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Taylor Approximation of Integral Manifolds

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Integral manifolds generalize invariant manifolds to nonautonomous ordinary differential equations. In this paper, we develop a method to calculate their Taylor approximation with respect to the state space variables. This is of decisive importance, e.g., in nonautonomous bifurcation theory or for an application of the reduction principle in a time-dependent setting.

KEY WORDS: Taylor approximation; integral manifold; nonautonomous bifurcation.

MATHEMATICS SUBJECT CLASSIFICATION: Primary 37D10; Secondary 37C60, 37G35.

1. INTRODUCTION

In the local analysis of (autonomous) dynamical systems, there are two canonical ways to simplify a given nonlinear problem: (1) eliminate the nonlinearity as far as possible, and (2) reduce its dimension. Both lines of approach are fairly classical and led to the rigorous development of mathematical techniques like normal forms and center manifold reduction, respectively. The normal form theory dates back to Poincaré in the late 19th century already, while the center manifold theorem in finite dimensions has been proved in [32] (cf. also [8, 11, pp. 317, 41, pp. 89–169]). These techniques are the most important, generally available methods in local investigations of dynamical systems, and they form the basic for, e.g., a local dynamic bifurcation theory.

Over the last decades, nonautonomous dynamical systems became a popular and important field of research, since they frequently arise in

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applications (e.g., in the development of more realistic models) and inner-mathematical areas (e.g., to investigate the behavior of equations close to fixed nonconstant reference solutions of, e.g., almost periodic, homoclinic and heteroclinic type). Nevertheless, until now, the two classical approaches to simplify dynamical systems mentioned above are still fairly underdeveloped for explicitly time-dependent problems. Actually, a normal form theory for such systems is of quite recent origin (cf. [40]). On the other hand, a center manifold reduction for nonautonomous systems splits into two problems: first of all, one needs an appropriate reduction principle, which states that the essential dynamics of a given system is captured by the behavior on the center manifold. Second, one needs to know the center manifold or at least a suitable approximation of it. A reduction principle for nonautonomous ordinary differential eqautions (ODEs) can be found in [3] (or see [36] for a corresponding nonautonomous center manifold theory). The paper at hand addresses the question how to determine a Taylor approximation of the occurring center manifolds.

More precisely, our setting is broader and not limited to center manifolds: We present an approach to compute higher order local approximations of pseudo-hyperbolic integral manifolds near steady states for nonautonomous ODEs (see also the so-called "branch manifolds" introduced in [36]). The integral manifolds under consideration canonically generalize invariant manifolds to explicitly time-dependent right-hand sides and include the full hierarchy of strongly stable, stable, center-stable, as well as the corresponding unstable manifolds as special cases. We point out that our approach is not limited to the periodic or almost-periodic case. In our time-dependent situation, the Taylor coefficients are determined by bounded solutions of a linear ODE in a multilinear mapping space. Furthermore, we provide an explicit expression for these solutions in terms of so-called Lyapunov–Perron integrals (cf. Theorem 4.1). The same technique has been used in [38] for the purpose of a smooth linearization for vector fields.

For autonomous ODEs, such approximations via Taylor expansions are widely studied, e.g., in the monograph [28, pp. 172–187, Section 5.4] or the papers [7, 16, 21]. In this case, the situation is simpler, since the Taylor coefficients of invariant manifolds are (uniquely) determined by algebraic equations, the so-called multilinear Sylvester equations. In a different context [18], use Taylor series to obtain algebraic (polynomial) approximations of global attractors.

The framework for our investigations are nonautonomous ODEs in Banach spaces. Even though their state space is allowed to be infinite dimensional, differing from abstract evolution equations, we make the restriction that the operators involved are bounded and, concerning their linear part, everywhere defined.

The outline of the paper is as follows. First, we establish our basic terminology and a crucial result on the existence of bounded solutions for linear ODEs in spaces of multilinear mappings. Section 3 sets up the necessary theoretical background on local pseudo-stable/-unstable integral manifolds; in particular, it addresses the question of their uniqueness. In Section 4, we derive a linear ODE for the Taylor coefficients of the integral manifolds and solve it analytically. We demonstrate our results in Section 5 by two examples. The first one is the celebrated Lorenz equation with nonautonomously perturbed parameters. We are able to prove that a nonautonomous bifurcation of pitchfork type occurs in the sense of pullback attractors. Second, we calculate approximations of the center-stable and center-unstable integral manifolds for a system occurring in the investigation of traveling wave solutions for a modified Korteweg-de Vries equation. For the reader's convenience, two appendices contain a theorem on global integral manifolds and an existence result for pullback attractors.

We close this introduction by reviewing different approaches to the numerical computation of invariant manifolds for autonomous ODEs: [19] is based on the graph transform method, [20] apply a PDE approach to obtain global invariant manifolds and [22] use invariant foliations. While differentiability of the right-hand side is essential in our approach and applications, the paper [26] provides a method to approximate non-smooth center manifolds based on a discretization of the Lyapunov–Perron operator. Furthermore, [15] work with subdivision techniques to obtain global approximations and [5, 6] generalize this to nonautonomous ODEs.

After all, we refer to [33] for related results and further references in the discrete case of nonautonomous difference equations, where the methods are partially parallel to the present paper, and consequently, we can shorten some proofs here. Nonetheless, because of the following reasons and distinctions, we think it is not legitimate to consider our present ODE treatment as direct consequence of the corresponding investigation in [33]:

- Due to the invertibility of transition operators for the ODEs under consideration, the present treatment of exponential dichotomies appears simpler. One does not need to assume invertibility of the linear part restricted to its pseudo-unstable invariant subspace, in order to obtain a robust nonautonomous hyperbolicity concept (in form of an exponential dichotomy).
- In addition, integral manifolds of ODEs need to satisfy certain continuity assumptions for their partial derivatives (cf. (H_4) in Section 4) w.r.t. the time variable. By the trivial topology of the integers, this is redundant for difference equations.

- Finally, the invariance equation for integral manifolds is a first-order partial differential equation (see (3.4)), while the corresponding theory of difference equations leads to a nonlinear functional equation. Hence, a priori it is not clear whether the methods developed for the discrete case simply carry over to our setting, or if different techniques have to be employed. Indeed, one needs supplementary tools (e.g., the Leibniz rule from Lemma 2.1) in our analysis, yielding another homological equation (see (4.3)), which is structurally different from the related discrete object (see [33, (4.4)]).

In addition, the difference equations paper [33] illustrates the obtained results via critical problems from stability theory; more precisely, we applied a nonautonomous reduction principle to various biological models. In this paper, however, our bias is different and on a nonautonomous bifurcation scenario in some parametrically perturbed Lorenz system.

To conclude this introduction, we think it is useful and interesting to show that the algebraic problems from the well-known autonomous theory become tasks related to perturbation theory of ODEs in a nonautonomous setting. Moreover, having a nonautonomous bifurcation theory available (cf. the approaches of [23–25, 30, 31, 34, 37]), it is our hope that the introduced methods in connection with, for instance [40], can be helpful to simplify problems.

2. PRELIMINARIES

Above all, let us introduce some notation. $\mathbb{N}$ stands for the set of positive integers, $\mathbb{R}$ for the real and $\mathbb{C}$ for the complex field. Throughout this paper, the real ($\mathbb{F}=\mathbb{R}$) or complex ($\mathbb{F}=\mathbb{C}$) Banach spaces $\mathcal{X}, \mathcal{Y}$ are allowed to be infinite dimensional, and their norm is denoted by $\|\cdot\|$. In such a normed space, B_ρ is the open ball with center 0 and radius $\rho > 0$; beyond that, $U_\rho(V) \subseteq \mathcal{X}$ is the open ρ-neighborhood of $V \subseteq \mathcal{X}$. Such a subset V is called *star-shaped* w.r.t. 0, if one has $\{hx \in \mathcal{X} : h \in [0, 1]\} \subseteq V$ for all $x \in V$.

We write $I_{\mathcal{X}}$ for the identity mapping on $\mathcal{X}$, and for an n-tuple of the same vector $x \in \mathcal{X}$ we use the abbreviation $x^n := (x, \dots, x) \in \mathcal{X}^n$. With $n \in \mathbb{N}, \mathcal{L}_n(\mathcal{X}; \mathcal{Y})$ is the Banach space of symmetric n-linear bounded operators from $\mathcal{X}^n$ to $\mathcal{Y}, \mathcal{L}_n(\mathcal{X}) := \mathcal{L}_n(\mathcal{X}; \mathcal{X})$ and $\mathcal{L}(\mathcal{X}) := \mathcal{L}_1(\mathcal{X})$; all these spaces are equipped with their canonical norm. For a mapping $X \in \mathcal{L}_n(\mathcal{X}; \mathcal{Y})$, we abbreviate $Xx_1, \dots, x_n := X(x_1, \dots, x_n)$. With a closed subspace $\mathcal{X}_1 \subseteq \mathcal{X}$ and $T \in \mathcal{L}_1(\mathcal{X}_1; \mathcal{X})$, we define $X_T \in \mathcal{L}_n(\mathcal{X}_1; \mathcal{Y})$ by

$$X_T x_1, \dots, x_n := X(Tx_1, \dots, Tx_n) \quad \text{for all } x_1, \dots, x_n \in \mathcal{X}_1$$

and obtain the norm estimate (cf. [1, p. 62])

$$\|X_T\| \leqslant \|T\|^n \|X\| \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

The range of T is denoted by $\mathcal{R}(T) := T(\mathcal{X}_1)$.

During the rest of the paper, $\mathbb{I}$ denotes a real interval unbounded above.

Let $U \subseteq \mathcal{X}$ be nonempty and open. We say a mapping $F : U \times \mathbb{I} \to \mathcal{Y}$ is *uniformly bounded* if it maps bounded subsets of U into bounded sets (uniformly in $\mathbb{I}$), i.e., if for any bounded $\Omega \subseteq U$, there exists an $M \geqslant 0$ such that $\|F(x,t)\| \leqslant M$ for all $x \in \Omega$, $t \in \mathbb{I}$. We write $D\bar{F}$ for the Fréchet derivative of a mapping $\bar{F} : U \to \mathcal{Y}$, and if $F : U \times \mathbb{I} \to \mathcal{Y}$ depends differentiably on the first variable, then its partial derivative is denoted by $D_1 F$. For integers $m \geqslant 0$, the higher order derivatives $D^m \bar{F}$ or $D_1^m F$ are defined inductively, and F is said to be *uniformly C^m-bounded*, if $D_1^m F$ is uniformly bounded and the functions $D_1^n F(0,\cdot) : \mathbb{I} \to \mathcal{L}_n(\mathcal{X};\mathcal{Y})$ are bounded for $n \in \{1, \dots, m-1\}$. Thus, for a set U star-shaped w.r.t. 0, the mean value theorem implies the uniform boundedness of $D_1^n F$ for $n \in \{1, \dots, m\}$.

Now we quote a version of the Leibniz (product) rule and the chain rule for higher order Fréchet derivatives. To achieve a compact and convenient notation, we use (ordered) partitions of finite sets. These partitions consist of tuples of subsets of a given finite set. More precisely, with $j, l \in \mathbb{N}$, we write

$$P_j(l) := \left\{ (N_1, \dots, N_j) \;\middle|\; \begin{array}{l} N_i \subseteq \{1, \dots, l\} \text{ for } i \in \{1, \dots, j\}, \\ N_1 \cup \dots \cup N_j = \{1, \dots, l\}, \\ N_i \cap N_k = \emptyset \text{ for } i \neq k,\ i,k \in \{1, \dots, j\} \end{array} \right\}$$

for the *partitions* of $\{1, \dots, l\}$ with *length* j. Moreover, the *ordered partitions* of $\{1, \dots, l\}$ with length j are given by

$$P_j^<(l) := \left\{ (N_1, \dots, N_j) \in P_j(l) \;\middle|\; \begin{array}{l} N_i \neq \emptyset \text{ for } i \in \{1, \dots, j\}, \\ \max N_i < \max N_{i+1} \text{ for } 1 \leqslant i < j \end{array} \right\}.$$

In case of a set $N = \{n_1, \dots, n_k\} \subseteq \{1, \dots, l\}$ for $k \in \mathbb{N}, k \leqslant l$, we abbreviate $D^k \bar{F}(x) x_N := D^k \bar{F}(x) x_{n_1}, \dots, x_{n_k}$ for $x \in U, x_1, \dots, x_l \in \mathcal{X}$, where $\bar{F} : U \to \mathcal{Y}$ is assumed to be k-times differentiable. At last, $\#N$ is the cardinality of a finite set $N \subset \mathbb{N}$.

Lemma 2.1. (Leibniz rule). *Given $m, n \in \mathbb{N}$, an open set $U \subseteq \mathcal{X}, x \in U$, Banach spaces $\mathcal{Y}_1, \dots, \mathcal{Y}_n, \mathcal{Z}$ and mappings $f_i : U \to \mathcal{Y}_i, i \in \{1, \dots, n\}$, which are m-times differentiable. Then for any bounded multilinear mapping $M : \mathcal{Y}_1 \times \dots \times \mathcal{Y}_n \to \mathcal{Z}$, also the mapping $g : U \to \mathcal{Z}, g(x) := M(f_1(x), \dots, f_n(x))$*

is m-times differentiable, and for $l \in \{1, \dots, m\}$, the derivatives possess the representation

$$D^l g(x)x_1, \dots, x_l = \sum_{(N_1,\dots,N_n)\in P_n(l)} M(D^{\#N_1} f_1(x)x_{N_1}, \dots, D^{\#N_n} f_n(x)x_{N_n})$$

for any $x_1, \dots, x_l \in \mathcal{X}$.

Proof. See [1, pp. 95–96 and p. 112, Example 2.4C]. □

To reveal the efficiency of this notation, we consider the following simple example, as well as Example 2.4 below.

Example 2.2. For $l=2$ and $n=3$, we obtain the partition

$$\begin{aligned} P_3(2) = \big\{ &(\emptyset, \emptyset, \{1,2\}), (\emptyset, \{1,2\}, \emptyset), (\{1,2\}, \emptyset, \emptyset), (\emptyset, \{1\}, \{2\}), \\ &(\emptyset, \{2\}, \{1\}), (\{1\}, \emptyset, \{2\}), (\{1\}, \{2\}, \emptyset), (\{2\}, \{1\}, \emptyset), \\ &(\{2\}, \emptyset, \{1\}) \big\} \end{aligned}$$

and in case of a bounded 3-linear mapping M, the above Lemma 2.1 yields

$$\begin{aligned} &D^2 g(x)x_1x_2x_3 \\ &\quad = M(f_1(x), f_2(x), D^2 f_3(x)x_1x_2) + M(f_1(x), D^2 f_2(x)x_1x_2, f_3(x)) \\ &\qquad + M(D^2 f_1(x)x_1x_2, f_2(x), f_3(x)) + M(f_1(x), Df_2(x)x_1, Df_3(x)x_2) \\ &\qquad + M(f_1(x), Df_2(x)x_2, Df_3(x)x_1) + M(Df_1(x)x_1, f_2(x), Df_3(x)x_2) \\ &\qquad + M(Df_1(x)x_1, Df_2(x)x_2, f_3(x)) + M(Df_1(x)x_2, Df_2(x)x_1, f_3(x)) \\ &\qquad + M(Df_1(x)x_2, f_2(x), Df_3(x)x_1) . \end{aligned}$$

Lemma 2.3. (chain rule). *Given $m \in \mathbb{N}$, open sets $U, V \subseteq \mathcal{X}, x \in U$ and mappings $f: V \to \mathcal{X}, g: U \to \mathcal{X}$, which are m-times differentiable and satisfy $g(U) \subseteq V$. Then the composition $f \circ g: U \to \mathcal{X}$ is m-times differentiable, and for $l \in \{1, \dots, m\}$, the derivatives possess the representation*

$$\begin{aligned} &D^l (f \circ g)(x)x_1, \dots, x_l \\ &\quad = \sum_{j=1}^{l} \sum_{(N_1,\dots,N_j)\in P_j^{<}(l)} D^j f(g(x)) D^{\#N_1} g(x)x_{N_1}, \dots, D^{\#N_j} g(x)x_{N_j} \end{aligned}$$

for any $x_1, \dots, x_l \in \mathcal{X}$.

Proof. See [35, Theorem 2]. □

Example 2.4. To clarify Lemma 2.3 in case, e.g., $l=3$, we obtain the ordered partitions

$$P_1^{<}(3) = \{(\{1,2,3\})\},$$
$$P_2^{<}(3) = \{(\{1\},\{2,3\}), (\{1,2\},\{3\}), (\{2\},\{1,3\})\},$$
$$P_3^{<}(3) = \{(\{1\},\{2\},\{3\})\}$$

and therefore, the third-order derivative of the composition $f \circ g$ reads as

$$\begin{aligned} &D^3(f\circ g)(x) \\ &\quad = Df(g(x))D^3g(x)x_1x_2x_3 + D^2f(g(x))Dg(x)x_1D^2g(x)x_2x_3 \\ &\qquad + D^2f(g(x))D^2g(x)x_1x_2Dg(x)x_3 + D^2f(g(x))Dg(x)x_2D^2g(x)x_1x_3 \\ &\qquad + D^3f(g(x))Dg(x)x_1Dg(x)x_2Dg(x)x_3\,. \end{aligned}$$

Since we are dealing with nonautonomous differential equations, it is advantageous to have some further notions available. Any subset $S \subseteq \mathbb{I}\times\mathcal{X}$ is called a *nonautonomous set*, and the sets $S(t) := \{x\in\mathcal{X} : (t,x)\in S\}$ for $t\in\mathbb{I}$ are its *t-fibers*.

For a differentiable function $\phi : I\to\mathcal{X}$, $I\subseteq\mathbb{I}$ is an interval, its derivative is denoted by $\dot\phi : I\to\mathcal{X}$. We use the notation

$$\dot x = f(x,t) \tag{2.2}$$

to denote the ODE $\dot x(t) = f(x(t),t)$ with *right-hand side* $f: U\times\mathbb{I}\to\mathcal{X}$, where U is an open subset of the Banach space $\mathcal{X}$ and f satisfies conditions guaranteeing existence and uniqueness of solutions. A differentiable function $\phi: I\to\mathcal{X}$ is said to solve (2.2) on $I\subseteq\mathbb{I}$ if $\phi(t)\in U$ and $\dot\phi(t)\equiv f(\phi(t),t)$ holds for all $t\in\mathbb{I}$. Let φ stand for the *general solution* of (2.2), i.e., $\varphi(\cdot;t_0,x_0)$ is the unique noncontinuable solution of (2.2) satisfying the initial condition $\varphi(t_0;t_0,x_0)=x_0$ for $t_0\in\mathbb{I}$, $x_0\in U$.

Given a continuous coefficient function $A:\mathbb{I}\to\mathcal{L}(\mathcal{X})$, we define the *transition operator* $\Phi(t,\tau)\in\mathcal{L}(\mathcal{X})$, $\tau,t\in\mathbb{I}$, of the linear ODE

$$\dot x = A(t)x \tag{2.3}$$

in $\mathcal{X}$ as solution of the $\mathcal{L}(\mathcal{X})$-valued initial value problem $\dot X = A(t)X$, $X(\tau) = I_{\mathcal{X}}$ (cf. [14, p. 101]). A projection-valued function $P_-:\mathbb{I}\to\mathcal{L}(\mathcal{X})$ is said to be an *invariant projector* of (2.3) supposed that

$$\Phi(t,\tau)P_-(\tau) = P_-(t)\Phi(t,\tau) \quad \text{for all } t,\tau\in\mathbb{I}. \tag{2.4}$$

The *complementary projector* $P_+ : \mathbb{I} \to \mathcal{L}(\mathcal{X})$, defined by $P_+(t) := I_{\mathcal{X}} - P_-(t)$ for all $t \in \mathbb{I}$, is also an invariant projector. One can show that invariant projectors are continuously differentiable, and they satisfy the linear ODE

$$\dot{P}_\pm(t) = A(t)P_\pm(t) - P_\pm(t)A(t) \quad \text{for all } t \in \mathbb{I}, \tag{2.5}$$

here the symbol $P_\pm$ simultaneously stands for P_+ or P_-, respectively. We proceed similarly with our further notation throughout the paper.

As mentioned in section 1, a crucial tool in our analysis will be a particular class of linear ODEs in spaces of n-linear mappings. The remaining section features some preliminaries on this aspect. More precisely, we are interested in linear differential equations in $\mathcal{L}_n(\mathcal{X})$ of the form

$$\dot{X}_{P_\pm(t)} = L_{A(t)} X_{P_\pm(t)} \tag{2.6}$$

with a coefficient function $L_T \in \mathcal{L}(\mathcal{L}_n(\mathcal{X}))$ given by (cf. [38, p. 1066])

$$(L_T X)x_1, \dots, x_n := TXx_1, \dots, x_n - \sum_{j=1}^{n} Xx_1, \dots, x_{j-1}Tx_jx_{j+1}, \dots, x_n$$

for $T \in \mathcal{L}(\mathcal{X})$ and vectors $x_1, \dots, x_n \in \mathcal{X}$. It is worth mentioning that these equations are not ODEs of the form (2.2), since the projectors $P_\pm(t)$ are noninvertible in general. It is easy to see that, given $\tau \in \mathbb{I}$ and initial state $\Xi \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_\mp(\tau))\big)$ with $\Xi_{P_\pm(\tau)} \equiv \Xi$,

$$\Lambda_\pm(t, \tau)\Xi := \Phi(t, \tau)\Xi_{\Phi(\tau,t)P_\pm(t)}, \quad \text{for all } t \in \mathbb{I}, \tag{2.7}$$

defines the uniquely determined solution $\Lambda_\pm(\cdot, \tau)\Xi$ of Eq. (2.6) satisfying the relation $(\Lambda_\pm(t, \tau)\Xi)_{P_\pm(t)} = \Lambda_\pm(t, \tau)\Xi$ for $t \in \mathbb{I}$.

In order to discuss integral manifolds of nonautonomous ODEs, we need an appropriate, i.e., robust hyperbolicity notion for their linear part.

Hypothesis. *Assume the continuous function $A : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ satisfies:*

(H_1) *Hypothesis on linear part: The linear ODE* (2.3) *possesses an exponential dichotomy, i.e., there exists an invariant projector $P_- : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ such that for all $t, \tau \in \mathbb{I}$, the estimates*

$$\begin{aligned} \|\Phi(t, \tau)P_+(\tau)\| &\leqslant K_+ e^{\alpha(t-\tau)}, \\ \|\Phi(\tau, t)P_-(t)\| &\leqslant K_- e^{\beta(\tau-t)} \end{aligned} \tag{2.8}$$

for all $\tau \leqslant t$ hold with real constants $K_+, K_- \geqslant 1, \alpha < \beta$.

Remark 2.5.

(1) In the autonomous case, i.e., if $A_0 := A(t)$ does not depend on $t \in \mathbb{I}$, it is sufficient to assume that the spectrum $\sigma(A_0) \subseteq \mathbb{C}$ of $A_0 \in \mathcal{L}(\mathcal{X})$ can be separated into a "pseudo-stable" spectral part σ_+ with $\Re\sigma_+ < \alpha$, and a disjoint "pseudo-unstable" part σ_- with $\beta < \Re\sigma_+$. Then $P_\pm$ are constant (in $t \in \mathbb{I}$) and given by the spectral projectors related to $\sigma_\pm$, respectively (cf. [14, pp. 72–73]).
(2) For a T-periodic differential equation (2.3), $T > 0$, an exponential dichotomy is implied by the fact that $\sigma(\Phi(\tau + T, \tau)), \tau \in \mathbb{I}$ fixed, can be separated into a "pseudo-stable" spectral part σ_+ with $|\sigma_+| < \alpha$, and a disjoint "pseudo-unstable" part σ_- satisfying $\beta < |\sigma_-|$ (cf. [14, p. 203, Theorem 2.1]).
(3) Further sufficient conditions for an exponential dichotomy can be found in [13].

Our first result deals with perturbations of linear systems (2.6) in $\mathcal{L}_n(\mathcal{X})$. For this, the notion of quasiboundedness is convenient. With reals γ and a fixed $\tau \in \mathbb{I}$, we say a function $\phi : \mathbb{I} \to \mathcal{X}$ is *γ-quasibounded* if

$$\|\phi\|_{\tau,\gamma} := \sup_{t \in \mathbb{I}} \|\phi(t)\| \, e^{\gamma(\tau - t)} < \infty,$$

holds. Obviously, 0-quasiboundedness coincides with the classical notion of boundedness.

Lemma 2.6. (quasibounded solutions). *Suppose that* (H_1) *is satisfied, let* $n \in \mathbb{N}, \tau \in \mathbb{I}, \gamma \in \mathbb{R}$, *and assume* $H^\pm : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ *is γ-quasibounded with* $H^\pm(t) \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_\mp(t))\big)$ *for* $t \in \mathbb{I}$. *Then for the ODE*

$$\dot{X}_{P_\pm(t)} = L_{A(t)} X_{P_\pm(t)} + H^\pm(t)_{P_\pm(t)} \tag{2.9}$$

in $\mathcal{L}_n(\mathcal{X})$, *the following holds:*

(a) *In case* $\gamma < \beta - n\alpha$, *there exists a unique γ-quasibounded solution* $\Gamma_+ : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ *of* (2.9) *with*

$$\Gamma_+(t) = \Gamma_+(t)_{P_+(t)} \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_-(t))\big) \quad \textit{for all } t \in \mathbb{I}, \tag{2.10}$$

given by

$$\Gamma_+(t) := -\int_t^\infty \Phi(t,s) H^+(s)_{\Phi(s,t)P_+(t)} \, ds \tag{2.11}$$

and satisfying the estimate $\|\Gamma_+\|_{\tau,\gamma} \leqslant \frac{K_- K_+^n}{\beta - \gamma + n\alpha} \left\|H^+\right\|_{\tau,\gamma}$.

(b) *In case* $\mathbb{I}=\mathbb{R}$ *and* $\gamma>\alpha-n\beta$, *there exists a unique* γ*-quasibounded solution* $\Gamma_-:\mathbb{R}\to\mathcal{L}_n(\mathcal{X})$ *of* (2.9) *with* $\Gamma_-(t)=\Gamma_-(t)_{P_-(t)}\in\mathcal{L}_n\big(\mathcal{X};\mathcal{R}(P_+(t))\big)$ *for all* $t\in\mathbb{R}$, *given by*

$$\Gamma_-(t):=\int_{-\infty}^{t}\Phi(t,s)H^-(s)_{\Phi(s,t)P_-(t)}\,ds$$

and satisfying the estimate $\|\Gamma_-\|_{\tau,\gamma}\leqslant\frac{K_+K_-^n}{\gamma+\alpha-n\beta}\left\|H^-\right\|_{\tau,\gamma}$.

Proof. Let $\tau\in\mathbb{I}$. We only prove the assertion (a), since (b) can be shown similarly.

(I) We first consider the special case $H^+(t)\equiv 0$ on $\mathbb{I}$. Then Eq. (2.9) coincides with (2.6). Let $\Gamma_+:\mathbb{I}\to\mathcal{L}_n(\mathcal{X})$ be a γ-quasibounded solution of (2.9) satisfying (2.10). Taking the limit $t\to\infty$ in the inequality

$$\begin{aligned}
\|\Gamma_+(\tau)\| &\overset{(2.7)}{=} \left\|\Phi(\tau,t)\Gamma_+(t)_{\Phi(t,\tau)P_+(\tau)}\right\|\\
&\overset{(2.10)}{\leqslant} \|\Phi(\tau,t)P_-(t)\|\left\|\Gamma_+(t)_{\Phi(t,\tau)P_+(\tau)}\right\|\\
&\overset{(2.1)}{\leqslant} \|\Phi(\tau,t)P_-(t)\|\,\|\Gamma_+(t)\|\,\|\Phi(t,\tau)P_+(\tau)\|^n\\
&\overset{(2.8)}{\leqslant} K_-K_+^n e^{(n\alpha+\gamma-\beta)(t-\tau)}\,\|\Gamma_+\|_{\tau,\gamma}
\end{aligned}$$

for all $t\geqslant\tau$ yields $\Gamma_+(\tau)=0$. Since $\tau\in\mathbb{I}$ was arbitrary, the zero solution of (2.9) is the only γ-quasibounded solution satisfying (2.10).

(II) We now omit the restriction on H^+ and note that the function Γ_+ from (2.11) is well-defined, since the estimate

$$\begin{aligned}
\|\Gamma_+(t)\| &\overset{(2.11)}{\leqslant} \int_t^\infty \left\|\Phi(t,s)P_-(s)H^+(s)_{\Phi(s,t)P_+(t)}\right\|\,ds\\
&\overset{(2.1)}{\leqslant} \int_t^\infty \|\Phi(t,s)P_-(s)\|\left\|H^+(s)\right\|\|\Phi(s,t)P_+(t)\|^n\,ds\\
&\overset{(2.8)}{\leqslant} K_-K_+^n e^{\gamma(t-\tau)}\left\|H^+\right\|_{\tau,\gamma}\int_t^\infty e^{(s-t)(\gamma+n\alpha-\beta)}\,ds\\
&= \frac{K_-K_+^n}{\beta-n\alpha-\gamma}\left\|H^+\right\|_{\tau,\gamma}e^{\gamma(t-\tau)}\quad\text{for all } t\in\mathbb{I},
\end{aligned}$$

holds, which in turn yields the claimed estimate for $\|\Gamma_+\|_{\tau,\gamma}$. Moreover, it is easy to see that Γ_+ satisfies (2.10). Γ_+ is a solution of (2.9), since the Leibniz rule (Lemma 2.1) yields

$$
\begin{aligned}
\dot{\Gamma}_+(t)_{P_+(t)} &\equiv \Phi(t,t)H^+(t)_{\Phi(t,t)P_+(t)} - A(t)\int_t^\infty \Phi(t,s)H^+(s)_{\Phi(s,t)P_+(t)}\,ds \\
&\quad - \sum_{j=1}^n \int_t^\infty \Phi(t,s)H^+(s)\Big(\Phi(s,t)P_+(t),\ldots,\underbrace{\frac{d}{dt}(\Phi(s,t)P_+(t))}_{j\text{th position}},\ldots,\Phi(s,t)P_+(t)\Big) \\
&\overset{(2.5)}{\equiv} H^+(t)_{P_+(t)} + A(t)\Gamma_+(t) - \sum_{j=1}^n \Gamma_+(t)\big(I_{\mathcal{X}},\ldots,\underbrace{A(t)}_{j\text{th position}},\ldots,I_{\mathcal{X}}\big) \\
&\equiv L_{A(t)}\Gamma_+(t)_{P_+(t)} + H^+(t)_{P_+(t)} \quad \text{on } \mathbb{I}.
\end{aligned}
$$

Finally, the uniqueness statement results from step (I), because the difference of any two γ-quasibounded solutions of (2.9) is a γ-quasibounded solution of (2.6) and therefore, identically vanishing. □

3. PROPERTIES OF INTEGRAL MANIFOLDS

In the following, we introduce and summarize some fundamental facts on integral manifolds of ODEs. For the autonomous and center manifold situation (e.g., [12, pp. 1–48, Chapter 1, 41, pp. 89–169) or 8] are good references), whereas the general nonautonomous setting is treated in [4] or [36, 39]. We consider nonautonomous ODEs of the form

$$
\dot{x} = A(t)x + F(x,t) \tag*{$(*)_F$}
$$

with a mapping $F: U_0 \times \mathbb{I} \to \mathcal{X}$, where $U_0 \subseteq \mathcal{X}$ is open and star-shaped w.r.t. $0 \in \mathcal{X}$.

Hypothesis. *Let $m \in \mathbb{N}$. Assume the continuous mapping $F: U_0 \times \mathbb{I} \to \mathcal{X}$ satisfies:*

(H$_2$) *Hypothesis on nonlinearity: F is m-times continuously Fréchet differentiable in the first argument, $F(0,t) \equiv 0$ on $\mathbb{I}$, we have the limit relation*

$$
\lim_{x \to 0} \|D_1 F(x,t)\| = 0 \quad \textit{uniformly in } t \in \mathbb{I} \tag{3.1}
$$

and F is uniformly C^m-bounded.

Remark 3.1. One typically obtains $(*)_F$ from (2.2) as *equation of perturbed motion*. Thereto, let $\phi: \mathbb{I} \to \mathcal{X}$ be a fixed reference solution of (2.2), and one is interested in the behavior close to ϕ. Hence, instead of (2.2), one investigates $(*)_F$ with

$$
A(t) := D_1 f(\phi(t), t),
$$

$$F(x,t) := f(x+\phi(t),t) - f(\phi(t),t) - D_1 f(\phi(t),t)x$$

and assumes that the partial derivatives $D_1^n f$ exist, are continuous and uniformly bounded for $n \in \{0, \dots, m\}$, and that one has

$$\lim_{x \to 0} \|D_1 f(x+\phi(t),t) - D_1 f(\phi(t),t)\| = 0, \quad \text{uniformly in } t \in \mathbb{I}.$$

Next, we introduce a nonautonomous version of an invariant manifold for $(*)_F$. Let $P_\pm : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ be the invariant projector of (2.3) from (H_1), $U \subseteq U_0$ is open star-shaped w.r.t. 0 and φ denotes the general solution to $(*)_F$. Assume $s^\pm : U \times \mathbb{I} \to \mathcal{X}$ is a mapping continuously Fréchet differentiable and satisfying

$$s^\pm(0,t) \equiv 0 \quad \text{on } \mathbb{I}, \lim_{x \to 0} \left\| D_1 s^\pm(x,t) \right\| = 0, \quad \text{uniformly in } t \in \mathbb{I}, \tag{3.2}$$

$$s^\pm(x,t) = s^\pm(P_\pm(t)x,t) \in \mathcal{R}(P_\mp(t)) \tag{3.3}$$

for all $t \in \mathbb{I}, x \in \mathcal{X}$. Then the nonautonomous set given by the graph

$$\mathcal{S}^\pm := \left\{ (\tau, \xi + s^\pm(\xi,\tau)) \in \mathbb{I} \times \mathcal{X} : \xi \in \mathcal{R}(P_\pm(\tau)) \cap U \right\}$$

is called a *local integral manifold* of the nonlinear ODE 3.1 if

$$(t_0, x_0) \in \mathcal{S}^\pm \quad \Rightarrow \quad (t, \varphi(t; t_0, x_0)) \in \mathcal{S}^\pm \quad \text{for all } t \in J_U(t_0, x_0)$$

holds, where $J_U(t_0, x_0) \subseteq \mathbb{I}$ is the maximal interval of existence for $\varphi(\cdot; t_0, x_0)$ w.r.t. U. One speaks of a C^m-integral manifold of $(*)_F$ if the partial derivatives $D_1^n s^\pm$ exist and are continuous for $n \in \{1, \dots, m\}$. In case $U = \mathcal{X}$, we say that $\mathcal{S}^\pm$ is a *global integral manifold* of $(*)_F$. Geometrically, conditions (3.2) and (3.3) imply that $\mathcal{S}^\pm$ contains the zero solution of $(*)_F$, and $\mathcal{S}^\pm$ is fiber-wise tangent to the vector bundle $\{(\tau, \xi) \in \mathbb{I} \times \mathcal{X} : \xi \in \mathcal{R}(P_\pm(\tau))\}$, while (3.3) yields that each t-fiber $\mathcal{S}^\pm(t)$ is a graph over $\mathcal{R}(P_\pm(t)) \cap U$.

Local integral manifolds satisfy the following nonlinear first-order partial differential equation, named as *invariance equation*

$$\begin{aligned} &A(t)s^\pm(\xi,t) + P_\mp(t)F(\xi + s^\pm(\xi,t),t) \\ &\quad = D_1 s^\pm(\xi,t)\left(A(t)\xi + P_\pm(t)F(\xi + s^\pm(\xi,t),t)\right) + D_2 s^\pm(\xi,t) \end{aligned} \tag{3.4}$$

for all $t \in \mathbb{I}, \xi \in \mathcal{R}(P_\pm(t)) \cap U$ such that $\xi + s^\pm(\xi,t) \in U_0$.

$\mathcal{S}^+$ and $\mathcal{S}^-$ are denoted as *pseudo-stable* and *pseudo-unstable* integral manifolds of $(*)_F$, respectively. To be more specific, $\mathcal{S}^+$ describes a *center-stable* integral manifold in case $\beta > 0$, a *stable* integral manifold in the hyperbolic situation $\alpha < 0 < \beta$ and a *strongly stable* integral manifold in case $\beta < 0$. Under the additional assumption $\mathbb{I} = \mathbb{R}$, $\mathcal{S}^-$ describes a *center-unstable* integral manifold in case $\alpha < 0$, an *unstable* integral manifold in the hyperbolic situation $\alpha < 0 < \beta$ and a *strongly unstable* integral manifold in case $\alpha > 0$. In the light of Remark 2.5, this terminology corresponds to the autonomous situation of invariant manifolds considered, e.g., in [12].

Concerning the existence of smooth local integral manifolds, due to our general Banach space setting, we have to impose the assumption that $\mathcal{X}$ is a C^m-Banach space; that is, the norm on $\mathcal{X}$ is of class C^m away from 0. A characterization of such spaces, as well as concrete examples, can be found in [27, pp. 127–152, Section 13]; e.g., Hilbert spaces are C^∞-Banach spaces. Then, on $\mathcal{X}$, there exists a C^m-*cut-off function* (or *bump function*) $\chi : \mathcal{X} \to [0, 1]$ with the properties

$$\chi(x) \equiv 1 \quad \text{on } x \in B_1, \qquad \chi(x) \equiv 0 \quad \text{on } x \in \mathcal{X} \setminus B_2 \tag{3.5}$$

(cf. [1, p. 473, Lemma 4.2.13]). This is of crucial importance for the proof of

***Theorem 3.2* (existence of local integral manifolds).** *Suppose* (H_1) *and* (H_2) *hold and that* $\mathcal{X}$ *is a* C^m*-Banach space. Then there exists a* $\rho_0 > 0$ *such that one has with* $U = B_{\rho_0}$:

(a) *Under the gap condition*

$$m\alpha < \beta \tag{3.6}$$

the ODE $(*)_F$ *possesses a local pseudo-stable* C^m*-integral manifold* $\mathcal{S}^+$,

(b) *for* $\mathbb{I} = \mathbb{R}$ *and under the gap condition*

$$\alpha < m\beta \tag{3.7}$$

the ODE $(*)_F$ *possesses a local pseudo-unstable* C^m*-integral manifold* $\mathcal{S}^-$,

(c) *for the corresponding mapping* $s^\pm : U \times \mathbb{I} \to \mathcal{X}$, *there exist reals* $\gamma_0, \dots, \gamma_m \geqslant 0$ *such that*

$$\left\| D_1^n s^\pm(x, t) \right\| \leqslant \gamma_n \quad \textit{for all } x \in U,\ t \in \mathbb{I},\ n \in \{0, \dots, m\} \tag{3.8}$$

(d) *if the* $(*)_F$ *mappings* A *and* F *are periodic in* t *with period* $\theta > 0$, *then*

$$s^\pm(x, t + \theta) = s^\pm(x, t) \quad \textit{for all } x \in \mathcal{X},\ t \in \mathbb{I}$$

and if is autonomous, then the mapping $s^\pm$ *is independent of* $t \in \mathbb{I}$, *i.e., the set* $\left\{ \xi + s^\pm(\xi) \in \mathcal{X} : \xi \in \mathcal{R}(P_\pm) \cap U \right\}$ *is a locally invariant manifold of* $(*)_F$.

Proof. One obtains the local integral manifolds as restriction of global integral manifolds by suitably modifying the ODE $(*)_F$ outside a small neighborhood of 0 with a C^m-cut-off function. Using this fairly standard cut-off technique, one can apply Theorem A.1 to the modified equation $(*)_F$ and deduce the assertions of the present Theorem 3.2. $\square$

The integral manifolds defined above are constructed as perturbations of the invariant vector bundles for the corresponding linear system (2.3), which are global objects. Therefore, it is reasonable that solutions of $(*)_F$ lying in the corresponding integral manifolds inherit the dynamical properties of the linearization at least locally. For instance, interpreting the stable integral manifold $\mathcal{S}^+$ as domain of attraction for the zero solution of $(*)_F$, it is desirable that the domain of definition for $s^+(\cdot, t)$ does not shrink to 0 for $t \to \infty$. Moreover, referring to an application of the reduction principle in critical stability problems, one needs the existence of a center-unstable integral manifold $\mathcal{S}^-$ on a set of the form $\mathbb{R} \times B_{\rho_0}$. However, one cannot expect that this uniformity in time persists if one weakens the limit relation (3.1) in assumption (H_2) to uniform convergence on compact sets, i.e.,

$$\lim_{x \to 0} \|D_1 F(x, t)\| = 0, \qquad \text{uniformly in } t \in K \tag{3.9}$$

for every compact subset $K \subset \mathbb{I}$. This is demonstrated by the following example.

Example 3.3. Let $\mathbb{I} := [1, \infty)$, and consider the nonautonomous ODE (2.2) with scalar right-hand side $f : \mathbb{R} \times \mathbb{I} \to \mathbb{R}$, given by

$$f(x, t) := -\tfrac{1}{2}x + F(x, t), \qquad F(x, t) := \int_0^x \psi(u, t)\, du\,,$$

where $\psi : \mathbb{R} \times \mathbb{I} \to \mathbb{R}$ is a bounded C^1-function, defined by

$$\psi(u, t) := \begin{cases} 1 & \text{for } |u| \geqslant 1, t, \\ \exp\left(-\left(\frac{1}{|u|} - t\right)^2\right) & \text{for } 0 < |u| < 1/t, \\ 0 & \text{for } u = 0. \end{cases}$$

The nonlinearity F does not fulfill (3.1) in assumption (H_2), whereas (3.9) holds, since one has $D_1 F(x, t) = \psi(x, t)$. Assume, there exists an $\eta > 0$ so that

$$\lim_{t \to \infty} \varphi(t + \tau; \tau, \eta) = 0 \quad \text{for all } \tau \in \mathbb{I}. \tag{3.10}$$

Due to $\lim_{t\to\infty}\int_0^\eta \psi(u,t)\,du=\eta$, there exists a $t_0=t_0(\eta)>1$ with $\int_0^\eta \psi(u,t)\,du\geqslant 2/3\eta$ for all $t\geqslant t_0$. This implies

$$f(t,\eta)\geqslant -\frac{1}{2}\eta+\frac{2}{3}\eta>0 \quad \text{for all } t\geqslant t_0\,.$$

Hence, one has $\varphi(t;\tau,\eta)\geqslant\eta$ for all $t\geqslant\tau\geqslant t_0$, which contradicts (3.10). Therefore, the trivial solution of (2.2) is not uniformly attractive, thus not even uniformly asymptotically stable in the sense of Lyapunov (see, e.g., [14, p. 279]), although the linearization $\dot{x}=-1/2x$ is actually exponentially stable. This example shows that notions of uniform stability do not persist under nonlinearities F satisfying only (3.9).

Remark 3.4. We point out that it is straight forward to set up a theory of integral manifolds for $(*)_F$, where the uniformity in limit relation (3.1) is relaxed to the fact for all $\varepsilon>0$, there exists a continuous $\Delta:\mathbb{I}\to(0,\infty)$ such that $x\in B_{\Delta(t)}$ implies $\|D_1F(x,t)\|<\varepsilon$ for all $t\in\mathbb{I}$. Using a time-dependent cut-off technique in the proof of Theorem 3.2, the fibers of the resulting integral manifolds are given as graphs of functions $s^\pm(\cdot,t)$ defined on open neighborhoods $U(t)\subseteq\mathcal{X}$ of 0, whereupon $U(t)$ is allowed to shrink to $\{0\}$ for $t\to\infty$. However, for the above reasons, we do not follow this approach.

It is well known that, even under Hypotheses (H_1) and (H_2), e.g., center-unstable integral manifolds are not unique in general (cf. [12, pp. 30–31, Example 3.5] for instance). Still, they can be obtained as restrictions of uniquely determined global integral manifolds of appropriately modified ODEs and calculated using Taylor expansions. Technically, this is guaranteed under our

Hypothesis. *Let $\mathcal{X}$ be a C^{m+1}-Banach space and assume*

(H_3) $A:\mathbb{I}\to\mathcal{L}(\mathcal{X})$ *is bounded.*

Proposition 3.5 (global integral manifolds). *Suppose* (H_1)–(H_3) *hold, and let $\mathcal{S}^\pm$ denote a C^{m+1}-integral manifold, where the corresponding mapping $s^\pm:U\times\mathbb{I}\to\mathcal{X}$ is uniformly C^{m+1}-bounded. In case $\mathcal{S}^+$ is considered, assume* (3.6) *holds, and in case of $\mathcal{S}^-$, assume $\mathbb{I}=\mathbb{R}$ and* (3.7). *Then there exist a real $\rho>0$, as well as mappings $F_\rho:\mathcal{X}\times\mathbb{I}\to\mathcal{X}$ and $s_\rho^\pm:\mathcal{X}\times\mathbb{I}\to\mathcal{X}$ such that the following holds:*

(a) *The graph $\mathcal{S}_\rho^\pm:=\left\{(\tau,\xi+s_\rho^\pm(\xi,\tau))\in\mathbb{I}\times\mathcal{X}:\xi\in\mathcal{R}(P_\pm(\tau))\right\}$ defines a global C^m-integral manifold of $(*)_{F_\rho}$,*

(b) *$F_\rho(x,t)\equiv F(x,t)$ on $B_\rho\times\mathbb{I}$,*

(c) *$s_\rho^\pm(x,t)\equiv s^\pm(x,t)$ on $B_\rho\times\mathbb{I}$, and $\mathcal{S}_\rho^\pm\cap(\mathbb{I}\times B_\rho)=\mathcal{S}^\pm\cap(\mathbb{I}\times B_\rho)$.*

Proof. First of all, for an arbitrary subset $V \subseteq \mathcal{X}$ we define

$$\mathcal{V}^{\pm}(V) := \{(\tau, \xi) \in \mathbb{I} \times \mathcal{X} : \xi \in \mathcal{R}(P_{\pm}(\tau)) \cap V\}$$

assume $\Omega \subseteq U_0$ is a neighborhood of zero and fix a C^{m+1}-cut-off function $\chi : \mathcal{X} \to [0, 1]$ satisfying (3.5) as introduced above. Choose a real number $r > 0$ so small that $B_{2r} \subseteq \Omega$ and $B_{3r} \subseteq U_0$. The following proof is divided into two parts:

(I) We start by proving a special case and suppose that $\mathcal{V}^{\pm}(\Omega)$ is a local integral manifold of $(*)_F$; that is, $\mathcal{V}^{\pm}(\Omega)$ is represented as graph of the mapping $s^{\pm} : \Omega \times \mathbb{I} \to \mathcal{X}$, $s^{\pm}(x, t) \equiv 0$. Then the invariance equation (3.4) for $(*)_F$ boils down to

$$P_{\mp}(t) F(\xi, t) = 0 \quad \text{for all } t \in \mathbb{I}, \ \xi \in \mathcal{R}(P_{\pm}(t)) \cap \Omega .$$

We define the extended mapping $F_r : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$ by

$$F_r(x, t) := \begin{cases} \chi\left(\frac{x}{r}\right) F(x, t) & \text{for } x \in B_{2r}, \\ 0, & \text{else} \end{cases} \tag{3.11}$$

and obtain

$$P_{\mp}(t) F_r(\xi, t) = 0 \quad \text{for all } t \in \mathbb{I}, \ \xi \in \mathcal{R}(P_{\pm}(t)) \tag{3.12}$$

since $P_{\mp}(t) F(\xi, t) = 0$ for $\xi \in \mathcal{R}(P_{\pm}(t)) \cap B_{2r}$ and $\chi\left(\frac{x}{r}\right) = 0$ for $\|x\| \geqslant 2r$ (cf. (3.5)). Since (3.12) coincides with the invariance equation for $s^{\pm}(x, t) \equiv 0$, the set $\mathcal{V}^{\pm}(\mathcal{X})$ is invariant under the modified ODE $(*)_{F_r}$. Moreover, due to (3.1), we are able to diminish $r > 0$ such that $\sup_{(x,t) \in \mathcal{X} \times \mathbb{I}} \|D_1 F_r(x, t)\|$ is sufficiently small to satisfy Hypothesis (ii) of Theorem A.1. This yields a unique global integral manifold $\mathcal{S}_r^{\pm}$ for $(*)_{F_r}$, representable as graph over the nonautonomous set $\mathcal{V}^{\pm}(\mathcal{X})$. Hence, $\mathcal{S}_r^{\pm} = \mathcal{V}^{\pm}(\mathcal{X})$, and furthermore, the assertions of Proposition 3.5 are evidently satisfied with $s_r^{\pm}(x, t) \equiv 0$ and $\rho = r$.

(II) Now consider the general situation, when $\mathcal{S}^{\pm}$ is a local integral manifold for $(*)_F$ given by a C^1-mapping $s^{\pm} : U \times \mathbb{I} \to \mathcal{X}$ satisfying (3.2) and (3.3). We first define the extended mapping $s_r^{\pm} : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$ by

$$s_r^{\pm}(x, t) := \begin{cases} \chi\left(\frac{x}{r}\right) s^{\pm}(x, t) & \text{for } x \in B_{2r}, \\ 0, & \text{else,} \end{cases}, \tag{3.13}$$

which, by assumption, possesses continuous globally bounded partial derivatives $D_1^n s_r^{\pm}$ for $n \in \{1, \ldots, m+1\}$; from (3.2) we obtain the limit relation

$$\lim_{r \searrow 0} \sup_{(x,t) \in \mathcal{X} \times \mathbb{I}} \left\| D_1 s_r^{\pm}(x, t) \right\| = 0 . \tag{3.14}$$

Particularly, it is possible to choose $r>0$ sufficiently small that

$$\left\| D_1 s_r^{\pm}(x,t) \right\| < 1/2 \quad \text{for all } x \in \mathcal{X},\ t \in \mathbb{I} \tag{3.15}$$

holds. Next, we define a C^{m+1}-diffeomorphism $\Psi_t : \mathcal{X} \to \mathcal{X}$ by

$$\Psi_t(x) := x - s_r^{\pm}(x,t) \quad \text{for all } t \in \mathbb{I},$$

whose inverse $\Psi_t^{-1} : \mathcal{X} \to \mathcal{X}$ is given by $\Psi_t^{-1}(x) = x + s_r^{\pm}(x,t)$. Under the change of variables $x \mapsto \Psi_t(x)$, the ODE $(*)_F$ takes the form $(*)_G$ with a continuous function $G : B_{2r} \times \mathbb{I} \to \mathcal{X}$, which is of class C^m in the first argument and given by

$$\begin{aligned} G(x,t) :=\ & A(t)s_r^{\pm}(x,t) + F(x + s_r^{\pm}(x,t), t) \\ & - D_1 s_r^{\pm}(x,t)\big(A(t)(x + s_r^{\pm}(x,t)) + F(x + s_r^{\pm}(x,t), t)\big) \\ & - D_2 s_r^{\pm}(x,t). \end{aligned} \tag{3.16}$$

Please note that $G(\cdot, t)$ is defined on B_{2r}, since the mean value inequality implies

$$\begin{aligned} \left\| x + s_r^{\pm}(x,t) \right\| & \overset{(3.2)}{\leqslant} \|x\| + \left\| s_r^{\pm}(x,t) - s_r^{\pm}(0,t) \right\| \\ & \overset{(3.15)}{\leqslant} \frac{3}{2}\|x\| < 3r \quad \text{for all } x \in B_{2r},\ t \in \mathbb{I} \end{aligned}$$

and therefore, the inclusion $x + s_r^{\pm}(x,t) \in U_0$ is fulfilled. Moreover, from the invariance equation (3.4) we have

$$\begin{aligned} G(x,t) =\ & F(x + s^{\pm}(x,t), t) - P_{\mp}(t)F(P_{\pm}(t)x + s^{\pm}(x,t), t) - D_1 s^{\pm}(x,t) \\ & \times A(t)\big(P_{\mp}(t)x + s^{\pm}(x,t)\big) - D_1 s^{\pm}(x,t)\big(F(x + s^{\pm}(x,t), t) \\ & - P_{\pm}(t)F(P_{\pm}(t)x + s^{\pm}(x,t), t)\big) \end{aligned} \tag{3.17}$$

for all $x \in B_r$ and $t \in \mathbb{I}$. Accordingly, (3.2) implies $G(0,t) \equiv 0$ on $\mathbb{I}$. Likewise, (H_3) and (3.1) and (3.2) lead to $\lim_{x \to 0} \|D_1 G(x,t)\| = 0$ uniformly in $t \in \mathbb{I}$. It is easy to see that in each t-fiber, we have $\Psi_t(\mathcal{S}^{\pm}(t)) \cap B_{2r} = \mathcal{V}^{\pm}(B_{2r})(t)$, and consequently, $\mathcal{V}^{\pm}(B_{2r})$ is a local integral manifold of $(*)_G$, and the results from Step (I) imply that $\mathcal{V}^{\pm}(\mathcal{X})$ is the unique global integral manifold of $(*)_{G_r}$ with $G_r : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$ given by

$$G_r(x,t) := \begin{cases} \chi\left(\frac{x}{r}\right) G(x,t) & \text{for } x \in B_{2r}, \\ 0 & \text{else,} \end{cases}$$

If we now apply the inverse transformation $x \mapsto \Psi_t^{-1}(x)$ to $(*)_{G_r}$, one gets an ODE of the form $(*)_{\bar{F}_r}$ with $\bar{F}_r : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$,

$$\begin{aligned}\bar{F}_r(x,t) := &-A(t)s_r^{\pm}(x,t) + G_r(x - s_r^{\pm}(x,t),t) \\ &+ D_1 s_r^{\pm}(x,t)\big(A(t)(x - s_r^{\pm}(x,t)) + G_r(x - s_r^{\pm}(x,t),t)\big) \\ &+ D_2 s_r^{\pm}(x,t).\end{aligned} \tag{3.18}$$

Due to the properties of G_r and $s_r^{\pm}$, we obtain from (H_3) that the partial derivatives $D_1^n \bar{F}_r, n \in \{1, \dots, m\}$, are continuous and globally bounded. Using (3.4) in order to rewrite (3.18) as in (3.17), we see that $\sup_{(x,t)\in\mathcal{X}\times\mathbb{I}} \|D_1 G_r(x,t)\|$ can be made smaller than any given positive number and it is possible to diminish $r > 0$ such that Hypothesis (ii) of Theorem A.1 is fulfilled w.r.t. $(*)_{\bar{F}_r}$.

Finally, choose a real $\rho \in (0, r)$ so small that the inclusion $B_\rho \subseteq \Psi_t^{-1}(B_r)$ holds for all $t \in \mathbb{I}$, which is possible due to (3.2) and the uniformity in $t \in \mathbb{I}$. Substituting (3.16) into (3.18) gives us the identity $\bar{F}_\rho(x,t) \equiv F(x,t)$ on $B_\rho \times \mathbb{I}$. From (3.13), it is obvious that $s_\rho^{\pm}(x,t) \equiv s^{\pm}(x,t)$ on $B_\rho \times \mathbb{I}$. Hence, $\mathcal{S}_\rho^{\pm} \cap (\mathbb{I}\times B_\rho) = \mathcal{S}^{\pm} \cap (\mathbb{I}\times B_\rho)$. Since $\mathcal{V}^{\pm}(\mathcal{X})$ is the unique global integral manifold of $(*)_{G_\rho}$ and $\Psi_t^{-1}(\mathcal{V}^{\pm}(\mathcal{X})(t)) = \mathcal{S}_\rho^{\pm}(t)$, the set $\mathcal{S}_\rho^{\pm}$ is invariant under $(*)_{\bar{F}_\rho}$. We apply Theorem A.1, which yields that $\mathcal{S}_\rho^{\pm}$ is the unique global integral manifold of $(*)_{F_r}$. This finishes the proof of this proposition. $\square$

Our next proposition states that all integral manifolds $\mathcal{S}^{\pm}$ of $(*)_F$ possess the same Taylor series w.r.t. their state space variable up to order m. Moreover, it enables us in the following Section 4 to calculate integral manifolds using approximate solutions of the invariance Eq. (3.4).

Proposition 3.6 (Taylor expansion). *Suppose that* (H_1)–(H_3) *hold and let* $\mathcal{S}^{\pm}$ *denote a* C^m*-integral manifold with corresponding uniformly* C^m*-bounded mapping* $s^{\pm}: U\times\mathbb{I}\to\mathcal{X}$*. In case* $\mathcal{S}^+$ *is considered, assume* (3.6) *holds, and in case of* $\mathcal{S}^-$*, assume* $\mathbb{I} = \mathbb{R}$ *and* (3.7)*. If a* C^1*-function* $\sigma: \mathcal{X}\times\mathbb{I}\to\mathcal{X}$ *is* $(m+1)$*-times continuously differentiable in the first variable, uniformly* C^{m+1}*-bounded and satisfies*

(i) $\sigma(0,t) \equiv 0$ *on* $\mathbb{I}, \lim_{x\to 0}\|D_1\sigma(x,t)\| \pm 0$ *uniformly in* $t \in \mathbb{I}$ *and* $\sigma(x,t) = \sigma(P_{\pm}(t)x, t) \in \mathcal{R}(P_{\mp}(t))$ *for* $t \in \mathbb{I}, x \in \mathcal{X}$,
(ii) *for reals* $r > 0$ *so small that* $x + \sigma(x,t) \in U_0$ *for all* $x \in B_r, t \in \mathbb{I}$, *the mapping* $\mathcal{M}_t\sigma : B_r \to \mathcal{X}$ *given by*

$$\begin{aligned}(\mathcal{M}_t\sigma)(x) := &\ A(t)\sigma(x,t) + P_{\mp}(t)F(P_{\pm}(t)x + \sigma(x,t),t) \\ &-D_1\sigma(x,t)\big(A(t)P_{\pm}(t)x + P_{\pm}(t)F(P_{\pm}(t)x + \sigma(x,t),t)\big) \\ &-D_2\sigma(x,t)\end{aligned}$$

fulfills $D^n(\mathcal{M}_t\sigma)(0) = 0$ *for all* $n \in \{1, \dots, m\}, t \in \mathbb{I}$,

then we have $D_1^n\sigma(0,t) = D_1^n s^{\pm}(0,t)$ *for all* $t \in \mathbb{I}, n \in \{0, \dots, m\}$.

Proof. Using Proposition 3.5, the proof can be done similarly to [33, Theorem 3.4]. □

4. TAYLOR APPROXIMATION

Since the integral manifold $\mathcal{S}^{\pm}$ of $(*)_F$ is graph of a function $s^{\pm}$ smooth in its state space variable, and with the aid of Proposition 3.6, it is natural to approximate $s^{\pm}$ by its Taylor expansion. In this section, we derive necessary equations, the corresponding Taylor coefficients need to satisfy, and prove that they are uniquely solvable if the gap conditions (3.6) or (3.7) on the linear part of $(*)_F$ are satisfied. Thanks to our compact notation, the actual derivation will be quite short.

For this, we assume that, in addition to (H_1) and (H_2), the following assumption is satisfied, which in particular holds if $A:\mathbb{I}\to\mathcal{L}(\mathcal{X})$ is a C^m-mapping and $F:U_0\times\mathbb{I}\to\mathcal{X}$ is of class C^{m+1} (cf. Theorem A.1).

Hypothesis. *Let $m\geqslant 2$ and suppose:*

(H_4) *The partial derivatives $D_1^n s^{\pm}(0,\cdot)$ are differentiable for every $n\in\{2,\dots,m\}$.*

We are interested in local approximations for the mapping $s^{\pm}$. Here Taylor's Theorem (cf. [1, Theorem 2.6.05, p. 93]) together with (3.2) implies the representation

$$s^{\pm}(x,t)=\sum_{n=2}^{m}\frac{1}{n!}s_n^{\pm}(t)x^n+R_m^{\pm}(x,t) \tag{4.1}$$

with coefficient functions $s_n^{\pm}:\mathbb{I}\to\mathcal{L}_n(\mathcal{X})$ given by $s_n^{\pm}(t):=D_1^n s^{\pm}(0,t)$ and a remainder $R_m^{\pm}$ satisfying $\lim_{x\to 0}\frac{R_m^{\pm}(x,t)}{\|x\|^m}=0$. Proposition 3.6 guarantees that the coefficient $s_n^{\pm}(t)$ is uniquely determined by the mapping from Theorem 3.2. Due to (3.8), the functions $s_n^{\pm}$ are bounded, i.e., one has $\|s_n^{\pm}(t)\|\leqslant\gamma_n$ for $t\in\mathbb{I}$ and $n\in\{2,\dots,m\}$. Before proceeding, we need some handy notational preparations:

- We introduce $S^{\pm}:U\times\mathbb{I}\to\mathcal{X}$, $S^{\pm}(x,t):=P_{\pm}(t)x+s^{\pm}(x,t)$ satisfying

$$D_1S^{\pm}(0,t)\overset{(3.2)}{=}P_{\pm}(t),\ D_1^nS^{\pm}(0,t)=D_1^ns^{\pm}(0,t)\quad\text{for all } t\in\mathbb{I}$$

and $n\in\{2,\dots,m\}$. Hence, for $S_n^{\pm}(t):=D_1^nS^{\pm}(0,t)$, we have the estimates

$$\left\|S_1^{\pm}(t)\right\|\overset{(2.8)}{\leqslant}K_{\pm},\ \left\|S_n^{\pm}(t)\right\|\overset{(3.8)}{\leqslant}\gamma_n\quad\text{for all } n\in\{2,\dots,m\}. \tag{4.2}$$

- We abbreviate $g^{\pm}(x,t) := A(t)P_{\pm}(t)x + P_{\pm}(t)F(S^{\pm}(x,t),t)$, and the chain rule from Lemma 2.3 yields that the partial derivatives $g_n^{\pm}(t) := D_1^n g^{\pm}(0,t)$ are given by (cf. (3.1) and (3.2))

$$g_1^{\pm}(t)x_1 \overset{(2.4)}{=} A(t)P_{\pm}(t)x_1,$$

$$g_n^{\pm}(t)x_1,\dots,x_n = \sum_{l=2}^{n} \sum_{(N_1,\dots,N_l)\in P_l^{<}(n)} P_{\pm}(t)D_1^l F(0,t)S^{\pm}_{\#N_1}(t)_{P_{\pm}(t)} x_{N_1},\dots,S^{\pm}_{\#N_l}(t)_{P_{\pm}(t)}x_{N_l}$$

  for all $x_1,\dots,x_n \in \mathcal{X}$ and $n \in \{2,\dots,m\}$. Moreover, (H_2)–(H_3) and the estimates (2.8) and (4.2) imply that $g_n^{\pm} : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ is a bounded function.

Directly from the invariance equation (3.4) and (3.3), we get

$$\begin{aligned} &A(t)s^{\pm}(x,t) + P_{\mp}(t)F(P_{\pm}(t)x + s^{\pm}(x,t),t) \\ &\quad = D_1 s^{\pm}(x,t)\big(A(t)P_{\pm}(t)x + P_{\pm}(t)F(P_{\pm}(t)x + s^{\pm}(x,t),t)\big) + D_2 s^{\pm}(x,t) \end{aligned}$$

and using the notation introduced above, this reads as

$$D_2 s^{\pm}(x,t) = A(t)s^{\pm}(x,t) + P_{\mp}(t)F(S^{\pm}(x,t),t) - D_1 s^{\pm}(x,t)g(x,t)$$

for all $t \in \mathbb{I}, x \in U$ with $S^{\pm}(x,t) \in U_0$. If we differentiate this identity using Lemmas 2.1 and 2.3 and set $x = 0$, one gets the equation

$$\begin{aligned} &\dot{s}_n^{\pm}(t)_{P_{\pm}(t)}x_1,\dots,x_n = A(t)s_n^{\pm}(t)_{P_{\pm}(t)}x_1,\dots,x_n \\ &\quad + P_{\mp}(t)\sum_{j=2}^{n}\sum_{(N_1,\dots,N_j)\in P_j^{<}(n)} D_1^j F(0,t)S^{\pm}_{\#N_1}(t)_{P_{\pm}(t)}x_{N_1},\dots,S^{\pm}_{\#N_j}(t)x_{N_j} \\ &\quad - \sum_{\substack{(N_1,N_2)\in P_2(n)\\ N_1,N_2\neq\emptyset}} s^{\pm}_{\#N_1+1}(t)_{P_{\pm}(t)}x_{N_1}\cdot g^{\pm}_{\#N_2}(t)_{P_{\pm}(t)}x_{N_2} \end{aligned}$$

for $n \in \{2,\dots,m\}$ and $x_1,\dots,x_n \in \mathcal{X}$. Therefore, the function $s_n^{\pm} : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ is a solution of the linear ODE

$$\dot{X}_{P_{\pm}(t)} = L_{A(t)}X_{P_{\pm}(t)} + H_n^{\pm}(t)_{P_{\pm}(t)} \tag{4.3}$$

in $\mathcal{L}_n(\mathcal{X})$, denoted as *homological equation* for $\mathcal{S}^{\pm}$ with inhomogeneities $H_n^{\pm} : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$,

$$
\begin{aligned}
H_n^{\pm}(t)x_1,\ldots,x_n &:= P_{\pm}(t)D_1^n F(0,t)x_1,\ldots,x_n \\
&\quad + P_{\mp}(t)\sum_{j=2}^{n-1}\sum_{(N_1,\ldots,N_j)\in P_j^<(n)} D_1^j F(0,t)S_{\#N_1}^{\pm}(t)x_{N_1},\ldots,S_{\#N_j}^{\pm}(t)x_{N_j} \\
&\quad - \sum_{\substack{(N_1,N_2)\in P_2(n)\\ 0<\#N_1<n-1\\ N_2\neq\emptyset}} s_{\#N_1+1}^{\pm}(t)x_{N_1}\cdot g_{\#N_2}^{\pm}(t)x_{N_2}.
\end{aligned}
\tag{4.4}
$$

Obviously, one has $H_2^{\pm}(t)=P_{\mp}(t)D_1^2F(0,t)$, and for $n\in\{3,\ldots,m\}$, the values $H_n^{\pm}(t)$ only depend on $s_2^{\pm},\ldots,s_{n-1}^{\pm}$. Therefore (4.3) represents a hierarchy of linear differential equations for the coefficients $s_n^{\pm}$. These equations have to be solved step by step, starting with $n=2$, and increasing n by 1 at each step. The solutions are given by the following

Theorem 4.1. *Suppose* (H$_1$)–(H$_4$) *hold, and consider a mapping* $s^{\pm}: U\times\mathbb{I}\to\mathcal{X}$ *from Theorem 3.2. Then the following holds:*

(a) *The coefficients* $s_n^{+}:\mathbb{I}\to\mathcal{L}_n(\mathcal{X}), n\in\{2,\ldots,m\}$, *in the Taylor expansion* (4.1) *of the mapping* $s^{+}:U\times\mathbb{I}\to\mathcal{X}$ *can be determined recursively from the Lyapunov–Perron integrals*

$$
s_n^{+}(t)=-\int_t^{\infty}\Phi(t,s)H_n^{+}(s)_{\Phi(s,t)P_+(t)}\,ds \quad \text{for all } n\in\{2,\ldots,m\}. \tag{4.5}
$$

(b) *In case* $\mathbb{I}=\mathbb{R}$, *the coefficients* $s_n^{-}:\mathbb{R}\to\mathcal{L}_n(\mathcal{X}), n\in\{2,\ldots,m\}$, *in the Taylor expansion* (4.1) *of the mapping* $s^{-}:U\times\mathbb{R}\to\mathcal{X}$ *can be determined recursively from the Lyapunov–Perron integrals*

$$
s_n^{-}(t)=\int_{-\infty}^{t}\Phi(t,s)H_n^{-}(s)_{\Phi(s,t)P_-(t)}\,ds \quad \text{for all } n\in\{2,\ldots,m\}. \tag{4.6}
$$

Proof. In the explanations preceding Theorem 4.1, we have seen that $s_n^{\pm}:\mathbb{I}\to\mathcal{L}_n(\mathcal{X})$ is a bounded solution of the homological equation (4.3). Moreover, it follows from (4.4), (2.8), (H_2), and (4.2) that each inhomogeneity $H_n^{\pm}$ is bounded, i.e., 0-quasibounded. Consequently, due to the gap conditions (3.6) and (3.7), it follows from Lemma 2.6 that $s_n^{\pm}$ possesses the claimed appearance. □

Remark 4.2.

(1) For an autonomous ODE $(*)_F$, the functions (4.5) and (4.6) are constant and given as stationary solutions of the homological

equation (4.3). Then (4.3) reduces to the algebraic problem discussed (e.g., in [7]).

(2) In general, the Lyapunov–Perron integrals from Theorem 4.1 can be evaluated only numerically. In order to achieve this, we briefly sketch the appropriate procedure:

   (i) It suffices to represent the mappings $H_n^{\pm}(t)$ as n-linear forms in the space $\mathcal{L}_n(\mathcal{R}(P_{\pm}(t)); \mathcal{X})$. Under the assumption $N := \dim \mathcal{X} < \infty$ and $k_{\pm} := \dim \mathcal{R}(P_{\pm}(t)) \leqslant N$ (referring to (2.4), note that $k_{\pm}$ is constant in time) we know that the space $\mathcal{L}_n(\mathcal{R}(P_{\pm}(t)); \mathcal{X})$ has dimension $K_{\pm} := N\binom{k_{\pm}+n-1}{n}$ and is canonically isomorphic to $\mathbb{F}^{K_{\pm}}$ (cf. [33] for details). Using this representation, instead of working with the sum over (ordered) partitions in (4.4), we recommend to derive the homological equation (4.3) for s_n directly from the invariance equation (3.4) by means of computer algebra to calculate the necessary derivatives.

   (ii) Since the integrands in (4.5) and (4.6) are exponentially decaying, at least in principle, it is not difficult to obtain error estimates for their finite-interval approximations.

However, in concrete examples, the crucial problem is to obtain the invariant projectors $P_{\pm}$ from Hypothesis (H_1). Provided they are known, as well as the transition operator $\Phi(t,s)$ of (2.3), a recursive scheme to approximate $s_n^{\pm}(t)$ can be implemented on a computer, according to the suggestions made above.

5. EXAMPLES

In this section, we illuminate our results using two examples. The first is a nonautonomous bifurcation problem to illustrate a reduction on a center-unstable integral manifold.

Example 5.1. We consider a nonautonomous version of the well-known Lorenz equations (cf., e.g., [29] or [28, pp. 188, 291]), given by a 3-dimensional system

$$\begin{aligned} \dot{x}_1 &= \sigma_\varepsilon(t)(x_2 - x_1), \\ \dot{x}_2 &= \rho_\varepsilon(t)x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= -\beta_\varepsilon(t)x_3 + x_1x_2, \end{aligned} \tag{5.1}$$

which obviously can be written in the form (2.2) with right-hand side

$$f_\varepsilon(x,t) = \begin{pmatrix} \sigma_\varepsilon(t)(x_2 - x_1) \\ \rho_\varepsilon(t)x_1 - x_2 - x_1x_3 \\ -\beta_\varepsilon(t)x_3 + x_1x_2 \end{pmatrix}.$$

As discussed in [29], the Lorenz equations are a simplified toy model of Raleigh–Bernard thermal convection. To incorporate an external forcing into this model, it is interesting when all three parameters $\sigma_\varepsilon, \rho_\varepsilon, \beta_\varepsilon$ are perturbed nonautonomously, i.e., we assume the functions $\sigma_\varepsilon, \rho_\varepsilon, \beta_\varepsilon : \mathbb{R} \to (0, \infty)$ are given by

$$\sigma_\varepsilon(t) = \sigma_0 + \varepsilon\sigma(t), \qquad \rho_\varepsilon(t) = 1 + \rho_0 + \varepsilon\rho(t), \qquad \beta_\varepsilon(t) = \beta_0 + \varepsilon\beta(t)$$

with real constants $\sigma_0, \beta_0 > 0$, $\rho_0 \in \mathbb{R}$, bounded C^3-functions σ, ρ, β, and $\varepsilon \in \mathbb{R}$, which will serve as bifurcation parameter. It is our goal to study the stability of the equilibrium $x = 0$ of (5.1) for different values of ε. From the linearization

$$D_1 f_0(0,t) = \begin{pmatrix} -\sigma_0 & \sigma_0 & 0 \\ 1+\rho_0 & -1 & 0 \\ 0 & 0 & -\beta_0 \end{pmatrix},$$

we see that in case $\varepsilon = 0$ the origin is asymptotically stable for $\rho_0 \in \left[-\left(\frac{\sigma_0+1}{2\sigma_0}\right)^2, 0\right)$ and unstable for $\rho_0 > 0$. More interesting is the nonhyperbolic case $\rho_0 = 0$, where a pitchfork bifurcation occurs as ρ_0 passes through 0. To mimic this situation, we assume $\rho_0 = 0$ from now on. Before proceeding, we augment the original system (5.1) by considering the parameter ε as an additional state space variable satisfying $\dot{\varepsilon} = 0$ and – to simplify our calculations – apply the linear transformation

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} := \begin{pmatrix} -\sigma_0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \varepsilon \end{pmatrix}$$

to the resulting equation $\dot{x} = f_\varepsilon(x,t), \dot{\varepsilon} = 0$. This implies the 4-dimensional system

$$\dot{y} = Ay + F(y,t) \tag{5.2}$$

with $A := \operatorname{diag}(-\sigma_0 - 1, -\beta_0, 0, 0)$ and the nonlinearity

$$F(y,t) := \begin{pmatrix} \frac{\sigma_0}{\sigma_0+1} y_1y_2 - \frac{\sigma(t)+\sigma_0(\sigma(t)+\rho(t))}{\sigma_0+1} y_1y_4 - \frac{1}{\sigma_0+1} y_2y_3 + \frac{\rho(t)}{\sigma_0+1} y_3y_4 \\ -\sigma_0 y_1^2 + (1-\sigma_0)y_1y_3 - \beta(t)y_2y_4 + y_3^2 \\ \frac{\sigma_0^2}{\sigma_0+1} y_1y_2 + \frac{\sigma(t)+\sigma_0(\sigma(t)-\sigma_0\rho(t))}{\sigma_0+1} y_1y_4 - \frac{\sigma_0}{\sigma_0+1} y_2y_3 + \frac{\sigma_0\rho(t)}{\sigma_0+1} y_3y_4 \\ 0 \end{pmatrix}.$$

Thus, we can apply Theorem 3.2 to (5.2) to show that there exists a center-unstable manifold $\mathcal{S}^- \subseteq \mathbb{R}\times\mathbb{R}^3$ with 2-dimensional fibers. The ansatz

$$s^-(y_3, y_4, t) = \sum_{i=0}^{2} y_3^{2-i} y_4^i \begin{pmatrix} s^1_{2-i,i}(t) \\ s^2_{2-i,i}(t) \end{pmatrix} + O\left(\sqrt{y_3^2+y_4^2}^{\,3}\right)$$

yields that Eq. (5.2) reduced to the center-unstable manifold $\mathcal{S}^-$ is given by

$$\dot{y}_3 = \frac{\sigma_0}{\sigma_0+1}\left(\varepsilon\rho(t)y_3 - s^2_{2,0}(t)y_3^3\right) + O(\varepsilon y_3^2, \varepsilon^2 y_3, y_3^4)\,.$$

Using Theorem 4.1, we obtain $s^2_{2,0}(t) \equiv \frac{1}{\beta_0}$, and consequently, the bifurcation equation is

$$\dot{y}_3 = \frac{\sigma_0}{\sigma_0+1}\left(\varepsilon\rho(t)y_3 - \tfrac{1}{\beta_0}y_3^3\right) + r(y_3, t, \varepsilon)\,, \tag{5.3}$$

where the remainder r satisfies the three limit relations

$$\lim_{y_3\to 0} \sup_{\varepsilon\in(-|y_3|^3, |y_3|^3)} \sup_{t\in\mathbb{R}} \frac{|r(y_3,t,\varepsilon)|}{|y_3|^3} = 0\,, \tag{5.4}$$

$$\lim_{\varepsilon\to 0} \sup_{y_3\in(-|\varepsilon|, |\varepsilon|)} \sup_{t\in\mathbb{R}} \frac{|r(y_3,t,\varepsilon)|}{|\varepsilon|^2} = 0\,, \tag{5.5}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \limsup_{y_3\to 0} \sup_{t\in\mathbb{R}} \frac{|r(y_3,t,\varepsilon)|}{|y_3|} = 0\,. \tag{5.6}$$

To motivate our nonautonomous bifurcation result, we again recapitulate the autonomous situation. Assuming the functions ρ, σ, β are constant and $\rho(t) \equiv \bar{\rho}$ for some $\bar{\rho} > 0$, we get that (5.3) has the form

$$\dot{y}_3 = \frac{\sigma_0}{\sigma_0+1}\left(\varepsilon\bar{\rho}y_3 - \tfrac{1}{\beta_0}y_3^3\right) + O(\varepsilon y_3^2, \varepsilon^2 y_3, y_3^4)\,. \tag{5.7}$$

It is easy to check that this equation admits a pitchfork bifurcation, i.e., the equilibrium 0 is stable for $\varepsilon \leqslant 0$ and unstable for $\varepsilon > 0$. For small $\varepsilon > 0$, there are two additional stable equilibria branching from the origin, denoted by $y_\varepsilon^- < 0 < y_\varepsilon^+$. The compact interval enclosed by these two equilibria forms an attractor of (5.7): let ϕ_ε denote the local flow generated by (5.7). Then $\left[y_\varepsilon^-, y_\varepsilon^+\right]$ is invariant, and there exists a $\gamma_\varepsilon > 0$ such that $\left[y_\varepsilon^-, y_\varepsilon^+\right] = \bigcap_{t\geqslant 0}\phi_\varepsilon(t, U_{\gamma_\varepsilon}([y_\varepsilon^-, y_\varepsilon^+]))$. Thus, in addition to the pitchfork bifurcation, the autonomous system undergoes an attractor transition from a nontrivial attractor to a trivial attractor in the limit $\varepsilon \searrow 0$.

To establish a nonautonomous generalization of this scenario, we omit the autonomous restriction on σ, ρ, β, i.e., they are allowed to be bounded C^3-functions, and we suppose

$$\liminf_{t \to -\infty} |\rho(t)| > 0. \tag{5.8}$$

We will show that under this assumption, the nonautonomous equation (5.3) admits a bifurcation of pullback attractors (see, also Appendix B). For our purpose, we make use of a local version of such a pullback attractor. A *local pullback attractor* of (2.2) is given by a nonempty nonautonomous set $A \subseteq \mathbb{R} \times \mathcal{X}$ fulfilling the following three properties:

- A is *invariant*, i.e., its fibers satisfy $\varphi(t; \tau, A(\tau)) = A(t)$ for all $t, \tau \in \mathbb{R}$.
- A is *compact*, i.e., $A(t)$ is compact for all $t \in \mathbb{R}$.
- A is *locally pullback attracting*, i.e., there exists a $\gamma > 0$ with

$$\lim_{t \to -\infty} d\big(\varphi\big(\tau; t, U_\gamma(A(t))\big), A(\tau)\big) = 0 \quad \text{for all } \tau \in \mathbb{R}.$$

Here, d means the *Hausdorff semi-distance*, which for nonempty sets $A, B \subseteq \mathcal{X}$ is defined by $d(A, B) := \sup_{x \in A} \inf_{y \in B} \|x - y\|$.

We obtain the subsequent nonautonomous bifurcation result.

Proposition 5.2. *For the bifurcation equation* (5.3)*, the following statements hold:*

(a) *In case* $\liminf_{t \to -\infty} \rho(t) > 0$*, there exists an* $\hat{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0, \hat{\varepsilon})$ *there exists a nontrivial local pullback attractor* A_ε *of* (5.3)*, and we have*

$$\lim_{\varepsilon \searrow 0} d(A_\varepsilon(t), \{0\}) = 0 \quad \textit{for all } t \in \mathbb{R}. \tag{5.9}$$

Furthermore, $\mathbb{R} \times \{0\}$ *is a local pullback attractor of* (5.3) *for* $\varepsilon \in (-\hat{\varepsilon}, 0]$.

(b) *In case* $\limsup_{t \to -\infty} \rho(t) < 0$*, there exists an* $\hat{\varepsilon} > 0$ *such that for all* $\varepsilon \in (-\hat{\varepsilon}, 0)$ *there exists a nontrivial local pullback attractor* A_ε *of* (5.3)*, and we have*

$$\lim_{\varepsilon \nearrow 0} d(A_\varepsilon(t), \{0\}) = 0 \quad \textit{for all } t \in \mathbb{R}.$$

Furthermore, $\mathbb{R} \times \{0\}$ *is a local pullback attractor of* (5.3) *for* $\varepsilon \in [0, \hat{\varepsilon})$.

Proof. We only treat case (a), since (b) can be shown analogously. Let φ_ε denote the general solution of the bifurcation equation (5.3). Due to (5.4) and the boundedness of ρ, there exists an $s_1 > 0$ such that

$$\left|\frac{\sigma_0 \varepsilon \rho(t) y_3}{\sigma_0 + 1} + r(y_3, t, \varepsilon)\right| \leqslant \frac{\sigma_0}{2\beta_0(\sigma_0+1)}|y_3|^3 \quad \text{for all } t \in \mathbb{R}, |y_3| \leqslant s_1$$

and $|\varepsilon| < |y_3|^3$. This implies

$$g\left(-\sqrt[3]{\varepsilon}, t, \frac{\varepsilon}{2}\right) > 0 > g\left(\sqrt[3]{\varepsilon}, t, \frac{\varepsilon}{2}\right) \quad \text{for all } t \in \mathbb{R}, \varepsilon \in (0, s_1^3), \tag{5.10}$$

where g denotes the right-hand side of (5.3). In the following, we fix $\varepsilon \in (0, \frac{1}{2}s_1^3)$ and define a nonautonomous set N_ε by its fibers

$$N_\varepsilon(t) := \left[-2\sqrt[3]{\varepsilon}, 2\sqrt[3]{\varepsilon}\right] \quad \text{for all } t \in \mathbb{R}.$$

Due to (5.10), the relation $\varphi_\varepsilon(\tau; \tau - t, N_\varepsilon(\tau - t)) \subseteq N_\varepsilon(\tau)$ for all $\tau \in \mathbb{R}, t \geqslant 0$ holds. Hence, Theorem B.1 implies the existence of a pullback attractor A_ε with attraction universe $\{N_\varepsilon\}$. It is easy to check that A_ε is also a local pullback attractor. Please note that $A_\varepsilon \subseteq N_\varepsilon$. In connection with the continuity of φ_ε, this guarantees the limit relation (5.9). We show now that the sets A_ε are nontrivial for sufficiently small $\varepsilon > 0$. Due to (5.8), there exist $\rho_- > 0$ and $t^- \in \mathbb{R}$ such that

$$\rho(t) > \rho^- \quad \text{for all } t \leqslant t^-.$$

Furthermore, due to (5.5), there exists an $\tilde{\varepsilon} \in (0, \frac{1}{2}s_1^3)$ such that

$$\left|-\frac{\sigma_0 y_3^3}{\beta_0(\sigma_0+1)} + r(y_3, t, \varepsilon)\right| \leqslant \frac{\sigma_0 \rho^-}{2(\sigma_0+1)}|\varepsilon|^2 \quad \text{for all } t \in \mathbb{R}, |\varepsilon| \leqslant \tilde{\varepsilon}$$

and $|y_3| < |\varepsilon|$. This implies

$$g\left(-\frac{\varepsilon}{2}, t, \varepsilon\right) < 0 < g\left(\frac{\varepsilon}{2}, t, \varepsilon\right) \quad \text{for all } t \leqslant t^-, \varepsilon \in (0, \tilde{\varepsilon}).$$

Hence, we have $A_\varepsilon(t) \supseteq \left[-\varepsilon/2, \varepsilon/2\right]$ for all $t \leqslant t^-$. Now we consider negative values of ε. Due to (5.6), there exist $\hat{\varepsilon} \in (0, \tilde{\varepsilon})$ such that for all $\varepsilon \in (-\hat{\varepsilon}, 0)$, there exists an $s_2 > 0$ with

$$|r(y_3, t, \varepsilon)| \leqslant -\frac{\sigma_0 \rho^- \varepsilon}{2(\sigma_0+1)}|y_3| \qquad \text{for all } |y_3| \leqslant s_2.$$

Therefore, we have

$$g(y_3, t, \varepsilon) < \frac{\sigma_0 \rho^- \varepsilon}{2(\sigma_0+1)} y_3 \quad \text{for all } y_3 \in (0, s_2), \ \ t \leqslant t^-$$

and

$$g(y_3, t, \varepsilon) > \frac{\sigma_0 \rho^- \varepsilon}{2(\sigma_0 + 1)} y_3 \quad \text{for all } y_3 \in (-s_2, 0), \ \ t \leqslant t^- .$$

This implies that $\mathbb{R} \times \{0\}$ is a local pullback attractor for $\varepsilon \in (-\hat{\varepsilon}, 0)$. With similar arguments, one can see that also in case $\varepsilon = 0$, this set is a local pullback attractor (note that this is the autonomous case and the origin of (5.1) is asymptotically stable as discussed earlier). This finishes the proof. □

Remark 5.3. Not only the reduced equation (5.3) admits a nonautonomous bifurcation of this type but also the Lorenz equation (5.1) itself. This is due to an asymptotic phase property of the center-unstable manifold (cf. Theorem A.1(b) or [3, Theorem 4] for a global version), i.e., every solution of (5.1) in a neighborhood of the manifold approaches exponentially a solution on the center-unstable manifold in forward time. Therefore, for small $\varepsilon > 0$, there also exists a local pullback attractor of (5.1) shrinking down to $\{0\}$ for $\varepsilon \searrow 0$.

Our second, relatively simple example is primarily of an illustrative nature.

Example 5.4. Consider the following modified Korteweg-de Vries equation

$$u_t + u_{xxx} + a(t) u^2 u_x = 0 ,$$

where we assume the coefficient $a : \mathbb{R} \to \mathbb{R}$ is a smooth bounded function. The investigation of traveling wave solutions having the structure $U(x - c^2 t) = u(x, t), c > 0$, leads to the 3-dimensional nonautonomous ODE

$$\dot{x}_1 = x_2 , \qquad \dot{x}_2 = x_3 , \qquad \dot{x}_3 = c^2 x_2 - a(t) x_1^2 x_2 . \tag{5.11}$$

Then its linearization along the trivial solution does not depend on t and has the eigenvalues $-c, 0, c$. Thus, (5.11) can be transformed into a system with decoupled linear part via a constant linear transformation. For simplicity, we consider this transformed system from now on. Theorem 3.2 is applicable, yielding center-stable and center-unstable manifolds $\mathcal{S}^+$ and $\mathcal{S}^-$ with 2-dimensional fibers, respectively. If we assume their representation

$$D^n_{(1,2)} s^{\pm}(0, 0, t) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}^n = \sum_{i=0}^{n} s^{\pm}_{j-i,i}(t) h_1^{j-i} h_2^i \qquad \text{for all } h_1, h_2 \in \mathbb{R}$$

then Theorem 4.1 yields $D^2_{(1,2)} s^{\pm}(0, 0, t) \equiv 0$ and the following expressions

$$s^+_{30}(t) = -\frac{1}{2c} \int_t^{\infty} e^{4c(t-s)} a(s) ds , \quad s^-_{30}(t) = 0 ,$$

$$s_{21}^{+}(t) = -\frac{1}{2c}\int_t^{\infty} e^{3c(t-s)}a(s)ds\,, \quad s_{21}^{-}(t) = -\frac{1}{2c}\int_{-\infty}^{t} e^{-2c(t-s)}a(s)ds\,,$$
$$s_{12}^{+}(t) = -\frac{1}{2c}\int_t^{\infty} e^{2c(t-s)}a(s)ds\,, \quad s_{12}^{-}(t) = -\frac{1}{2c}\int_{-\infty}^{t} e^{-3c(t-s)}a(s)ds\,,$$
$$s_{03}^{+}(t) = 0\,, \qquad s_{03}^{-}(t) = -\frac{1}{2c}\int_{-\infty}^{t} e^{-4c(t-s)}a(s)ds$$

for the third-order Taylor coefficients of s^+ and s^-. Higher-order Taylor coefficients can be obtained successively. Choosing $c=1$ and an oscillatory damping $a(t)=e^{-|t|}\sin(t)$ – purely to avoid numerical integrations – we have computed a sixth-order approximation of the center-stable manifold $\mathcal{S}^+$ and center-unstable manifold $\mathcal{S}^-$ for the system (5.11). Note, that the special structure of (5.11) allows to apply our results for only continuous functions a. Figure 1 visualizes the fibers $\mathcal{S}^{\pm}(t)$ for $t\in\{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$ in the box $[-2, 2]^3$.

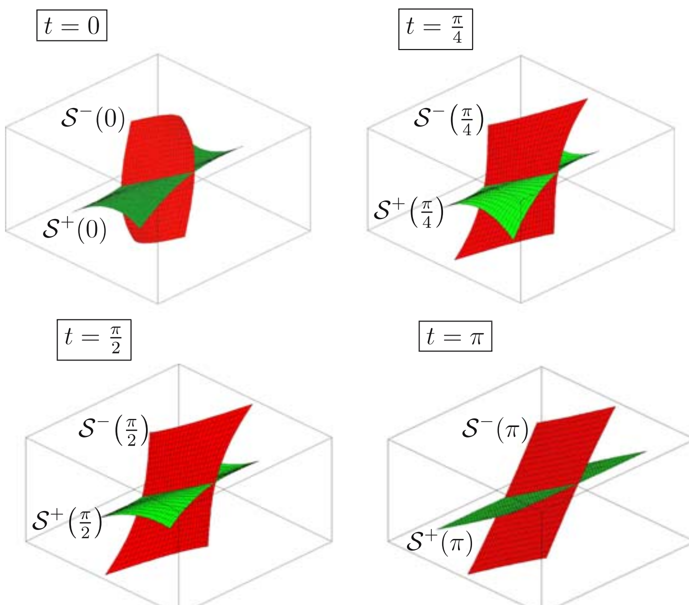

Figure 1. It is apparent that $\mathcal{S}^{\pm}(t)$ becomes smoother as t evolves. This is due to our particular choice for the function a to be exponentially decaying. Therefore, (5.11) is asymptotically autonomous and asymptotically linear. The fibers $\mathcal{S}^{\pm}(t)$ approach the center-unstable and center-stable subspaces for the linearization of (5.11) as $t\to\pm\infty$.

APPENDIX A: GLOBAL INTEGRAL MANIFOLDS

To make our approach more accessible to readers not familiar with the nonautonomous theory, we state our global existence theorem for integral manifolds in this appendix. It can be considered as abstraction of the classical Hadamard–Perron theorem and is quoted often in the main text. We remark that its general assumptions make it a quite flexible tool.

Theorem A.1 **(global existence of integral manifolds).** *Let $m \in \mathbb{N}$ and $\mathbb{I} \subseteq \mathbb{R}$ be an interval unbounded above. Assume the continuous functions $A : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ and $F : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$ satisfy:*

(i) *The linear ODE $\dot{x} = A(t)x$ possesses an exponential dichotomy, i.e., there exists an invariant projector $P_- : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ such that for all $t, \tau \in \mathbb{I}$ the estimates*

$$\|\Phi(t,\tau)P_+(\tau)\| \leqslant K_+ e^{\alpha(t-\tau)}, \qquad \|\Phi(\tau,t)P_-(t)\| \leqslant K_- e^{\beta(\tau-t)}$$

for all $\tau \leqslant t$ hold with real constants $K_+, K_- \geqslant 1, \alpha < \beta$.

(ii) *We have the identity $F(0,t) \equiv 0$ on $\mathbb{I}$ and the partial derivatives $D_1^n F$ exist and are continuous for $n \in \{1, \dots, m\}$ with globally bounded partial derivatives*

$$|F|_n := \sup_{(x,t) \in \mathcal{X} \times \mathbb{I}} \left\| D_1^n F(x,t) \right\| < \infty.$$

Moreover, with $K := K_+ + K_- + K_+ K_- \max\{K_+, K_-\}$ we require

$$|F|_1 < \frac{\beta - \alpha}{4K},$$

choose a fixed $\delta \in \left(2K\,|F|_1, \frac{\beta-\alpha}{2}\right)$ and define $\Gamma := (\alpha + \delta, \beta - \delta)$.

Then, denoting the general solution of the ODE

$$\dot{x} = A(t)x + F(x,t) \tag{A.1}$$

by φ, the following holds for all $\gamma \in \Gamma$:

(a) *The global pseudo-stable manifold of* (A.1), *given by*

$$\mathcal{S}^+ := \left\{ (\tau, x_0) \in \mathbb{I} \times \mathcal{X} : \sup_{\tau \leqslant t} \|\varphi(t;\tau,x_0)\|\, e^{\gamma(\tau - t)} < \infty \right\}$$

is independent of $\gamma \in \Gamma$ and possesses the representation

$$\mathcal{S}^+ = \left\{ (\tau, \xi + s^+(\xi,\tau)) \in \mathbb{I} \times \mathcal{X} : \tau \in \mathbb{I}, \xi \in \mathcal{R}(P_+(\tau)) \right\}$$

with a C^1-mapping $s^+ : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$. It satisfies:

(a_1) *Under the gap condition*

$$m\alpha < \beta,$$

the partial derivatives $D_1^n s^+ : \mathcal{X}\times\mathbb{I}\to\mathcal{X}$ *exist and are continuous with*

$$\sup_{(x,t)\in\mathcal{X}\times\mathbb{I}} \left\| D_1^n s^+(x,t) \right\| < \infty \quad \text{for all } n\in\{1,\dots,m\},$$

if furthermore the derivatives $D^{m-1}A, D_2^k D_1^n F$ *exist and are continuous for* $0\leqslant k < m, 0\leqslant k+n\leqslant m$, *then* s^+ *is m-times continuously differentiable,*

(a_2) *in case* $\mathbb{I}=\mathbb{R}$, *the nonautonomous set* $\mathcal{S}^+$ *possesses an asymptotic (backward) phase, i.e., there exists a continuous mapping* $\pi^+ : \mathcal{X}\times\mathbb{R}\to\mathcal{X}$ *such that for all* $\tau\in\mathbb{R}, \pi^+(\cdot,\tau) : \mathcal{X}\to\mathcal{S}^+(\tau)$ *is a retraction onto* $\mathcal{S}^+(\tau)$ *with*

$$\left\| \varphi(t;\tau,x_0) - \varphi(t;\tau,\pi^+(x_0,\tau)) \right\| \leqslant C^+ \|x_0\| e^{\gamma(t-\tau)}$$

for all $t\leqslant\tau$, $x_0\in\mathcal{X}$ *and some real* $C^+\geqslant 0$.

(b) *In case* $\mathbb{I}=\mathbb{R}$, *the global pseudo-unstable manifold of* (A.1)*, given by*

$$\mathcal{S}^- := \left\{ (\tau,x_0)\in\mathbb{I}\times\mathcal{X} : \sup_{t\leqslant\tau} \|\varphi(t;\tau,x_0)\| e^{\gamma(\tau-t)} < \infty \right\}$$

is independent of $\gamma\in\Gamma$ *and possesses the representation*

$$\mathcal{S}^- = \left\{ (\tau, \xi + s^-(\xi,\tau)) \in \mathbb{I}\times\mathcal{X} : \tau\in\mathbb{I}, \xi\in\mathcal{R}(P_-(\tau)) \right\}$$

with a C^1*-mapping* $s^- : \mathcal{X}\times\mathbb{I}\to\mathcal{X}$. *It satisfies:*

(b_1) *Under the gap condition*

$$\alpha < m\beta$$

the partial derivatives $D_1^n s^- : \mathcal{X}\times\mathbb{I}\to\mathcal{X}$ *exist and are continuous with*

$$\sup_{(x,t)\in\mathcal{X}\times\mathbb{I}} \left\| D_1^n s^-(x,t) \right\| < \infty \quad \text{for all } n\in\{1,\dots,m\};$$

if furthermore the derivatives $D^{m-1}A, D_2^k D_1^n F$ *exist and are continuous for* $0\leqslant k < m, 0\leqslant k+n\leqslant m$, *then* s^- *is m-times continuously differentiable,* (b$_2$) *the nonautonomous set $\mathcal{S}^-$ possesses an asymptotic (forward) phase, i.e., there exists a continuous mapping $\pi^-:\mathcal{X}\times\mathbb{R}\to\mathcal{X}$ such that for all $\tau\in\mathbb{R}, \pi^-(\cdot,\tau): \mathcal{X}\to\mathcal{S}^-(\tau)$ is a retraction onto $\mathcal{S}^-(\tau)$ with*

$$\left\|\varphi(t;\tau,x_0)-\varphi(t;\tau,\pi^-(x_0,\tau))\right\|\leqslant C^-\,\|x_0\|\,e^{\gamma(t-\tau)}$$

for all $\tau\leqslant t$, $x_0\in\mathcal{X}$ and some real $C^-\geqslant 0$.

(c) *The nonautonomous set $\mathcal{S}^\pm$ is invariant in the sense that its fibers satisfy $\mathcal{S}^\pm(t)=\varphi(t;\tau,\mathcal{S}^\pm(\tau))$ for all $t,\tau\in\mathbb{R}$, one has the representation*

$$s^\pm(x_0,\tau)=s^\pm(P_\pm(\tau)x_0,\tau)\in\mathcal{R}(P_\mp(\tau))\quad \textit{for all } \tau\in\mathbb{I},\ x_0\in\mathcal{X}$$

and the invariance equation

$$P_\mp(t)\varphi(t;\tau,x_0)=s^\pm(P_\pm(t)\varphi(t;\tau,x_0),t)\quad \textit{for all } t\in\mathbb{I},\ (\tau,x_0)\in\mathcal{S}^\pm.$$

(d) *One has the identity $s^\pm(0,\tau)\equiv 0$ on $\mathbb{I}$, and in case $\mathbb{I}=\mathbb{R}$, only the zero solution of* (A.1) *is contained both in $\mathcal{S}^+$ and $\mathcal{S}^-$, i.e.,*

$$\mathcal{S}^+\cap\mathcal{S}^-=\mathbb{R}\times\{0\};$$

hence, the zero solution is the only γ-quasibounded solution of (A.1).

(e) *If the ODE* (A.1) *and P_- are T-periodic for some $T>0$, then also $s^\pm(\cdot,x_0)$ and the fibers $\mathcal{S}^\pm(\cdot)$ are T-periodic for all $x_0\in\mathcal{X}$. In particular, for an autonomous equation* (A.1)*, the fibers $\mathcal{S}^\pm(t)$ are constant in $t\in\mathbb{I}$ and each $\mathcal{S}^\pm(t)$ is an invariant manifold of* (A.1).

Proof. The verification of Theorem A.1 is technically involved, in particular concerning the smoothness assertions. We therefore, give only a sketch how the main ingredients can be assembled. In addition, we restrict to the literature on nonautonomous equations. Prototypes of Theorem A.1 can be found in [36] for almost periodic equations and [4] for measurable time-dependence. Concerning the asymptotic phase of $\mathcal{S}^\pm$, we refer to the work of [3, Theorem 4]. A comprehensive account to the differentiability properties can is given in [39]; smoothness in the state space is also considered by Chicone and Latushkin [10]. Both [4, 39] suppose the linear part of (A.1) is in block diagonal form, whereas we — similarly to [10] – make the exponential dichotomy assumption (i). Hence, the smoothness claims, as well as the asymptotic phase properties, can be obtained by combining the methods in [10, 39]. □

APPENDIX B. PULLBACK ATTRACTORS

Since 1990s, the attractivity of nonautonomous sets is intensively discussed. In particular, the notion of *pullback attractor* has been introduced (see, e.g., [9]). Please note that the so-called *random attractors* are closely related to pullback attractors (see, e.g., [2, 17]).

In Example 5.1, we used the notion of a local pullback attractor, which has been developed in [34]. Local pullback attractors are special cases of pullback attractors with an attraction universe. For the reader's convenience, the definition and an existence result for such pullback attractors are presented.

Let $\mathcal{D}$ be a collection of nonautonomous sets (often, $\mathcal{D}$ consists of fiberwise-constant bounded nonautonomous sets). Then a *pullback attractor* of (2.2) with *attraction universe* $\mathcal{D}$ is given by a nonempty nonautonomous set $A \subseteq \mathbb{R} \times \mathcal{X}$ fulfilling the three properties:

- A is *invariant*, i.e., its fibers satisfy $\varphi(t; \tau, A(\tau)) = A(t)$ for all $t, \tau \in \mathbb{R}$.
- A is *compact*, i.e., $A(t)$ is compact for all $t \in \mathbb{R}$.
- A is *pullback attracting w.r.t.* $\mathcal{D}$, i.e., for all $D \in \mathcal{D}$, we have

$$\lim_{t \to -\infty} d\big(\varphi(\tau; t, D(t)), A(\tau)\big) = 0 \quad \text{for all } \tau \in \mathbb{R}.$$

Here, d means the Hausdorff semi-distance.

It is easy to see that a nonempty nonautonomous set A is a local pullback attractor if and only if there exists a $\gamma > 0$ such that with the nonautonomous set D, defined by its t-fibers $D(t) := U_\gamma(A(t))$, A is pullback attractor with attraction universe $\{D\}$.

The following existence result for pullback attractors plays a crucial role in proof of Proposition 5.2.

Theorem B.1. *Consider a collection of nonautonomous sets $\mathcal{D}$, and let $B \subset \mathbb{R} \times \mathcal{X}$ be a compact pullback absorbing set, i.e., all fibers of B are compact and for all $D \in \mathcal{D}$ and $\tau \in \mathbb{R}$, there exists a $t^* < \tau$ such that $\varphi(\tau; t, D(t)) \subseteq B(\tau)$ for all $t \leqslant t^*$. Then, there exists a pullback attractor A with attraction universe $\mathcal{D}$, which fulfills the representation*

$$A(\tau) = \bigcap_{t^* \leqslant \tau} \overline{\bigcup_{t \leqslant t^*} \varphi(\tau; t, B(t))} \qquad \textit{for all } \tau \in \mathbb{R}.$$

If, in addition, $A \in \mathcal{D}$, then A is uniquely determined. In case $B \in \mathcal{D}$, the relation $A \subseteq B$ is fulfilled.

Proof. See [17, Theorem 3.5]. □

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