Journal of Dynamics and Differential Equations, Vol. 18, No. 3, July 2006 (© 2006)
DOI: 10.1007/s10884-006-9024-3

Perturbation Theory for Approximation of Lyapunov Exponents by QR Methods

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Received December 31 2005

Motivated by a recently developed backward error analysis for QR methods, we consider the error in the Lyapunov exponents of perturbed triangular systems. We consider the case of stable and distinct Lyapunov exponents as well as the case of stable but not necessarily distinct exponents. We illustrate our analytical results with a numerical example.

KEY WORDS: Lyapunov exponents; integral separation; QR methods.

AMS SUBJECT CLASSIFICATION: 65L.

1. INTRODUCTION

Lyapunov exponents are often employed in the numerical study of nonlinear dynamical systems and are probably the most widely used quantities for detecting chaos, estimating dimensions of attractors, entropy (e.g., see [2, 3, 7, 21]). However, there is little error analysis of the techniques used to approximate Lyapunov exponents; the works [8, 11, 17, 19] are the only works of which we know dealing precisely with error analysis for approximation of Lyapunov exponents. In this paper, we provide quantifiable error bounds for Lyapunov exponents approximated by QR techniques.

We recall that, to approximate Lyapunov exponents for a linear nonautonomous system $\dot{x} = A(t)x$, the basic idea of QR methods consists in first triangularizing (via the QR factorization) an underlying fundamental matrix solution X: $X = QR$, and then extracting the Lyapunov exponents

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from the diagonal of the transformed triangular system: $\dot{R} = BR$, with B triangular. This basic approach is common in analytical works on the subject (see [12, 15, 20]) and it is also the most common approach for numerical methods to approximate Lyapunov exponents (see [4, 8, 9] and references therein).

In the recent work [11], we gave a backward error analysis for QR methods used to approximate Lyapunov exponents. Our analysis showed that, by QR methods (i.e., by a numerical realization of QR methods), one obtains an **exact** triangularization of a fundamental matrix solution of a perturbed triangular problem with coefficient function $B+E$, instead of B. We were also able to give quantitative bounds on the perturbation E and showed that in principle this perturbation can be made arbitrarily small by controlling the accuracy of the computation. Now, for systems with stable Lyapunov exponents (a necessary condition for trying to approximate them), small perturbations reflect in small errors in the Lyapunov exponents (e.g., see [1] for necessary and sufficient conditions for the stability of Lyapunov exponents). In this paper, we clarify, and quantify, the error induced by a small perturbation E on the Lyapunov exponents.

We consider both the case of stable distinct and stable but not distinct Lyapunov exponents. In the former case, there are essentially two steps involved in our analysis: first, we restrict to consider perturbed diagonal problems as opposed to perturbed triangular problems, then we show the existence of a uniformly (in time) near identity orthogonal change of variables that transforms the perturbed diagonal problem to triangular. This allows us to bound the error between the true Lyapunov exponents and those of the perturbed problem. At leading order in the size of the perturbation, we bound this error by a multiple of the size of the perturbation, the key factors contributing to this multiple depending on the degree of integral separation in the system and the condition number of the diagonalizing transformation. In this light, our result may be viewed as a time dependent analogue of the Bauer–Fike Theorem for perturbation of eigenvalues of non-normal matrices (see [13]). In the case of stable and not distinct exponents, we proceed along similar lines, but in a block sense.

This paper is outlined as follows. In Section 2, we review the basics of Lyapunov exponents and QR methods. We also recall the backward error result of [11] and further – under the assumption of integral separation – specialize to diagonal and perturbed diagonal systems. Our main result for the case of stable distinct Lyapunov exponents is in Section 3. We first show that the perturbed diagonal system can be transformed to triangular by an orthogonal change of variables that stays uniformly close to the identity and then quantify the perturbation in the Lyapunov exponents. In Section 4, we present an analysis for the case of stable but not distinct Lyapunov exponents. The technique is based upon a block version of the argument for distinct Lyapunov exponents with some key differences together with a Gronwall type bound due to Vinograd to control the Lyapunov exponents within a block which gives equal exponents. Finally, in Section 5, we illustrate our analysis with a numerical result.

2. BACKGROUND

Consider the non-autonomous linear system

$$\dot{x} = A(t)x, \quad t \geqslant 0, \tag{2.1}$$

where we will assume that the function A: $I\!R^{+} \to I\!R^{n\times n}$ is bounded. Let X be a fundamental matrix solution of (2.1) and consider

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \ln ||X(t)e_i||, \quad i = 1, \ldots, n, \tag{2.2}$$

where the e_i's are the standard unit vectors. Here, and everywhere else in this work, the norm is the 2-norm for vectors and the induced norm for matrices. When $\sum_{i=1}^{n} \lambda_i$ is minimized with respect to all possible fundamental matrix solutions, then the λ_i's are called Lyapunov exponents, and the corresponding fundamental matrix solution is called normal (see [16]).

The Lyapunov exponents are said to be stable if they are continuous with respect to perturbations in the coefficient matrix. That is, if "for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\sup_{t\in I\!R^{+}} ||F(t)|| < \delta(\epsilon)$ implies

$$|\lambda_i - \hat{\lambda}_i| < \epsilon, \quad i = 1, \ldots, n, \tag{2.3}$$

where the $\hat{\lambda}_i$'s are the (ordered) Lyapunov exponents of the perturbed system $\dot{x} = [A(t) + F(t)]x$".

Our aim in this paper is to determine the dependence of δ on ϵ.

If the Lyapunov exponents are distinct, then the exponents are stable (see [1]) if and only if X is an *integrally separated* fundamental matrix solution. X is integrally separated if for $i = 1, \ldots, n-1$, there exist $a > 0$ and $1 \geqslant d > 0$ such that

$$\frac{||X(t)e_i||}{||X(s)e_i||} \cdot \frac{||X(s)e_{i+1}||}{||X(t)e_{i+1}||} \geqslant d e^{a(t-s)}, \tag{2.4}$$

for all t, s: $t \geqslant s \geqslant 0$.

A commonly used technique for the approximation of Lyapunov exponents involves determining a time-dependent othogonal change of variables Q that transforms the given fundamental matrix solution (equivalently, the associated coefficient matrix function) to upper triangular. It has been shown in [9, 10] that stable Lyapunov exponents may be determined as appropriate time averages of the diagonal elements of the upper triangular coefficient matrix function. It is well known (e.g., see [9]) that such Q is unique and satisfies

$$\dot{Q} = Q(t)S(Q, A), \qquad Q(0) = Q_0, \tag{2.5}$$

where we have set $S := Q^T(t)\dot{Q}(t)$, with entries

$$S(Q(t), A(t))_{ij} = \begin{cases} (Q^T(t)A(t)Q(t))_{ij}, & i > j, \\ 0, & i = j, \\ -(Q^T(t)A(t)Q(t))_{ji}, & i < j. \end{cases} \tag{2.6}$$

So, if Q is known, then R satisfies the transformed system

$$\dot{R} = B(t)R, \qquad R(0) = R_0, \tag{2.7}$$

where we have set

$$B(t) := Q^T(t)A(t)Q(t) - Q^T(t)\dot{Q}(t) \tag{2.8}$$

and B is upper triangular by the way that S has been defined. Stable Lyapunov exponents are then obtained (see [10]) from the diagonal $D(t)$ of $B(t)$ as

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t}\int_0^t B_{ii}(s)ds. \tag{2.9}$$

Finally, we notice that integral separation (2.4) can be rephrased in terms of integral separation of the diagonal D of the coefficients B:

$$\int_s^t \big(D_{ii}(\tau) - D_{i+1,i+1}(\tau)\big)\, d\tau \geqslant a(t-s) + \ln(d), \quad t \geqslant s, \tag{2.10}$$

where $a > 0$ and $0 < d \leqslant 1$, for all $t, s \ : \ t \geqslant s \geqslant 0$.

QR-based methods for the approximation of Lyapunov exponents (see [8]) are a numerical realization of the transformation to the form (2.7). Unavoidably, this cannot be done exactly, since the relevant differential equations cannot be integrated exactly. Furthermore, it is unreasonable to expect that the differential equations can be integrated so to obtain globally accurate approximations (recall that we are interested in the limit $t \to +\infty$). So, we can only expect to be able to control the *local error* on each step. Recall that this is the error committed on one single integration step. Here below, we let η be a bound on the local error encurred upon during numerical integration of the differential equations.

The backward error result obtained in [11] is summarized next. For precise statements we refer to the original work, here we are content with the general flavor of the result.

Summary 2.1. *With a numerical realization of the QR methods, in the limit as $t \to +\infty$, we are not obtaining the Lyapunov exponents of the triangular system* (2.7 *and* 2.8)*, but rather the Lyapunov exponents of the perturbed system*

$$\dot{\hat{R}} = (B(t) + F(t))\,\hat{R}, \tag{2.11}$$

where B is given in (2.8)*, and F is bounded as*

$$\|F\| \leqslant c\eta + O(\eta^2) \tag{2.12}$$

with the main contribution to the magnification factor c being the departure from normality of the exact triangular factor R.

Next, consider the unperturbed and perturbed triangular systems

$$\dot{R} = B(t)R, \qquad \dot{\hat{R}} = [B(t) + F(t)]\hat{R}, \tag{2.13}$$

where we will assume that $\|F\| \leqslant \delta$. Now, write $R = R_D + R_U$, where R_D is the diagonal part of R and R_U is the (strictly) upper part of R, so that $R = (I + R_U R_D^{-1})R_D =: ZR_D$. Accordingly, we have the unperturbed and perturbed diagonal systems

$$\dot{R}_D = D(t)R_D, \qquad \dot{\hat{R}}_D = [D(t) + E(t)]\hat{R}_D, \tag{2.14}$$

where $D(t) = \mathrm{diag}(B(t))$ and $E = Z^{-1}FZ$.

Lemma 2.1. *If the systems in* (2.14) *are integrally separated, then the Lyapunov exponents of the systems with $A(t) = B(t)$ and $A(t) = D(t)$ are equal and the Lyapunov exponents of the systems with $A(t) = B(t) + F(t)$ and $A(t) = D(t) + E(t)$ are also equal.*

Proof. The first statement is a consequence of [9, Theorem 5.1]. The second statement follows from the fact that $Z(t)$ is a Lyapunov transformation. □

In the next section, we quantify the error in the Lyapunov exponents by working with the diagonal systems (2.14). We let $\omega := \|E\|$, and observe that

$$\omega \leqslant \|F\|\mathrm{cond}(Z) \leqslant \delta \ \mathrm{cond}(Z) \quad \text{where} \quad \mathrm{cond}(Z) = \sup_{t\geqslant 0} \|Z(t)\| \cdot \|Z^{-1}(t)\|. \tag{2.15}$$

Remark 2.1. A uniform bound on $\|Z(t)\|$ can be obtained using results in [10]. Consider the upper triangular system in (2.13) with coefficient matrix function B. Suppose $|B_{ij}(t)| \leqslant M$ for $i < j$ and suppose for $k - i = j \geqslant 1$,

$$\int_s^t B_{ii}(r) - B_{kk}(r)dr \geqslant a_j(t-s) - d_j, \quad t \geqslant s. \tag{2.16}$$

Let $K_j = e^{d_j}/a_j$. Then using Lemma 4.1 and Theorem 4.2 of [10], if $j = k - i$, then the change of variables $Z(t)$ enjoys the following uniform bound

$$|Z_{ik}(t)| \leqslant K_j[M + M^2 K_{j-1} + \cdots + M^j K_1] =: \omega_j \tag{2.17}$$

and for $\overline{\omega} := \max\{1, \omega_1, \ldots, \omega_{n-1}\}$, $\|Z(t)\| \leqslant \overline{\omega}$. A similar bound may be obtained for $\|Z^{-1}(t)\|$.

3. FORWARD ERROR ANALYSIS – THE STABLE DISTINCT CASE

Consider the time dependent, homogeneous linear ODE with piecewise continuous coefficient matrix function,

$$\dot{x} = A(t)x, \qquad A(t) = D(t) + E(t), \tag{3.1}$$

where $D(t) = \mathrm{diag}(D_{11}(t), \ldots, D_{nn}(t))$, $\sup_t ||D(t)|| \leqslant M$, and $\sup_t ||E(t)|| \leqslant \omega$. We will assume that the diagonal matrix function D is integrally separated (see (2.10)) and for $i < j$ we set

$$\int_s^t D_{ii}(\tau) - D_{jj}(\tau) \geqslant a_{ij}(t-s) - d_{ij}, \quad t \geqslant s, \tag{3.2}$$

where $a_{ij} > 0$ and $d_{ij} \geqslant 0$. Define $a > 0$ and $d \geqslant 0$ as a_{ij} and d_{ij}, respectively, that maximize $e^{d_{ij}}/a_{ij}$ for $i < j$.

Let $\{\mu_i\}_{i=1}^n$ be the Lyapunov exponents of (3.1) and $\{\lambda_i\}_{i=1}^n$ the Lyapunov exponents of the unperturbed system $\dot{x} = D(t)x$. We show that there exists an orthogonal change of variables to upper triangular satisfying (2.5) with $A(t) = D(t) + E(t)$ that remains, under reasonable conditions, a small perturbation of the identity given the initial condition $Q(0) = Q_0 = I$. Of course, we notice that if $||E|| \leqslant \omega$, then $||Q^T E Q|| \leqslant \omega$.

Lemma 3.1. *If $\omega < \omega_+(\alpha, K, M)$, then $|Q_{ij}(t)| \leqslant \rho$ for $i \neq j$ and all $t \geqslant 0$, where $\rho = \beta \cdot \omega$, $\beta = \alpha K$, $\alpha > 1$, $K = e^d/a$, and*

$$\omega_+(\alpha, K, M) := \left(\sqrt{a_1^2 + 4(\alpha - 1)a_2} - a_1\right)/(2a_2), \tag{3.3}$$

where $a_2 = n^2\beta^2[M\beta + 2]$ and $a_1 = n\beta[2M\beta + 1]$.

Proof. For $i < j$ we have

$$\begin{aligned}\dot{Q}_{ij} &= -Q_{ij}[D_{ii} - D_{jj}] + \left(Q_{ij}[D_{ii} - D_{jj}] + e_i^T (Q[S(Q, D) + S(Q, E)])e_j\right)\\ &=: -Q_{ij}[D_{ii} - D_{jj}] + q_{ij}(t, Q, \omega)\end{aligned} \tag{3.4}$$

and a similar formula for $i > j$. We want to show that if the conditions of the theorem are satisfied and $Q(0) = I$, then $|Q_{ij}(t)| \leqslant \rho$ for all $i \neq j$ and $t \geqslant 0$. The proof involves applying [14, Theorem IV.2.1].

Using the non-linear variation of constants formula we have for $Q(0) = I$ and $i < j$,

$$\begin{aligned}Q_{ij}(t) &= \int_0^t e^{-\int_\tau^t (D_{ii}(r) - D_{jj}(r))dr} q_{ij}(\tau, Q(\tau), \omega)d\tau\\ &\leqslant \int_0^t e^{-a(t-\tau)+d} |q_{ij}(\tau, Q(\tau), \omega)|d\tau.\end{aligned} \tag{3.5}$$

Thus, $\sup_t |Q_{ij}(t)| \leqslant K \sup_t |q_{ij}(t, Q(t), \omega)|$ where $K = e^d/a$. We have

$$\begin{aligned}|q_{ij}(t, Q, \omega)| &\leqslant |q_{ij}(t, Q, \omega) - q_{ij}(t, I, \omega)| + |q_{ij}(t, I, \omega)|\\ &\leqslant \eta(\rho, \omega)\rho + N(\omega),\end{aligned} \tag{3.6}$$

where since $S(I, D) = 0$ and $S(I, E) = E_L - E_L^T$ where E_L is the strict lower triangular portion of E, $N(\omega) \leqslant \omega$. To bound $\eta(\rho, \omega)$ write

$$\begin{aligned}q_{ij}(t, Q, \omega) &= q_{ij}^D(t, Q, \omega) + q_{ij}^E(t, Q, \omega)\\ &:= \left(Q_{ij}[D_{ii} - D_{jj}] + e_i^T QS(Q, D)e_j\right) + e_i^T QS(Q, E)e_j\end{aligned} \tag{3.7}$$

and first consider $q^D_{ij}(t, Q, \omega)$. We have, writing $Q(t)=[q_1(t)|\cdots|q_n(t)]$,

$$
\begin{aligned}
& q^D_{ij}(t, Q, \omega) - q^D_{ij}(t, I, \omega) \\
&\quad = q^D_{ij}(t, Q, \omega) = Q_{ij}[D_{ii} - D_{jj}] + \left[-\sum_{k=1}^{j-1} Q_{ik} q_k^T + \sum_{k=j+1}^{n} Q_{ik} q_k^T \right] D q_j \\
&\quad = Q_{ij}\left[D_{ii} - D_{jj}\right] + \sum_{l=1}^{n} D_{ll} \left[Q_{lj} \cdot \left\{ -\sum_{k=1}^{j-1} Q_{ik} Q_{lk} + \sum_{k=j+1}^{n} Q_{ik} Q_{lk} \right\}\right] \\
&\quad = D_{ii} \left[Q_{ij} \cdot \left\{ 1 - \sum_{k=1}^{j-1} Q_{ik}^2 + \sum_{k=j+1}^{n} Q_{ik}^2 \right\}\right] \\
&\qquad + D_{jj} \left[-Q_{ij} + Q_{jj} \cdot \left\{ -\sum_{k=1}^{j-1} Q_{ik} Q_{jk} \sum_{k=j+1}^{n} Q_{ik} Q_{jk} \right\}\right] \\
&\qquad + \sum_{l \neq i,j} D_{ll} \left[Q_{lj} \cdot \left\{ -\sum_{k=1}^{j-1} Q_{ik} Q_{lk} + \sum_{k=j+1}^{n} Q_{ik} Q_{lk} \right\}\right]. \qquad (3.8)
\end{aligned}
$$

By orthogonality we have

$$
1 = \sum_{k=1}^{n} Q_{ik}^2 \quad \text{and} \quad Q_{ij} Q_{jj} = -\sum_{k \neq j} Q_{ik} Q_{jk}, \qquad (3.9)
$$

so for $i < j$,

$$
\begin{aligned}
q^D_{ij}(t, Q, \omega) = {} & D_{ii} \left[Q_{ij} \left(Q_{ij}^2 + 2 \sum_{k=j+1}^{n} Q_{ik}^2 \right)\right] \\
& + D_{jj} \left[-Q_{ij}(1 - Q_{jj}^2) + 2 Q_{jj} \sum_{k=j+1}^{n} Q_{ik} Q_{jk} \right] \\
& + \sum_{l \neq i,j} D_{ll} \left[Q_{lj} \left(-\sum_{k=1}^{j-1} Q_{ik} Q_{lk} + \sum_{k=j+1}^{n} Q_{ik} Q_{lk} \right)\right]. \qquad (3.10)
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\left|q_{ij}^D(t,Q,\omega)-q_{ij}^D(t,I,\omega)\right| \\
&\quad\leqslant M\rho\Big[(2(n-j)+1)\rho^2+(n-1)\rho^2+2(n-j)\rho \\
&\qquad+(n-2)(n-1)\rho^2\Big]\rho \\
&\quad\leqslant M\rho\Big[(n^2-2j+2)\rho^2+2(n-j)\rho\Big] \\
&\quad\leqslant M\rho\left(n^2\rho^2+2n\rho\right).
\end{aligned} \tag{3.11}$$

For the term $q_{ij}^E(t,Q,\omega)$, using (2.6), we have for $i<j$ (and similarly for $i>j$)

$$\begin{aligned}
q_{ij}^E(t,Q,\omega)-q_{ij}^E(t,I,\omega) &= [QS(Q,E)-S(I,E)]_{ij} \\
&= -\sum_{k=1}^{j-1} Q_{ik}S(Q,E)_{jk}+\sum_{k=j+1}^{n} Q_{ik}S(Q,E)_{kj}+E_{ji} \\
&= E_{ji}(1-Q_{jj}+\sum_{k\neq i,k=1}^{n} Q_{ik}^2 Q_{jj}) \\
&\quad-\ Q_{ii}\sum_{(l,m)\neq(j,i),l,m=1}^{n} Q_{lj}E_{lm}Q_{mi} \\
&\quad+\sum_{k=j+1}^{n} Q_{ik}S(Q,E)_{kj}.
\end{aligned} \tag{3.12}$$

Thus,

$$\begin{aligned}
|q_{ij}^E(t,Q,\omega)-q_{ij}^E(t,I,\omega)| &\leqslant \rho\omega[1+(n-1)\rho+(n^2-1)\rho+n-j] \\
&\leqslant \rho\omega n[1+(n+1)\rho] \\
&\leqslant \rho\omega n[1+2n\rho]=:\rho\eta_{ij}^E.
\end{aligned} \tag{3.13}$$

So, we have

$$\eta(\rho,\omega)\leqslant M\rho[n^2\rho+2n]+\omega n[1+2n\rho] \tag{3.14}$$

and finally from (3.6) we obtain

$$\sup_t |Q_{ij}(t)|\leqslant K\,(\omega+\eta(\rho,\omega)\rho).$$

Theorem IV.2.1 of [14] may be applied if $K[\eta(\rho,\omega)\rho + N(\omega)] < \rho$. Using the bound on $\eta(\rho,\omega)$ in (3.14) and the form for $\rho = \beta\omega$, this condition is equivalent to $a_2\omega^2 + a_1\omega + (1-\alpha) < 0$ or $\omega < \omega_+(\alpha, K, M)$ with $\omega_+(\alpha, K, M)$ given in (3.3). □

Remark 3.1. If we fix $\alpha = 2$ (as suggested by a small calculation), we obtain the value of $\omega_+(\alpha, K, M)$

$$\omega_+ = \frac{1}{4nKX\beta}\left(\sqrt{1+4X/(4M\beta+1)}-1\right), \qquad X = \frac{M\beta+2}{2M\beta+1}. \quad (3.15)$$

The asymptotic regimes are of interest.

- If $M\beta \approx 0$ (i.e., $K \approx 0$, that is a is large) then $X \approx 2$ and $\omega_+ \approx 1/4nK$, that is ω_+ can be large. This is the case when there is strong separation in the diagonal of the coefficient matrix B. It is the most benign case.
- If $M\beta \gg 1$, then $X \approx 1/2$, and $\omega_+ \approx \frac{1}{(2nK)(4M\beta+1)} \ll 1$. This is the case when either the coefficients are very large or there is weak integral separation in the diagonal of B. This is the hardest case.
- The case of $M\beta \approx 1$ gives $\omega_+ \approx 1/2nK$, which may be either large or not depending on K (and n).

In all cases, there is a $O(1/n)$ dependence in ω_+. This is unavoidable given the global point of view we adopted. Perhaps, a different analysis using integral separation constants a_k, d_k that correspond to the integral separation between diagonal elements i and j with $k = i - j$ could be performed to remove this dependence on the dimension of the problem.

Using Lemma 3.1, we obtain our main result.

Theorem 3.2. *Assume that the principal matrix solutions associated to both unperturbed and perturbed systems in* (3.1) *are integrally separated. Then, with same notation and assumptions of Lemma 3.1, if* $\omega < \omega_+(\alpha, K, M)$ *where* ω_+ *is given* (3.3)*, then*

$$|\mu_i - \lambda_i| \leqslant \rho^2 \sum_{k\neq i} \gamma_{ik} + \omega, \quad (3.16)$$

where $\rho = \beta\cdot\omega$*,* $\beta = \alpha K$*,* $K = e^d/a$*, and* $\gamma_{ik} = \sup_t |D_{ii}(t) - D_{kk}(t)|$.

Proof. We have for $i = 1, \ldots, n$ that

$$(Q^TAQ)_{ii} - D_{ii} = (Q_{ii}^2 - 1)\cdot D_{ii} + \sum_{k\neq i} Q_{ki}^2 D_{kk} + (Q^TEQ)_{ii}. \quad (3.17)$$

Then

$$
\begin{aligned}
\mu_i - \lambda_i &:= \limsup_{t\to\infty} \frac{1}{t}\int_0^t (Q^T A Q)_{ii}(\tau)d\tau - \limsup_{t\to\infty} \frac{1}{t}\int_0^t D_{ii}(\tau)d\tau \\
&\leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left[(Q_{ii}^2 - 1)\cdot D_{ii} + \sum_{k\neq i} Q_{ki}^2 D_{kk} + (Q^T E Q)_{ii} \right] d\tau \\
&= \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left[\sum_{k\neq i} Q_{ki}^2 (D_{kk} - D_{ii}) + (Q^T E Q)_{ii} \right] d\tau
\end{aligned} \tag{3.18}
$$

and the result follows since $||Q^T E Q|| = \|E\| \leqslant \omega$. □

4. THE NON-INTEGRALLY SEPARATED CASE

Here we consider the case of non-distinct Lyapunov exponents. As a starting point we employ block analogues of the previous results. The arguments are even more technical than before, in part because the characterization of stability for the exponents is more complicated than in the case of stable and distinct exponents. We will review this characterization next.

We need some definitions before stating the theorem due to Bylov and Izobov [5] and Millionshchikov [18] on stability of Lyapunov exponents.

Definition 4.1. [1]. Bounded, measurable functions, $l(t)$ and $u(t)$, defined on $I\!R^+$, are said to be lower and upper functions for (3.1) if for any solution x of (3.1) and any $\epsilon > 0$ there exist positive constants $d_{l,\epsilon}$ and $D_{u,\epsilon}$ such that

$$
d_{l,\epsilon} \exp\left(\int_s^t (l(\tau) - \epsilon)d\tau\right) \leqslant \frac{||x(t)||}{||x(s)||} \leqslant D_{u,\epsilon} \exp\left(\int_s^t (u(\tau) + \epsilon)d\tau\right) \tag{4.1}
$$

for $t \geqslant s \geqslant 0$ and the quantities $d_{l,\epsilon}$, $D_{u,\epsilon}$ are independent of t and s.

For (3.1), we define the following two quantities:

$$
\Omega = \inf_u \left\{ \limsup_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds \right\}, \tag{4.2}
$$

where the infimum is taken over all upper functions, called upper central exponent in [1], and

$$
\bar{\omega} = \sup_l \left\{ \limsup_{t\to\infty} \frac{1}{t}\int_0^t l(s)ds \right\}, \tag{4.3}
$$

where the supremum is taken over all lower functions.

We are ready to state the stability theorem for Lyapunov exponents in the case of non-distinct Lyapunov exponents.

Theorem 4.1. [5, 18], [1, Theorem 5.4.9]. *The Lyapunov exponents of* $\dot{x} = A(t)x$ *are stable if and only if there exists a Lyapunov transformation* T *that transforms* $\dot{x} = A(t)x$ *to the block diagonal form*

$$\dot{z} = \mathrm{diag}[B_{11}(t), \ldots, B_{mm}(t)]z, \tag{4.4}$$

where each $B_{kk}(t)$ *is upper triangular of dimension* n_k, $k = 1, \ldots, m$. *Moreover, for each block system* $\dot{z}_k = B_{kk}(t)z_k$, $k = 1, \ldots, m$, *we have:*

(i) *all solutions of the block have the same Lyapunov exponents,* Λ_k, *and* $\bar{\omega}_k = \Omega_k = \Lambda_k$;
(ii) *for any* b_i *an arbitrary diagonal element of* B_{ii} *and* b_{i+1} *an arbitrary diagonal element of* $B_{i+1,i+1}$, b_i *and* b_{i+1} *are integrally separated.*

Now, recall that we are interested in studying the difference in the exponents of the unperturbed and perturbed triangular systems (2.13):

$$\dot{R} = B(t)R, \qquad \dot{\hat{R}} = [B(t) + F(t)]\hat{R}, \qquad \|F\| \leqslant \delta.$$

Let us partition R, B, and F, in a block way, with the partitioning inherited by the integral separation in the system, and furthermore write $R = R_D + R_U$, where R_D is the block diagonal part of R and R_U is the block upper part of R. That is: $R_D = \mathrm{diag}[R_{11}(t), \ldots, R_{mm}(t)]$, and so forth. Again, we can write $R = (I + R_U R_D^{-1})R_D =: ZR_D$. Accordingly, we have the unperturbed and perturbed diagonal systems as we did in (2.14):

$$\dot{R}_D = D(t)R_D, \qquad \dot{\hat{R}}_D = [D(t) + E(t)]\hat{R}_D, \tag{4.5}$$

where $D(t) = \mathrm{diag}[B_{11}(t), \ldots, B_{mm}(t)]$ (see (4.4)). In the present context, we replace the condition of integral separation (2.4) by the following block condition (which follows easily from point (ii) of Theorem 4.1)

$$\|R_{ii}^{-1}(t)R_{ii}(s)\| \; \|R_{i+1,i+1}^{-1}(s)R_{i+1,i+1}(t)\| \leqslant e^{\tilde{d}}e^{-\tilde{a}(t-s)}, \quad t \geqslant s, \tag{4.6}$$

where $\tilde{a} > 0$ and $\tilde{d} \geqslant 0$, for all $t, s \; : \; t \geqslant s \geqslant 0$, and $i = 1, \ldots, m-1$.

With these preparations, the block analog of Lemma 2.1 still holds, and again we have that $\omega := \|E\|$ is bounded as in (2.15), the difference being that Z is now a block matrix.

Next, we want to show that there is an orthogonal change of variables that brings $D + E$ (where D is upper triangular blocks on the diagonal) to block upper triangular, though not necessarily anymore with triangular diagonal blocks. The basic idea to achieve our goal is to use a block form of the construction that we used in the integrally separated case together with a careful choice for certain entries of the skew-symmetric matrix function $S(Q, D)$.

Write the equations for the orthogonal change of variables in block form, where the size of the blocks is determined by the integral separation in the system, as

$$\dot{Q}^{(ij)} = \sum_k Q^{(ik)} S^{(kj)}, \quad Q^{(ij)}, S^{(ij)} \in I\!R^{n_i \times n_j}. \tag{4.7}$$

For $j < i$, with obvious notation, we have

$$\begin{aligned} S^{(ij)} \equiv S_D^{(ij)} + S_E^{(ij)} &:= [Q^T D Q]^{(ij)} + S_E^{(ij)} \\ &= \sum_k \left(Q^T\right)^{(ik)} D^{(kk)} Q^{(kj)} + S_E^{(ij)} \\ &= \sum_k \left(Q^{(ki)}\right)^T D^{(kk)} Q^{(kj)} + S_E^{(ij)} \end{aligned} \tag{4.8}$$

and for $i < j$, $S^{(ij)}$ is determined by skew-symmetry; that is: $S^{(ij)} = -S^{(ji)^T}$, for $i < j$. We have yet to determine the $S^{(jj)}$. The obvious choice would be to set $S^{(jj)} = 0$, but below we will adopt a more useful choice.

For $i < j$ we write the equation for $\dot{Q}^{(ij)}$ as

$$\begin{aligned} \dot{Q}^{(ij)} &= -\left(D^{(ii)T} Q^{(ij)} - Q^{(ij)} D^{(jj)T}\right) \\ &\quad + \left[\left(D^{(ii)T} Q^{(ij)} - Q^{(ij)} D^{(jj)T}\right) + \sum_k Q^{(ik)} S^{(kj)}\right] \end{aligned} \tag{4.9}$$

and for $i > j$ we write

$$\begin{aligned} \dot{Q}^{(ij)} &= -\left(Q^{(ij)} D^{(jj)} - D^{(ii)} Q^{(ij)}\right) + \Big[\left(Q^{(ij)} D^{(jj)} - D^{(ii)} Q^{(ij)}\right) \\ &\quad + \sum_k Q^{(ik)} S^{(kj)}\Big]. \end{aligned} \tag{4.10}$$

Next we consider the block analogues of (3.8) and (3.10) and the orthogonality condition (3.9) with the key difference in this block case being that the term $D_{jj}\left[-Q_{ij}(1 - Q_{jj}^2)\right]$ in (3.10) is replaced by the term

$$-Q^{(ij)}\left[D^{(jj)T}-Q^{(jj)T}D^{(jj)T}Q^{(jj)}\right]$$
$$=Q^{(ij)}\left[D^{(jj)T}\left(Q^{(jj)}-I\right)+\left(Q^{(jj)T}-I\right)D^{(jj)T}\right.$$
$$\left.+\left(Q^{(jj)T}-I\right)D^{(jj)T}\left(Q^{(jj)}-I\right)\right]. \tag{4.11}$$

To ensure that this term is sufficiently small, we must have $Q^{(jj)}\approx I$ uniformly in t and this motivates finding a choice $S^{(jj)}\neq 0$. To understand the choice we make in (4.13) below, observe that for all $p=1,\dots,m$, we have

$$\frac{d}{dt}\left(Q^{(pp)}-I\right)=-\left(Q^{(pp)}-I\right)+\left[(Q^{(pp)}-I)+Q^{(pp)}S^{(pp)}\right.$$
$$\left.+\sum_{k\neq p}Q^{(pk)}S^{(kp)}\right]. \tag{4.12}$$

Now, we want the term in brackets in (4.12) to be of $O(\rho^2)$ if the terms $Q^{(pk)}$ and $(Q^{(pp)}-I)$ are $O(\rho)$. We notice that by the form of $S^{(kj)}$ in (4.8) the term $\sum_{k\neq p}Q^{(pk)}S^{(kp)}$ in (4.12) is $O(\rho^2)$, while if the terms $S^{(jj)}$ and $Q^{(ij)}$ are $O(\rho)$, then the terms $Q^{(ij)}S^{(jj)}$ in (4.9) and (4.10) are also $O(\rho^2)$. Therefore, the term that requires attention is the first term in brackets in (4.12): $(Q^{(pp)}-I)+Q^{(pp)}S^{(pp)}$.

We are ready to select a useful choice for $S^{(pp)}$. For $k>l$, define

$$S_{kl}^{(pp)}=-Q_{kk}^{(pp)}Q_{kl}^{(pp)} \tag{4.13}$$

with the remaining portion of $S^{(pp)}$ determined by skew-symmetry. To see why this is a judicious choice, from (4.12) we see that in order to ensure $Q^{(pp)}\approx I$ uniformly in t we need to show that if $|Q_{ij}^{(pp)}|\leqslant\rho$ for all $i\neq j$, then $|[(Q^{(pp)}-I)+Q^{(pp)}S^{(pp)}]_{kl}|\leqslant C\rho^2$ for all k,l and some constant C. Now, we have

$$[Q^{(pp)}(I+S^{(pp)})]_{kl}=\sum_j Q_{kj}^{(pp)}\left(I+S^{(pp)}\right)_{jl}$$
$$=\sum_{j>l}Q_{kj}^{(pp)}\left(I+S^{(pp)}\right)_{jl}+Q_{kl}^{(pp)}\left(I+S^{(pp)}\right)_{ll}$$

$$
\begin{aligned}
&+\sum_{j<l} Q_{kj}^{(pp)}\left(I+S^{(pp)}\right)_{jl} \\
&= -\sum_{j=l+1}^{n_p} Q_{kj}^{(pp)} Q_{jj}^{(pp)} Q_{jl}^{(pp)} + Q_{kl}^{(pp)} \\
&+\sum_{j=1}^{l-1} Q_{kj}^{(pp)} Q_{ll}^{(pp)} Q_{lj}^{(pp)}
\end{aligned} \tag{4.14}
$$

There are three cases to consider: $k>l$, $k=l$, and $k<l$. If $k>l$, then

$$
\begin{aligned}
\left[Q^{(pp)}(I+S^{(pp)})\right]_{kl} &= Q_{kl}^{(pp)}\left(1-(Q_{kk}^{(pp)})^2\right) \\
&- \sum_{j=l+1,\, j\neq k}^{n_p} Q_{kj}^{(pp)} Q_{jj}^{(pp)} Q_{jl}^{(pp)} \\
&+ Q_{ll}^{(pp)} \sum_{j=1}^{l-1} Q_{kj}^{(pp)} Q_{lj}^{(pp)}.
\end{aligned} \tag{4.15}
$$

If $k=l$, then

$$
\begin{aligned}
\left[Q^{(pp)}\left(I+S^{(pp)}\right)\right]_{kk} &= 1+\left(Q_{kk}^{(pp)}-1\right) \\
&- \sum_{j=k+1}^{n_p} Q_{kj}^{(pp)} Q_{jj}^{(pp)} Q_{jk}^{(pp)} \\
&+ Q_{kk}^{(pp)} \sum_{j=1}^{k-1} \left(Q_{kj}^{(pp)}\right)^2
\end{aligned} \tag{4.16}
$$

and observe that for $Q_{kk}^{(pp)} \neq -1$, $Q_{kk}^{(pp)} - 1 = ((Q_{kk}^{(pp)})^2 - 1)/(Q_{kk}^{(pp)} + 1)$. If $k<l$, using the orthogonality condition (used relatively to the entire Q)

$$
\sum_{j=1}^{n} Q_{kj} Q_{lj} = 0, \tag{4.17}
$$

we have

$$\left[Q^{(pp)}(I+S^{(pp)})\right]_{kl} = Q^{(pp)}_{kl}\left(1-(Q^{(pp)}_{ll})^2\right) - \sum_{j=l+1}^{n_p} Q^{(pp)}_{kj}Q^{(pp)}_{jj}Q^{(pp)}_{jl} - Q^{(pp)}_{ll} \sum_{\substack{j=1 \\ Q_{kj}\neq Q^{(pp)}_{kj} \\ Q_{lj}\neq Q^{(pp)}_{lj}}}^{n} Q_{kj}Q_{lj}. \tag{4.18}$$

Formulas (4.15, 4.16, 4.18) will be needed in the proof of Theorem 4.3. Also the following Lemma will be needed in the proof, in which case the matrices A and B of the lemma will be the triangular matrices which are the diagonal blocks of the coefficient matrix D in (4.5).

Lemma 4.2. *Consider*

$$\dot{W}(t) = -[A(t)W(t) - W(t)B(t)] + F(t), \tag{4.19}$$

where $W(t), F(t) \in I\!R^{p\times q}$, $A(t) \in I\!R^{p\times p}$, *and* $B(t) \in I\!R^{q\times q}$ *with* A *and* B *both bounded, piecewise continuous. If there exists* $\tilde{a} > 0$ *and* $\tilde{d} \geqslant 0$ *such that* (*recall* (4.6))

$$||X^{-1}(t)\ X(s)|| \cdot ||Y^{-1}(s)\ Y(t)|| \leqslant e^{\tilde{d}-\tilde{a}(t-s)}, \quad t \geqslant s \tag{4.20}$$

for fundamental matrix solutions X, Y *satisfying* $\dot{X} = XA$ *and* $\dot{Y} = YB$, *then there exists a solution to* (4.19) *such that* $\sup_{t\geqslant 0} \|X^{-1}(t)[\int_0^t X(s)F(s) Y^{-1}(s)ds]Y(t)\| \leqslant K \cdot \sup_t \|F(t)\|$ *where* $K = e^{\tilde{d}}/\tilde{a}$.

Proof. By the variation of constants formula with $W(0) = 0$ we have

$$\begin{aligned} \|W(t)\| &= \left\|X^{-1}(t)\left[\int_0^t X(s)F(s)Y^{-1}(s)ds\right]Y(t)\right\| \\ &\leqslant \left\|X^{-1}(t)\right\|\left[\int_0^t \|X(s)\|\,\|F(s)\|\,\left\|Y^{-1}(s)\right\|\right]\|Y(t)\| \\ &\leqslant \int_0^t e^{\tilde{d}-\tilde{a}(t-s)}ds \sup_t \|F(t)\|. \end{aligned} \tag{4.21}$$

□

Remark 4.1. The constants $\tilde{a}$ and $\tilde{d}$ can be bounded in terms of the integral separation constants between the diagonal elements of different blocks as in part (ii) of Theorem 4.1 and the quantities $\epsilon, d_{l,\epsilon}, D_{u,\epsilon}$ that bound the upper and lower functions for the blocks. Using Theorems 5.1 and 5.2 of [10] we can quantify the quantities that characterize the upper and lower functions in terms of so-called Lillo conditions and bounds on the off diagonal elements within the block.

We are now ready to state and prove a theorem providing the existence of an orthogonal change of variables that stays uniformly close to the identity and brings the perturbed diagonal system in (4.5) to block upper triangular, though the diagonal blocks are not necessarily triangular.

Theorem 4.3. *For $i, j = 1, \ldots, m$, let*

$$\kappa_{ij}(t) = \left\| Q^{(ij)}(t) \right\|, \quad \text{for} \quad i \neq j, \qquad \text{and} \quad \kappa_{jj}(t) = \left\| Q^{(jj)}(t) - I \right\|. \tag{4.22}$$

Let $\beta = \alpha K$, with $\alpha > 1$, $K = \max\{e^{\tilde{d}}/\tilde{a}, 1\}$ with $\tilde{a}$ and $\tilde{d}$ as in (4.20), $\rho = \beta \cdot \omega$, where ω is a bound on $\sup_t \|E(t)\|$, E in (4.5), and let $M_{jj} = \sup_t \|D^{(jj)}(t)\|$, and $M = \max_j M_{jj}$. Finally, let $a_1 = \beta[c_1\beta + m]$, and $a_2 = \beta^2[2m^2 + \beta c_2]$, where $c_1 = \max\left(4M(m-1) + (n_j^{\max} - 1),\ 2M(m-1) + 2nn_j^{\max})\right)$, where $n_j^{\max} = \max_j n_j$, and $c_2 = M(m-1)\max(7,\ m)$.

Then, for $Q(0) = I$, if

$$\omega < \omega_+ := \left(\sqrt{a_1^2 + 4(\alpha - 1)a_2} - a_1\right) / (2a_2) \tag{4.23}$$

then $\kappa_{ij}(t) < \rho$ for all $t \geqslant 0$ and all $i, j = 1, \ldots, m$.

Proof. Using the equation for $\dot{Q}^{(ij)}$ in (4.7) together with the definition for $S^{(ij)}$ in (4.8) and the definition for $S^{(jj)}$ in (4.13), we have for $i < j$ using (4.9) (and similarly for $i > j$ using (4.10)),

$$\dot{Q}^{(ij)} = -\left(D^{(ii)T} Q^{(ij)} - Q^{(ij)} D^{(jj)T}\right) + q^{(ij)}(t, Q, \omega), \tag{4.24}$$

where

$$q^{(ij)}(t, Q, \omega) = \left(D^{(ii)T} Q^{(ij)} - Q^{(ij)} D^{(jj)T}\right) + \sum_k Q^{(ik)} S^{(kj)}. \tag{4.25}$$

For $i = j$, using (4.12), we have

$$\frac{d}{dt}\left(Q^{(jj)} - I\right) = -\left(Q^{(jj)} - I\right) + q^{(jj)}(t, Q, \omega), \tag{4.26}$$

where

$$q^{(jj)}(t, Q, \omega) = \left(Q^{(jj)} - I\right) + \sum_k Q^{(jk)} S^{(kj)}. \tag{4.27}$$

By the non-linear variation of constants formula for $i<j$ (and similarly for $i>j$),

$$
\begin{aligned}
Q^{(ij)}(t) &= Y^{(ii)}(t)\left[\int_0^t (Y^{(ii)}(\tau))^{-1} q^{(ij)}(\tau, Q(\tau), \omega)(Z^{(jj)}(\tau))^{-1} d\tau\right] Z^{(ii)}(t)\\
&\leqslant K_{ij} \sup_t \left\| q^{(ij)}(t, Q(t), \omega)\right\|, \qquad (4.28)
\end{aligned}
$$

where $Y^{(ii)}$ is a fundamental matrix solution for $\dot{Y}^{(ii)} = Y^{(ii)} D^{(ii)T}$, $Z^{(jj)}$ is a fundamental matrix solution for $Z^{(jj)} = Z^{(jj)} D^{(jj)T}$, and $K_{ij} \leqslant e^{\tilde{d}}/\tilde{a}$ by Lemma 4.2.

For $i=j$, from (4.12) we have

$$
Q^{(jj)}(t) - I = e^{-t}\int_0^t e^{\tau} q^{(jj)}(\tau, Q(\tau), \omega) d\tau \leqslant K_{jj} \sup_t \|q^{(jj)}(t, Q(t), \omega)\| \qquad (4.29)
$$

with $K_{jj} \leqslant 1$.

Let $K = \max_{i,j} K^{(ij)}$.

For $i<j$, and similarly for $i>j$, we can write

$$
\begin{aligned}
\left\| q^{(ij)}(t, Q, \omega)\right\| &\leqslant \left\| q^{(ij)}(t, Q, \omega) - q^{(ij)}(t, I, \omega)\right\|\\
&\quad + \left\| q^{(ij)}(t, I, \omega)\right\| \leqslant \eta(\rho, \omega)\rho + N(\omega), \qquad (4.30)
\end{aligned}
$$

where $N(\omega) \leqslant \omega$. To bound $\eta(\rho, \omega)$, write

$$
\begin{aligned}
q^{(ij)}(t, Q, \omega) &= q_D^{(ij)}(t, Q, \omega) + q_E^{(ij)}(t, Q, \omega)\\
&:= \left[(D^{(ii)T} Q^{(ij)} - Q^{(ij)} D^{(jj)T}) + \sum_k Q^{(ik)} S_D^{(kj)} \right]\\
&\quad + \sum_k Q^{(ik)} S_E^{(kj)}, \qquad (4.31)
\end{aligned}
$$

where $S_D^{(kj)}$ and $S_E^{(kj)}$ are as in (4.8). Then, analogously to (3.8)–(3.10), for $i < j$ we have

$$\begin{aligned}
&q_D^{(ij)}(t, Q, \omega) \\
&\quad = \left(\left[I - \sum_{k=1}^{j-1} Q^{(ik)} Q^{(ik)T}\right] D^{(ii)T} + \left[\sum_{k=j+1}^{m} Q^{(ik)} Q^{(ik)T}\right] D^{(ii)}\right) Q^{(ij)} \\
&\qquad - Q^{(ij)} \left[D^{(jj)T} - Q^{(jj)T} D^{(jj)T} Q^{(jj)}\right] \\
&\qquad + \sum_{k=j+1}^{m} Q^{(ik)} Q^{(ik)T} \left(D^{(jj)T} + D^{(jj)}\right) Q^{(jj)} \\
&\qquad - \sum_{l \neq i,j} \left[\sum_{k=1}^{j-1} Q^{(ik)} Q^{(lk)T} D^{(ll)T} Q^{(lj)} - \sum_{k=j+1}^{m} Q^{(ik)} Q^{(lk)T} D^{(ll)} Q^{(lj)}\right] \\
&\qquad + Q^{(ij)} S_D^{(jj)}.
\end{aligned} \tag{4.32}$$

Using (4.11),

$$\begin{gathered}
I - \textstyle\sum_{k=1}^{j-1} Q^{(ik)} Q^{(ik)T} = Q^{(ij)} Q^{(ij)T} + \sum_{k=j+1}^{m} Q^{(ik)} Q^{(ik)T} \quad \text{and} \\
Q^{(jj)} = \left(Q^{(jj)} - I\right) + I
\end{gathered} \tag{4.33}$$

and so we have

$$\begin{aligned}
\left\| q_D^{(ij)}(t, Q, \omega) \right\| \leqslant{}& \left[(\kappa_{ij}^2 + 2 \sum_{k=j+1}^{m} \kappa_{ik}^2) M_{ii} + 2\kappa_{jj} M_{jj}\right] \kappa_{ij} \\
&+2 \sum_{k=j+1}^{m} \kappa_{ik}^2 M_{jj} (1 + \kappa_{jj}) \\
&+ \sum_{l \neq i,j} \left[(1 + \kappa_{ii}) \kappa_{li} \kappa_{lj} + (1 + \kappa_{ll}) \kappa_{il} \kappa_{lj} + \sum_{\substack{k=1 \\ k \neq i,l,j}}^{m} \kappa_{ik} \kappa_{lk} \kappa_{lj}\right] \\
&\times M_{ll} + \kappa_{ij} s_j,
\end{aligned} \tag{4.34}$$

where $s_j = \sup_t \| S_D^{(jj)}(t) \|$.

For $i=j$, using (4.27) and (4.8), we have

$$\begin{aligned} q_D^{(jj)}(t,Q,\omega) &= \left[Q^{(jj)}(I+S_D^{(jj)})-I\right]+\sum_{k\neq j} Q^{(jk)}S_D^{(kj)} \\ &= \left[Q^{(jj)}(I+S_D^{(jj)})-I\right] \\ &\quad -\sum_l \left[\sum_{k=1}^{j-1} Q^{(jk)}Q^{(lk)T}D^{(ll)T}Q^{(lj)}\right. \\ &\qquad \left. -\sum_{k=j+1}^{m} Q^{(jk)}Q^{(lk)T}D^{(ll)}Q^{(lj)}\right]. \end{aligned} \tag{4.35}$$

Thus,

$$\begin{aligned} \left\|q_D^{(jj)}(t,Q,\omega)\right\| &\leqslant \left\|[Q^{(jj)}(I+S_D^{(jj)})-I]\right\| \\ &\quad +M_{jj}\left(1+\kappa_{jj}\right)\sum_{k\neq j}\kappa_{jk}^2+\sum_{l\neq j}\kappa_{jl}\left(1+\kappa_{ll}\right)M_{ll}\kappa_{lj} \\ &\quad +\sum_{l\neq j}\sum_{k\neq j,l}\kappa_{jk}\kappa_{lk}M_{ll}\kappa_{lj}. \end{aligned} \tag{4.36}$$

Next we find bounds for s_j and, using (4.15)–(4.18), for $\|Q^{(jj)}(I+S_D^{(jj)})-I\|$. If $\|Q^{(jj)}-I\|\leqslant\rho$, given the way the entries of $S^{(jj)}$ have been defined (see (4.13)), we easily obtain $s_j\leqslant(n_j-1)\rho$. The bound on $\|Q^{(jj)}(I+S_D^{(jj)})-I\|$ is trickier, and it is convenient to consider the various contributions within this term separately.

For the lower entries $(k>l)$ we can use (4.15) to obtain a bound on each entry as

$$\left|\left(Q^{(jj)}\left(I+S_D^{(jj)}\right)\right)_{kl}\right|\leqslant 2\rho^2+(n_j-2)\rho^2=n_j\rho^2$$

and thus a bound on all the lower part of $Q^{(jj)}(I+S_D^{(jj)})$ is

$$\sum_{k>l}\left(Q^{(jj)}\left(I+S_D^{(jj)}\right)\right)_{kl}^2\leqslant n_j^2\rho^4\frac{n_j(n_j-1)}{2}.$$

For the diagonal entries $(k=l)$, we can use (4.16) and (4.17) to obtain that

$$\left|\left(Q^{(jj)}(I+S_D^{(jj)})-I\right)_{kk}\right|\leqslant(n-1)\rho^2+(n_j-1)\rho^2=(n+n_j-2)\rho^2$$

so that

$$\sum_k \left(Q^{(jj)}(I+S_D^{(jj)})\right)_{kk}^2 \leqslant n_j\rho^4(n+n_j-2)^2 .$$

Finally, for the upper part ($k<l$) we can use (4.18) and (4.17) to obtain a bound on each entry as

$$\left|\left(Q^{(jj)}(I+S_D^{(jj)})\right)_{kl}\right| \leqslant 2\rho^2+(n_j-l)\rho^2+(n-n_j)\rho^2 \leqslant n\rho^2$$

and thus a bound on all the upper part of $Q^{(jj)}(I+S_D^{(jj)})$ is

$$\sum_{k<l} \left(Q^{(jj)}(I+S_D^{(jj)})\right)_{kl}^2 \leqslant n^2\rho^4\frac{n_j(n_j-1)}{2} .$$

Thus, we obtain

$$\left\|Q^{(jj)}\left(I+S_D^{(jj)}\right)-I\right\| \leqslant \rho^2\left[n_j(n+n_j-2)^2+(n^2+n_j^2)n_j(n_j-1)/2\right]^{1/2}$$

and since

$$n_j(n+n_j-2)^2+n^2n_j(n_j-1)/2 \leqslant 4n^2n_j(n_j+1)/2$$

and

$$\left[n_j^2n_j(n_j-1)/2+4n^2n_j(n_j+1)/2\right]^{1/2} \leqslant 2nn_j,$$

we have

$$\left\|Q^{(jj)}(I+S_D^{(jj)})-I\right\| \leqslant 2\rho^2nn_j . \tag{4.37}$$

If $\kappa_{ij}\leqslant\rho$ and $M_{ij}\leqslant M$, we have for $i<j$ from (4.34),

$$\begin{aligned}
\left\|q_D^{(ij)}(t,Q,\omega)\right\| &\leqslant \rho^2M[\rho+2(m-j)\rho+2+2(m-j)(1+\rho) \\
&\qquad +2(m-2)(1+\rho)+(m-3)\rho]+(n_j-1)\rho^2 \\
&= \rho^2M[\rho(7m-4j-6)+(4m-2j-2)]+(n_j-1)\rho^2 \\
&\leqslant \rho^2M(7\rho+4)(m-1)+(n_j-1)\rho^2 \\
&=: \eta_{ij}^D(\rho,\omega)\rho
\end{aligned} \tag{4.38}$$

that is $\eta_{ij}^D(\rho,\omega)=\rho M(7\rho+4)(m-1)+(n_j-1)\rho$.

For $i=j$, instead, from (4.36) and (4.37) we have

$$\begin{aligned}
\left\|q_D^{(jj)}(t,Q,\omega)\right\| &\leqslant \rho^2\left[2nn_j+M(m-1)\left(2(1+\rho)+(m-2)\rho\right)\right] \\
&=: \eta_{jj}^D(\rho,\omega)\rho.
\end{aligned} \tag{4.39}$$

Now, let $\eta^D(\rho,\omega) := \max_{i,j} \eta_{ij}^D(\rho,\omega)$ and note that

$$\eta^D(\rho,\omega) \leqslant \rho^2 c_2 + \rho c_1,$$

where $c_1 = \max\big(4M(m-1)+(n_j-1),\ 2M(m-1)+2nn_j\big)$ and $c_2=(m-1)M \max(7,m)$.

So, using the bound for $\|q_E^{(ij)}(t,Q,\omega) - q_E^{(ij)}(t,I,\omega)\|$ from (3.13) we set

$$\eta(\rho,\omega) = \eta^D(\rho,\omega) + \omega m[1+2m\rho] \tag{4.40}$$

and the result follows by applying Theorem IV.2.1 of [14] provided $K[\eta(\rho,\omega)\rho + N(\omega)] < \rho$. Using the bound on $\eta(\rho,\omega)$ in (4.40) and the form for $\rho=\beta\omega$, this condition is equivalent to $a_2\omega^2 + a_1\omega + (1-\alpha) < 0$ or $\omega < \omega_+(\alpha,K,M)$ with $\omega_+(\alpha,K,M)$ given in (4.23). □

Following [6] (see Theorems 5.1.2 and 5.1.3 in [1]) we have the following result which bounds the perturbation in the exponents within each of the diagonal blocks.

Theorem 4.4. *Suppose for an upper triangular block T, the assumptions of Theorem 4.1 hold. Let $\epsilon>0$ be given, define $\overline{D}_\epsilon := \max\{D_{u,\epsilon}, 1/d_{l,\epsilon}\}$ and $\delta := \epsilon/(4\overline{D}_\epsilon)$. Consider the systems $\dot u = T(t)u$ with n Lyapunov exponents equal to λ and $\dot u = [T(t)+E(t)]u$ with $\sup_t \|E(t)\| \leqslant \delta$ and Lyapunov exponents $\{\mu^{(i)}\}_{i=1}^n$. Then*

$$\left|\mu^{(i)} - \lambda\right| \leqslant 4\overline{D}_\epsilon \delta \tag{4..41}$$

for $i=1,\dots,n$.

Now we state our main perturbation result in the case of non-distinct Lyapunov exponents.

Theorem 4.5. *Assume that $\omega < \omega_+(\alpha,K,M)$ where ω_+ is given (4.23). Consider the ith block, if $n_i>1$, then*

$$\left|\mu_i^{(j)} - \lambda_i\right| \leqslant 4\overline{D}_{i,\epsilon}\left[M_{ii}(\rho^2+2\rho) + \sum_{k\neq i} M_{kk}\rho^2 + \omega + s_i\right] \tag{4.42}$$

for $j=1,\dots,n_i$ where $\rho=\beta\cdot\omega$, $\beta=\alpha K$, $K=\max\{e^{\tilde d}/\tilde a, 1\}$. If $n_i=1$, then

$$|\mu_i^{(1)} - \lambda_i| \leqslant \rho^2 \sum_{k\neq i}[M_{kk}+M_{ii}] + \omega. \tag{4.43}$$

Proof. By Theorem 4.2 in [10] there exists a Lyapunov transformation that transforms the block upper triangular system to block diagonal without changing the diagonal blocks.

We have for $i=1,\ldots,m$ that

$$\begin{aligned}(Q^TAQ)^{(ii)}-D^{(ii)} = {} & Q^{(ii)T}D^{(ii)}Q^{(ii)}-D^{(ii)} \\ & +\sum_{k\neq i} Q^{(ki)T}D^{(kk)}Q^{(ki)} \\ & +(Q^TEQ)^{(ii)}-S^{(ii)}. \end{aligned} \tag{4.44}$$

If $n_i=1$, then (4.43) follows by the argument in the proof of Theorem 3.2 but with the term $\sum_{k\neq i} Q_{ki}^2(D_{kk}-D_{ii})$ replaced by

$$\sum_{k\neq i} Q^{(ki)T}[D^{(kk)}Q^{(ki)}-Q^{(ki)}D^{(ii)}]. \tag{4.45}$$

If $n_i>1$, then result follows from Theorem 4.4 using (4.44) by writing

$$\begin{aligned}Q^{(ii)T}D^{(ii)}Q^{(ii)}-D^{(ii)} = {} & (Q^{(ii)T}-I)D^{(ii)}(Q^{(ii)}-I) \\ & +(Q^{(ii)T}-I)D^{(ii)}+D^{(ii)}(Q^{(ii)}-I). \end{aligned} \tag{4.46}$$

□

5. NUMERICAL EXAMPLE

We build an example where we vary the strength of the integral separation, whether the exponents there are all distinct or some are equal, and the departure from normality of the exact triangular factor. Take the following upper triangular function $B(t)=D(t)+U(t)$, with

$$D(t)=\mathrm{diag}(D_{11}(t), D_{22}(t), D_{33}(t), D_{44}(t)), \tag{5.1}$$

where we take $D_{11}(t)=10+\sin(t)$, $D_{22}(t)=\zeta\cos(t)$, $D_{33}(t)=\lambda_3-\zeta\cos(t)$, $D_{44}(t)=-10+\sin(t)$, and

$$U(t)=\kappa\begin{pmatrix}0 & \cos(t) & \sin(t) & \cos(t)\\ 0 & 0 & \cos(t) & \sin(t)\\ 0 & 0 & 0 & \cos(t)\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{5.2}$$

We will adjust the parameter κ to change the degree to which there is non-normality in the upper triangular part and the parameters λ_3 and ζ to change the degree to which there is integral separation as well as allowing for the case of non-distinct exponents.

Then, we rotate B, and consider the linear system (2.1) with

$$A(t) = Q(t)B(t)Q^T(t) + \dot{Q}(t)Q^T(t)$$

and

$$Q(t) = \mathrm{diag}(1, Q_\beta(t), 1)\cdot \mathrm{diag}(Q_\eta(t), Q_\eta(t)).$$

We set

$$Q_\gamma(t) = \begin{pmatrix} \cos(\gamma t) & \sin(\gamma t) \\ -\sin(\gamma t) & \cos(\gamma t) \end{pmatrix}, \quad \eta = 1, \quad \beta = \sqrt{2}.$$

Regardless of the value of κ in (5.2), this is a regular system with stable Lyapunov exponents given by the limits of

$$\lambda_i(t) := \frac{1}{t}\int_0^t D_{ii}(s)ds, \quad i = 1, 2, 3, 4 \quad \text{i.e.: } \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{10, 0, \lambda_3, -10\}.$$

We will consider the case of distinct exponents by setting $\lambda_3 = -1$ and the case of two equal exponents by setting $\lambda_3 = 0$. All results on this problem have been obtained using the code `leslis`, which we wrote and is public domain and can be downloaded from our websites: `http://www.math.gatech.edu/~dieci` and `http://www.math.ku.edu/~evanvleck`. In particular, we employ the continuous QR method using the projected fifth-order scheme (`IPAR(8)=0` in `LESLIS`), with local error control on the Q-factor and the exponents (`IPAR(10)=10` in `LESLIS`), and `TOL` is the value of the local error tolerance.

In Table I, we report on experiments that have been carried out up to $T = 10^4$, and we show the errors $e_i := |\lambda_i - \lambda_i^c(T)|$ where $\lambda_i^c(T)$ are the computed values at T of $\lambda_i(T)$, $i = 1, 2, 3, 4$. Scientific notation is used throughout. Given that we use `TOL`=1.E-4, and that we compute up to $T = 10^4$, we cannot expect to see errors better than about 1.E-4, and this would correspond to the case in which the errors are indeed of size `TOL`. [Increasing the length of the interval on which we compute does not increase the accuracy, as we observed in [11].]

We observe that if the coefficient matrix is normal (that is, $\kappa = 0$), and there is a sufficient degree of integral separation (that is, $\zeta = 5$), then the exponents are accurate regardless of whether or not they are distinct ($\lambda_3 = -1$ or $\lambda_3 = 0$). The exponents remain reasonably accurate even with $\kappa = 10$, as long as there is sufficient integral separation ($\zeta = 5$), but are much more accurate when $\lambda_3 = 0$ than when $\lambda_3 = -1$. This illustrates very clearly the difference between having distinct exponents in a system with weak integral separation versus having equal exponents with strong integral separation in a block sense. When we further weaken integral separation ($\zeta = 10$) also in the block sense, then the exponents are no longer all

Table I. Error in the exponents changing the degree of non-normality, integral separation, and distinct or non-distinct.

κ	λ_3	ζ	e_1	e_2	e_3	e_4
0	−1	5	2.4E-4	1.1E-4	1.3E-4	1.3E-4
0	−1	10	2.4E-4	1.5E0	1.5E0	1.4E-4
10	−1	5	2.6E-4	8.5E-1	8.5E-1	6.8E-5
10	−1	10	2.5E-4	3.8E0	3.8E0	4.5E-5
0	0	5	2.4E-4	8.5E-3	8.5E-3	1.2E-4
0	0	10	2.3E-4	2.0E0	2.0E0	1.4E-4
10	0	5	2.6E-4	2.6E-4	1.9E-4	5.9E-5
10	0	10	2.4E-4	2.4E0	2.4E0	1.7E-4
10	0	20	3.4E-1	8.2E0	7.3E0	1.2E0

$\mathrm{TOL} = 1.\mathrm{E}\text{-}4$, $T = 10^4$.

accurate, regardless of whether or not they are distinct, although the first and last exponents remain accurate. This betrays the fact that it may be possible to do a more refined component-wise analysis of the error in each exponent, based upon the varying degrees of integral separation within the upper triangular system. Finally, in the last row of Table 1 we weaken the integral separation to the point that all the exponents are poorly approximated.

6. CONCLUSION

The backward error analysis result of [11] said that – by QR methods – one will compute the Lyapunov exponents of a perturbed triangular system. Here, we examined the impact of this perturbation on the accuracy of the exponents. First, we performed a reduction from perturbed triangular to perturbed diagonal systems, then, under the assumption of stable Lyapunov exponents, we proved the existence of near identity orthogonal change of variables to upper triangular form. This allowed us to obtain precise bounds on the error in the Lyapunov exponents. The numerical results suggest the importance of further improvements of this type of analysis to provide a componentwise analysis of the error in the exponents based upon the varying degrees of integral separation within the given system (see also Remark 3.1).

ACKNOWLEDGMENTS

This work was supported in part under NSF Grants DMS/FRG-0139895, DMS/FRG-0139824, and DMS-0513438.

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