + u(t) \tag{1.1}$$

for a signal emitted at time $t-s$ and arriving after reflection at time t. The corresponding computed position may be given by

$$p = \frac{c\,s}{2}, \tag{1.2}$$

since this yields the correct position at least if $u(t) = u(t-s)$. The equations of motion are

$$u' = v, \tag{1.3}$$

$$v' = a(p, p') \tag{1.4}$$

with a suitable acceleration function $a:\mathbb{R}^2 \to \mathbb{R}$, which should satisfy

$$a(0,0) = 0$$

(no acceleration in landing position with speed zero).

Let us agree that soft landing is given by solutions (u, v) with

$$u(t) > 0 \quad \text{for} \quad t < t_0 \leq \infty \quad \text{and} \quad \lim_{t \nearrow t_0} (u(t), v(t)) = (0,0). \tag{1.5}$$

Solutions with $u(t_0) = 0$ for some t_0 and negative speed $v(t_0) < 0$ correspond to collisions, which are to be avoided. The main result of this paper is that for the model (1.1–4) soft landing occurs and is not rare: we find an open set of initial data for which solutions satisfy the relations (1.5). This domain of soft landing will contain data with u strictly positive and v strictly negative, representing initial descent.

Let us describe an obvious obstacle on the way to the open domain of soft landing. For a local analysis it would certainly be good that the state of interest, given by $u=0=v$, be an interior point of the domain on which one has a semiflow. In the model considered, neighborhoods of the equilibrium given by $u=0=v$ should contain states with $u<0$ (we are not yet precise here what states are). For such, Eq. (1.1) would yield negative signal travel times s, and the equations of motion (1.3-4) would contain advanced arguments of u (and v) if p (and p') are replaced using the right hand sides of Eqs. (1.1 and 2). For differential equations with both delayed and advanced arguments, however, there seem no basic existence-and-uniqueness results available, not to speak of linearization and local invariant manifolds. An immediate idea how to avoid this difficulty is to replace the term on the right hand side of Eq. (1.1) by

$$|u(t-s)|+|u(t)|.$$

This would yield correct, nonnegative signal travel times also for motion with $u<0$, but lack of smoothness (only Lipschitz continuity properties) would preclude the application of results on linearization and local invariant manifolds from [12].

The paper is organized as follows. In Section 2, the choice of the acceleration function a is discussed. The simplest of all possible models for soft landing, without any delay, are linear vectorfields $(u,v)\mapsto(v,a(u,v))$ on the plane. If there is an open domain of soft landing for such a model then the same may be expected for the model given by the system (1.1–4), because due to Eq. (1.1) the state-dependent delay is small for positions close to equilibrium. It should be noticed that even with a linear function a the model (1.1–4) is nonlinear.

Section 3 recalls basic facts from [12,20,21] about semiflows for differential equations with state-dependent delay and reformulates the model (1.1–1.4) appropriately, namely as an initial value problem for an Equation of the form

$$x'(t)=f(x_t).$$

Here $f:U\to\mathbb{R}^2$ is a functional on an open subset U of the Banach space

$$C_2^1=C^1([-r,0],\mathbb{R}^2)$$

with suitable $r>0$, and

$$x_t(s)=x(t+s).$$

The associated initial value problem yields a semiflow of continuously differentiable solution operators only for initial data in a positively invariant, infinite-dimensional submanifold $X_f\subset U$. The problems mentioned above, concerning solutions close to equilibrium, are reflected in the fact that the point $0 \in C^1_2$ is not even contained in the domain U; it lies on the boundary of U and is a limit point of X_f.

Another aspect is that the definition of f in Section 3 involves a decision how to interpret the term p' in Eq. (1.4). Our choice and its consequences for the original model are discussed in Remark 3.1.

In Section 4, the problem how to describe the dynamics close to $0 \in C^1_2$ is overcome by an extension $g: V \to \mathbb{R}^2$, $V \supset U$, of f so that the associated manifold $X = X_g \subset V \subset C^1_2$ on which one has a nice semiflow contains the stationary point 0. The extension employs odd continuation of functions on the initial interval $[-r, 0]$ to functions on $[-r, r]$.

Section 5 deals with the position of the tangent space $T_0 X$ in C^1_2, with spectral properties of a linearized solution operator on $T_0 X$, and with decompositions of $T_0 X$. These preparations are used in Section 6 where a locally positively invariant *fast manifold* W in X helps to find solutions of the extended equation from Section 4 with soft landing properties similar to (1.5). The manifold W can be constructed following the steps toward stable manifolds in Section 3.5 of [12].

In Section 7, the result on the extended equation yields a nonempty open domain of soft landing for the model equation from Section 3. This domain contains initial data $\psi = (\phi, \eta) \in C^1_2$ with ϕ strictly positive and ϕ' strictly negative, also η strictly negative, which correspond to descent during the whole initial time interval $[-r, 0]$.

Related models were proposed and analyzed in [1,2,18–20,23]. Versions of Eq. (1.1) occur also in models for the classical two-body-problem of electrodynamics, see Driver's work [4–10]. Other results on local invariant manifolds for equations with state-dependent delay were obtained by Krishnan [13,14] (unstable manifolds) and by Krisztin in [12,15,16] (unstable manifolds, center manifolds).

2. THE ACCELERATION FUNCTION

In order to understand for which acceleration functions a in Eq. (1.4) there is hope to obtain the desired open domain of soft landing we neglect the control by echo here and restrict attention to linear systems

$$u' = v, \tag{2.1}$$

$$v' = a(u, v), \tag{2.2}$$

where

$$a(u, v) = \alpha u + \beta v \tag{2.3}$$

with real constants α, β. Trajectories cross the upper part of the vertical axis from left to right.

The characteristic equation for the eigenvalues of the system matrix

$$\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$$

is

$$\lambda^2 - \beta\lambda - \alpha = 0$$

with the solutions

$$\lambda_1 = \frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} + \alpha}, \ \lambda_2 = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \alpha}.$$

The only parameter configurations for which there exists an open domain of soft landing, i. e., an open set of inital data so that the corresponding solutions $(u, v) : [0, \infty) \to \mathbb{R}^2$ satisfy

$$u(t) > 0 \quad \text{for all } t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} (u(t), v(t)) = (0, 0)$$

are the following two. For

$$\beta < 0 \quad \text{and} \quad -\frac{\beta^2}{4} < \alpha < 0, \tag{2.4}$$

both eigenvalues $\lambda_1 < \lambda_2$ are negative. The eigenvectors

$$(1, \lambda_j), \ j \in \{1, 2\},$$

associated with λ_1, λ_2, respectively, point into the fourth quadrant, with $(1, \lambda_1)$ lower than $(1, \lambda_2)$. Trajectories cross the right part of the horizontal axis from top to bottom. The domain of soft landing contains the convex subset of the right halfplane above the ray $\mathbb{R}^+(1, \lambda_1)$; the ray $\mathbb{R}^+(1, \lambda_2)$ in the other eigenspace, in which convergence to $(0, 0)$ is weaker, belongs to this subset.

For

$$\beta < 0 \quad \text{and} \quad \alpha = -\frac{\beta^2}{4},$$

both eigenvalues $\lambda_1 = \lambda_2 = \lambda < 0$ coincide, and there is only a one-dimensional eigenspace $\mathbb{R}(1, \lambda)$, with $(1, \lambda)$ pointing into the fourth quadrant. The same set as before, with $(1, \lambda)$ in place of $(1, \lambda_1)$, is contained in the domain of soft landing.

In both cases initial data in the open right halfplane below the ray $\mathbb{R}^+(1, \lambda_1)$ produce collisions, i.e., the associated trajectories cross the vertical axis $(u = 0)$ with negative speed $(v < 0)$ at some $t > 0$.

By Eq. (1.1), signal travel times are small for positions close to landing. It may be expected that close to the stationary point given by $u = 0 = v$ the dynamics of the full model is strongly influenced by that of the simple system (2.1 and 2). For this reason and in order to keep the model simple, we restrict attention from now on to the case that the acceleration function is linear and given by (2.3) and that (2.4) holds, neglecting the other, less generic of the two preceding cases.

3. THE MODEL

It seems reasonable to restrict attention to positions bounded by

$$|u| < b$$

for some $b > 0$. Then Eq. (1.1) yields the a priori bound

$$\frac{2b}{c}$$

for the signal travel times s, which is the maximal delay in the system.

For $r \geq \frac{2b}{c}$, let

$$C = C([-r, 0], \mathbb{R}) \quad \text{and} \quad C_2 = C([-r, 0], \mathbb{R}^2)$$

denote the Banach spaces of continuous functions on the initial interval $[-r, 0]$ with values in $\mathbb{R}$ and in $\mathbb{R}^2$, respectively. The norms are given by

$$\|\phi\|_C = \max_{-r \leq t \leq 0} |\phi(t)| \quad \text{and} \quad \|\psi\|_{C_2} = \|\phi\|_C + \|\eta\|_C,$$

respectively, where $\phi \in C$ and $\eta \in C$ are the components of $\psi = (\phi, \eta) \in C_2$. Analogously,

$$C^1 = C^1([-r, 0], \mathbb{R}) \quad \text{and} \quad C^1_2 = C^1([-r, 0], \mathbb{R}^2)$$

denote the Banach spaces of continuously differentiable functions on the initial interval $[-r, 0]$ with values in $\mathbb{R}$ and in $\mathbb{R}^2$, respectively. The norms are given by

$$\|\phi\|_{C^1} = \|\phi\|_C + \|\phi'\|_C, \|\psi\|_{C^1_2} = \|\phi\|_{C^1} + \|\eta\|_{C^1},$$

respectively, where $\psi = (\phi, \eta) \in C^1_2$ has first component $\phi \in C^1$ and second component $\eta \in C^1$. The space C^2_2 and its norm are defined accordingly. The vectorspace of functions $[-r, 0] \to \mathbb{R}$ is denoted by

$$\mathbb{R}^{[-r,0]}.$$

Later we shall set

$$r=\frac{4b}{c}.$$

Let us recall the convention in functional differential equations that for a map $x: I\to\mathbb{R}^n$ with $[t-r,t]\subset I$ for some $t\in\mathbb{R}$ the segment $x_t : [-r,0]\to\mathbb{R}^n$ is defined by

$$x_t(s)=x(t+s).$$

We need some basic facts about semiflows generated by delay differential equations

$$x'(t)=f(x_t)$$

with a functional $f:U\to\mathbb{R}^2$, $U\subset C_2^1$ open. Suppose
(S) f is continuously differentiable, each derivative $Df(\psi):C_2^1\to\mathbb{R}^2$, $\psi\in U$, extends to a linear map $D_e f(\psi):C_2\to\mathbb{R}^2$, and the map

$$U\times C_2\ni(\psi,\chi)\mapsto D_e f(\psi)\chi\in\mathbb{R}^2$$

is continuous.

Assume also that the set

$$X_f=\{\psi\in U:\psi'(0)=f(\psi)\}$$

is nonempty.

Then X_f is a C^1-submanifold of C_2^1 of codimension 2, and the continuously differentiable maximal solutions $x=x^{\psi}$, $x:[-r,t_{\psi})\to\mathbb{R}^2$, $0<t_{\psi}\le\infty$, of the initial value problems

$$x'(t)=f(x_t),\ x_0=\psi\in X_f,$$

constitute a continuous semiflow F_f on X_f, by

$$F_f(t,\psi)=x_t^{\psi}\text{ for }\psi\in X_f\text{ and }0\le t<t_{\psi}.$$

All solution operators $F_f(t,\cdot)$ on non-empty domains are continuously differentiable, with

$$D_2F_f(t,\psi):T_{\psi}X_f\to T_{F_f(t,\psi)}X_f$$

given by

$$D_2F_f(t,\psi)\chi=v_t^{\psi,\chi},$$

where $v = v^{\psi,\chi}$ is the continuously differentiable solution of the initial value problem

$$v'(t) = Df(F_f(t,\psi))v_t, \;\; v_0 = \chi \in T_\psi X_f$$

for the linear variational equation along the solution x^ψ.

The tangent spaces of the manifold X_f are given by

$$T_\psi X_f = \{\chi \in C_2^1 : \chi'(0) = Df(\psi)\chi\}.$$

For more details and proofs of the preceding statements, see Section 3 in [12] and the references given there, notably [21,22].

We return to the model (1.1–4), with signal speed $c > 0$ given and the acceleration function as specified in Section 2. Let $b > 0$ and $r \geq \frac{2b}{c}$. In order to rewrite the model in the abstract form above we set

$$U = \{(\phi,\eta) \in C_2^1 : 0 < \phi(t) < b \quad \text{and} \quad -\frac{c}{3} < \eta(t) \text{ on } [-r,0], \|\phi'\|_C < c\}.$$

Solution segments, or initial data $\psi = (\phi,\eta)$ in this open convex subset of C_2^1 correspond to motion which is bounded by b and slower than the signal speed c. The role of the condition on η will become apparent when in the next section a functional g is defined.

Notice that, due to the contraction mapping principle, each $\phi \in C^1$ with $0 \leq \phi(t) < b$ on $[-r,0]$ and $\|\phi'\|_C < c$ uniquely determines a solution $s = \sigma(\phi) \in [0,r)$ of the equation

$$c\,s = \phi(-s) + \phi(0).$$

The resulting map

$$\sigma : \{\phi \in C^1 : 0 \leq \phi(t) < b \text{ on } [-r,0] \text{ and } \|\phi'\|_C < c\} \to \mathbb{R}$$

has range in $[0,r)$, satisfies

$$\sigma(0) = 0,$$

is Lipschitz continuous and is, moreover, continuously differentiable on the open subset

$$\{\phi \in C^1 : 0 < \phi(t) < b \text{ on } [-r,0] \text{ and } \|\phi'\|_C < c\}$$

of the space C^1, as one can show by means of the Implicit Function Theorem. For details, see [21].

On the way to a definition of a functional $f : U \to \mathbb{R}^2$ corresponding to the equations (1.1–4) we can now use Equation (1.2) to replace the term p in Equation (1.4) by $\frac{c}{2}\sigma(u_t)$. With regard to the term p' in Equation (1.4) and in view of Eq. (1.2) we are interested in the derivatives of the functions

$$\sigma_u : (0, t_0) \ni t \mapsto \sigma(u_t) \in \mathbb{R},$$

for given continuously differentiable functions $u : [-r, t_0) \to \mathbb{R}$, $0 < t_0 \leq \infty$, which satisfy $0 \leq u(t) < b$ and $|u'(t)| < c$ everywhere. Applications of the Implicit Function Theorem to the equation

$$0 = u(t - s) + u(t) - c\,s$$

in neighborhoods of the solutions $(t_1, \sigma(u_{t_1}))$, $0 < t_1 < t_0$, yield that the function σ_u is continuously differentiable on $(0, t_0)$. Hence

$$0 = u'(t - \sigma_u(t))(1 - \sigma_u'(t)) + u'(t) - c\,\sigma_u'(t)$$

or

$$\sigma_u'(t) = \frac{u'(t - \sigma(u_t)) + u'(t)}{u'(t - \sigma(u_t)) + c} \tag{3.1}$$

on $(0, t_0)$. If $t - \sigma(u_t) \geq 0$ (which holds for $t \geq r$) and if there is a differentiable function $v : [-r, t_0) \to \mathbb{R}$ so that Equation (1.3) holds on $[0, t_0)$ then we have

$$\sigma_u'(t) = \frac{v(t - \sigma(u_t)) + v(t)}{v(t - \sigma(u_t)) + c}, \tag{3.2}$$

which in contrast to Equation (3.1) does not contain derivatives on the right hand side.

Guided by Eq. (1.1–1.4) and (3.2) we define $f : U \to \mathbb{R}^2$ by

$$f(\phi, \eta) = \left(\eta(0), a\left(\frac{c}{2}\sigma(\phi), \frac{c}{2}\frac{\eta(-\sigma(\phi)) + \eta(0)}{\eta(-\sigma(\phi)) + c} \right) \right). \tag{3.3}$$

Remark 3.1.

(i) The component η in the right hand side of the preceding definition (3.3), instead of the derivative ϕ' as suggested by Eq. (3.1), makes it possible to establish the smoothness property (S) for the functional f, see proof of Proposition 4.3.

(ii) Concerning the motion under control by echo the choice of η instead of ϕ' means that for initial data $\psi = (\phi, \eta)$, solutions $x = x^\psi = (u, v)$ and times $t \in [0, r)$ with $t - \sigma(u_t) < 0$ and $\eta(t - \sigma(u_t)) \neq \phi'(t - \sigma(u_t))$ the value of the acceleration a is not given by the pair (p, p') (computed position and its derivative, both at time t) but by a pair (p, p^*) where p^* may differ from p' and depends on $\eta(t-\sigma(u_t))$, i.e., on the second component of the initial state, whose physical role is not clear for $\eta(t-\sigma(u_t)) \neq \phi'(t-\sigma(u_t))$. The chosen functional f can be interpreted as a feedback control mechanism which is just a bit different from the one described in Section 1. Notice that along solutions $x=(u, v)$ of the initial value problem

$$x'(t)=f(x_t) \quad \text{for} \quad t>0, \quad x_0=(\phi, \eta)\in X_f$$

both mechanisms coincide at all segments x_t with $t \geq r$, due to $u_t' = v_t$. Along solutions starting from special initial data in X_f with $\phi' = \eta$ both mechanisms coincide for all $t \geq 0$.

(iii) The following scenario how the control mechanism may be started was suggested by the anonymous referee. Let $-T < -r$ and assume $u(-T)$ and $u'(-T)$ are known. Choose a smooth steering function $a_s : [-T, \infty) \to \mathbb{R}$. Then the equation

$$u''(t)=a_s(t)$$

and the initial data determine position and speed of the modeled object for $t \geq -T$. Assume that for $|t| \leq r$ we have

$$0<u(t)<b \quad \text{and} \quad -\frac{c}{3}<u'(t)<c.$$

Suppose that at $t=0$ the object begins to send signals. After reflection, the returning first signal is received at $t=t_0=\sigma(u_{t_0})$ given by the unique solution of the equation

$$c\, t_0 = u(0)+u(t_0),$$

we have $0<t_0<r$. The returning first signal triggers the automatic control mechanism: for $t \geq t_0$, the motion of the object is no longer given by $u''=a_s$, but by the solution of the initial value problem

$$x'(t)=f(x_t) \quad \text{for} \quad t \geq t_0, \quad x_{t_0}=(u_{t_0}, u'_{t_0}),$$

as long as the first component of x remains positive.

(iv) Another possibility to obtain a functional with property (S) from the original model (1.1–4) is to replace p' in Eq. (1.4) by a difference quotient, at the cost of an additional constant time lag. As we are mainly interested in the dynamical properties of position control by echo we prefer to avoid additional delays.

Proposition 3.2.

(i) $f(\psi) \to 0$ *as* $\psi \to 0$,
(ii) $X_f \neq \emptyset$.

Proof. (i) For $U \ni \psi = (\phi, \eta) \to 0$, $\sigma(\phi) \to \sigma(0) = 0$, by continuity. Combine this with the definition of f and $\eta \to 0$ to complete the proof.

(ii) Pick $\phi \in C^1$ with $0 < \phi(t) < b$ on $[-r, 0], \|\phi'\|_C < c$ and $-\frac{c}{3} < \phi'(0)$. Then $\sigma(\phi) > 0$. Consider all $\eta \in C^1$ with $\eta(0) = \phi'(0) \in (-\frac{c}{3}, c)$. Using $\sigma(\phi) > 0$ it is easy to find such a function η which in addition satisfies

$$\eta'(0) = a\left(\frac{c}{2}\sigma(\phi), \frac{c}{2}\frac{\eta(-\sigma(\phi)) + \eta(0)}{\eta(-\sigma(\phi)) + c}\right) = \frac{c}{2}\alpha\sigma(\phi) + \frac{c}{2}\beta\frac{\eta(-\sigma(\phi)) + \eta(0)}{\eta(-\sigma(\phi)) + c}$$

and $-\frac{c}{3} < \eta(t)$ on $[-r, 0]$. Then, $\psi = (\phi, \eta) \in U$ and $\psi'(0) = f(\psi)$, which means $\psi \in X_f$. □

Remark 3.3. The problem that values $\eta(t) \neq \phi'(t)$, $-r \leq t < 0$, for initial data $\psi = (\phi, \eta) \in X_f$ have no physical meaning can be avoided by restricting attention to initial data of the form

$$\psi = (\phi, J(\phi)) \in X_f$$

with a fixed map

$$J : \{\phi \in C^1 : 0 < \phi(t) < b \text{ on } [-r, 0],\ \|\phi'\|_C < c\} \to C^1.$$

Existence of such maps should be obvious from proof of Proposition 3.2 (ii).

4. EXTENSION

We need the vectorspace $\mathbb{R}^{[-r,r]}$ and the Banach spaces $C([-r, r], \mathbb{R})$ and $C^1([-r, r], \mathbb{R})$, which are analogous to the spaces $\mathbb{R}^{[-r,0]}$, C, C^1 in the preceding section, as well as the linear evaluation functional

$$ev_0 : \mathbb{R}^{[-r,0]} \ni \phi \mapsto \phi(0) \in \mathbb{R}$$

and the nonlinear evaluation map

$$ev : \mathbb{R}^{[-r,r]} \times [-r, r] \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R},$$

which has a continuously differentiable restriction to the open subset $C^1([-r,r],\mathbb{R}) \times (-r,r)$ of the Banach space $C^1([-r,r],\mathbb{R}) \times \mathbb{R}$, with partial derivatives given by

$$D_1 ev(\phi,t)\tilde{\phi} = \tilde{\phi}(t) \quad \text{and} \quad D_2 ev(\phi,t)1 = \phi'(t).$$

Notice also that the map

$$C([-r,r],\mathbb{R}) \times [-r,r] \stackrel{ev}{\to} \mathbb{R}$$

is continuous, due to the estimate

$$|\phi(t) - \phi_0(t_0)| \le |\phi(t) - \phi_0(t)| + |\phi_0(t) - \phi_0(t_0)| \le \|\phi - \phi_0\| + |\phi_0(t) - \phi_0(t_0)|.$$

In order to extend the functional f to a larger domain which contains $0 \in C_2^1$ we begin with the signal travel time functional σ. First, consider the linear operator

$$E : \mathbb{R}^{[-r,0]} \to \mathbb{R}^{[-r,r]}$$

of odd extension which is given by

$$(E\phi)(t) = 2\phi(0) - \phi(-t) \quad \text{for} \quad 0 < t \le r$$

and

$$(E\phi)(t) = \phi(t) \quad \text{for} \quad -r \le t \le 0.$$

The induced maps

$$C \stackrel{E}{\to} C([-r,r],\mathbb{R})$$

and

$$C^1 \stackrel{E}{\to} C^1([-r,r],\mathbb{R})$$

are continuous, both with norm equal to 3. Fix

$$r = \frac{4b}{c}$$

from now on. For any given $\phi \in C^1$ with $\|\phi\|_C < b$ and $\|\phi'\|_C < c$ the equation

$$c\,t = (E\phi)(-t) + \phi(0) \tag{4.1}$$

has a unique solution $t = \tau(\phi) \in (-r, r)$. This follows as in proof of Proposition 8 in [21] by means of the Contraction Mapping Principle since

$$|(E\phi)'(-t)| \le \|\phi'\|_C < c$$

and

$$|(E\phi)(-t) + \phi(0)| \le 4\|\phi\|_C < 4b.$$

We obtain a map

$$\tau : \{\phi \in C^1 : \|\phi\|_C < b \text{ and } \|\phi'\|_C < c\} \quad \to \quad (-r, r) \subset \mathbb{R},$$

which extends σ and is Lipschitz continuous.

Proposition 4.1.

(i) $\tau(\phi) = 0$ *if and only if* $\phi(0) = 0$, *in particular,* $\tau(0) = 0$.
(ii) τ *is continuously differentiable, with*

$$D\tau(\phi)\tilde{\phi} = \frac{(E\tilde{\phi})(-\tau(\phi)) + \tilde{\phi}(0)}{(E\phi)'(-\tau(\phi)) + c}.$$

Proof. The proof of the equivalence in assertion (i) is immediate from Eq. (4.1). To prove assertion (ii) we shall apply the Implicit Function Theorem to Eq. (4.1) in the form

$$0 = (E\phi)(-t) + \phi(0) - c\,t = G(\phi, t),$$

with

$$G : \{\phi \in C^1 : \|\phi\|_C < b \text{ and } \|\phi'\|_C < c\} \times (-r, r) \to \mathbb{R}$$

given by

$$ev \circ ((E \circ pr_1) \times (-\,\mathrm{id} \circ pr_2)) + ev_0 \circ pr_1 - c\, pr_2,$$

where pr_1 and pr_2 denote the projections onto the first and second component of elements in $\mathbb{R}^{[-r,0]} \times \mathbb{R}$, respectively. It is almost obvious that the map G is continuously differentiable. At each zero $(\phi_0, t_0) = (\phi_0, \tau(\phi_0))$ of G we have

$$\begin{aligned} D_2G(\phi_0, t_0)1 &= \lim_{h \searrow 0} \frac{1}{h}((E\phi_0)(-t_0 - h) - (E\phi_0)(-t_0)) - c \\ &= -(E\phi_0)'(-t_0) - c < 0, \end{aligned}$$

so $D_2G(\phi_0, t_0)$ is an isomorphism, and the Implicit Function Theorem yields that for ϕ close to ϕ_0 the unique solution $\tau(\phi)$ of Eq. (4.1) is given by a continuously differentiable function. The formula for the derivative follows from

$$\begin{aligned} 0 &= D_1G(\phi,\tau(\phi))\tilde{\phi} + D_2G(\phi,\tau(\phi))[D\tau(\phi)\tilde{\phi}] \\ &= D_1G(\phi,\tau(\phi))\tilde{\phi} + D\tau(\phi)\tilde{\phi}\cdot D_2G(\phi,\tau(\phi))1 \\ &= (E\tilde{\phi})(-\tau(\phi)) + ev_0(\tilde{\phi}) + D\tau(\phi)\tilde{\phi}[-(E\phi)'(-\tau(\phi)) - c]. \qquad \square \end{aligned}$$

Consider the linear extensions

$$D_e\tau(\phi): C\to\mathbb{R}, \quad \phi\in C^1 \quad \text{with} \quad \|\phi\|_C < b \quad \text{and} \quad \|\phi'\|_C < c,$$

given by

$$D_e\tau(\phi)\hat{\phi} = \frac{(E\hat{\phi})(-\tau(\phi)) + \hat{\phi}(0)}{(E\phi)'(-\tau(\phi)) + c}.$$

The next result shows that τ has the smoothness property (S).

Proposition 4.2. *The map*

$$\{\phi\in C^1 : \|\phi\|_C < b, \|\phi'\|_C < c\}\times C \ni (\phi,\hat{\phi}) \mapsto D_e\tau(\phi)\hat{\phi}\in\mathbb{R}$$

is continuous.

Proof. Use continuity of the maps $C([-r,r],\mathbb{R})\times(-r,r)\stackrel{ev}{\to}\mathbb{R}$, $C\stackrel{E}{\to} C([-r,r],\mathbb{R})$, τ: $C^1\stackrel{E}{\to} C^1([-r,r],\mathbb{R})$, and

$$C^1([-r,r],\mathbb{R})\ni\phi\mapsto\phi'\in C([-r,r],\mathbb{R}). \qquad \square$$

Now consider the convex open neighborhood

$$V = \{(\phi,\eta)\in C^1_2 : \|\phi\|_C < b, \|\phi'\|_C < c, -\frac{c}{3} < \eta(t) \quad \text{on} \quad [-r,0]\}$$

of $0\in C^1_2$. Notice that the condition

$$-\frac{c}{3} < \eta(t) \quad \text{on} \quad [-r,0]$$

guarantees $0\neq(E\eta)(-\tau(\phi)) + c$ for $(\phi,\eta)\in V$. The functional $g: V\to\mathbb{R}^2$ given by

$$g(\phi,\eta) = \left(\eta(0), a\left(\frac{c}{2}\tau(\phi), \frac{c}{2}\frac{(E\eta)(-\tau(\phi)) + \eta(0)}{(E\eta)(-\tau(\phi)) + c}\right)\right)$$

satisfies

$$g(\psi) = f(\psi) \quad \text{on} \quad U\subset V.$$

Obviously,

$$g(0)=0$$

and

$$0 \in X_g = \{\psi \in V : \psi'(0) = g(\psi)\}.$$

Incidentally, notice that for $\psi = (\phi, \eta) \in X_g$,

$$\phi'(0) = \eta(0) \in \left(-\frac{c}{3}, c\right).$$

Proposition 4.3. *The functionals g and f have property (S).*

Proof. 1. Consider g. We only show that the map

$$h : V \ni (\phi, \eta) \quad \mapsto \quad \frac{(E\eta)(-\tau(\phi)) + \eta(0)}{(E\eta)(-\tau(\phi)) + c} \in \mathbb{R}$$

has property (S), as the remaining parts of the proof that g has property (S) are almost obvious. Continuous differentiability of h is a consequence of the chain and quotient rules combined with the following facts: The map $C_2^1 \ni (\phi, \eta) \mapsto \eta(0) \in \mathbb{R}$ is linear and continuous, $C^1([-r, r], \mathbb{R}) \times (-r, r) \overset{ev}{\to} \mathbb{R}$ is continuously differentiable, $C^1 \overset{E}{\to} C^1([-r, r], \mathbb{R})$ is linear and continuous, τ is continuously differentiable. To compute the derivatives of h, consider first the map

$$k : V \to \mathbb{R}$$

given by

$$k(\phi, \eta) = (E\eta)(-\tau(\phi)) = (ev \circ ((E \circ \text{ pr}_2) \times (-\tau \circ \text{ pr}_1)))(\phi, \eta),$$

where pr_1 and pr_2 denote the projections onto the first and second component of elements in $V \subset C_2^1$, respectively. The map k is continuously differentiable, and for $(\phi, \eta) \in V$ and $(\tilde{\phi}, \tilde{\eta}) \in C_2^1$ we have

$$\begin{aligned} D_1 k(\phi, \eta)\tilde{\phi} &= -(E\eta)'(-\tau(\phi))(D\tau(\phi)\tilde{\phi}), \\ D_2 k(\phi, \eta)\tilde{\eta} &= (E\tilde{\eta})(-\tau(\phi)), \end{aligned}$$

hence

$$\begin{aligned} Dk(\phi, \eta)(\tilde{\phi}, \tilde{\eta}) &= D_1 k(\phi, \eta)\tilde{\phi} + D_2 k(\phi, \eta)\tilde{\eta} \\ &= -(E\eta)'(-\tau(\phi))(D\tau(\phi)\tilde{\phi}) + (E\tilde{\eta})(-\tau(\phi)). \end{aligned}$$

It follows that

$$Dh(\phi,\eta)(\tilde{\phi},\tilde{\eta})$$

$$= \frac{[Dk(\phi,\eta)(\tilde{\phi},\tilde{\eta})+\tilde{\eta}(0)][k(\phi,\eta)+c]-[k(\phi,\eta)+\eta(0)]Dk(\phi,\eta)(\tilde{\phi},\tilde{\eta})}{(k(\phi,\eta)+c)^2}$$
$$= [(E\eta)(-\tau(\phi))+c]^{-2}\{[(E\tilde{\eta})(-\tau(\phi))-(E\eta)'(-\tau(\phi))(D\tau(\phi)\tilde{\phi})+\tilde{\eta}(0)]$$
$$\times[(E\eta)(-\tau(\phi))+c]-[(E\eta)(-\tau(\phi))+\eta(0)]$$
$$\times[(E\tilde{\eta})(-\tau(\phi))-(E\eta)'(-\tau(\phi))(D\tau(\phi)\tilde{\phi})]\}.$$

For each $(\phi,\eta)\in V$ consider the linear extension

$$D_eh(\phi,\eta):C_2\to\mathbb{R}$$

given by

$$D_eh(\phi,\eta)(\hat{\phi},\hat{\eta})$$
$$=[(E\eta)(-\tau(\phi))+c]^{-2}\{[(E\hat{\eta})(-\tau(\phi))-(E\eta)'(-\tau(\phi))(D_e\tau(\phi)\hat{\phi})+\hat{\eta}(0)]$$
$$\times[(E\eta)(-\tau(\phi))+c]-[E\eta)(-\tau(\phi))+\eta(0)][(E\hat{\eta})(-\tau(\phi))$$
$$-(E\eta)'(-\tau(\phi))(D_e\tau(\phi)\hat{\phi})]\}$$

for $(\hat{\phi},\hat{\eta})\in C_2$. Continuity of the map

$$V\times C_2\ni(\phi,\eta,\hat{\phi},\hat{\eta})\mapsto D_eh(\phi,\eta)(\hat{\phi},\hat{\eta})\in\mathbb{R}$$

is a consequence of Proposition 4.2 combined with continuity of the maps

$$C\overset{ev_0}{\to}\mathbb{R}\quad\text{and}\quad C^1\overset{ev_0}{\to}\mathbb{R},k,$$

$$\{\phi\in C^1:\|\phi\|_C<b,\|\phi'\|_C<c\}\times C\ni(\phi,\hat{\eta})\mapsto(E\hat{\eta})(-\tau(\phi))\in\mathbb{R}$$

and

$$V\ni(\phi,\eta)\mapsto(E\eta)'(-\tau(\phi))\in\mathbb{R}.$$

Here continuity of the preceding two maps follows by means of continuity of the evaluation map

$$C([-r,r],\mathbb{R})\times(-r,r)\overset{ev}{\to}\mathbb{R}.$$

2. Property (S) of f follows from property (S) of g and $f=g|U$. □

We obtain that the set

$$X=X_g=\{\psi\in V:\psi'(0)=g(\psi)\}$$

is a continuously differentiable submanifold of the space C_2^1, with codimension 2, and $0 \in X$. Let $F = F_g$ denote the semiflow on X generated by the maximal continuously differentiable solutions $x^{\psi} : [-r, t_{\psi}) \to \mathbb{R}^2$ of the equation

$$x'(t) = g(x_t), \tag{4.2}$$

which start at data $x_0 = \psi \in X$.

Also,

$$X_f = \{\psi \in U : \psi'(0) = f(\psi)\} = X \cap U$$

is a continuously differentiable submanifold of C_2^1, with codimension 2.

5. LINEARIZATION

This section deals with the tangent space

$$T_0 X = \{\psi \in C_2^1 : \psi'(0) = Dg(0)\psi\}$$

of X at the stationary point $0 \in X$ of the semiflow F, and with spectral properties of the derivative $D_2 F(r, 0) : T_0 X \to T_0 X$. In the next section these spectral properties will be used to find a locally positively invariant manifold of the semiflow which helps to separate soft landing from collisions.

Using the map h from the proof of Proposition 4.3, we read off the derivative

$$\begin{aligned} Dg(0,0)(\phi,\eta) &= \left(\eta(0), \alpha\frac{c}{2}D\tau(0)\phi + \beta\frac{c}{2}Dh(0,0)(\phi,\eta)\right) \\ &= \left(\eta(0), \alpha\frac{c}{2}2\phi(0)\frac{1}{c} + \beta\frac{c}{2}\frac{2\eta(0)c}{c^2}\right) \\ &= (\eta(0), \alpha\phi(0) + \beta\eta(0)). \end{aligned}$$

Hence

$$T_0 X = \left\{\psi \in C_2^1 : \psi'(0) = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \psi(0)\right\}.$$

Let $L \subset C_2^1$ denote the two-dimensional subspace of all linear data ψ,

$$\psi(t) = t(a, b)$$

with $(a, b) \in \mathbb{R}^2$.

Proposition 5.1.

$$C_2^1 = T_0X \oplus L$$

and the projection $P : C_2^1 \to C_2^1$ *along* L *onto* T_0X *is given by*

$$(P\psi)(t) = \psi(t) - t\left(\psi'(0) - \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}\psi(0)\right).$$

If $P\psi = \hat{\psi}$ *then* $\hat{\psi}(0) = \psi(0)$.

Proof. 1. Let $\psi \in T_0X \cap L$. By $\psi \in L$, $\psi(0) = 0$. By the equation defining T_0X, $\psi'(0) = 0$. By linearity, $\psi = 0$. Hence $T_0X \cap L = \{0\}$, and $\dim L = 2 = \operatorname{codim} T_0X$ gives the asserted direct sum decomposition.

2. For $\psi \in C_2^1$, define $\psi^L \in L$ by

$$\psi^L(t) = t\left(\psi'(0) - \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}\psi(0)\right).$$

Then

$$(\psi - \psi^L)'(0) - \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}(\psi - \psi^L)(0)$$
$$= \psi'(0) - \left(\psi'(0) - \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}\psi(0)\right) - \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}(\psi(0) - 0) = 0,$$

hence $\psi - \psi^L \in T_0X$. It follows that $P\psi = \psi - \psi^L$, which shows the assertion about P. □

Remark 5.2. *It can be shown that for all* $\psi \in X$, $C_2^1 = T_\psi X \oplus L$, *but we shall not need this in the sequel.*

Later we shall use that P defines a manifold chart at $0 \in X$, i.e., a diffeomorphism from an open neighbourhood N of 0 in X onto an open neighborhood of 0 in the subspace T_0X of C_2^1.

Before turning to properties of the derivative $D_2F(r,0)$ it is convenient to have a look at the closed subspace

$$Z = \{\psi \in C_2^1 : \psi(0) = 0\}$$

of codimension 2 and the closed subspace

$$Q = T_0X \cap Z$$

of T_0X.

Proposition 5.3.

$$PZ = Q \quad \textit{and} \quad Z = Q \oplus L.$$

Proof. The first equation follows from $Q \subset T_0X = PC^1_2$ in combination with the last assertion of Proposition 5.1. We have $Q \cup L \subset Z$ and $Q \cap L \subset T_0X \cap L = \{0\}$. In order to have also $Z \subset Q + L$, notice that for $\psi \in Z$,

$$\psi = P\psi + (\mathrm{id} - P)\psi, \quad \text{with} \quad P\psi \in PZ = Q, \quad (\mathrm{id} - P)\psi \in L. \qquad \square$$

The linear variational equation along the zero solution, namely

$$y'(t) = Dg(0)y_t$$

or equivalently, with $y = (w, z)$, the system

$$w'(t) = z(t), \tag{5.1}$$

$$z'(t) = \alpha\, w(t) + \beta\, z(t), \tag{5.2}$$

shows that the space $Q \subset T_0X$ is positively invariant under the operators $D_2F(t, 0)$, $t \geq 0$, as for initial data in Q trivial continuation

$$y(t) = 0 \quad \text{for all} \quad t \geq 0,$$

yields a continuously differentiable solution $y : [-r, \infty) \to \mathbb{R}^2$.

Remark 5.4. *In terminology of linear retarded functional differential equations with state space C_2, solutions which decay to zero for $t \to \infty$ faster than any exponential are called small solutions; solutions of the present variational equation with segments $y_t \in Q$ are small solutions.*

Recall that in Section 2 we made the assumption that (2.4) holds. For $j \in \{1, 2\}$ define $\psi_j \in C^1_2$ by

$$\psi_j(t) = e^{\lambda_j t}(1, \lambda_j)$$

and set

$$G_j = \mathbb{R}\psi_j \subset C^1_2.$$

From here on, the reader may find it convenient to draw schematic figures of subspaces and invariant manifolds (Sections 6, and 7) contained in T_0X.

Proposition 5.5. *For $j \in \{1, 2\}$ we have*

$$\psi_j \in T_0X, \; D_2F(r, 0)\psi_j = e^{\lambda_j r}\psi_j, \; \ker D_2F(r, 0) = Q,$$

and

$$T_0X = Q \oplus G_1 \oplus G_2.$$

Proof. 1. The results of Section 2 in combination with the explicit characterization of T_0X and the fact that the linear variational equation at the zero solution reduces to the ODE (5.1 and 5.2) yield the first two parts of the assertion. Recall that for initial data $\psi \in Q$, $D_2F(t,0)\psi = 0$ for all $t \geq r$. In particular, $Q \subset \ker D_2F(r,0)$. The reversed inclusion is obvious since $D_2F(r,0)\psi = 0$ implies ($\psi \in T_0X$ and) $\psi(0) = (D_2F(r,0)\psi)(-r) = 0$.

2. Proof of the decomposition. The assertion follows easily from the inclusion

$$T_0X \subset Q \oplus (\mathbb{R}\psi_1 + \mathbb{R}\psi_2)$$

in combination with the equation

$$Q \cap (\mathbb{R}\psi_1 + \mathbb{R}\psi_2) = \{0\}.$$

To obtain the preceding inclusion, notice that for $\psi \in T_0X$ given the system

$$c_1\psi_1(0)^{tr} + c_2\psi_2(0)^{tr} = \psi(0)^{tr}$$

has a solution $(c_1, c_2)^{tr} \in \mathbb{R}^2$ since

$$\det(\psi_1(0)^{tr} \;\; \psi_2(0)^{tr}) = \det\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \lambda_2 - \lambda_1 \neq 0.$$

Then

$$\psi = (\psi - c_1\psi_1 - c_2\psi_2) + c_1\psi_1 + c_2\psi_2$$

with the term in brackets in Q, and the inclusion follows.

Next, suppose $\psi \in Q \cap (\mathbb{R}\psi_1 + \mathbb{R}\psi_2)$. Then $\psi = c_1\psi_1 + c_2\psi_2$ with reals c_1, c_2, and

$$0 = \psi(0)^{tr} = c_1\psi_1(0)^{tr} + c_2\psi_2(0)^{tr} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix};$$

as the determinant of the matrix does not vanish, we get $c_1 = 0 = c_2$, and thereby $\psi = 0$. □

In order to describe the desired open domain of soft landing we shall use a somewhat different decomposition of the tangent space. This is prepared in the final result of this section.

Proposition 5.6. *There exists* $\psi^* = (\phi^*, \eta^*) \in T_0X$ *with* $\phi^*(0) = 0 < \eta^*(0)$, $\|\psi^*\|_{C_2^1} = 1$, *and*

$$T_0X = Q \oplus G_1 \oplus \mathbb{R}\psi^*.$$

Proof. For $\psi=(\phi,\eta)\in C_2^1$ with $\phi(0)=0<\eta(0)$, $\psi^*=(\phi^*,\eta^*)=P\psi\in T_0X$ satisfies $\phi^*(0)=0<\eta^*(0)$, due to the last assertion of Proposition 5.1. We may assume $\|\psi^*\|_{C_2^1}=1$. It remains to show that $\psi^*\notin Q\oplus G_1$. Suppose this is false. Then

$$\psi^*=\psi^Q+c\psi_1 \quad \text{with} \quad \psi^Q\in Q,\ c\in\mathbb{R},$$

$$\psi^*(0)=0+c\psi_1(0)=c\begin{pmatrix}1\\ \lambda_1\end{pmatrix}.$$

Either $c=0$ and $\eta^*(0)=0$, or $c\neq 0$ and $\phi^*(0)\neq 0$, which both contradict the properties of ψ^*. □

6. THE FAST MANIFOLD AND APPROACH FROM ABOVE

In this section we find solutions of Eq.(4.2) with the extended functional g which have soft landing properties similar to (1.5). We need a continuously differentiable and locally positively invariant manifold $W\subset N\subset X$ of the semiflow with tangent space

$$T_0W=Q\oplus G_1.$$

Such a manifold can be constructed following the procedure in Section 3.5 of [12], beginning with the invariant decomposition in Proposition 5.5 and an estimate

$$\|D_2F(r,0)^j\psi\|_{C_2^1}\le c\rho^j\|\psi\|_{C_2^1}$$

for all $\psi\in Q\oplus G_1$ and all $j\in\mathbb{N}$, with constants $\rho\in(e^{\lambda_1 r},e^{\lambda_2,r})\subset(0,1)$ and $c>0$. The next step is an analog of Proposition 3.5.1 of [12], now about a locally positively invariant manifold $W_<\subset T_0X$ with $T_0W_<=Q\oplus G_1$ for the map $F(r,\cdot)$ in the local coordinates which are given by the projection P. Infinitesimal generators which occur at the beginning of Section 3.5 in [12] are not convenient for the present purpose, and can be avoided.

The *fast manifold* W which we obtain as just indicated has the property that for each $\psi\in W$, $t_\psi=\infty$ and the flowline $F(\cdot,\psi)$ decays to 0 for $t\to\infty$ faster than the exponential $t\mapsto e^{\lambda_2 t}$. For ψ in some neighborhood of 0 in W, $F(t,\psi)\in W$ for all $t\ge 0$. Also, for each $\lambda\in(\lambda_1,\lambda_2)$ given there is a constant $c_\lambda>0$ so that data $\psi\in X$ with $t_\psi=\infty$ and

$$\|F(t,\psi)\|_{C_2^1}e^{-\lambda t}\le c_\lambda \text{ on } [0,\infty) \tag{6.1}$$

belong to W. Fix some $\lambda\in(\lambda_1,\lambda_2)$.

The spectral properties of $D_2F(r,0)$ also imply that the stationary point $0 \in X$ is stable and exponentially attracting. This can be shown by a procedure which is simpler than the construction of the manifold W as sketched above, or by an application of the Principle of Linearized Stability from Section 3.6 in [12].

In addition we need to consider solutions x^{ψ} which start from data $\psi \in Z \cap X$. The next proposition establishes among others that the segments of such solutions merge into the stationary point at $t=r$ (they are special small solutions of the nonlinear equation (4.2)), and that a neighborhood of 0 in $Z \cap X$ is contained in the manifold W.

Using the result mentioned in Remark 5.2 one can also show that the inclusion map $Z \hookrightarrow C^1_2$ intersects X transversally, which implies that $Z \cap X$ is a continuously differentiable submanifold of Z and of C^1_2. As this will not be employed in the sequel, we omit the proof here.

Proposition 6.1. *For $\psi \in Z \cap X$, $t_{\psi} = \infty$, and for all $t \geq 0$,*

$$x^{\psi}(t) = 0, \quad x^{\psi}_t \in Z \cap X, \quad \textit{and} \quad \|x^{\psi}_t\|_{C^1_2} \leq \|\psi\|_{C^1_2}.$$

There exists an open neighborhood $N_1 \subset N$ of 0 in X so that $Z \cap N_1 \subset W$ and

$$Q \cap PN_1 = P(Z \cap N_1) \subset PW.$$

Proof. 1. For any $\psi = (\phi, \eta) \in Z \cap V$, $\phi(0) = 0$, and thereby $\tau(\phi) = 0$. Moreover, $g(\psi) = 0$. Now let $\psi \in Z \cap X$ be given. Then $\psi'(0) = g(\psi) = 0$. It follows that $x_0 = \psi$ and $x(t) = 0 \in \mathbb{R}^2$ for all $t > 0$ define a continuously differentiable function $x : [-r, \infty) \to \mathbb{R}^2$, $x = (u, v)$, with

$$\|u_t\|_C < b, \quad \|u'_t\|_C < c, \quad -\frac{c}{3} < v_t(s) \quad \text{on} \quad [-r, 0]$$

for all $t \geq 0$. Consequently, $x_t \in V \cap Z$ and $x'(t) = 0 = g(x_t)$ for all $t \geq 0$. Now the assertion on x^{ψ} becomes obvious.

2. The first statement of the proposition shows that for initial data $\psi \in Z \cap X$ with $\|\psi\|_{C^1_2}$ sufficiently small the estimate (6.1) holds. This

implies $Z \cap N_1 \subset W$ for some open neighborhood $N_1 \subset N$ of 0 in X. Moreover,

$$Q \cap PN_1 = (Z \cap T_0 X) \cap PN_1 = PZ \cap PN_1 = P(Z \cap N_1) \subset PW. \qquad \square$$

In order to define sets above and below W we use the manifold chart $N \xrightarrow{P} PN \subset T_0 X$ and consider the submanifold $PW \subset T_0 X$ with tangent space

$$T_0 PW = DP(0) T_0 W = P(T_0 W) = P(Q \oplus G_1) = Q \oplus G_1.$$

For $\delta > 0$ let Q_δ and $G_{1,\delta}$ denote the open balls of radius δ and center 0 in the subspaces $Q \subset T_0 X \subset C^1_2$ and $G_1 \subset T_0 X \subset C^1_2$, respectively. There exist $\delta > 0$ and $\epsilon > 0$ so that the open box

$$B = Q_\delta + G_{1,\delta} + (-2\epsilon, 2\epsilon)\psi^* \subset T_0 X$$

has the following properties: $B \subset PN$, $F([0,\infty) \times \{\psi\}) \subset W$ for all $\psi \in W$ with $P\psi \in B$, and

$$PW \cap B = \{\psi + w(\psi) \in T_0 X : \psi \in Q_\delta + G_{1,\delta}\}$$

for a continuously differentiable map

$$w : Q_\delta + G_{1,\delta} \to (-\epsilon, \epsilon)\psi^*$$

which satisfies $w(0) = 0$ and $Dw(0) = 0$. Due to the last assertion of Proposition 6.1 we may assume $Q_\delta \subset PW$, and obtain

$$w(\psi) = 0 \quad \text{for} \quad \psi \in Q_\delta. \tag{6.2}$$

The complement $B \setminus PW$ is the union of two nonempty disjoint open subsets $w_<$ and $w_>$, with $-\epsilon\psi^* \in w_<$ and $\epsilon\psi^* \in w_>$, which are both connected (due to the bound for w which is half the radius of the $\mathbb{R}\psi^*$-component of B). Using (6.2) and connectedness we infer

$$Q_\delta + (-2\epsilon, 0)\psi^* \subset w_< \text{ and } Q_\delta + (0, 2\epsilon)\psi^* \subset w_>.$$

The subset

$$w_{>>} = \{\psi = (\phi, \eta) \in w_> : \phi(0) > 0\} \subset B \subset T_0 X$$

is open as $C^1_2 \ni (\phi, \eta) \mapsto \phi(0) \in \mathbb{R}$ is continuous. Let $N_{>>}$ denote the open subset of $N \subset X$ given by $PN_{>>} = w_{>>}$.

By asymptotic stability, there exists an open neighborhood $N_0 \subset N_1$ of 0 in X so that for $\psi \in N_0$, $t_\psi = \infty$ and $F(t, \psi) \in N_1$ for all $t \geq 0$, and

$F(t,\psi)\to 0$ as $t\to\infty$. We can also achieve $PF(t,\psi)\in B\subset T_0X$ for all $t\geq 0$ and $\psi\in N_0$.

The open set

$$N_0\cap N_{>>}\subset N\subset X$$

is nonempty. This follows easily from the fact that for every $\tau\in(0,1]$ there exists $\rho_\tau>0$ with $\tau\epsilon\psi^*+\rho\psi_1\in w_{>>}$ for all $\rho\in(0,\rho_\tau)$.

Proposition 6.2. *(Soft landing properties for initial data above the fast manifold). For $\psi\in N_0\cap N_{>>}$ the solution $x=x^\psi=(u,v)$ satisfies $u(0)>0$, $u(t)\geq 0$ for all $t\geq 0$ and $x(t)\to 0$ as $t\to\infty$.*

Proof. Let $\psi\in N_0\cap N_{>>}$, $x=x^\psi=(u,v)$, $y_t=Px_t$ for all $t\geq 0$. Then $x(t)\to 0$ as $t\to\infty$ and $y_t\in B$ for all $t\geq 0$, and $y_0\in w_{>>}$. Proposition 5.1 yields $(P\psi)(0)=\psi(0)=(\phi(0),\eta(0))$, and we infer $u(0)=\phi(0)>0$. Asssume $u(t_0)<0$ for some $t_0>0$. Then there is a smallest $t_1\in(0,t_0)$ with $u(t_1)=0$; on $[0,t_1)$, $u(t)>0$.

Case I: $v(t_1)>0$. Then $u'(t_1)=v(t_1)>0$, hence $u(t)<0$ for some $t\in(0,t_1)$, and we have a contradiction.

Case II: $v(t_1)=0$. Then $F(t_1,\psi)=x_{t_1}\in Z\cap N_1$. By Proposition 6.1, $x_t\in Z$ for all $t\geq t_1$ which contradicts $u(t_0)<0$.

Case III: $v(t_1)<0$. Then $\hat{\psi}=(\hat{\phi},\hat{\eta})=y_{t_1}=Px_{t_1}$ satisfies $\hat{\phi}(0)=u(t_1)=0>v(t_1)=\hat{\eta(0)}$, according to Proposition 5.1. We have

$$\hat{\psi}-\frac{\hat{\eta}(0)}{\eta^*(0)}\psi^*\in Z\cap T_0X=Q,$$

hence

$$\hat{\psi}\in Q+\frac{\hat{\eta}(0)}{\eta^*(0)}\psi^*$$

and

$$\frac{\hat{\eta}(0)}{\eta^*(0)}<0.$$

Using this and $\hat{\psi}=y_{t_1}\in B$ we get

$$y_{t_1}=\hat{\psi}\in Q_\delta+(-2\epsilon,0)\psi^*\subset w_<.$$

As $y_0\in w_>$, continuity yields that for some $t_2\in(0,t_1)$,

$$y_{t_2}\in PW\cap B.$$

By the choice of B, $x_t\in W$ for all $t\geq t_2$. It follows that $y_{t_1}=Px_{t_1}\in PW\cap B$, which contradicts $y_{t_1}\in w_<$. □

7. INITIAL DESCENT AND SOFT LANDING

In this section, it is first verified that the set $N_0 \cap N_{>>} \subset X$ contains data $\psi = (\phi, \eta)$ which satisfy

$$\phi(t) > 0 > \phi'(t) \quad \text{and} \quad 0 > \eta(t) \quad \text{for} \quad -r \leq t \leq 0. \tag{7.1}$$

The inequalities for ϕ and ϕ' describe sinking motion above ground during the whole initial interval. Then we shall deduce from Proposition 6.2 that for such initial data the solutions of the model equation

$$x'(t) = f(x_t)$$

with $f : U \to \mathbb{R}^2$ defined by (3.3) have the soft landing properties (1.5).

Choose $s_0 > 0$ so that for $|s| < s_0$, $s\,\psi_2 \in B \cap PN_0$. For each $s \in (-s_0, s_0)$ define $c(s) \in N_0 \subset N \subset X$ by $Pc(s) = s\psi_2$.

Proposition 7.1. *There exists $s_1 \in (0, s_0)$ so that for each $s \in (0, s_1) c(s) = (\phi, \eta)$ satisfies (7.1).*

Proof. The continuously differentiable curve $c : (-s_0, s_0) \ni s \mapsto c(s) \in C^1_2$ with trace in $N \subset X$ satisfies $c(0) = 0$ and $c'(0) = Dc(0)1 \in T_0X$. It follows that

$$\psi_2 = (P \circ c)'(0) = Pc'(0) = c'(0).$$

Consequently, for $\epsilon_1 > 0$ with $0 < \epsilon_1 < \min \phi_2$ and $\max \phi_2' = \max \eta_2 < -\epsilon_1 < 0$ there exists $s_1 \in (0, s_0)$ so that for $0 < s < s_1$ and for all $t \in [-r, 0]$ the point $c(s) = (\phi, \eta)$ satisfies

$$\begin{aligned} \epsilon_1 |s| &\geq \|c(s) - c(0) - s\,c'(0)\| \\ &= \|c(s) - s\psi_2\|_{C^1_2} \\ &\geq |\phi(t) - s\,\phi_2(t)| + |\phi'(t) - s\,\phi_2'(t)| + |\eta(t) - s\,\eta_2(t)|. \end{aligned}$$

Dividing by s we infer from the choice of ϵ_1 and from the properties of ψ_2 that $\phi(t) > 0 > \phi'(t)$ and $0 > \eta(t)$. □

Let $Pr : T_0X \to T_0X$ be the projection along $\mathbb{R}\psi^*$ onto $Q \oplus G_1$.

Proposition 7.2.

$$(\mathrm{id} - Pr)\psi_2 = \frac{\lambda_2 - \lambda_1}{\eta^*(0)}\psi^*,$$

$$\frac{\lambda_2 - \lambda_1}{\eta^*(0)} > 0 \quad \textit{and} \quad Pr\,\psi_2 \neq 0.$$

Proof. For some $\psi^Q \in Q$ and reals a, a^*,

$$\psi_2 = \psi^Q + a\,\psi_1 + a^*\,\psi^*.$$

Using $\phi_2(0) = 1$, $\psi^Q = (\phi^Q, \eta^Q)$, and $\phi^Q(0) = 0$, $\phi_1(0) = 1$, $\phi^*(0) = 0$, we infer $a = 1$. Also, $\eta_2(0) = \lambda_2$, $\eta^Q(0) = 0$, $\eta_1(0) = \lambda_1$, $\eta^*(0) > 0$. It follows that $\lambda_2 = 0 + 1 \cdot \lambda_1 + a^*\eta^*(0)$, or

$$a^* = \frac{\lambda_2 - \lambda_1}{\eta^*(0)} > 0.$$

We have $Pr\,\psi_2 \neq 0$ since otherwise $\psi_2 \in \mathbb{R}\psi^*$, hence $\phi_2(0) = \phi^*(0) = 0$, contradicting $\phi_2(0) = 1$. □

Proposition 7.3. *There exists $s_2 \in (0, s_1)$ so that for each $s \in (0, s_2)$ and for all $t \in [0, 1]$,*

$$t(\mathrm{id} - Pr)s\,\psi_2 + (1-t)s\,\psi_2 \in B \setminus PW.$$

Proof. 1. The preceding proposition shows that the slope

$$sl(\psi_2) = \frac{\|(\mathrm{id} - \mathrm{Pr})\psi_2\|_{C_2^1}}{\|\mathrm{Pr}\,\psi_2\|_{C_2^1}}$$

is positive. Using $w(0) = 0$ and $Dw(0) = 0$ we find $\delta_1 \in (0, \delta)$ so that for each $\psi \in Q_{\delta_1} + G_{1,\delta_1}$ with $\psi \neq 0$ we have

$$sl(\psi + w(\psi)) = \frac{\|(\mathrm{id} - Pr)(\psi + w(\psi))\|_{C_2^1}}{\|Pr(\psi + w(\psi))\|_{C_2^1}} = \frac{\|w(\psi)\|_{C_2^1}}{\|\psi\|_{C_2^1}} < sl(\psi_2).$$

Hence $\psi + w(\psi) \notin \mathbb{R}\psi_2$ for such ψ. Choose $s_2 \in (0, s_1)$ so that $Pr(s\,\psi_2) \in Q_{\delta_1} + G_{1,\delta_1}$ for $|s| < s_2$. Let $s \in (0, s_2)$ be given.

2. Proof of $t(\mathrm{id} - Pr)s\,\psi_2 + (1-t)s\,\psi_2 \in B$ for $0 \leq t \leq 1$. As $s\psi_2 \in B$, $s\,\psi_2 = \psi^Q + a\,\psi_1 + a^*\psi^*$ with $\|\psi^Q\|_{C_2^1} < \delta$, $\|a\psi_1\|_{C_2^1} < \delta$ and $|a^*| < 2\epsilon$. It follows that $(\mathrm{id} - Pr)s\,\psi_2 = a^*\psi^* \in B$. Convexity of B implies the assertion.
3. Proof of $t(\mathrm{id} - Pr)s\,\psi_2 + (1-t)s\,\psi_2 \notin PW$.

3.1. The case $t < 1$. Then

$$Pr(t(\mathrm{id} - Pr)s\,\psi_2 + (1-t)s\,\psi_2) = (1-t)Pr(s\,\psi_2) \neq 0,$$

$$
\begin{aligned}
sl(t(\mathrm{id}-Pr)s\,\psi_2+(1-t)s\,\psi_2) &= \frac{\|(\mathrm{id}-Pr)(t(\mathrm{id}-Pr)s\,\psi_2+(1-t)s\,\psi_2)\|_{C^1_2}}{\|Pr(t(\mathrm{id}-Pr)s\,\psi_2+(1-t)s\,\psi_2)\|_{C^1_2}}\\
&= \frac{\|t(\mathrm{id}-Pr)s\,\psi_2+(1-t)(\mathrm{id}-Pr)(s\,\psi_2)\|_{C^1_2}}{\|(1-t)Pr(s\,\psi_2)\|_{C^1_2}}\\
&= \frac{1}{1-t}\frac{\|(\mathrm{id}-Pr)(s\,\psi_2)\|_{C^1_2}}{\|Pr(s\,\psi_2)\|_{C^1_2}}\\
&= \frac{1}{1-t}sl(s\,\psi_2)=\frac{1}{1-t}sl(\psi_2)>sl(\psi_2)
\end{aligned}
$$

and

$$
\begin{aligned}
Pr(t(\mathrm{id}-Pr)s\,\psi_2+(1-t)s\,\psi_2) = (1-t)Pr(s\,\psi_2)\in(1-t)(Q_{\delta_1}\\
+G_{1,\delta_1})\subset Q_{\delta_1}+G_{1,\delta_1}.
\end{aligned}
$$

By the choice of δ_1, $sl(\psi)<sl(\psi_2)$ for all $\psi\in(B\cap PW)\setminus\{0\}$ with $Pr\,\psi\in Q_{\delta_1}+G_{1,\delta_1}$. The preceding inequalities and the result of part 2 combined yield the assertion.

3.2 The case $t=1$. Then

$$
\begin{aligned}
t(\mathrm{id}-Pr)s\,\psi_2+(1-t)s\,\psi_2 &= (\mathrm{id}-Pr)(s\,\psi_2)\\
&= s(\mathrm{id}-Pr)\psi_2=s\frac{\lambda_2-\lambda_1}{\eta^*(0)}\psi^*\in\mathbb{R}\psi^*\setminus\{0\}.
\end{aligned}
$$

By $w(0)=0$, $(B\cap PW)\cap\mathbb{R}\psi^*=\{0\}$. Part 2 combined with the preceding relations yields the assertion.

Corollary 7.4. *For* $0<s<s_2$, $c(s)=(\phi,\eta)$ *satisfies* (7.1) *and belongs to* $N_0\cap N_{>>}$.

Proof. Let $s\in(0,s_2)$ be given. Recall $c(s)\in N_0$. By Proposition 7.1, $c(s)=(\phi,\eta)$ satisfies (7.1). By Proposition 7.3, $s\,\psi_2$ and $(\mathrm{id}-Pr)s\,\psi_2\in\mathbb{R}\psi^*$ belong to the same connected component of $B\setminus PW$. Since

$$
(\mathrm{id}-Pr)s\psi_2=s(\mathrm{id}-Pr)\psi_2=s\frac{\lambda_2-\lambda_1}{\eta^*(0)}\psi^*
$$

and

$$
s\frac{\lambda_2-\lambda_1}{\eta^*(0)}>0
$$

we have $(\mathrm{id}-Pr)s\,\psi_2\in w_>$. It follows that $s\,\psi_2\in w_>$. Moreover, $s\,\psi_2=s(\phi_2,\eta_2)$ and $\phi_2(0)=1>0$, and we obtain $s\,\psi_2\in w_{\gg}$. Hence $c(s)\in N_{\gg}$.

□

Consider $f = g|U$. Recall that for $\psi = (\phi, \eta) \in U$,

$$\phi(t) > 0 \quad \text{for all} \quad t \in [-r, 0],$$

and

$$X_f = X \cap U.$$

Let

$$\Sigma = \{\psi \in N_0 \cap N_{>>} : (7.1) \quad \text{holds}\}.$$

Corollary 7.5. *(Open domain of soft landing) The set Σ is a nonempty open subset of the manifold $X_f \subset U$, and the maximal continuously differentiable solutions $x : [-r, t_e) \to \mathbb{R}^2$, $x = (u, v)$, of the initial value problems*

$$x'(t) = f(x_t), \; x_0 = \psi \in \Sigma, \tag{7..2}$$

satisfy $u(t) > 0$ on $[-r, t_e)$ and

$$\lim_{t \nearrow t_e} (u(t), v(t)) = (0, 0).$$

Proof. 1. By Corollary 7.4, $\Sigma \neq \emptyset$. Proof that Σ is an open subset of X_f: Σ is an open subset of the manifold X. We have $\Sigma \subset X$ and

$$\Sigma \subset \{(\phi, \eta) \in V : 0 < \phi(t) \text{ on } [-r, 0]\} = U,$$

hence

$$\Sigma \subset X \cap U = X_f.$$

This implies the assertion.

2. Let $\psi \in \Sigma$. Proposition 6.2 says that the maximal continuously differentiable solution $x = (u, v)$ of the initial value problem

$$x'(t) = g(x_t), \quad x_0 = \psi,$$

satisfies $u(0) > 0$, $u(t) \geq 0$ on $[0, \infty)$, and $\lim_{t \to \infty} x(t) = 0$. In case $u(t) > 0$ on $[0, \infty)$ we infer from (7.1) that $u(t) > 0$ on $[-r, \infty)$. Thereby, $x_t \in U$ for all $t \geq 0$. Consequently, $x'(t) = g(x_t) = f(x_t)$ for all $t \geq 0$, and x is also a maximal continuously differentiable solution of the initial value problem (7.2). In the remaining case there exists $t_0 > 0$ with $u(t) > 0$ on $[0, t_0)$ and $u(t_0) = 0$. By (7.1), $u(t) > 0$ on $[-r, t_0)$. As u is nonnegative, $u'(t_0) = 0$, and Equation (4.2) yields

$$v(t_0) = u'(t_0) = 0.$$

We obtain $x_t \in U$ on $[0, t_0)$ and $\lim_{t \nearrow t_0}(u(t), v(t)) = (0, 0)$. As $u(t_0) = 0$, $x_{t_0} \notin U$. It follows that $x|[-r, t_0)$ is the maximal continuously differentiable solution of the initial value problem (7.2). □

Remark 7.6. The soft landing model is given by the maximal continuously differentiable solutions of the initial value problem (7.2), all of which have strictly positive first components. If we enlarge the model a bit and include all continuously differentiable solutions $x = (u, v)$ of the initial value problems

$$x'(t) = g(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \psi = (\phi, \eta) \in X, \quad \phi(t) \geq 0 \quad \text{on} \quad [-r, 0]$$

as long as u remains nonnegative then all solutions which correspond to soft landing in finite time, i.e., with

$$u(t_0) = 0 = v(t_0) \quad \text{and} \quad u(t) > 0 \quad \text{on} \quad [-r, t_0) \quad \text{for some} \quad t_0 \geq 0,$$

satisfy $u(t) = 0 = v(t)$ for $t \geq t_0$ (see Proposition 6.1). So the control mechanism does not lead to new motion after landing.

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