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# A Non-Newtonian Fluid with Navier Boundary Conditions

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We consider in this paper the equations of motion of third grade fluids on a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ with Navier boundary conditions. Under the assumption that the initial data belong to the Sobolev space $H^2$, we prove the existence of a global weak solution. In dimension two, the uniqueness of such solutions is proven. Additional regularity of bidimensional initial data is shown to imply the same additional regularity for the solution. No smallness condition on the data is assumed.

**KEY WORDS:** Global weak solutions; Navier boundary conditions; non-newtonian; third grade fluid.

## 1. INTRODUCTION

Recently, the class of non-Newtonian fluids of differential type has received a special attention, mainly because it includes the family of second grade fluids which are very interesting for several reasons. First of all, these equations were deduced by Dunn and Fosdick [9] from physical principles. Later on, another interpretation was found by Camassa and Holm [7], see also [11, 12]: the one-dimensional version of these equations can be used as a model for shallow water and the generalization to higher dimension uses an interesting geometric property involving geodesics, similar to the one that is well-known for the Euler equations. Finally, these equations were found to be useful in turbulence theory (see [8]).

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Fluids of grade three are a generalization of second grade fluids and constitute the next step in the modeling of fluids of differential type. Roughly speaking, if for second grade fluids the stress tensor is polynomial of degree two in the first two Rivlin–Ericksen tensors (see [16]), for third grade fluids the stress tensor is polynomial of degree three in the first three Rivlin–Ericksen tensors. The particular form of the stress tensor was deduced from physical principles by Fosdick and Rajagopal [10] and the associated partial differential equation can be written under the following form:

$$\begin{aligned}&\partial_t(u-\alpha_1\Delta u)-\nu\Delta u+\operatorname{curl}(u-\alpha_1\Delta u)\wedge u\\&\quad-(\alpha_1+\alpha_2)\left(A\Delta u+2\operatorname{div}\left[\nabla u(\nabla u)^t\right]\right)-\beta\operatorname{div}(|A|^2A)=f-\nabla_p,\ \operatorname{div}\ u=0.\end{aligned}\tag{1}$$

Here, $\wedge$ denotes the exterior product, $u(t,x)$ is the velocity vector field, $f(t,x)$ is the forcing applied to the fluid, $p(t,x)$ is a scalar function representing the pressure, $A=(a_{ij})_{i,j}$ is the matrix whose coefficients are given by $a_{ij}(u)=\partial_i u_j+\partial_j u_i$, $|A|^2=\sum_{i,j}a_{ij}^2$ and $\nu,\alpha_1,\alpha_2,\beta$ are some material coefficients which must satisfy the following hypotheses:

$$\nu\geqslant 0,\quad \alpha_1>0,\quad \beta\geqslant 0\quad\text{and } |\alpha_1+\alpha_2|\leqslant(24\nu\beta)^{1/2}.\tag{2}$$

We refer to Fosdick and Rajagopal [10] for further details concerning the modeling of this equation. Note that the case $\beta=0$ corresponds to the equation of second grade fluids. We also observe that, as in [5], the last inequality in (2) will not be used here.

Here, we consider Eq. (1) on a smooth bounded domain $\Omega$ of $\mathbb{R}^2$ or $\mathbb{R}^3$ and we supplement it with the following Navier boundary conditions:

$$u\cdot n=0\quad\text{and } (An)_{\tan}=0\quad\text{on } \partial\Omega,\tag{3}$$

where $n$ denotes the exterior unitary normal to the boundary and $(An)_{\tan}$ is the tangential part of the vector $An$. The Navier boundary conditions can be traced back to the original paper of Navier [15], are mentioned in the work of Serrin [18] and were used (in a slightly different form) to model a free boundary for the Navier–Stokes equations (see [19–21] and the references therein). We also mention that these conditions were also obtained by Jäger and Mikelić [13, 14] by means of homogenization over a rough boundary. Let us finally note that second grade fluids with Navier boundary conditions were studied in [6].

There are several works on the mathematical theory of third grade fluids on bounded domains (see [2, 3, 17]). These results consider the case of homogeneous Dirichlet boundary conditions and prove global existence and uniqueness of solutions for small initial data in $H^3$ or $W^{2,r}$ with $r > 3$, and local existence and uniqueness for large data.

In [5], see also [4], the authors took advantage of the observation that the nonlinear term $-\operatorname{div}(|A|^2 A)$ has a good sign and is more regularizing than the viscosity term $-\Delta u$. Nevertheless, since this term is nonlinear, its derivatives do not have the same special structure. Consequently, it is not trivial to use this term in higher order energy estimates, like for example the $H^2$ estimates. However, in the absence of boundaries, some special integrations by parts were performed in [4, 5] and it was possible to exploit the symmetry of the term $-\operatorname{div}(|A|^2 A)$. This resulted in a global existence theorem without any smallness assumption and, moreover, for less regular initial data ($H^2$ instead of $H^3$ as in the bounded domain case). Uniqueness and additional regularity in dimension two was also proved. Unfortunately, the proofs from [4, 5] do not extend to the bounded domain case since the integrations by parts performed yield some boundary terms which are not vanishing and cannot be estimated in a satisfactory manner.

Here we are able to extend the approach of [4, 5] to bounded domains in the case of Navier boundary conditions. More precisely, we prove the following theorem:

***Theorem 1*** (Existence, uniqueness and regularity). *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, $u_0 \in H^2(\Omega)$ a divergence free vector field verifying the Navier boundary conditions* (3), *$f \in L^2_{\text{loc}}([0,\infty); L^2(\Omega))$ and suppose that $\beta > 0$. Then there exists a global solution $u \in L^\infty_{\text{loc}}([0,\infty); H^2(\Omega))$ with initial data $u_0$. Furthermore, if the space dimension is two, then this solution is unique. Finally, also in the case of the dimension two, if $u_0 \in H^3(\Omega)$ and $f \in L^2_{\text{loc}}([0,\infty); H^1(\Omega))$ then this additional regularity of the initial data is preserved, i.e., $u \in L^\infty_{\text{loc}}([0,\infty); H^3(\Omega))$.*

From a technical point of view, the advantage of the Navier boundary conditions over the Dirichlet ones is that if $u$ verifies (3), then, as it was observed in [6], $\Delta u$ is almost tangent to the boundary in the sense that it can be expressed in terms of derivatives of order 1 of $u$. This is very important if we want to make $H^2$ estimates. Indeed, making $H^2$ estimates requires to multiply equation (1) by $\Delta u$ and if we do so we end up with a nonvanishing pressure term $\int_\Omega \nabla p \cdot \Delta u$. If we want to avoid estimating the pressure (which we do not know how to estimate), then we need $\Delta u$ to be tangent to the boundary in order to conclude that the pressure term vanishes. As noticed above, this is almost true for Navier boundary conditions but definitely wrong in the case of Dirichlet boundary conditions. For this reason, we cannot prove Theorem 1 in the case of Dirichlet boundary conditions. The global existence of solutions for large data in this case remains a very interesting open problem.

We finally note that although in the statement of Theorem 1 only the $H^3$ regularity is shown to be propagated by the equation, it is easy to show that other regularities are propagated, too. Indeed, once we have the control over the $H^3$ norm we have the control over the Lipschitz norm of the solution and this easily implies that other regularities are propagated.

The structure of the paper is the following. In the next section, we introduce the notation and prove some identities and inequalities related to the Navier boundary conditions. Section 2 contains the proof of the global existence of $H^2$ solutions. And in Sections 3 and 4, we consider the case of the dimension two and prove first the uniqueness of $H^2$ solutions, and second the propagation of the $H^3$ regularity of the initial data.

## 2. NOTATIONS AND PRELIMINARY RESULTS

The partial derivative with respect to $x_i$ is denoted by $\partial_i$. We generically denote by $C$ a constant that may change its value from one line to another. The constants $C_1, C_2, \dots,$ and $K_1, K_2, \dots,$ are fixed once introduced. In the following, we will denote by $n: \overline{\Omega} \to \mathbb{R}^d$ some smooth extension to $\overline{\Omega}$ of the exterior unitary normal to $\partial\Omega$. Consequently, the notation for the normal derivative $\partial_n = \sum_i n_i \partial_i$ makes sense not only on the boundary but also in the interior of the domain. For a vector field $u$ we denote by $L = L(u)$ it is gradient matrix $L = L(u) = \nabla u = (\partial_j u_i)_{i,j}$ so that $A = A(u) = L(u) + L(u)^t$. We will use the notations

$$|\nabla^2 u|^2 = \sum_{i,j,k} (\partial_j \partial_k u_i)^2 \quad \text{and} \quad |\nabla A(u)|^2 = \sum_{i,j,k} |\partial_i a_{jk}(u)|^2.$$

We denote by $\mathscr{D}(u) = (u, \nabla u)$ the vector of $\mathbb{R}^{d^2+d}$ whose components are the components of $u$ and the first-order derivatives of these components. Similarly, $\mathscr{D}^k(u) = (u, \nabla u, \dots, \nabla^k u)$, is the vector of $\mathbb{R}^{d^{k+1}+\dots+d^2+d}$ whose components are the components of $u$ together with the derivatives of order up to $k$ of these components. We say that a function $F = F(\mathscr{D}^k(u))$ (possibly vector-valued) is of *form k* if it can be expressed as a linear combination of the components of $\mathscr{D}^k(u)$ with coefficients polynomials in $n$ and its derivatives.

The equivalence sign $\simeq$ applies to two quantities such that the ratio lies between two strictly positive constants depending only on the domain $\Omega$.

The divergence of a matrix $M = (m_{ij})$ is the vector whose $i$th component is given by $(\mathrm{div} M)_i = \sum_j \partial_j m_{ij}$. We denote by $W^{k,p}(\Omega)$ the standard Sobolev space of functions whose derivatives up to the order $k$ belong to

$L^p$ and set $H^k = W^{k,2}$. The operator $\mathbb{P}$ denotes the Leray projector, i.e., the orthogonal projection in $(L^2(\Omega))^3$ on the subspace of divergence free vector fields tangent to the boundary.

We will use in the sequel the following three versions of the Sobolev norms $H^1, H^2$, and $H^3$:

$$|||u|||_{H^1} = \left(\|u\|_{L^2}^2 + 2\alpha_1 \|D(u)\|_{L^2}^2\right)^{1/2},$$

$$|||u|||_{H^2} = \left(|||u|||_{H^1}^2 + \|\mathbb{P}(u - \alpha_1 \Delta u)\|_{L^2}^2\right)^{1/2}$$

and

$$|||u|||_{H^3} = (|||u|||_{H^1}^2 + \|\operatorname{curl}(u - \alpha_1 \Delta u)\|_{L^2}^2)^{1/2}.$$

Here, $D(u) = \frac{1}{2} A(u)$ denotes the deformation tensor. Note that the norms $\|\cdot\|_{H^1}$ and $|||\cdot|||_{H^1}$ are equivalent by the Korn inequality, while the norm $\|\cdot\|_{H^2}$, respectively $\|\cdot\|_{H^3}$, is equivalent to the norm $|||\cdot|||_{H^2}$, respectively $|||\cdot|||_{H^3}$, as a consequence of Corollary 6.

## 2.1. Some Identities

We prove now some identities related to the Navier boundary conditions. First recall that it was proved in [6, Proposition 2] that

$$v \cdot n|_{\partial\Omega} = F_1(\mathscr{D}(u))|_{\partial\Omega}, \quad v = u - \alpha_1 \Delta u \tag{4}$$

for some function $F_1$ of *form* 1. Next, we show the following lemma:

***Lemma 2.*** *Let $u$ be a divergence free vector field verifying the Navier boundary conditions* (3) *and define $\lambda, \mu : \overline{\Omega} \to \mathbb{R}$ by:*

$$\lambda = \langle A(u)n, n\rangle, \quad \mu = \partial_n(u \cdot n). \tag{5}$$

*Then the following relations hold true:*

$$An = \lambda n, \quad \nabla(u \cdot n) = \mu n, \quad \partial_n u = (\lambda - \mu)n + \sum_j u_j \nabla n_j \quad \text{on } \partial\Omega. \tag{6}$$

*Moreover, there exist a finite number of functions $G_\ell$ and $H_\ell$ of form* 1 *such that*

$$\partial_n(|A|^2) = \sum_\ell G_\ell(\mathscr{D}(u)) H_\ell(\mathscr{D}(u)) \quad \text{on} \quad \partial\Omega. \tag{7}$$

**Proof.** Since $(An)_{\tan}=0$ on $\partial\Omega$, we know that there exists some $\tilde{\lambda}$: $\partial\Omega\to\mathbb{R}^d$ such that $An=\tilde{\lambda}n$ on $\partial\Omega$. Taking the scalar product with $n$ we get $\langle An, n\rangle=\lambda$ which implies at once that $\lambda=\tilde{\lambda}$ on the boundary. Similarly, we know that $u\cdot n\big|_{\partial\Omega}=0$ which implies that $\nabla(u\cdot n)\big|_{\partial\Omega}$ is normal to the boundary. As above we obtain that $\nabla(u\cdot n)=\mu n$ on the boundary. The relation for $\partial_n u$ follows at once from the first two relations together with the following identity that holds true for an arbitrary vector field $u$:

$$\partial_n u = An - \nabla(u\cdot n) + \sum_j u_j \nabla n_j. \tag{8}$$

This completes the proof of relation (6). To prove (7), observe first that, by the symmetry of $A$,

$$\partial_n(|A|^2)=4\sum_{i,j} a_{ij}\partial_n(\partial_i u_j)=4\sum_{i,j} a_{ij}\partial_i(\partial_n u_j)-4\sum_{i,j,k} a_{ij}\partial_i n_k\partial_k u_j.$$

The last term is of the required form. Next, if we introduce $F_0(u)=\sum_j u_j\nabla n_j$, then we saw above that

$$\partial_n u_j-(\lambda-\mu)n_j-F_{0,j}(u)=0 \quad \text{on} \quad \partial\Omega,$$

where $F_{0,j}$ denotes the $j$th component of $F_0$. The gradient of the above function is therefore normal to the boundary and we infer that there exists $\gamma_j$: $\partial\Omega\to\mathbb{R}$ such that

$$\nabla[\partial_n u_j-(\lambda-\mu)n_j-F_{0,j}(u)]=\gamma_j n \quad \text{on} \quad \partial\Omega. \tag{9}$$

Taking the scalar product with $n$ yields

$$\gamma_j=\partial_n[\partial_n u_j-(\lambda-\mu)n_j-F_{0,j}(u)] \quad \text{on} \quad \partial\Omega. \tag{10}$$

Using (9) and (6) we obtain that, on the boundary of $\Omega$,

$$\begin{aligned}\sum_{i,j} a_{ij}\partial_i(\partial_n u_j) &= \sum_{i,j} a_{ij}[\gamma_j n_i+\partial_i(\lambda-\mu)n_j]+\sum_{i,j} a_{ij}[(\lambda-\mu)\partial_i n_j+\partial_i F_{0,j}(u)]\\ &= \lambda[n\cdot\gamma+\partial_n(\lambda-\mu)]+\sum_{i,j} a_{ij}[(\lambda-\mu)\partial_i n_j+\partial_i F_{0,j}(u)],\end{aligned}$$

where we used that $An=\lambda n$ on $\partial\Omega$. The last sum is obviously the sum of products of two functions of *form* 1. According to what is proved above, to complete the proof it suffices to show that $n\cdot\gamma+\partial_n(\lambda-\mu)$ can be expressed on the boundary as a function of *form* 1. From (10) we get that

$$n\cdot\gamma=n\cdot\partial_n^2 u-|n|^2\partial_n(\lambda-\mu)-\frac{\lambda-\mu}{2}\partial_n(|n|^2)-n\cdot\partial_n F_0(u) \quad \text{on } \partial\Omega,$$

so

$$n \cdot \gamma + \partial_n(\lambda - \mu) = n \cdot \partial_n^2 u - \frac{\lambda - \mu}{2} \partial_n(|n|^2) - n \cdot \partial_n F_0(u) \quad \text{on } \partial\Omega$$

It remains to prove that $n \cdot \partial_n^2 u\big|_{\partial\Omega}$ can be expressed as a function of *form* 1 and this follows at once from (4). Indeed, suppose for example, that the space dimension is two and let $\tau = (n_2, -n_1)$ be the tangential vector. It is a simple calculation to show that $(\Delta - \partial_n^2 - \partial_\tau^2)u$ can be expressed as a function of *form* 1, so in order to conclude it suffices to show that $n \cdot \partial_\tau^2 u\big|_{\partial\Omega}$ can be expressed as a function of *form* 1. This is obvious as it is clear that $[n \cdot \partial_\tau^2 u - \partial_\tau^2(u \cdot n)]\big|_{\partial\Omega}$ can be expressed as a function of *form* 1 and $\partial_\tau^2(u \cdot n)\big|_{\partial\Omega} \equiv 0$ since $u$ is tangent to the boundary and $\partial_\tau$ is a tangential derivative. This completes the proof. □

The following lemma is a simple exercise of differential geometry.

***Lemma 3.*** *Let $u : \overline{\Omega} \to \mathbb{R}^d$ be a vector field tangent to the boundary of $\Omega$. Then the vector field $(u \cdot \nabla)n - \sum_j u_j \nabla n_j$ is normal to the boundary.*

**Proof.** Let $x_0 \in \partial\Omega$. Since $\partial\Omega$ is a hypersurface, there exist a neighborhood $V$ of $x_0$ and a smooth function $\phi : V \to \mathbb{R}$ such that $\phi\big|_{\partial\Omega \cap V} \equiv 0$ and there exists some smooth scalar function $\delta : \partial\Omega \cap V \to \mathbb{R}$ such that $n = \delta \nabla\phi$. Note that since $u$ is tangent to the boundary, $u \cdot \nabla$ is a tangential derivative, so $(u \cdot \nabla)n\big|_{\partial\Omega}$ is independent of the extension of $n$. Moreover, choosing another extension $n$ of the unitary exterior normal results in adding a multiple of the normal to the boundary to each of the $\nabla n_j$, so the tangential part of the vector field $\sum_j u_j \nabla n_j$ is independent of the extension of $n$. Therefore, the tangential part of the vector field $(u \cdot \nabla)n - \sum_j u_j \nabla n_j$ is independent of the extension of the exterior normal. We choose to extend $n$ to $V$ by $n = \tilde{\delta}\nabla\phi$, where $\tilde{\delta}$ is an arbitrary smooth extension of $\delta$ initially defined only on the boundary. We can now write

$$\begin{aligned}(u \cdot \nabla)n - \sum_j u_j \nabla n_j &= (u \cdot \nabla)(\tilde{\delta}\nabla\phi) - \sum_j u_j \nabla(\tilde{\delta}\partial_j\phi) \\ &= [(u \cdot \nabla)\tilde{\delta}]\nabla\phi - \nabla\tilde{\delta}(u \cdot \nabla)\phi.\end{aligned}$$

The first term is obviously normal to $\partial\Omega$. The second one vanishes since $u \cdot \nabla$ is a tangential derivative and $\phi$ vanishes on the boundary. We infer that the vector field $(u \cdot \nabla)n - \sum_j u_j \nabla n_j$ is normal to the boundary on $\partial\Omega \cap V$, in particular in $x_0$. Since $x_0$ was arbitrary, the conclusion follows. □

We finally recall the following Green formula (see [6, Lemma 3]):

$$\int_\Omega \Delta u \cdot \widetilde{u} = -2 \int_\Omega D(u) \cdot D(\widetilde{u}), \tag{11}$$

where $u$ and $\widetilde{u}$ are two divergence free vector fields such that $u$ verifies the Navier boundary conditions (3) and $\widetilde{u}$ is tangent to the boundary.

## 2.2. Some Inequalities

First observe that the following identity

$$2\partial_j\partial_k u_i = \partial_j a_{ik}(u) + \partial_k a_{ij}(u) - \partial_i a_{jk}(u)$$

holds for any vector field $u$. Consequently,

$$|\nabla^2 u| \leqslant \frac{3}{2}|\nabla A(u)|. \tag{12}$$

Next, let us recall the Korn inequality (see, for instance [22]): for every $p \in (1, \infty)$, there exists a constant $K_0(p, \Omega)$ such that for every vector field $u$ we have that

$$\|u\|_{W^{1,p}} \leqslant K_0(p, \Omega)(\|u\|_{L^p} + \|A(u)\|_{L^p}). \tag{13}$$

We prove now the following lemma.

***Lemma 4.*** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^2$. There exist constants $K_1 = K_1(\Omega)$ and $K_2 = K_2(\Omega)$ such that for all $f \in H^2(\Omega)$ one has that*

$$\|f\|_{L^\infty(\Omega)} \leqslant \frac{K_1}{\sqrt{\varepsilon}}\|f\|^{1-\varepsilon}_{H^1(\Omega)}\|f\|^{\varepsilon}_{H^2(\Omega)} \quad \text{for all } \varepsilon \in (0, 1]$$

*and*

$$\|f\|_{L^4(\Omega)} \leqslant K_2\|f\|^{1/2}_{L^2(\Omega)}\|f\|^{1/2}_{H^1(\Omega)}.$$

**Proof.** Let $E$ be an extension operator $E: H^2(\Omega) \to H^2(\mathbb{R}^2)$ such that there exists a constant $C_1'$ such that for all $h \in H^2(\Omega)$,

$$E(h)\big|_\Omega = h, \quad \|E(h)\|_{H^2(\mathbb{R}^2)} \leqslant C_1'\|h\|_{H^2(\Omega)}, \quad \|E(h)\|_{H_1(\mathbb{R}^2)} \leqslant C_1'\|h\|_{H^1(\Omega)}$$
$$\text{and} \quad \|E(h)\|_{L^2(\mathbb{R}^2)} \leqslant C_1'\|h\|_{L^2(\Omega)}.$$

The existence of such an extension operator is well-known (see for instance [1, Theorem 4.26]). Let us also recall that the embedding $H^{1+\varepsilon}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ holds with norm $\leqslant \frac{C_2'}{\sqrt{\varepsilon}}$, for some constant $C_2'$ independent of $\varepsilon$ (for a simple proof see [4, Proposition 1]). Using also the standard interpolation inequality

$$\|\cdot\|_{H^{1+\varepsilon}(\mathbb{R}^2)} \leqslant \|\cdot\|^{1-\varepsilon}_{H^1(\mathbb{R}^2)}\|\cdot\|^{\varepsilon}_{H^2(\mathbb{R}^2)},$$

we can write

$$\begin{aligned}\|f\|_{L^\infty(\Omega)} \leqslant \|E(f)\|_{L^\infty(\mathbb{R}^2)} &\leqslant \frac{C_2'}{\sqrt{\varepsilon}}\|E(f)\|_{H^{1+\varepsilon}(\mathbb{R}^2)}\\ &\leqslant \frac{C_2'}{\sqrt{\varepsilon}}\|E(f)\|_{H^1(\mathbb{R}^2)}^{1-\varepsilon}\|E(f)\|_{H^2(\mathbb{R}^2)}^{\varepsilon}\\ &\leqslant \frac{C_1'C_2'}{\sqrt{\varepsilon}}\|f\|_{H^1(\Omega)}^{1-\varepsilon}\|f\|_{H^2(\Omega)}^{\varepsilon}.\end{aligned}$$

Similarly, using the embedding $H^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ with norm denoted by $C_3'$, we obtain that

$$\begin{aligned}\|f\|_{L^4(\Omega)} &\leqslant \|E(f)\|_{L^4(\mathbb{R}^2)} \leqslant C_3'\|E(f)\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \leqslant C_3'\|E(f)\|_{L^2(\mathbb{R}^2)}^{1/2}\|E(f)\|_{H^1(\mathbb{R}^2)}^{1/2}\\ &\leqslant C_1'C_3'\|f\|_{L^2(\Omega)}^{\frac{1}{2}}\|f\|_{H^1(\Omega)}^{1/2}.\end{aligned}$$ □

***Lemma 5.*** *Let $u:\overline{\Omega}\to\mathbb{R}^d$ be a smooth divergence free vector field verifying the Navier boundary conditions* (3) *and set $v=u-\alpha_1\Delta u$. Then for all $r\in(1,\infty)$, there exist constants $K_3=K_3(r,\Omega)$ and $K_4=K_4(r,\Omega)$ independent of the vector field $u$ and such that*

$$\|v-\mathbb{P}v\|_{L^r} \leqslant K_3\|u\|_{W^{1,r}} \tag{14}$$

*and*

$$\|v-\mathbb{P}v\|_{W^{1,r}} \leqslant K_4\|u\|_{W^{2,r}}. \tag{15}$$

**Proof.** From the definition of the Leray projector, we know there is some $\phi$ such that $v-\mathbb{P}v=\nabla\phi$. Taking the divergence of this relation we obtain that

$$\Delta\phi=\mathrm{div}\nabla\phi=\mathrm{div}(v-\mathbb{P}v)=0.$$

Taking now the scalar product with $n$, restricting to the boundary and using relation (4) we get

$$\partial_n\phi\big|_{\partial\Omega}=n\cdot\nabla\phi\big|_{\partial\Omega}=n\cdot v\big|_{\partial\Omega}=F_1(\mathscr{D}(u))\big|_{\partial\Omega}.$$

The two required estimates follow now immediately from standard trace estimates and the regularity theory for the Neumann problem for the laplacian. We also observe that the explicit expression for $F_1(\mathscr{D}(u))$ obtained in [6] involves only tangential derivatives of $u$ on the boundary and not normal derivatives. Indeed, in the case $d=2$ we have from [6, Proposition 1] that $F_1(\mathscr{D}(u))=2\alpha_1\partial_\tau(u\cdot\partial_\tau n)$, $\partial_\tau=n_1\partial_2-n_2\partial_1$, while in the case $d=3$ the explicit formula contained in the proof of [6, Proposition 2] can be written under the form $F_1(\mathscr{D}(u))=\alpha_1 n\cdot[\nabla_\tau\times(2n\times\sum_{i=1}^3 u_i\nabla_\tau n_i)]+$

$\alpha_1(\nabla_\tau \cdot u)(\nabla_\tau \cdot n)$, where $\nabla_\tau$ is the following vector of tangential derivatives: $\nabla_\tau = n \times \nabla$. Since $F_1(\mathscr{D}(u))$ involves only tangential derivatives of $u$, we see that the trace of $F_1(\mathscr{D}(u))$ on $\partial\Omega$ is well defined if $u \in W^{1,r}(\Omega)$. We finally note that inequality (14) can also be obtained from a straightforward integration by parts. □

We obtain immediately the following corollary.

***Corollary 6.*** *Let $u: \overline{\Omega} \to \mathbb{R}^d$ be a smooth divergence free vector field verifying the Navier boundary conditions* (3) *and set $v = u - \alpha_1 \Delta u$. We have the following equivalent quantities for the $H^2$ and $H^3$ norms:*

$$\|u\|_{H^2} \simeq \|u\|_{H^1} + \|\mathbb{P}v\|_{L^2} \tag{16}$$

*and*

$$\|u\|_{H^3} \simeq \|u\|_{H^1} + \|\text{curl } v\|_{L^2}. \tag{17}$$

*Moreover, there exists a constant $K_5 = K_5(\Omega)$ independent of $u$ such that*

$$\|u\|_{W^{1,12}} \leqslant K_5(\|u\|_{H^1} + \|A(u)\|_{L^{12}}). \tag{18}$$

**Proof.** We know from [6, Proposition 3], see also [21], that $\|u\|_{H^2} \simeq \|v\|_{L^2}$. On the other hand

$$\|v\|_{L^2} \leqslant \|\mathbb{P}v\|_{L^2} + \|v - \mathbb{P}v\|_{L^2} \leqslant \|\mathbb{P}v\|_{L^2} + K_3(2, \Omega)\|u\|_{H^1},$$

where we have used (14). This proves (16) since the reverse inequality follows trivially using that $\mathbb{P}$ is an orthogonal projection in $L^2$. Next, we note that (17) is proved in [6, Proposition 6]. To prove (18), we use the Korn inequality (13) for $p = 12$ to write

$$\|u\|_{W^{1,12}} \leqslant K_0(\|u\|_{L^{12}} + \|A\|_{L^{12}}). \tag{19}$$

If the dimension is 2, then (18) simply follows from the embedding $H^1(\Omega) \hookrightarrow L^{12}(\Omega)$. In dimension 3, an additional step is necessary. We use the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{1,12}(\Omega) \hookrightarrow L^\infty(\Omega)$ with norms $C_4'$, respectively, $C_5'$ to deduce that

$$\begin{aligned} K_0\|u\|_{L^{12}} &\leqslant K_0\|u\|_{L^6}^{1/2}\|u\|_{L^\infty}^{1/2} \leqslant K_0 C_4'^{1/2} C_5'^{1/2}\|u\|_{H^1}^{1/2}\|u\|_{W^{1,12}}^{1/2} \\ &\leqslant \tfrac{1}{2}\|u\|_{W^{1,12}} + \frac{K_0^2 C_4' C_5'}{2}\|u\|_{H^1}. \end{aligned}$$

Plugging this relation in (19) implies at once (18). □

## 3. GLOBAL EXISTENCE FOR LARGE $H^2$ DATA

In order to get $H^2$ estimates for $u$, the natural way would be to multiply (1) by $u-\alpha_1\Delta u$ and to integrate. Unfortunately, this does not work as the pressure term will not vanish. Therefore, one has to multiply by $\mathbb{P}(u-\alpha_1\Delta u)$ instead and this results in estimates on $\|\mathbb{P}(u-\alpha_1\Delta u)\|_{L^2}$ only. In view of (16), we also need to estimate the $H^1$ norm of $u$.

Let us recall that the equation for the velocity can be written under the following equivalent form (see [5])

$$\partial_t v - \nu\Delta u + (u\cdot\nabla)v + \sum_j v_j \nabla u_j - (\alpha_1+\alpha_2)\operatorname{div}(A^2) + \beta K(u) = f - \nabla p', \tag{20}$$

where we used the notations

$$v = u - \alpha_1\Delta u, \quad K(u) = -\operatorname{div}(|A|^2 A) \quad \text{and} \quad A = A(u).$$

The first step in making $H^2$ estimates are the $H^1$ estimates.

### 3.1. $H^1$ a *priori* Estimates

Let us multiply (20) by $u$ and integrate in space to obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|u\|_{L^2}^2 + 2\alpha_1\|D(u)\|_{L^2}^2\right) + 2\nu\|D(u)|_{L^2}^2 + \beta\int_\Omega K(u)\cdot u \\ &= -\int_\Omega (u\cdot\nabla)v\cdot u - \sum_j\int_\Omega v_j\nabla u_j\cdot u + \int_\Omega(\alpha_1+\alpha_2)\operatorname{div}(A^2)\cdot u + \int_\Omega f\cdot u,\end{aligned}$$

where we used the Green formula (11). We classically have that

$$\int_\Omega (u\cdot\nabla)v\cdot u + \sum_j\int_\Omega v_j\nabla u_j\cdot u = \int_\Omega\sum_{i,j}\partial_i(u_iu_jv_j) = \int_{\partial\Omega} n\cdot u\ \ u\cdot v = 0,$$

where we used the Stokes formula together with the fact that $u$ is divergence free and tangent to the boundary. Next, an integration by parts shows that

$$\begin{aligned}\int_\Omega K(u)\cdot u &= -\int_\Omega \operatorname{div}(|A|^2A)\cdot u = \int_\Omega |A|^2 A\cdot\nabla u \\ &\quad - \int_{\partial\Omega}|A|^2(An)\cdot u = \frac{1}{2}\int_\Omega |A|^4,\end{aligned}$$

where we used the symmetry of the matrix $A$ and the Navier boundary conditions to deduce that the boundary term vanishes. Similarly

$$\begin{aligned}(\alpha_1+\alpha_2)\int_\Omega \operatorname{div}(A^2)\cdot u &= -(\alpha_1+\alpha_2)\int_\Omega A^2\cdot\nabla u+(\alpha_1+\alpha_2)\int_{\partial\Omega}(A^2 n)\cdot u\\ &\leqslant |\alpha_1+\alpha_2|\|A^2\|_{L^2}\|\nabla u\|_{L^2}\\ &\leqslant \frac{\beta}{4}\int_\Omega |A|^4+C(\alpha_1,\alpha_2,\beta)\|u\|_{H^1}^2.\end{aligned}$$

Finally,

$$\int_\Omega f\cdot u\leqslant \|f\|_{L^2}\|u\|_{L^2}\leqslant \tfrac{1}{2}\|f\|_{L^2}^2+\tfrac{1}{2}\|u\|_{H^1}^2.$$

We conclude that the following differential inequality holds

$$\frac{d}{dt}|||u|||_{H^1}^2+\frac{\beta}{2}\|A\|_{L^4}^4\leqslant \|f\|_{L^2}^2+C_1|||u|||_{H^1}^2 \tag{21}$$

for some constant $C_1$. The Gronwall lemma now implies that

$$|||u|||_{H^1}^2+\frac{\beta}{2}\int_0^t\|A\|_{L^4}^4\leqslant e^{C_1 t}\left(|||u_0|||_{H^1}^2+\int_0^t\|f\|_{L^2}^2\right).$$

We infer that

$$|||u(t)|||_{H^1}\leqslant e^{\frac{c_1 t}{2}}\left(|||u_0|||_{H^1}+\|f\|_{L^2((0,t)\times\Omega)}\right)\stackrel{\text{def}}{=} M_0(t) \tag{22}$$

and

$$\|A\|_{L^4((0,t)\times\Omega)}\leqslant\left(\frac{2}{\beta}\right)^{1/4}e^{\frac{c_1 t}{4}}\left(|||u_0|||_{H^1}+\|f\|_{L^2((0,t)\times\Omega)}\right)^{1/2}.$$

From the Sobolev embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$ with norm constant denoted by $C_2$ we get that

$$\|u\|_{L^4((0,t)\times\Omega)}\leqslant C_2\|u\|_{L^4(0,t;H^1)}\leqslant C_2 t^{1/4}e^{\frac{c_1 t}{2}}\left(|||u_0|||_{H^1}+\|f\|_{L^2((0,t)\times\Omega)}\right).$$

The Korn inequality (13) together with the two previous relations now imply that

$$\begin{aligned}\|u\|_{L^4(0,t;W^{1,4})} &\leqslant K_0(4,\Omega)e^{\frac{c_1 t}{4}}\left[\left(\frac{2}{\beta}\right)^{1/4}+C_2 t^{1/4}e^{\frac{c_1 t}{4}}\right]\\ &\quad\times\left(1+|||u_0|||_{H^1}+\|f\|_{L^2((0,t)\times\Omega)}\right)\\ &\stackrel{\text{def}}{=} M_1(t)\end{aligned} \tag{23}$$

and from the Sobolev embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ with norm constant $C_3$ we get that

$$\|u\|_{L^1(0,t;L^\infty)} \leqslant t^{3/4}\|u\|_{L^4(0,t;L^\infty)} \leqslant C_3 t^{3/4} M_1(t) \stackrel{\text{def}}{=} M_2(t). \tag{24}$$

### 3.2. A *priori* Estimate for $\|\mathbb{P}v\|_{L^2}$

The heart of the matter is now the estimate for $\mathbb{P}v$. Let us multiply equation (20) by $\mathbb{P}v$ and integrate in space to obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbb{P}v\|_{L^2}^2 + \beta\int_\Omega K(u)\cdot\mathbb{P}v = \nu\int_\Omega \Delta u\cdot\mathbb{P}v - \sum_j \int_\Omega v_j \nabla u_j \cdot \mathbb{P}v \\ +(\alpha_1+\alpha_2)\int_\Omega \operatorname{div}(A^2)\cdot\mathbb{P}v - \int_\Omega (u\cdot\nabla)\cdot\mathbb{P}v \\ +\int_\Omega f\cdot\mathbb{P}v. \end{aligned} \tag{25}$$

The most important point in these *a priori* estimates is the estimate of the $K$-term. It is precisely this part of the proof that allows us to obtain the global existence for large data.
**Estimate of the $K$-term.** We first write

$$\int_\Omega K(u)\cdot\mathbb{P}v = \underbrace{\int_\Omega K(u)\cdot v}_{I_1} + \underbrace{\int_\Omega K(u)\cdot(\mathbb{P}v - v)}_{I_2}. \tag{26}$$

To estimate $I_1$, we start with an integration by parts:

$$\begin{aligned} I_1 = -\int_\Omega \operatorname{div}(|A|^2 A)\cdot v &= \int_\Omega |A|^2 A\cdot\nabla v - \int_{\partial\Omega} |A|^2 A n\cdot v \\ &= \frac{1}{2}\int_\Omega |A|^2 A\cdot A(v) - \int_{\partial\Omega} |A|^2 A n\cdot v \end{aligned}$$

Since, $A(v) = A - \alpha_1 \Delta A$, one has that

$$\int_\Omega |A|^2 A\cdot A(v) = \int |A|^4 - \alpha_1\int_\Omega |A|^2 A\cdot\Delta A.$$

A second integration by parts shows that

$$\int_\Omega |A|^2 A\cdot\Delta A = -\sum_i \int_\Omega \partial_i(|A|^2 A)\cdot\partial_i A + \int_{\partial\Omega} |A|^2 A\cdot\partial_n A$$

But it is just a simple computation to note that

$$\sum_i \int_\Omega \partial_i(|A|^2 A)\cdot\partial_i A = \int_\Omega |A|^2|\nabla A|^2 + \frac{1}{2}\int_\Omega |\nabla(|A|^2)|^2.$$

Putting together the above relations we infer that

$$\begin{aligned} I_1 &= \frac{1}{2}\int_\Omega |A|^4 + \frac{\alpha_1}{2}\int_\Omega |A|^2|\nabla A|^2 + \frac{\alpha_1}{4}\int_\Omega |\nabla(|A|^2)|^2 \\ &\quad -\frac{\alpha_1}{2}\underbrace{\int_{\partial\Omega}|A|^2 A\cdot\partial_n A}_{I_{11}} - \underbrace{\int_{\partial\Omega}|A|^2 An\cdot v}_{I_{12}}. \end{aligned} \tag{27}$$

We now have to estimate the boundary terms $I_{11}$ and $I_{12}$. To bound $I_{12}$, we use the Navier boundary conditions together with relations (4) and (6) to write

$$I_{12} = \int_{\partial\Omega}|A|^2 An\cdot v = \int_{\partial\Omega}|A|^2\lambda n\cdot v = \int_{\partial\Omega}|A|^2\lambda F_1(\mathscr{D}(u)), \tag{28}$$

where $\lambda$ is given in relation (5). By the Stokes formula, we can return to an integral on $\Omega$ and write

$$\begin{aligned} I_{12} &= \int_{\partial\Omega} n\cdot\Big[n|A|^2\lambda F_1(\mathscr{D}(u))\Big] = \int_\Omega \operatorname{div}\Big[n|A|^2\lambda F_1(\mathscr{D}(u))\Big] \\ &= \int_\Omega \operatorname{div}\, n|A|^2\lambda F_1(\mathscr{D}(u)) + \int_\Omega \partial_n(|A|^2)\lambda F_1(\mathscr{D}(u)) \\ &\quad + \int_\Omega |A|^2\partial_n\lambda F_1(\mathscr{D}(u)) + \int_\Omega |A|^2\lambda\partial_n F_1(\mathscr{D}(u)). \end{aligned}$$

We observe that each of the integrands above can be expressed as a sum of terms of two types:

- either a product of two components of $A$ times a function of *form* 1 times a function of *form* 2;
- or a product of some component of $A$ times a second order derivative of $u$ times two functions of *form* 1.

Consequently, one can bound

$$|I_{12}| \leqslant C\int_\Omega \Big(|A|^2|\mathscr{D}(u)|^2 + |A||\nabla^2 u||\mathscr{D}(u)|^2\Big).$$

By the Sobolev embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ we can further write

$$\begin{aligned} C\int_\Omega |A|^2|\mathscr{D}(u)|^2 &\leqslant C\||A|^2\|_{L^3}\|\mathscr{D}(u)\|_{L^6}\|\mathscr{D}(u)\|_{L^2} \\ &\leqslant C\||A|^2\|_{L^6}\|\mathscr{D}(u)\|_{L^6}\|\mathscr{D}(u)\|_{L^2} \leqslant \varepsilon\|A\|_{L^{12}}^4 + C(\varepsilon)\|u\|_{H^2}^2\|u\|_{H^1}^2, \end{aligned}$$

where $\varepsilon$ is a sufficiently small parameter to be chosen later. Next,

$$\begin{aligned}
C\int_\Omega |A||\nabla^2 u||\mathscr{D}(u)|^2 &\leqslant C\||A||\nabla^2 u|\|_{L^2}\|\mathscr{D}(u)\|_{L^{12}}\|\mathscr{D}(u)\|_{L^{\frac{12}{5}}}\\
&\leqslant C\||A||\nabla^2 u|\|_{L^2}\left(\|u\|_{H^1}+\|A\|_{L^{12}}\right)\|\mathscr{D}(u)\|_{L^3} \qquad (29)\\
&\leqslant C\||A||\nabla^2 u|\|_{L^2}\left(\|u\|_{H^1}+\|A\|_{L^{12}}\right)\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}\\
&\leqslant \varepsilon\||A||\nabla^2 u|\|_{L^2}^2+\varepsilon\|A\|_{L^{12}}^4+\varepsilon\|u\|_{H^1}^4+C(\varepsilon)\|u\|_{H^1}^2\|u\|_{H^2}^2,
\end{aligned}$$

where we used relation (18), the interpolation inequality $\|\cdot\|_{L^3}\leqslant\|\cdot\|_{L^2}^{1/2}\|\cdot\|_{L^6}^{1/2}$, the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and the Young inequality $xyz\leqslant\frac{x^2}{2}+\frac{y^4}{4}+\frac{z^4}{4}$. Combining the two previous inequalities results in the following bound for $I_{12}$:

$$|I_{12}|\leqslant\varepsilon\||A||\nabla^2 u|\|_{L^2}^2+2\varepsilon\|A\|_{L^{12}}^4+\varepsilon\|u\|_{H^1}^4+C(\varepsilon)\|u\|_{H^1}^2\|u\|_{H^2}^2. \qquad (30)$$

To estimate $I_{11}$ we simply use (7) and write

$$-\frac{\alpha_1}{2}I_{11}=-\frac{\alpha_1}{4}\sum_\ell\int_{\partial\Omega}|A|^2G_\ell(\mathscr{D}(u))H_\ell(\mathscr{D}(u)).$$

The right-hand side is quite similar to the last term in (28), so the estimate for $I_{11}$ is exactly the same as for $I_{12}$:

$$\frac{|\alpha_1|}{2}|I_{11}|\leqslant\varepsilon\||A||\nabla^2 u|\|_{L^2}^2+2\varepsilon\|A\|_{L^{12}}^4+\varepsilon\|u\|_{H^1}^4+C(\varepsilon)\|u\|_{H^1}^2\|u\|_{H^2}^2. \quad (31)$$

To complete the estimate for the $K$-term, it remains to estimate the integral $I_2$. We have that

$$\begin{aligned}
|I_2|=\left|\int_\Omega \operatorname{div}(|A|^2A)\cdot(v-\mathbb{P}v)\right| &\leqslant C\int_\Omega |A|^2|\nabla^2 u||v-\mathbb{P}v)|\\
&\leqslant CK_3(12/5,\Omega)\||A||\nabla^2 u|\|_{L^2}\|A\|_{L^{12}}\|u\|_{W^{1,\frac{12}{5}}},
\end{aligned}$$

where we used the Hölder inequality together with relation (14) for $r=12/5$. The last term is entirely similar with an intermediate term from relations (29), so the same estimate holds for $I_2$:

$$|I_2|\leqslant\varepsilon\||A||\nabla^2 u|\|_{L^2}^2+\varepsilon\|A|_{L^{12}}^4+\varepsilon\|u|_{H^1}^4+C(\varepsilon)\|u|_{H^1}^2\|u|_{H^2}^2. \qquad (32)$$

The final estimate for the $K$-term now follows from relations (26), (27), (30)–(32) and reads

$$\begin{aligned}
\int_\Omega K(u)\cdot\mathbb{P}v \geqslant{}& \frac{1}{2}\int_\Omega |A|^4+\frac{\alpha_1}{2}\int_\Omega |A|^2|\nabla A|^2+\frac{\alpha_1}{4}\int_\Omega|\nabla(|A|^2)|^2\\
&-3\varepsilon\||A||\nabla^2 u|\|_{L^2}^2-5\varepsilon\|A\|_{L^{12}}^4-3\varepsilon\|u\|_{H^1}^4-C(\varepsilon)\|u\|_{H^1}^2\|u\|_{H^2}^2
\end{aligned} \qquad (33)$$

for some constant $C(\varepsilon)$.

Let us now estimate the other terms in (25). First,

$$(\alpha_1+\alpha_2)\int_\Omega \operatorname{div}(A^2)\cdot \mathbb{P}v \\ \leqslant |\alpha_1+\alpha_2|\|\operatorname{div}(A^2)\|_{L^2}\|\mathbb{P}v\|_{L^2} \leqslant \varepsilon\||A||\nabla^2 u|\|_{L^2}^2 + C(\varepsilon)\|v\|_{L^2}^2. \quad (34)$$

Next,

$$-\sum_j \int_\Omega v_j \nabla u_j \cdot \mathbb{P}v = -\sum_j \int_\Omega v_j \nabla u_j \cdot v + \sum_j \int_\Omega v_j \nabla u_j \cdot (v-\mathbb{P}v).$$

The first term can be estimated as in [5, Relations (8)–(13)] by

$$-\sum_j \int_\Omega v_j \nabla u_j v = -\int_\Omega (v\cdot\nabla)u\cdot v \leqslant \varepsilon\||A||\nabla A|\|_{L^2}^2 + C(\varepsilon)\|v\|_{L^2}^2.$$

To bound the second term, we use (14) together with Hölder's inequality

$$\sum_j \int_\Omega v_j \nabla u_j \cdot (v-\mathbb{P}v) \leqslant \sum_j \|v_j\|_{L^2}\|\nabla u_j\|_{L^4}\|\mathbb{P}v-v\|_{L^4} \leqslant C\|v\|_{L^2}^2 + C\|u\|_{W^{1,4}}^4.$$

Therefore,

$$-\sum_j \int_\Omega v_j \nabla u_j \cdot \mathbb{P}v \leqslant 4\varepsilon\||A||\nabla^2 u|\|_{L^2}^2 + C(\varepsilon)\|v\|_{L^2}^2 + C\|u\|_{W^{1,4}}^4. \quad (35)$$

We go to the following term to estimate. One has that

$$-\int_\Omega (u\cdot\nabla)v\cdot\mathbb{P}v = \int_\Omega (u\cdot\nabla)(\mathbb{P}v-v)\cdot v \leqslant \|u\|_{L^\infty}\|\nabla(\mathbb{P}v-v)\|_{L^2}\|v\|_{L^2} \\ \leqslant C\|u\|_{L^\infty}\|u\|_{H^2}^2, \quad (36)$$

where we used (15).

We finally estimate

$$\int_\Omega f\cdot\mathbb{P}v \leqslant \|f\|_{L^2}\|\mathbb{P}v\|_{L^2} \leqslant \tfrac{1}{2}\|f\|_{L^2}^2 + \tfrac{1}{2}\|v\|_{L^2}^2 \quad (37)$$

and

$$\nu\int_\Omega \Delta u\cdot\mathbb{P}v \leqslant \nu\|\Delta u\|_{L^2}\|\mathbb{P}v\|_{L^2} \leqslant C\|u\|_{H^2}^2. \quad (38)$$

Collecting estimates (25), (33)–(38) results in

$$\frac{1}{2}\frac{d}{dt}\|\mathbb{P}v\|_{L^2}^2+\frac{\beta}{2}\int_\Omega |A|^4+\frac{\alpha_1\beta}{2}\int_\Omega |A|^2|\nabla A|^2+\frac{\alpha_1\beta}{4}\int_\Omega |\nabla|(|A|^2)|^2$$
$$\leqslant \varepsilon(3\beta+5)\int |A|^2|\nabla^2 u|^2+5\varepsilon\beta\|A\|_{L^{12}}^4$$
$$+C(\varepsilon)(1+\|u\|_{L^\infty}+\|u\|_{H^1}^2)\|u\|_{H^2}^2+C\|u\|_{W^{1,4}}^4+\frac{1}{2}\|f\|_{L^2}^2. \tag{39}$$

Observe next that if we denote by $C_4$ the constant from the Sobolev embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$, then we can further bound

$$\|A\|_{L^{12}}^4=\||A|^2\|_{L^6}^2\leqslant C_4^2\||A|^2\|_{H^1}^2=C_4^2\int_\Omega |A|^4+C_4^2\int_\Omega |\nabla(|A|^2)|^2.$$

We now choose

$$\varepsilon=\min\left(\frac{1}{20C_4^2},\frac{\alpha_1}{40C_4^2},\frac{\alpha_1\beta}{9(3\beta+5)}\right)$$

and note that, according to (12) and to the above inequalities, for this choice of $\varepsilon$ one has that

$$\varepsilon(3\beta+5)\int |A|^2|\nabla^2 u|^2\leqslant \frac{\alpha_1\beta}{4}\int_\Omega |A|^2||\nabla A|^2$$

and

$$5\varepsilon\beta\|A\|_{L^{12}}^4\leqslant \frac{\beta}{4}\int_\Omega |A|^4+\frac{\alpha_1\beta}{8}\int_\Omega |\nabla(|A|^2)|^2.$$

Using these bounds in (39) and adding the result to (21) yields the following differential inequality for the $H^2$ norm of $u$:

$$\frac{d}{dt}|||u|||_{H^2}^2+\beta\int_\Omega |A|^4+\frac{\alpha_1\beta}{2}\int_\Omega |A|^2|\nabla A|^2+\frac{\alpha_1\beta}{4}\int_\Omega |\nabla(|A|^2)|^2$$
$$\leqslant C_5\left(1+\|u\|_{L^\infty}+|||u|||_{H^1}^2\right)|||u|||_{H^2}^2+C_5\|u\|_{W^{1,4}}^4+2\|f\|_{L^2}^2$$

for some constant $C_5$. Gronwall's lemma now implies that

$$|||u(t)|||_{H^2}^2+\min\left(\beta,\frac{\alpha_1\beta}{4}\right)\int_0^t \||A|^2\|_{H^1}^2+\frac{\alpha_1\beta}{2}\int_0^t\int_\Omega |A|^2|\nabla A|^2$$
$$\leqslant e^{C_5(t+\int_0^t\|u\|_{L^\infty}+\int_0^t|||u|||_{H^1}^2)}\left(|||u_0|||_{H^2}^2+2\int_0^t\|f\|_{L^2}^2+C_5\int_0^t\|u\|_{W^{1,4}}^4\right) \tag{40}$$
$$\leqslant e^{C_5\left(t+M_2(t)+tM_0^2(t)\right)}\left(|||u_0|||_{H^2}^2+2\int_0^t\|f\|_{L^2}^2+C_5M_1^4(t)\right)$$
$$\stackrel{\text{def}}{=} M_3(t),$$

where we used the notation introduced in relations (22)–(24). The above bound is an *a priori* $H^2$ estimate. These estimates imply the global existence of a weak solution of (3) which belongs to $L^\infty_{\rm loc}([0,\infty); H^2)$ in the same way as in [5] with the obvious modifications specific to bounded domains with Navier boundary conditions as was done in [6] (in particular, one has to replace the Friedrichs approximation procedure from [5] with the Galerkin method with a special basis from [6]).

## 4. UNIQUENESS IN DIMENSION TWO

To prove uniqueness of solutions we follow the same approach as in [5]. The difficulty here is to show that the boundary terms that show up in the integrations by parts can be controlled. In fact, we will show that they all vanish. Let $u$ and $\tilde{u}$ be two solutions belonging to $L^\infty_{\rm loc}\left([0,\infty); H^2\right)$ with the same initial data. It was observed in [5] that the equation of motion of a third grade fluid can be written under the following form

$$\partial_t(u-\alpha_1\Delta u)+(u\cdot\nabla)u-\nu\Delta u+\operatorname{div}N(u)+\beta K(u)=f-\nabla p',$$

where

$$N(u)=-\alpha_1(u\cdot\nabla A+L^tA+AL)-\alpha_2 A^2.$$

We will use in the following the notations

$$w=u-\tilde{u},\quad A=A(u),\quad \tilde{A}=A(\tilde{u}),\quad L=L(u),\quad \tilde{L}=L(\tilde{u}).$$

Subtracting the equations for $u$ and $\tilde{u}$ and multiplying the result by $w$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||w|||^2_{H^1}+2\nu\|D(w)\|^2_{L^2}-\beta\int_\Omega \operatorname{div}(|A|^2A-|\tilde{A}|^2\tilde{A})\cdot w\\ =-\int_\Omega[(u\cdot\nabla)u-(\tilde{u}\cdot\nabla)\tilde{u}]\cdot w-\int_\Omega\operatorname{div}[N(u)-N(\tilde{u})]\cdot w.\qquad(41)\end{aligned}$$

Since $(|A|^2A-|\tilde{A}|^2\tilde{A})n\big|_{\partial\Omega}$ is proportional to $n$ and $w\big|_{\partial\Omega}$ is orthogonal to $n$, we see that

$$\begin{aligned}-\int_\Omega\operatorname{div}(|A|^2A-|\tilde{A}|^2\tilde{A})\cdot w &= \int_\Omega(|A|^2A-|\tilde{A}|^2\tilde{A})\cdot\nabla w\\ &\quad+\int_{\partial\Omega}[(|A|^2)A-|\tilde{A}|^2\tilde{A})n]\cdot w\\ &=\frac{1}{2}\int_\Omega(|A|^2A-|\tilde{A}|^2\tilde{A})\cdot A(w)\\ &=\frac{1}{4}\int_\Omega(|A|^2-|\tilde{A}|^2)^2+\frac{1}{4}\int_\Omega|A(w)|^2(|A|^2+|\tilde{A}|^2).\end{aligned}$$

The first term on the right-hand side of (41) can be estimated as in [5]:

$$\int_\Omega [(u\cdot\nabla)u-(\tilde u\nabla)\tilde u]\cdot w=\int_\Omega (w\cdot\nabla)u\cdot w\leqslant \|w\|_{L^4}^2\|\nabla u\|_{L^2}\leqslant C\|w\|_{H^1}^2\|u\|_{H^2}.$$

The last term in (41) is integrated by parts as follows:

$$-\int_\Omega \operatorname{div}[N(u)-N(\tilde u)]\cdot w=\int_\Omega [N(u)-N(\tilde u)]\cdot\nabla w-\int_{\partial\Omega}[(N(u)-N(\tilde u))n]\cdot w].$$

Since $A^2n\big|_{\partial\Omega}$ is proportional to $n$ and $w\big|_{\partial\Omega}$ is tangent to $\partial\Omega$, we have that

$$[N(u)n]\cdot w\big|_{\partial\Omega}=-\alpha_1\left[\big((u\cdot\nabla)A+L^tA+AL\big)n\right]\cdot w. \tag{42}$$

We now show that the above boundary terms vanish. First note that since $An=\lambda n$ on $\partial\Omega$, we have that

$$\begin{aligned}(L^tAn)_i&=\lambda(L^tn)_i=\lambda\sum_j\partial_iu_jn_j=\lambda\partial_i(u\cdot n)-\lambda\sum_j u_j\partial_in_j\\&=[\lambda\mu n-\lambda F_0(u)]_i\quad\text{on }\partial\Omega,\end{aligned}$$

where we used (6) and the notation $F_0(u)=\sum_j u_j\nabla n_j$. On the other hand, we see immediately that $Ln=\partial_n u$, so

$$ALn=A\partial_n u=A[(\lambda-\mu)n+F_0(u)]=\lambda(\lambda-\mu)n+AF_0(u)\quad\text{on }\partial\Omega.$$

We deduce that

$$(L^tA+AL)n=\lambda^2n+(A-\lambda I)F_0(u)\quad\text{on }\partial\Omega. \tag{43}$$

Next, we write

$$\begin{aligned}[(u\cdot\nabla)A]n&=(u\cdot\nabla)(An)-A(u\cdot\nabla)n\\&=(u\cdot\nabla)(An-\lambda n)+(u\cdot\nabla)(\lambda n)-A(u\cdot\nabla)n\end{aligned}$$

on the boundary. Now, $An-\lambda n=0$ on $\partial\Omega$ and since $u$ is tangent to the boundary, $u\cdot\nabla$ is a tangential derivative so $(u\cdot\nabla)(An-\lambda n)=0$ on the boundary. Moreover, $(u\cdot\nabla)(\lambda n)=\lambda(u\cdot\nabla)n+n(u\cdot\nabla)\lambda$. We therefore deduce that

$$[(u\cdot\nabla)A]n=n(u\cdot\nabla)\lambda-(A-\lambda I)(u\cdot\nabla)n. \tag{44}$$

Collecting relations (42)–(44) we find that

$$[N(u)n]\cdot w\big|_{\partial\Omega}=-\alpha_1[\lambda^2+(u\cdot\nabla)\lambda]n\cdot w-\alpha_1\{(A-\lambda I)[F_0(u)-(u\cdot\nabla)n]\}\cdot w.$$

The right-hand side above vanishes: the first term is zero since $w$ is tangent to the boundary and the second term vanishes since by Lemma 3 and the Navier boundary conditions we can deduce that the vector field $(A-\lambda I)[F_0(u)-(u\cdot\nabla)n]$ is normal to the boundary. This shows that there are no boundary terms when integrating by parts the term coming from $N(u)$. Once this fact proved, one can continue the proof exactly like in [5] starting from Eq. (28) of [5]. Indeed, the only other integrations by parts performed after relation (28) in [5] require only the condition of tangency to the boundary. It would be useless to reproduce those estimates here, so we refer to Busuioc and Iftimie [5] for what is left in the proof of the uniqueness.

## 5. ADDITIONAL $H^3$ REGULARITY IN DIMENSION TWO

In the same spirit as in [4], we prove now that in dimension 2, the $H^3$ regularity of the initial data is propagated.

Let us apply the curl operator to the equation of $v$ under the form given in (20) and take the $L^2$ scalar product with curl $v$ to obtain that

$$\frac{1}{2}\frac{d}{dt}\|\operatorname{curl} v\|_{L^2}^2=\nu\int_\Omega \operatorname{curl} v\ \ \Delta\operatorname{curl} u-\beta\int_\Omega \operatorname{curl} v\operatorname{curl} K(u)+\int_\Omega \operatorname{curl} f\operatorname{curl} v$$
$$\underbrace{-\int_\Omega \operatorname{curl}[(u\cdot\nabla)v]\operatorname{curl} v-\sum_j\int_\Omega \operatorname{curl}(v_j\nabla u_j)\ \operatorname{curl} v+(\alpha_1+\alpha_2)\int_\Omega \operatorname{curl}[\operatorname{div}(A^2)]\operatorname{curl} v.}_{I}$$

We remark that all the integrands composing the part $I$ above can be written as a sum of two type of terms:

- either a function of *form* 1 times two functions of *form 3*,
- or a function of *form* 3 times two functions of *form 2*,

plus one term in which we can find fourth-order derivatives of $u$. This additional term is $\int_\Omega (u\cdot\nabla)$ curl $v$ curl $v$. However, this term vanishes by a well-known cancellation property together with the fact that $u$ is tangent to the boundary. Consequently, we can bound

$$\begin{aligned}|I| &\le C\int_\Omega |\mathscr{D}(u)||\mathscr{D}^3(u)||\mathscr{D}^3(u)|+C\int_\Omega |\mathscr{D}^2(u)||\mathscr{D}^2(u)||\mathscr{D}^3(u)|\\
&\le C\|\mathscr{D}(u)\|_{L^\infty}\|u\|_{H^3}^2+C\|\mathscr{D}^2(u)\|_{L^4}^2\|u\|_{H^3}\\
&\le \frac{C}{\sqrt{\varepsilon}}\|u\|_{H^2}^{1-\varepsilon}\|u\|_{H^3}^{2+\varepsilon}+C\|u\|_{H^2}\|u\|_{H^3}^2\\
&\le \frac{C}{\sqrt{\varepsilon}}\|u\|_{H^2}^{1-\varepsilon}\|u\|_{H^3}^{2+\varepsilon},\end{aligned}$$

where $\varepsilon\in(0,1]$ is to be chosen later, the constant $C$ is independent of $\varepsilon$ and we have used Lemma 4. Next,

$$\nu \int_\Omega \text{curl } v \ \Delta \text{ curl } u \le \nu \|u\|_{H^3}^2$$

and

$$\int_\Omega \text{curl } f \text{ curl } v \le \| \text{ curl } f\|_{L^2} \| \text{ curl } v\|_{L^2} \le \frac{1}{2} \| \text{ curl } f\|_{L^2}^2 + \frac{1}{2}\|u\|_{H^3}^2.$$

We now estimate the trilinear term. After expanding curl $K(u) = -$ curl div$(|A|^2 A)$ we observe that we can bound

$$-\beta \int_\Omega \text{curl } K(u) \text{ curl } v \le C \int_\Omega |A|^2 |\mathscr{D}^3(u)|^2 + C\int_\Omega |A| |\mathscr{D}^2(u)|^2 |\mathscr{D}^3(u)|.$$

As above, we use Lemma 4 and the fact that $H^2(\Omega)$ is an algebra to deduce that

$$\begin{aligned} &C \int_\Omega |A| |\mathscr{D}^2(u)|^2 |\mathscr{D}^3(u)| \\ &\leqslant C \| A \|_{L^\infty} \| \mathscr{D}^2(u) \|_{L^4}^2 \| \mathscr{D}^3(u) \|_{L^2} \leqslant \frac{C}{\sqrt{\varepsilon}} \| u \|_{H^2}^{2-\varepsilon} \| u \|_{H^3}^{2+\varepsilon} \end{aligned}$$

and

$$\begin{aligned} &C \int_\Omega |A|^2 |\mathscr{D}^3(u)|^2 \leqslant \| A^2 \|_{L^\infty} \| u \|_{H^3}^2 \\ &\leqslant \frac{C}{\sqrt{\varepsilon}} \| A^2 \|_{H^1}^{1-\frac{\varepsilon}{2}} \| A^2 \|_{H^2}^{\frac{\varepsilon}{2}} \| u \|_{H^3}^2 \leqslant \frac{C}{\sqrt{\varepsilon}} \| A^2 \|_{H^1}^{1-\frac{\varepsilon}{2}} \| u \|_{H^3}^{2+\varepsilon}. \end{aligned}$$

Putting together all the above relations, we conclude that

$$\begin{aligned} \frac{d}{dt} \| \text{curl } v \|_{L^2}^2 &\leqslant \frac{C}{\sqrt{\varepsilon}} \| u \|_{H^3}^{2+\varepsilon} \left( \| u \|_{H^2}^{1-\varepsilon} + \| u \|_{H^2}^{2-\varepsilon} + \| A^2 \|_{H^1}^{1-\frac{\varepsilon}{2}} \right) \\ &\quad + (1+2\nu) \| u \|_{H^3}^2 + \| \text{curl} f \|_{L^2}^2 \\ &\leqslant \frac{C}{\sqrt{\varepsilon}} (1 + \| u \|_{H^3}^2)^{1+\frac{\varepsilon}{2}} \left( 1 + \| u \|_{H^2}^2 + \| A^2 \|_{H^1} \right) + \| \text{curl} f \|_{L^2}^2 \end{aligned}$$

for some constant $C$ independent of $\varepsilon$.

Adding this relation to (21) we get the following differential inequality for the $H^3$ norm of $u$:

$$\frac{d}{dt} |\!|\!| u |\!|\!|_{H^3}^2 \leqslant \frac{C_6}{\sqrt{\varepsilon}} (1 + |\!|\!| u |\!|\!|_{H^3}^2)^{1+\frac{\varepsilon}{2}} \left( 1 + |\!|\!| u |\!|\!|_{H^2}^2 + \| A^2 \|_{H^1} \right) + \| f \|_{H^1}^2$$

for some constant $C_6$ independent of $\varepsilon$. Let

$$B(t) = \frac{C_6}{2} (1 + |\!|\!| u |\!|\!|_{H^2}^2 + \| A^2 \|_{H^1}) \quad \text{and} \quad h(t) = 1 + |\!|\!| u |\!|\!|_{H^3}^2.$$

From (40) we infer that the following bound holds for the time integral of $B$:

$$\int_0^t B(\tau)d\tau \leqslant \frac{C_6}{2}\left[t + tM_3(t) + \left(\frac{tM_3(t)}{\min(\beta, \frac{\alpha_1\beta}{4})}\right)^{1/2}\right] \overset{\text{def}}{=} M_4(t).$$

Since $h$ verifies the differential inequality

$$h' \leqslant 2\frac{B(t)}{\sqrt{\varepsilon}}h^{1+\frac{\varepsilon}{2}} + \| f \|_{H^1}^2,$$

one has that

$$\left(h^{-\frac{\varepsilon}{2}}\right)' \geqslant -\frac{\varepsilon}{2h^{1+\frac{\varepsilon}{2}}}\left(2\frac{B(t)}{\sqrt{\varepsilon}}h^{1+\frac{\varepsilon}{2}} + \| f \|_{H^1}^2\right) \geqslant -\sqrt{\varepsilon}(B + \| f \|_{H^1}^2).$$

After integration

$$h^{-\frac{\varepsilon}{2}}(t) \geqslant h^{-\frac{\varepsilon}{2}}(0) - \sqrt{\varepsilon}\left(M_4(t) + \int_0^t \| f(\tau) \|_{H^1}^2 \, d\tau\right). \tag{45}$$

We now fix $t$ and choose $\varepsilon_0 = \varepsilon_0(t)$ such that

$$\varepsilon_0 \leqslant 1 \quad \text{and} \quad \varepsilon_0^{-\frac{1}{2}}(1 + |||u_0|||_{H^3}^2)^{-\frac{\varepsilon_0}{2}} \geqslant 2\left(M_4(t) + \int_0^t \| f(\tau) \|_{H^1}^2 \, d\tau\right).$$

Note that such an $\varepsilon_0$ exists as the limit when $\varepsilon \to 0$ of the left-hand side is $+\infty$. Moreover, $\varepsilon_0$ can be made explicit but this is not very useful here. In view of (45), we deduce that

$$h^{-\frac{\varepsilon_0}{2}}(t) \geqslant \frac{h^{-\frac{\varepsilon_0}{2}}(0)}{2},$$

that is,

$$|||u(t)|||_{H^3}^2 \leqslant 4^{\frac{1}{\varepsilon_0}}\left(1 + |||u_0|||_{H^3}^2\right).$$

These are $H^3$ *a priori* estimates for $\| u \|_{H^3}$. As in Section 2, one can consider the Galerkin method with the special basis adapted to the Navier boundary conditions. The above $H^3$ *a priori* estimates will hold true for the sequence of approximate solutions. Therefore, the approximating solutions will be bounded in $L^\infty_{\text{loc}}([0, \infty); H^3(\Omega))$ and the limit solution must also belong to this class. We conclude that there exists a (unique) global solution belonging to $L^\infty_{\text{loc}}([0, \infty); H^3(\Omega))$.

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