Journal of Dynamics and Differential Equations, Vol. 18, No. 4, October 2006 (© 2006)
DOI: 10.1007/s10884-006-9046-x

Entire Solutions with Merging Fronts to Reaction–Diffusion Equations

Yoshihisa Morita[1,2] and Hirokazu Ninomiya[1]

Received January 30, 2006

We deal with a reaction–diffusion equation $u_t = u_{xx} + f(u)$ which has two stable constant equilibria, $u = 0, 1$ and a monotone increasing traveling front solution $u = \phi(x + ct)$ $(c > 0)$ connecting those equilibria. Suppose that $u = a$ $(0 < a < 1)$ is an unstable equilibrium and that the equation allows monotone increasing traveling front solutions $u = \psi_1(x + c_1 t)$ $(c_1 < 0)$ and $\psi_2(x + c_2 t)$ $(c_2 > 0)$ connecting $u = 0$ with $u = a$ and $u = a$ with $u = 1$, respectively. We call by an entire solution a classical solution which is defined for all $(x, t) \in \mathbb{R}^2$. We prove that there exists an entire solution such that for $t \approx -\infty$ it behaves as two fronts $\psi_1(x + c_1 t)$ and $\psi_2(x + c_2 t)$ on the left and right x-axes, respectively, while it converges to $\phi(x + ct)$ as $t \to \infty$. In addition, if $c > -c_1$, we show the existence of an entire solution which behaves as $\psi_1(-x + c_1 t)$ in $x \in (-\infty, (c_1 + c)t/2]$ and $\phi(x + ct)$ in $x \in [(c_1 + c)t/2, \infty)$ for $t \approx -\infty$.

KEY WORDS: reaction–diffusion equation; entire solution; traveling front wave; bistable nonlinearity; merging fronts.

AMS 2000 SUBJECT CLASSIFICATION: 35K57; 35B05; 35B40.

1. INTRODUCTION

We are concerned with the following reaction–diffusion equation on $\mathbb{R}$:

$$u_t = u_{xx} + f(u), \tag{1.1}$$

where $f(u)$ is C^2 on an open interval containing $[0, 1]$ and

$$f(0) = f(a) = f(1) = 0, \qquad f'(0)f'(a) \neq 0, \quad f'(1) < 0. \tag{1.2}$$

[1] Department of Applied Mathematics and Informatics, Ryukoku University, Seta Otsu 520-2194, Japan. E-mails: morita@rins.ryukoku.ac.jp, ninomiya@math.ryukoku.ac.jp
[2] To whom correspondence should be addressed.

The condition (1.2) implies that constant solutions $u=0, a, 1$ are nondegenerate equilibria of (1.1) and $u=1$ is stable. If, in addition to (1.2), the condition

$$\begin{aligned} &f'(0)<0, \qquad f'(a)>0, \\ &f(u)\neq 0 \quad \text{for} \quad u\in(0,a)\cup(a,1) \end{aligned} \tag{1.3}$$

is satisfied, then (1.2) is called a bistable reaction–diffusion equation and it is used as a simple model describing propagation of species in population biology or propagation of nerve excitation. It is also known for the bistable case that it allows a traveling wave solution with monotone profile, called traveling front solution, under the condition

$$\int_0^1 f(u)du\neq 0$$

(see [1, 2, 4, 7]).

Throughout this paper, we assume that (1.1) with (1.2) possesses a monotone increasing traveling front $u=\phi(x+ct)$ with $c>0$, connecting the two equilibria $u=0$ and $u=1$, that is, $\phi(x+ct)$, $\xi=x+ct$ is solution of the following boundary value problem of an ordinary differential equation:

$$\begin{aligned} &\phi''-c\phi'+f(\phi)=0, \qquad \phi'(\xi)>0 \ \ (\xi\in\mathbb{R}), \\ &\lim_{\xi\to-\infty}\phi(\xi)=0, \qquad \lim_{\xi\to\infty}\phi(\xi)=1, \end{aligned} \tag{1.4}$$

where $'=d/d\xi$, $''=d^2/d\xi^2$. We note that the reflected one $\phi(-x+ct)$ is a traveling front solution with a monotone decreasing configuration. We also give a remark that $c>0$ is realized by the condition

$$\int_0^1 f(u)du>0, \tag{1.5}$$

if the solution of (1.4) exists. A specific example of the reaction–diffusion equation satisfying the above assumptions is given by

$$u_t=u_{xx}+u(1-u)(u-a), \quad 0<a<1/2, \tag{1.6}$$

which is often called the Nagumo equation in propagation of nerve impulse, or the Allen–Cahn equation in a phase transition problem.

In addition to the existence and stability of traveling front solutions, we can discuss the dynamics for a solution with two fronts to (1.1). For instance the asymptotic behavior of diverging fronts for an appropriate initial condition is studied in [7] for a bistable reaction-diffusion equation. On the other hand the annihilation of two facing fronts is easily proved by the comparison principle. Since these asymptotic phenomena are observed for a large class of initial data, it is natural to suspect that there is a special solution describing these phenomena. More precisely we may think if there are solutions defined for all $(x,t)\in\mathbb{R}^2$, which correspond to such phenomena. We call a classical solution defined for all $(x,t)\in\mathbb{R}^2$ by an entire solution. In fact for (1.1) with (1.2) and (1.3) the existence of an entire solutions with diverging fronts is proved in [9] while the existence of an entire solution with the annihilation is proved in [10, 16] (see also [5, 9]). We note that it is difficult to describe analytically the whole dynamics of the entire solutions. For instance the precise analytical description of the annihilation is still open. Nonetheless, once we notice such entire solutions are characterized by the asymptotic behavior as $t\to-\infty$, we can prove the existence of the desired entire solution.

In addition to the entire solutions mentioned above we can consider other entire solutions. Recall the study of [11] where the Fisher-KPP equation, that is, a monostable reaction–diffusion equation also allows entire solutions with annihilation (see also [10]). In the Fisher KPP equation there are a continuous family of traveling front solutions with different speeds, thus we can construct a family of entire solutions by a combination of traveling front solutions with different speeds. Coming back to the bistable case (1.1) with (1.2) and (1.3), if we restrict $f(u)$ in the interval $[0,a]$, the reaction–diffusion equation has the constant solution $u=0$ as a unique stable equilibrium. Namely the equation can be regarded as a Fisher-KPP equation in $u\in[0,a]$. Thus there exists a family of traveling front solutions $\{\psi_1(x+c_1t)\}$ connecting $u=0$ and $u=a$ ([1, 14]). If the ψ_1 is monotone increasing, we see $c_1<0$ and there is the maximum speed $c_{1,\max}$. This traveling front solution $\psi_1(\xi),\xi=x+c_1t$ satisfies

$$\begin{aligned}&\psi_1''-c_1\psi_1'+f(\psi_1)=0, \quad \psi_1'(\xi)>0 \ \ (\xi\in\mathbb{R}),\\ &\lim_{\xi\to-\infty}\psi_1(\xi)=0, \qquad \lim_{\xi\to\infty}\psi_1(\xi)=a.\end{aligned} \tag{1.7}$$

Similarly there exists a family of traveling front solutions $\{\psi_2(x+c_2t)\}$ which are given by solving

$$\begin{aligned}&\psi_2''-c_2\psi_2'+f(\psi_2)=0, \quad \psi_2'(\xi)>0 \ \ (\xi\in\mathbb{R}),\\ &\lim_{\xi\to-\infty}\psi_2(\xi)=a, \qquad \lim_{\xi\to\infty}\psi_2(\xi)=1.\end{aligned} \tag{1.8}$$

In this case there is the minimum speed $c_{2,min}>0$. Applying the results of [11] (or [10]) yields the existence of entire solutions which converges to $\psi_1(x+c_1t)$ and $\psi_1(-x+\tilde{c}_1t)$ in the left x-axis and in the right x-axis as $t\to-\infty$, respectively. We can also see that there are entire solutions which converges to $\psi_2(-x+\tilde{c}_2t)$ and $\psi_2(x+c_2t)$ and in the left x-axis and in the right x-axis as $t\to-\infty$, respectively. We note that any combination of the speeds is allowed and these entire solution have the annihilation dynamics as time goes forward.

In this paper, we explore new types of entire solutions to (1.1). Consider a combination of the traveling fronts $\psi_1(x+c_1t)$ and $\psi_2(x+c_2t)$ and suppose that the two fronts emerge from the left axis and right axis, respectively. Then for the bistable case we can see from a numerical simulation that two fronts merge and turn to be a single front with the same configuration of $\phi(x+ct)$ (see the snapshots of such a solution in Fig. 1). Moreover if $c > -c_1$, we can also suspect if there is an entire solution with a combination of $\psi_1(-x+c_1t)$ and $\phi(x+ct)$. In this case, again for the bistable case, a numerical simulation suggests that the faster front $\phi(x+ct)$ eventually catches up $\psi_1(-x+c_1t)$ and they merge (see Fig. 2).

The next theorem establishes the existence of entire solutions with such behaviors as $t \to -\infty$.

Theorem 1.1. *Consider* (1.1) *under the conditions* (1.2). *Let* $\phi(x+ct)$ *be a solution of* (1.4) *with* $c>0$ *and let* $\psi_1(x+c_1t)$ *and* $\psi_2(x+c_2t)$ *be solutions to* (1.7) *with* $c_1<0$ *and* (1.8) *with* $c_2>0$, *respectively.*

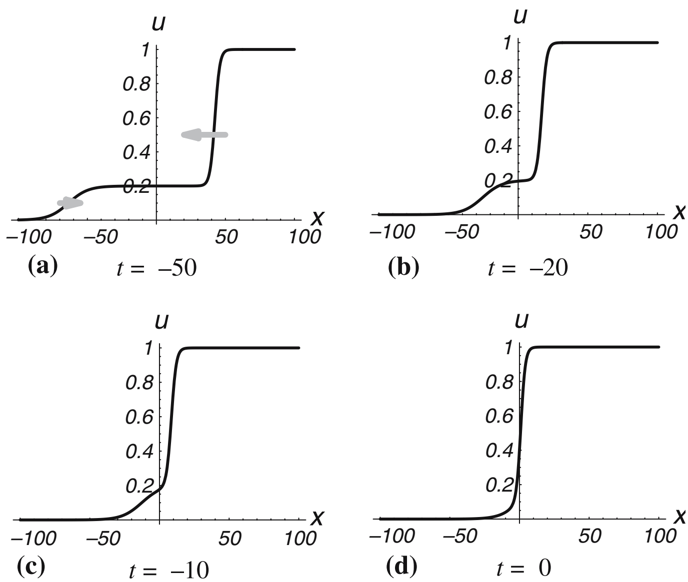

Figure 1. Four snapshots of the dynamics of the exact solution to (1.6) given in Remark 1.3.

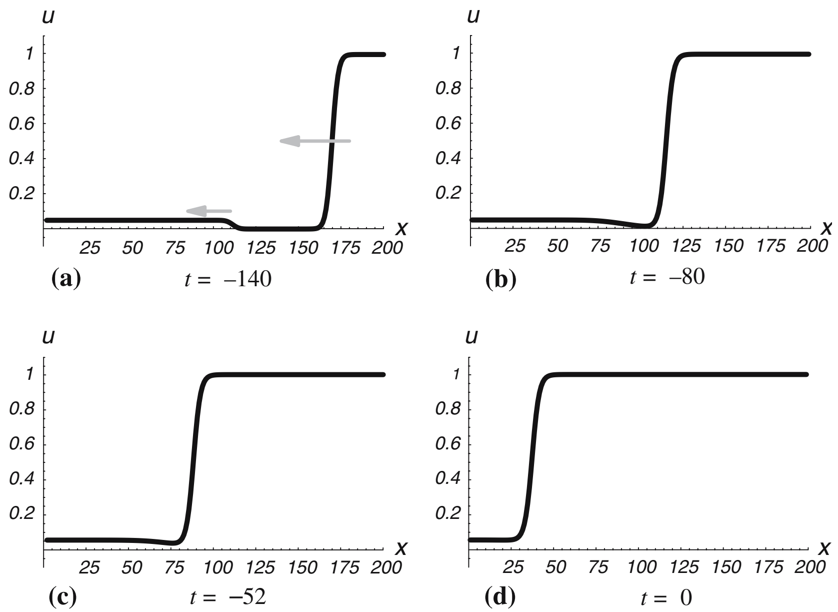

Figure 2. Four snapshots of the dynamics of the numerical solution to (1.1) in Theorem 1.1(ii).

(i) *There exists an entire solution* $\Psi_{12}(x,t)$ *satisfying*

$$\lim_{t\to-\infty}\left\{\sup_{-\infty<x\le(c_1+c_2)t/2}|\Psi_{12}(x,t)-\psi_1(x+c_1t)| + \sup_{(c_1+c_2)t/2\leqslant x<\infty}|\Psi_{12}(x,t)-\psi_2(x+c_2t)|\right\}=0. \tag{1.9}$$

Moreover if, in addition, $f'(0)<0$, *then there is a number* θ *such that*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|\Psi_{12}(x,t)-\phi(x+ct+\theta)|=0. \tag{1.10}$$

(ii) *If* $0<-c_1<c$, *there exists an entire solution* $\Phi_{10}(x,t)$ *satisfying*

$$\lim_{t\to-\infty}\left\{\sup_{-\infty<x\leqslant(c_1+c)t/2}|\Phi_{10}(x,t)-\psi_1(-x+c_1t)| + \sup_{(c_1+c)t/2\leqslant x<\infty}|\Phi_{10}(x,t)-\phi(x+ct)|\right\}=0 \tag{1.11}$$

and

$$\lim_{t\to\infty}\inf_{x\in\mathbb{R}}\Phi_{10}(x,t)=a,\qquad \lim_{t\to\infty}\sup_{x\in[-K,\infty)}|\Phi_{10}(x,t)-1|=0 \tag{1.12}$$

for arbitrarily given number $K>0$.

We remark that the assertion for the asymptotic behavior as $t\to\infty$ of (ii) in Theorem 1.1 is weak compared with (1.10). It is fortunate for (i) that one can directly apply the result in [7] to determine the asymptotic behavior as $t\to\infty$. As mentioned above the entire solution Φ_{10} seems to converges to a front solution $\psi_2(x+c_2t)$ in the bistable case. Although we have many results for the convergence of a solution to the traveling wave solution if the initial data in the interval $[a,1]$ (see [3, 8, 12, 14, 15]), the previous results cannot apply to the present case because the solution takes the values out of the range $[a,1]$. We will not discuss the convergence of Φ_{10} to the front solution as $t\to\infty$ in the present paper, though it is an interesting problem.

Next we are going to the specific equation (1.6). Then the traveling front solution $\phi(x+ct)$ is explicitly written as

$$\begin{aligned}\phi(x+ct) &= \frac{\exp\left[x/\sqrt{2}+(1/2-a)t\right]}{1+\exp\left[x/\sqrt{2}+(1/2-a)t\right]}\\ &= \frac{1}{2}+\frac{1}{2}\tanh\left(\frac{x+ct}{2\sqrt{2}}\right),\qquad c=\sqrt{2}\left(\frac{1}{2}-a\right).\end{aligned}$$

In addition since the minimum speed of traveling front solutions $\psi_1(-x+c_1t)$ is given by $2\sqrt{f'(a)}=2\sqrt{a-a^2}$, we can determine the condition in (ii) of Theorem 1.1. The result is as follows:

Corollary 1.2. *If* $0<a<1-\sqrt{6}/3$, *then* (1.6) *possesses entire solutions as in Theorem* 1.1 (ii).

Remark 1.3. It is known that (1.6) also allows exact traveling front solutions

$$\begin{aligned}\tilde{\psi}_1(x+c_1t) &= \frac{a\exp\left[ax/\sqrt{2}-(a-a^2/2)t\right]}{1+\exp\left[ax/\sqrt{2}-(a-a^2/2)t\right]}\\ &= \frac{a}{2}+\frac{a}{2}\tanh\left[\frac{a(x+c_1t)}{2\sqrt{2}}\right],\qquad c_1=-\frac{2-a}{\sqrt{2}}\end{aligned}$$

and

$$\tilde{\psi}_2(x+c_2t) = \frac{a+\exp\left[(1-a)x/\sqrt{2}+(1-a^2)t/2\right]}{1+\exp\left[(1-a)x/\sqrt{2}+(1-a^2)t/2\right]}$$

$$= \frac{a+1}{2}+\frac{1-a}{2}\tanh\left[\frac{(1-a)(x+c_2t)}{2\sqrt{2}}\right], \qquad c_2=\frac{1+a}{\sqrt{2}}.$$

Moreover Kawahara–Tanaka [13] found an exact solution of the entire solution combining $\tilde{\psi}_1$ and $\tilde{\psi}_2$ as

$$u(x,t)=\frac{\exp\left[x/\sqrt{2}+(1/2-a)t\right]+a\exp\left[ax/\sqrt{2}-(a-a^2/2)t\right]}{1+\exp\left[x/\sqrt{2}+(1/2-a)t\right]+\exp\left[ax/\sqrt{2}-(a-a^2/2)t\right]}$$

$$=\frac{\exp\left[(1-a)x/\sqrt{2}+(1-a^2)t/2\right]+a}{\exp\left[-ax/\sqrt{2}+(a-a^2/2)t\right]+\exp\left[(1-a)x/\sqrt{2}+(1-a^2)t/2\right]+1}.$$

This exact solution certainly exhibits the asymptotic behaviors stated in (i) of Theorem 1.1. We, however, note that such an expression only allowed for this specific equation or the specific traveling wave solutions $\tilde{\psi}_1$ and $\tilde{\psi}_2$. In addition, as mentioned above, the equation under consideration admits infinitely many traveling wave solutions ψ_1 (resp. ψ_2) connecting between 0 and a (resp. between a and 1). Hence the result (i) of Theorem 1.1 guarantees the existence of the entire solution with any combination of two fronts ψ_1 and ψ_2 for the general $f(u)$.

We finally give a remark on the role of entire solutions in the global dynamics. The study for entire solutions is crucial in the following sense: firstly it helps us for the mathematical understanding of transient dynamics. As mentioned above, some transient dynamics can be characterized by the behavior of the past $t \approx -\infty$, even though we cannot describe the whole transient behavior. On the other hand our result has the implication that dynamics of two solutions can have distinct histories in the configuration, though their asymptotic profiles as $t \to \infty$ coincide. Secondly from the dynamical system point of view, determining of the dynamical structure of the maximal invariant set (or the global attractor) is one of the ultimate goal. To carry out it, we need to seek all the entire solutions or provide a useful classification of entire solutions as a step. The entire solutions found in the present paper exhibit characteristic dynamical behaviors which never arise in the finite interval case. We believe that our study will be a contribution to the study in this direction.

We organize the rest of the paper as follows: in the next section, we define a function linking two front solutions of (1.1). In Section 3, using such a function, we propose a supersolution and a subsolution defined for every $(x,t)\in\mathbb{R}\times(-\infty,0]$, and establish the existence of an entire solution sandwiched between the supersolution and the subsolution. In Section 4, we explicitly provide the super-subsolution pairs and apply the result of Section 3 to the two cases of Theorem 1.1. In the last section, we give remarks on some related works and on the uniqueness of the entire solutions.

2. SOME FUNCTION LINKING TWO-FRONTS DYNAMICS

Let $\phi_j=\phi_j(x+v_jt)$ $(j=1,2)$ be traveling wave solutions of (1.1) given by solutions of

$$\begin{array}{ll}\phi_j''(\xi)-v_j\phi_j'(\xi)+f(\phi_j(\xi))=0, & \xi\in\mathbb{R}\\ \phi_j(-\infty)=\alpha_j, & \phi_j(\infty)=\omega_j\end{array}\quad (j=1,2), \tag{2.1}$$

where $\alpha_j\in\{0,a,1\}$ and $\omega_j\in\{0,a,1\}$. Assume ϕ_j $(j=1,2)$ are strictly monotone, namely, there exists $\ell_j\in\{0,1\}$ $(j=1,2)$ such that

$$(-1)^{\ell_j}\phi_j'(\xi)>0\quad(\xi\in\mathbb{R},\ j=1,2).$$

To connect the two fronts solutions, we may assume

$$\omega_1=\alpha_2. \tag{2.2}$$

Namely we consider the combination of the fronts so that ϕ_1 is left to ϕ_2. We easily see possible combinations of (α_1,ω_2) and (α_2,ω_2) are

$$\begin{aligned}(\alpha_1,\omega_1,\alpha_2,\omega_2)\in\{&(0,a,a,1),(0,a,a,0),(0,1,1,a),(0,1,1,0),\\ &(a,0,0,a),(a,0,0,1),(a,1,1,a),(a,1,1,0),\\ &(1,a,a,1),(1,a,a,0),(1,0,0,a),(1,0,0,1)\}.\end{aligned}$$

Considering the direction of traveling wave solutions and the symmetry with respect to reflection, we can reduce the above cases to

$$\{(0,a,a,1),(1,a,a,1),(1,0,0,1),(0,a,a,0),(a,0,0,1)\}. \tag{2.3}$$

Although the former three cases in (2.3) were already studied in the previous paper [10] (see also [5, 11]), the argument below can be applied to all the cases of (2.3).

By the nondegenerate condition on f there are positive constants λ_j, μ_j $(j=1,2)$ and $M>0$ such that

$$
\begin{aligned}
&|\phi_j'(x+p)| \leqslant M \exp(\lambda_j(x+p)) \quad (x \leqslant -p),\\
&|\phi_j'(x+p)| \leqslant M \exp(-\mu_j(x+p)) \quad (x \geqslant -p).
\end{aligned} \tag{2.4}
$$

Moreover there is a positive constant ρ such that

$$
\begin{aligned}
&\frac{|\phi_j'(x+p)|}{|\phi_j(x+p)-\alpha_j|} \geqslant \rho \quad (x \leqslant -p),\\
&\frac{|\phi_j'(x+p)|}{|\phi_j(x+p)-\omega_j|} \geqslant \rho \quad (x \geqslant -p).
\end{aligned} \tag{2.5}
$$

Instead of (1.1), we consider

$$
U_t = U_{xx} - \overline{v} U_x + f(U), \quad x \in \mathbb{R}, \ \ \overline{v} = \frac{v_1+v_2}{2}. \tag{2.6}
$$

This equation is equivalent to (1.1) through the transformation $u(x,t) = U(x+\overline{v}t, t)$. We easily see that (2.6) has traveling wave solutions

$$
U = \phi_1(x - c_0 t), \quad \phi_2(x + c_0 t), \ \ c_0 := \frac{v_2 - v_1}{2}.
$$

Set

$$
\mathscr{F}[U] := U_t - U_{xx} + \overline{v} U_x - f(U)
$$

and define the rectangle region $D \subset \mathbb{R}^2$ as

$$
\begin{aligned}
D &:= D_c \backslash \{(\alpha_1, \omega_2)\},\\
D_c &:= \Big[\min\{\alpha_1,\omega_1\}, \ \max\{\alpha_1,\omega_1\}\Big] \times \Big[\min\{\alpha_2,\omega_2\}, \ \max\{\alpha_2,\omega_2\}\Big] \subset \mathbb{R}^2.
\end{aligned}
$$

For $q_j(t)$ $(j=1,2)$ and $Q(y,z) \in C^3(D)$, which are specified later, we put

$$
U = Q(\phi_1,\phi_2), \qquad \phi_1 = \phi_1(x - q_1(t)), \qquad \phi_2 = \phi_2(x + q_2(t)).
$$

We abbreviate

$$
Q_y = \frac{\partial Q}{\partial y}(\phi_1,\phi_2), \qquad Q_z = \frac{\partial Q}{\partial z}(\phi_1,\phi_2), \qquad Q_{yy} = \frac{\partial^2 Q}{\partial y^2}(\phi_1,\phi_2),
$$

etc. Then

$$
\mathscr{F}[Q(\phi_1,\phi_2)] = -Q_y \phi_1'(\dot{q}_1 - c_0) + Q_z \phi_2'(\dot{q}_2 - c_0) - B(\phi_1,\phi_2) - G(\phi_1,\phi_2), \tag{2.7}
$$

where

$$B(\phi_1,\phi_2) := Q_{yy}\{\phi_1'\}^2 + 2Q_{yz}\phi_1'\phi_2' + Q_{zz}\{\phi_2'\}^2, \tag{2.8}$$

$$G(\phi_1,\phi_2) := f(Q) - Q_y f(\phi_1) - Q_z f(\phi_2). \tag{2.9}$$

Lemma 2.1. *If $Q(y,z)\in C^3(D)$ allows the expressions*

$$Q(y,z) = \begin{cases} y + (y-\alpha_1)(z-\alpha_2)R_1(y,z) \\ z + (y-\omega_1)(z-\omega_2)R_2(y,z) \end{cases} \quad (y,z)\in D \tag{2.10}$$

for appropriate $R_1, R_2 \in C^1(D)$, then

$$\begin{array}{ll} Q_y(y,\alpha_2)=1, & Q_y(y,\omega_2)=0, \\ Q_z(\alpha_1,z)=0, & Q_z(\omega_1,z)=1 \end{array}$$

and

$$Q_{yy}(y,\alpha_2) = Q_{yy}(y,\omega_2) = Q_{zz}(\alpha_1,z) = Q_{zz}(\omega_1,z) = 0$$

hold for $(y,\alpha_j), (y,\omega_j), (\alpha_j,z), (\omega_j,z)\in D$ $(j=1,2)$. Moreover, there exist functions $\tilde{Q}_{11i}$, $\tilde{Q}_{22i}\in C^1(D)$ $(i=1,2)$ satisfying

$$\begin{array}{l} Q_{yy}(y,z) = (z-\alpha_2)\tilde{Q}_{111}(y,z) = (z-\omega_2)\tilde{Q}_{112}(y,z), \\ Q_{zz}(y,z) = (y-\alpha_1)\tilde{Q}_{221}(y,z) = (y-\omega_1)\tilde{Q}_{222}(y,z). \end{array} \tag{2.11}$$

We omit the proof of this lemma because of a simple and elementary computation. With the aid of Lemma 2.1 the function G defined in (2.9) satisfies

$$G(\phi_1,\alpha_2) = G(\alpha_1,\phi_2) = G(\phi_1,\omega_2) = G(\omega_1,\phi_2) = 0.$$

We thereby have the expressions

$$\begin{aligned} G(y,z) &= (y-\alpha_1)(z-\alpha_2)G_1(y,z) \\ &= (y-\omega_1)(z-\alpha_2)G_2(y,z) \\ &= (y-\omega_1)(z-\omega_2)G_3(y,z) \end{aligned} \tag{2.12}$$

for some continuous functions G_j $(j=1,2,3)$ in D.

Then we have the next lemma.

Lemma 2.2. *Let $\phi_1=\phi_1(x-q_1)$ and $\phi_2=\phi_2(x+q_2)$ be the traveling front solutions of* (2.6) *with the condition* (1.2). *Given $Q(y,z)$ of* (2.10), *suppose that there are positive constants $\tau>0, \delta_1$ and δ_2 such that*

$$A(\phi_1,\phi_2) := (-1)^{\ell_1}\phi_1' Q_y + (-1)^{\ell_2}\phi_2' Q_z > 0 \quad \textit{for} \quad q_1, q_2 \leqslant -\tau, \tag{2.13}$$

$$\left.\begin{aligned} &Q_y(\phi_1,\phi_2) \geqslant \delta_1 \\ &A(\phi_1,\phi_2) \geqslant \frac{|\phi_1'|}{2} Q_y(\phi_1,\phi_2) \end{aligned}\right. \quad \textit{for} \quad x\leqslant 0,\ q_2\leqslant -\tau \tag{2.14}$$

and

$$\left.\begin{aligned} &Q_z(\phi_1,\phi_2) \geqslant \delta_2 \\ &A(\phi_1,\phi_2) \geqslant \frac{|\phi_2'|}{2} Q_z(\phi_1,\phi_2) \end{aligned}\right. \quad \textit{for} \quad x\geqslant 0,\ q_1\leqslant -\tau \tag{2.15}$$

hold. In addition, assume that there exists a positive constant C_1 *satisfying*

$$|Q_{yz}(\phi_1,\phi_2)|,\ \ |\tilde{Q}_{11i}(\phi_1,\phi_2)|,\ \ |\tilde{Q}_{22i}(\phi_1,\phi_2)|\leqslant C_1 \quad \textit{for} \quad q_1,q_2\leqslant -\tau, \tag{2.16}$$

where $\tilde{Q}_{11i}, \tilde{Q}_{22i}$ $(i=1,2)$ *are as in* (2.11)*. Then there is a constant* $K>0$ *such that*

$$\left|\frac{G(\phi_1,\phi_2)+B(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| \leqslant \begin{cases} K|\phi_2'(x+q_2)|, & x\leqslant 0, \\ K|\phi_1'(x-q_1)|, & x\geqslant 0 \end{cases} \tag{2.17}$$

holds.

Proof. First consider the two cases: $x\leqslant q_1$ and $q_1\leqslant x\leqslant 0$. For $x\leqslant q_1$ we estimate

$$\begin{aligned} \left|\frac{G(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| &= \frac{|(\phi_1-\alpha_1)(\phi_2-\alpha_2)G_1|}{A(\phi_1,\phi_2)} \\ &\leqslant \frac{2|\phi_2-\alpha_2||G_1|}{Q_y|\phi_1'/(\phi_1-\alpha_1)|} \leqslant \frac{2C_1}{\rho^2\delta_1}|\phi_2'|. \end{aligned} \tag{2.18}$$

Similarly for $q_1\leqslant x\leqslant 0$,

$$\begin{aligned} \left|\frac{G(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| &= \frac{|(\phi_1-\omega_1)(\phi_2-\alpha_2)G_2|}{A(\phi_1,\phi_2)} \\ &\leqslant \frac{2|\phi_2-\alpha_2||G_2|}{Q_y|\phi_1'/(\phi_1-\omega_1)|} \leqslant \frac{2C_1}{\rho^2\delta_1}|\phi_2'|. \end{aligned} \tag{2.19}$$

Thus (2.18) and (2.19) yield

$$\left|\frac{G(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| \leqslant K_1|\phi_2'(x+q_2)| \quad (x\leqslant 0) \tag{2.20}$$

for a constant $K_1>0$.

Next consider the case $0\leqslant x\leqslant -q_2$.

$$\left|\frac{G(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| = \frac{|(\phi_1-\omega_1)(\phi_2-\alpha_2)G_2|}{A(\phi_1,\phi_2)} \leqslant \frac{2|\phi_1-\omega_1||G_2|}{Q_z|\phi_2'/(\phi_2-\omega_2)|}$$
$$\leqslant \frac{2|G_2||\phi_1-\omega_1|}{\rho\delta_2} \leqslant \frac{2C_1}{\rho^2\delta_2}|\phi_1'|. \tag{2.21}$$

Similarly for $x\geqslant -q_2$,

$$\left|\frac{G(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| = \frac{|(\phi_1-\omega_1)(\phi_2-\omega_2)G_3|}{A(\phi_1,\phi_2)}$$
$$\leqslant \frac{2|\phi_1-\omega_1||G_3|}{Q_z|\phi_2'/(\phi_2-\omega_2)|} \leqslant \frac{2C_1}{\rho^2\delta_2}|\phi_1'|. \tag{2.22}$$

Combining (2.21) and (2.22), we obtain

$$\left|\frac{G(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| \leqslant K_1|\phi_1'(x-q_1)| \quad (x\geqslant 0). \tag{2.23}$$

We estimate B/A. For $x\leqslant 0$,

$$\left|\frac{B(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| \leqslant \frac{2\Big(|\phi_1'|^2|Q_{yy}|+2|\phi_1'\phi_2'Q_{yz}|+|\phi_2'|^2|Q_{zz}|\Big)}{|\phi_1'|Q_y}$$
$$\leqslant \frac{2\Big(|\phi_1'||Q_{yy}|+2|\phi_2'||Q_{yz}|+|\phi_2'|^2|Q_{zz}|/|\phi_1'|\Big)}{\delta_1}.$$

Applying (2.11) with $y=\phi_1$ and $z=\phi_2$ an the right-hand side of this inequality, we see that there is a constant $C_2>0$ such that

$$\left|\frac{B(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| \leqslant \frac{2C_2}{\delta_1}|\phi_2'|. \tag{2.24}$$

Similarly we can obtain the inequality for $x\geq 0$. Hence we get to

$$\left|\frac{B(\phi_1,\phi_2)}{A(\phi_1,\phi_2)}\right| \leqslant \begin{cases} K_2|\phi_2'(x+q_2)| & (x\leqslant 0), \\ K_2|\phi_1'(x-q_1)| & (x\geqslant 0). \end{cases} \tag{2.25}$$

The desired conclusion follows from (2.20), (2.23) and (2.25). ∎

3. SUPER-SUBSOLUTIONS

Consider the following ordinary differential equation:

$$\begin{aligned} &\dot{p} = c_0 + Le^{\sigma p}, \quad -\infty < t < 0, \\ &p(0) = p_0, \end{aligned} \tag{3.1}$$

where L is a positive constant and $\sigma = \min\{\lambda_2, \mu_1\}$. A simple computation yields a solution to (3.1) as

$$p(t) = c_0 t - \frac{1}{\sigma} \log \left\{ e^{-\sigma p_0} + \frac{L(1 - e^{\sigma c_0 t})}{c_0} \right\} \tag{3.2}$$

with the asymptotics

$$\lim_{t \to -\infty} (p(t) - c_0 t) = -\frac{1}{\sigma} \log \left(e^{-\sigma p_0} + \frac{L}{c_0} \right). \tag{3.3}$$

Similarly we solve the equation

$$\begin{aligned} &\dot{r} = c_0 - Le^{\sigma r}, \quad -\infty < t < 0, \\ &r(0) = r_0 < \frac{1}{\sigma} \log \frac{c_0}{L} \end{aligned} \tag{3.4}$$

to obtain the solution

$$r(t) = c_0 t - \frac{1}{\sigma} \log \left\{ e^{-\sigma r_0} - \frac{L(1 - e^{\sigma c_0 t})}{c_0} \right\} \tag{3.5}$$

with the asymptotics

$$\lim_{t \to -\infty} (r(t) - c_0 t) = -\frac{1}{\sigma} \log \left(e^{-\sigma r_0} - \frac{L}{c_0} \right). \tag{3.6}$$

By virtue of (3.3) and (3.6)

$$\lim_{t \to -\infty} (p(t) - r(t)) = 0$$

implies

$$\begin{aligned} &p_0 = -\frac{1}{\sigma} \log \left(e^{-\sigma r_0} - \frac{2L}{c_0} \right), \\ &r_0 < \frac{1}{\sigma} \log \frac{c_0}{2L + c_0}. \end{aligned} \tag{3.7}$$

Furthermore we can easily verify that under the condition (3.7)

$$0 < p(t) - r(t) \leqslant N_1 e^{\sigma_1 t}, \qquad \sigma_1 := c_0 \sigma \quad (t \leqslant 0) \tag{3.8}$$

holds for a positive number N_1.

Next we consider a system of ordinary differential equations

$$\begin{aligned}\dot{p}_1 &= c_0 - Le^{\sigma p_1}\\ \dot{p}_2 &= c_0 + Le^{\sigma p_1}\end{aligned} \quad (p_1(0), p_2(0)) = (p_{10}, p_{20}), \quad t \leqslant 0. \tag{3.9}$$

The solutions are given by

$$\begin{aligned} p_1(t) &= c_0 t - \frac{1}{\sigma}\log\left\{e^{-\sigma p_{10}} - \frac{L(1-e^{\sigma c_0 t})}{c_0}\right\},\\ p_2(t) &= c_0 t + \frac{1}{\sigma}\log\left\{e^{-\sigma p_{10}} - \frac{L(1-e^{\sigma c_0 t})}{c_0}\right\} + p_{10} + p_{20}\end{aligned} \tag{3.10}$$

with the asymptotics

$$\begin{aligned} \lim_{t\to-\infty}(p_1(t) - c_0 t) &= -\frac{1}{\sigma}\log\left(e^{-\sigma p_{10}} - \frac{L}{c_0}\right),\\ \lim_{t\to-\infty}(p_2(t) - c_0 t) &= \frac{1}{\sigma}\log\left(e^{-\sigma p_{10}} - \frac{L}{c_0}\right) + p_{10} + p_{20}.\end{aligned} \tag{3.11}$$

In this case there is a number $N_2 > 0$ such that

$$0 < p_2(t) - p_1(t) \leqslant N_2 e^{\sigma_1 t} \quad (t \leqslant 0) \tag{3.12}$$

holds if

$$\begin{aligned} &\frac{\sigma}{2}(p_{10} + p_{20}) + \log\left(e^{-\sigma p_{10}} - \frac{L}{c_0}\right) = 0,\\ &p_{10} < \frac{1}{\sigma}\log\frac{c_0}{L}\end{aligned} \tag{3.13}$$

are satisfied.

Now we provide a lemma for the existence of supersolutions and subsolutions to (2.6).

Lemma 3.1. *Consider all the cases of* (2.3) *under the same assumptions in Lemma 2.2. Assume* $c_0 = (v_2 - v_1)/2 > 0$. *Let* $p(t), r(t)$, *and* $(p_1(t), p_2(t))$ *be solutions to* (3.1), (3.4), *and* (3.9), *respectively, and assume* $r_0 < p_0 < -\tau$ *and* $p_{10} < p_{20} < -\tau$. *Take* L *large so that* $L > KM$ *holds, where* M *and* K *are as in* (2.4) *and* (2.17), *respectively.*

(i) *For* $(\alpha_1, \omega_1, \alpha_2, \omega_2) = (a, 0, 0, 1)$, *the functions defined by*

$$\begin{aligned} \overline{U}(x,t) &:= Q(\phi_1(x - p(t)), \phi_2(x + p(t))),\\ \underline{U}(x,t) &:= Q(\phi_1(x - r(t)), \phi_2(x + r(t)))\end{aligned} \tag{3.14}$$

are a supersolution and a subsolution to (2.6) *for* $t\leqslant 0$, *respectively. If* (3.7) *is satisfied, then*

$$\begin{aligned} &\underline{U}(x,t) < \overline{U}(x,t) && (x\in\mathbb{R}, t\leqslant 0),\\ &\sup_{x\in\mathbb{R}}(\overline{U}(x,t)-\underline{U}(x,t))\leqslant \kappa e^{\sigma_1 t} && (t\leqslant 0) \end{aligned} \tag{3.15}$$

hold for some constants $\kappa>0$, *where* σ_1 *is as in* (3.8). *In addition the cases* $(\alpha_1,\omega_1,\alpha_2,\omega_2)=(1,0,0,1),(1,a,a,1)$ *allow the same assertions for the functions as defined by* (3.14).

(ii) *For* $(\alpha_1,\omega_1,\alpha_2,\omega_2)=(0,a,a,1)$, *the functions*

$$\begin{aligned} &\overline{U}(x,t) := Q(\phi_1(x-p_1(t)),\phi_2(x+p_2(t))),\\ &\underline{U}(x,t) := Q(\phi_1(x-p_2(t)),\phi_2(x+p_1(t))) \end{aligned} \tag{3.16}$$

are a supersolution and a subsolution to (2.6) *for* $t\leqslant 0$, *respectively. If* (3.13) *is satisfied, then* (3.15) *holds.*

(iii) *For* $(\alpha_1,\omega_1,\alpha_2,\omega_2)=(0,a,a,0)$, *the function defined by*

$$\begin{aligned} &\overline{U}(x,t) := Q(\phi_1(x-r(t)),\phi_2(x+r(t))),\\ &\underline{U}(x,t) := Q(\phi_1(x-p(t)),\phi_2(x+p(t))) \end{aligned} \tag{3.17}$$

are a supersolution and a subsolution to (2.6) *for* $t\leqslant 0$, *respectively. If* (3.7) *is satisfied, then* (3.15) *holds.*

Proof. We prove that $\overline{U}$ of (3.14) is a supersolution. In this case $\ell_1=1,\ell_2=0$. From (2.4) we see

$$\begin{aligned} &|\phi_2'(x+p)|\leqslant M\exp(\lambda_1(x+p))\leqslant M\exp(\lambda_1 p) \quad (x\leqslant 0),\\ &|\phi_1'(x-p)|\leqslant M\exp(-\mu_2(x-p))\leqslant M\exp(\mu_2 p) \quad (x\geqslant 0). \end{aligned}$$

Using this and Lemma 2.2 yields

$$|B(\phi_1,\phi_2)+G(\phi_1,\phi_2)|\leqslant A(\phi_1,\phi_2)M\exp(\sigma p). \tag{3.18}$$

By (2.7), (3.1), and (3.18) we obtain

$$\begin{aligned} \mathscr{F}[\overline{U}] &= -Q_y\phi_1'(\dot{p}-c_0)+Q_z\phi_2'(\dot{p}-c_0)-B(\phi_1,\phi_2)-G(\phi_1,\phi_2)\\ &\geq A(\phi_1,\phi_2)(Le^{\sigma p}-KMe^{\sigma p})\geq 0. \end{aligned}$$

Similarly $\mathscr{F}[\underline{U}]\leq 0$ for $\overline{U}$ of (3.14).

Next we show (3.15) in (i). We verify

$$\begin{aligned} \overline{U}-\underline{U} &= Q(\phi_1(x-p),\phi_2(x+p))-Q(\phi_1(x-r),\phi_2(x+r))\\ &= \int_0^1 A(\phi_1(x-\theta p-(1-\theta)r),\phi_2(x+\theta p+(1-\theta)r))d\theta\cdot(p-r)\\ &\geq 0 \quad (t\leqslant 0) \end{aligned}$$

since $p(t) \geqslant r(t)$ $(t \leqslant 0)$. The second inequality follows from

$$\overline{U} - \underline{U} = \int_0^1 A(\phi_1(x - \theta p - (1-\theta)r), \phi_2(x + \theta p + (1-\theta)r)) d\theta \cdot (p - r)$$

and (3.8).

Next consider $\overline{U}$ of (3.16). Notice $\ell_1 = \ell_2 = 0$. We easily obtain

$$\begin{aligned} |\phi_2'(x + p_2)| &\leqslant M \exp(\lambda_1 p_2) \quad (x \leqslant 0), \\ |\phi_1'(x - p_1)| &\leqslant M \exp(\mu_2 p_1) \quad (x \geqslant 0) \end{aligned}$$

and then

$$|B(\phi_1, \phi_2) + G(\phi_1, \phi_2)| \leqslant A(\phi_1, \phi_2) M \exp(\sigma p_1)$$

by virtue of $p_1(t) < p_2(t) < 0$. This yields

$$\begin{aligned} \mathscr{F}[\overline{U}] &= -Q_y \phi_1'(\dot{p}_1 - c_0) + Q_z \phi_2'(\dot{p}_2 - c_0) - B(\phi_1, \phi_2) - G(\phi_1, \phi_2) \\ &\geq A(\phi_1, \phi_2)(L e^{\sigma p_1} - K M e^{\sigma p_1}) \geq 0. \end{aligned}$$

Similarly, as for $\underline{U}$ of (3.16), we easily verify $\mathscr{F}[\underline{U}] \leq 0$. The inequality (3.15) can be shown in the same way as done for the former case by using (3.12). We omit the detail.

We leave the proof for the other cases to the readers, since those can be easily shown by the similar argument. ∎

The next proposition follows from applying the same argument found in [9, 10].

Proposition 3.2. *Let* $\underline{U}$ *and* $\overline{U}$ *be a super-subsolutions pair with* (3.15) *of Lemma* 3.1. *Then there exists a unique entire solution* $\tilde{u}(x,t)$ *to* (1.1) *and* (1.2) *satisfying*

$$\underline{U}(x + \overline{v}t, t) < \tilde{u}(x,t) < \overline{U}(x + \overline{v}t, t), \quad x \in \mathbb{R}, \ t \leqslant 0, \tag{3.19}$$

where $\overline{v} = (v_1 + v_2)/2$.

4. PROOF OF THEOREM 1.1

We first prove the first part of Theorem 1.1 by providing $Q(y,z)$ explicitly. We apply (ii) of Lemma 3.1 and Proposition 3.2 with $\phi_1(x + v_1 t) = \psi_1(x + c_1 t)$ and $\phi_2(x + v_2 t) = \psi_2(x + c_2 t)$. Set

$$Q(y,z) := \frac{(1-a)yz}{y(z-a) + a(1-z)} = \frac{(1-a)yz}{yz + a(1-y-z)}. \tag{4.1}$$

Let $(\alpha_1, \omega_1) = (0, a), (\alpha_2, \omega_2) = (a, 1)$. Namely $\phi_1(\xi), \xi = x + v_1 t$, and $\phi_2(\xi), \xi = x + v_2 t$ are monotone increasing. Then we can write

$$Q(y,z) = y + y(z-a)\left\{\frac{1-y}{y(z-a)+a(1-z)}\right\}$$

$$= z + (y-a)(z-1)\left\{\frac{-z}{y(z-a)+a(1-z)}\right\}$$

and

$$Q_y(y,z) = \frac{a(1-a)z(1-z)}{\{y(z-a)+a(1-z)\}^2}$$

$$Q_z(y,z) = \frac{a(1-a)y(1-y)}{\{y(z-a)+a(1-z)\}^2}$$

from which $A(\phi_1, \phi_2) > 0$ $(0 < \phi_1, \phi_2 < 1)$ follows.

By

$$0 < \phi_2(x+q_2) - a \leqslant \frac{M}{\rho} e^{\lambda_2 q_2} \quad (x \leqslant 0),$$

there exist numbers $r, \delta_1 > 0$ such that

$$Q_y \geqslant \delta_1 \quad (x \leqslant 0) \text{ for } \quad q_2 \leqslant -\tau,$$

while by

$$0 < a - \phi_1(x - q_1) \leqslant \frac{M}{\rho} e^{\mu_1 q_1} \quad (x \geqslant 0),$$

there exist $r, \delta_2 > 0$ such that

$$Q_z \geqslant \delta_2 \quad (x \geqslant 0) \text{ for } \quad q_1 \leqslant -\tau.$$

We also easily check

$$Q_{yy} = (\phi_2 - a)(\phi_2 - 1)\frac{2a(1-a)\phi_2}{\{\phi_1(\phi_2 - a) + a(1-\phi_2)\}^3},$$

$$Q_{yz} = \frac{a(1-a)\{(2a-1)\phi_1\phi_2 + a(1-\phi_1-\phi_2)\}}{\{\phi_1(\phi_2 - a) + a(1-\phi_2)\}^3},$$

$$Q_{zz} = \phi_1(\phi_1 - a)\frac{2a(1-a)(\phi_1 - 1)}{\{\phi_1(\phi_2 - a) + a(1-\phi_2)\}^3}$$

so that $\tilde{Q}_{11i}, \tilde{Q}_{22i}$ and Q_{yz} are bounded, because

$$\{(\phi_1(x-q_1), \phi_2(x+q_2)) \mid x \in \mathbb{R},\ q_1, q_2 \leq -\tau\}$$

is included compactly in D. This implies that the assertion of Proposition 3.2 holds for the super-subsolution pair through (4.1), hence we obtain (1.10).

The asymptotic behavior as $t \to \infty$ of (1.9) immediately follows from Theorem 3.1 of [7].

Next we prove the second part (ii) of Theorem 1.1. We apply (i) of Lemma 3.1 and Proposition 3.2 with $\phi_1(x+v_1t) = \psi_1(-x+c_1t)$ and $\phi_2(x+v_2t) = \phi(x+ct)$. In this case $(\alpha_1, \omega_1) = (a, 0)$, $(\alpha_2, \omega_2) = (0, 1)$, thus $\ell_1 = \ell_2 = 0$. In fact $\phi_1(\xi), \xi = x + v_1t\ \ (v_1 < 0)$, and $\phi_2(\xi), \xi = x + v_2t\ \ (v_2 > 0)$ are monotone decreasing and increasing, respectively.

Set

$$Q(y,z) := \frac{a(y+z) - (1+a)yz}{a - yz}. \tag{4.2}$$

Then we can write

$$\begin{aligned} Q(y,z) &= y + (y-a)z\left\{\frac{y-1}{a-yz}\right\} \\ &= z + y(z-1)\left\{\frac{z-a}{a-yz}\right\} \end{aligned}$$

and

$$Q_y(y,z) = \frac{a(a-z)(1-z)}{(a-yz)^2},$$

$$Q_z(y,z) = \frac{a(a-y)(1-y)}{(a-yz)^2}.$$

Since $0 < \phi_1 < a$ and $0 < \phi_2 < 1$, $Q_z > 0$. It is not clear that $Q_y > 0$. However by

$$0 < \phi_2 \leqslant \frac{M}{\rho} e^{\lambda_2(x+q_2)} \leqslant \frac{M}{\rho} e^{\lambda_2 q_2} \quad (x \leqslant 0)$$

there are positive constants r, δ_1 such that

$$Q_y \geqslant \delta_1 \quad (x \leqslant 0) \text{ for } \quad q_2 \leqslant -r.$$

On the other hand, by

$$0 < \phi_1 \leqslant \frac{M}{\rho} e^{-\mu_1(x-q_1)} \leqslant \frac{M}{\rho} e^{\mu_1 q_1} \quad (x \geqslant 0)$$

there are positive constants r, δ_2 such that

$$Q_z \geqslant \delta_2 \quad (x \geqslant 0) \text{ for } \quad q_1 \leqslant -r.$$

We may assume $\phi_2(0) = a$. Then we can check

$$A(\phi_1, \phi_2) \geqslant \begin{cases} -\phi_1' Q_y & (x \leqslant 0), \\ \phi_2' Q_z & (0 \leqslant x \leqslant -q_2), \\ \frac{1}{2}\phi_2' Q_z & (x \geqslant -q_2), \end{cases}$$

where we used $|\phi_1'| \leqslant M \exp(\mu_1 p_1)$ $(x \geqslant 0)$ and $1 - \phi_2 \leqslant \rho \phi_2'$ $(x \geqslant -q_2)$ to obtain the last inequlity. Moreover we have

$$Q_{yy} = (\phi_2 - 1)\phi_2 \frac{2a(\phi_2 - a)}{(a - \phi_1\phi_2)^3},$$

$$Q_{yz} = \frac{-a\{(\phi_1 - a)(\phi_2 - a) + a(\phi_1 - 1)(\phi_2 - 1)\}}{(a - \phi_1\phi_2)^3},$$

$$Q_{zz} = (\phi_1 - 1)\phi_1 \frac{2a(\phi_1 - a)}{(a - \phi_1\phi_2)^3}.$$

We can take r so that $0 < a - \phi_1\phi_2 < 1/2$, which leads us to the required conditions on Q. Hence the existence of the entire solution enjoying (1.12) was proved.

To prove asymptotic behavior (1.11), we notice that there are numbers $s_j (j = 1, 2)$ such that

$$\underline{U}(x, T) > \phi_1(x - c_0T + s_1), \quad \phi_2(x + c_0T + s_2)$$

holds for some $T \ll -1$. This implies that ϕ_1, ϕ_2 are subsolutions which bound the entire solution from below for all $t \geqslant T$. The assertion of (1.11) immediately follows from this fact by considering the asymptotic behavior of the traveling fronts ϕ_1, ϕ_2.

5. CONCLUDING REMARKS

The existence of a super-subsolution pairs satisfying (3.15) for the cases $(\alpha_1, \omega_1,\ \alpha_2, \omega_2) = (1, 0, 0, 1), (1, a, a, 1), (0, a, a, 0)$ in Lemma 3.1 was proved in the previous results; for instance [11] for the Fisher-KPP equation [9, 16] for the Nagumo equation (1.6) and [5, 10] for the general case of $f(u)$ satisfying (1.2). (Note that by the transformation $U = 1 - V$ the last case $(0, a, a, 0)$ is converted into the case $(1, b, b, 1), b = 1 - a$ if $f'(0) < 0$.) However, we emphasize that our result of Lemma 3.1 provides a systematic way of the construction for the super-subsolution pairs; it covers the old cases as well as the ones studied here.

The readers might suspect if the invariant manifold theory could be applied to the existence of the entire solutions instead of the comparison principle. If the equation is bistable, this approach was carried out in [16] to prove the existence of the entire solution for the case $(1, 0, 0, 1)$. We can also see that some additional argument to the result in [6] leads us to the existence of such an entire solution. However, since their approaches crucially depends on the nondegenerate condition of the linearized stability problem around the traveling wave solution, it is difficult to apply their results if a traveling wave solution connects an unstable equilibrium with the stable one.

The uniqueness of the entire solution in Proposition 3.2 is only established in the class of functions sandwiched between the super-subsolutions. On the other hand Chen-Guo [5] established the uniqueness in quite a larger class of functions if $f'(0) < 0$ holds in addition to (1.2) (see also [16]). Unfortunately their argument cannot directly apply to the present cases. Indeed if $u = a$ is unstable equilibrium, both traveling fronts ψ_1, ψ_2 are solutions to Fisher-KPP equations as mentioned in the introduction. Therefore to prove the similar uniqueness as in [5], we need more careful consideration. It would be a future work.

ACKNOWLEDGMENT

The second author was supported in part by the Grant-in-Aid for Scientific Research (C) No. 18540147, Japan Society for the Promotion of Science.

REFERENCES

1. Aronson, D. G., and Weinberger, H. F. (1975). Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In Goldstein J. A. (ed.), *Partial Differential Equations and Related Topics. Lecture Notes in Math.*, Vol. **446**, Springer. Berlin, pp. 5–49.
2. Aronson, D. G., and Weinberger, H. F. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Adv. Math.* **30**, 33–76.
3. Bramson, M. (1983). Convergence of solutions of the Kolmogorov equation to traveling waves, *Mem. Am. Math. Soc.* **44**, (285).
4. Chen, X. (1997). Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations. *Adv. Diff. Eq.* **2**, 125–160.
5. Chen, X., and Guo, J.-S. (2005). Existence and uniqueness of entire solutions for a reaction-diffusion equation, *J. Diff. Eq.* **212**, 62–84.
6. Ei, S. (2000). The motion of weakly interacting pulses in reaction-diffusion systems. *J. Dynam. Diff. Eq.* **14**, 85–137.

7. Fife, P. C., and McLeod, J. B. (1977). The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal*. **65**, 335–361.
8. Freidlin, M. (1985). Limit theorems for large deviations and reaction-diffusion equations. *Ann. Probab*. **13**, 639–675.
9. Fukao, Y., Morita, Y., and Ninomiya, H. (2004). Some entire solutions of the Allen-Cahn equation. *Taiwan. J. Math*. **8**, 15–32.
10. Guo, J.-S., and Morita, Y. (2005). Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations *Discrete Contin. Dyn. Syst*. **12**, 193–212.
11. Hamel, F., and Nadirashvili, N. (1999). Entire solutions of the KPP equation. *Commun. Pure Appl. Math*. **52**, 1255–1276.
12. Kametaka, Y. (1976). On the nonlinear diffusion equation of Kolmogorov-Petrovskii-Piskunov type. *Osaka J. Math*. **13**, 11–66.
13. Kawahara, T., and Tanaka, M. (1983). Interactions of traveling fronts: An exact solutions of a nonlinear diffusion equations. *Phys. Lett*. **97A**, 311–314.
14. Kolmogorov, A., Petrovsky, I., and Piskunov, N. (1937). Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bjul. Moskowskogo Gos. Univ. Serv. Int. Secur. A* **1**, 1–26.
15. Uchiyama, K. (1978). The behavior of solutions of some nonlinear diffusion equations for large time. *J. Math. Kyoto Univ*. **18**, 453–508.
16. Yagisita, H. (2003). Backward global solutions characterizing annihilation dynamics of travelling fronts. *Public Res. Inst. Math. Sci*. **39**, 117–164.