Journal of Dynamics and Differential Equations, Vol. 16, No. 4, October 2004 (© 2004)

Traveling Waves in Diffusive Random Media*

Wenxian Shen[1]

Received April 11, 2004; revised September 9, 2004

The current paper is devoted to the study of traveling waves in diffusive random media, including time and/or space recurrent, almost periodic, quasiperiodic, periodic ones as special cases. It first introduces a notion of traveling waves in general random media, which is a natural extension of the classical notion of traveling waves. Roughly speaking, a solution to a diffusive random equation is a traveling wave solution if both its propagating profile and its propagating speed are random variables. Then by adopting such a point of view that traveling wave solutions are limits of certain wave-like solutions, a general existence theory of traveling waves is established. It shows that the existence of a wave-like solution implies the existence of a critical traveling wave solution, which is the traveling wave solution with minimal propagating speed in many cases. When the media is ergodic, some deterministic properties of average propagating profile and average propagating speed of a traveling wave solution are derived. When the media is compact, certain continuity of the propagating profile of a critical traveling wave solution is obtained. Moreover, if the media is almost periodic, then a critical traveling wave solution is almost automorphic and if the media is periodic, then so is a critical traveling wave solution. Applications of the general theory to a bistable media are discussed. The results obtained in the paper generalize many existing ones on traveling waves.

KEY WORDS: diffusive random media; recurrence; almost periodicity; almost automorphy; traveling wave solution; wave-like solution; random equilibrium; random fixed point.
AMS Subject Classification: 35K55, 35K57, 35B50.

1. INTRODUCTION

In this paper, we study traveling wave solutions of the following reaction-diffusion equation in random media,

---

* Dedicated to Professor Shui-Nee Chow on the occasion of his 60th birthday.

[1] Department of Mathematics, Auburn University, AL 36849, USA. E-mail: wenxish@auburn.edu

1040-7294/04/1000-1011/0

$$\partial_t u = \partial_x^2 u + F(\theta_{t,x}\omega, u), \quad x \in \mathbb{R}, \tag{1.1}$$

where $\omega \in \Omega$, $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, $((\Omega, \mathscr{F}, \mathbb{P}), \{\theta_{t,x}\}_{t,x,\in\mathbb{R}})$ is a metric dynamical system, and $F: \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable and for each fixed $\omega \in \Omega$, $F(\theta_{t,x}\omega, u)$ is Hölder continuous in t, x and Lipschitz continuous in u (see (**H1**) in Section 2 for detail).

Equation (1.1) includes the cases that the media is homogeneous (that is, Ω is a singleton, in this case, $F(\theta_{t,x}\omega, u) = f(u)$, time periodic (that is, $\Omega = \{\theta_{t,0}\omega_0 | 0 \leqslant t \leqslant T\}$ for some ω_0, where $\theta_{t,x}\omega_0 = \theta_{t,0}\omega_0$ and $\theta_{0,0}\omega_0 = \theta_{T,0}\omega_0$, in this case, $F(\theta_{t,x}\omega_0, u) = F(\theta_{t,0}\omega_0) = f(t, u)$ is of period T in t), space periodic (i.e., $\Omega = \{\theta_{0,x}\omega_0 | 0 \leqslant x \leqslant p\}$ for some ω_0, where $\theta_{t,x}\omega_0 = \theta_{0,x}\omega_0$ and $\theta_{0,0}\omega_0 = \theta_{0,p}\omega_0$, in this case, $F(\theta_{t,x}\omega, u) = F(\theta_{0,x}\omega_0, u) = f(x, u)$ is of period p in x), etc. On the other hand, a general time–space dependent reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(t, x, u), \quad x \in \mathbb{R}, \tag{1.2}$$

where f is a bounded and uniformly continuous function, can be embedded into (1.1) with $\Omega = H(f) = cl\{f_{t,y}(\cdot,\cdot,\cdot) | \tau, y \in \mathbb{R}, f_{\tau,y}(t, x, u) = f(t + \tau, x + y, u)\}$, where the closure is taken under the compact open topology, $\mathscr{F} = \mathscr{B}_\Omega$ ($\mathscr{B}_\Omega$ is the Borel σ-algebra of Ω with respect to the compact open topology), $\theta_{t,x}\omega(\cdot,\cdot,\cdot) = \omega(t + \cdot, x + \cdot, \cdot)$, $\mathbb{P}$ a $\theta_{t,x}$-invariant measure on Ω (the existence of such measure is guaranteed by the Krylov–Bogoliubov theorem), and with $F(\theta_{t,x}\omega, u) = \omega(t, x, u)$ for $\omega \in H(f)$.

Equation (1.1) serves as mathematical models for many applied problems, for example, population genetics, gene development, phase transition, signal propagation, chemical kinetics, combustion, etc. (see [6,11,23, 24,26,35,45,55] and references). One of the central problems about (1.1) is the traveling wave solution problem. In homogenous media, classically a traveling wave solution is a solution $u(t, x)$ with a fixed profile $\phi(\cdot)$ and a constant speed c, that is, $u(t, x) = \phi(x - ct)$, and has been studied for a long time (see for example [6,12–14,23,24,33,35,37,43–45,51,53, 59,60]). However, the study of traveling wave solutions in inhomogeneous media has begun more recently. Nevertheless, there have been established some basic theoretical foundations for traveling wave solutions in time periodic media, space periodic media, time almost periodic media, general time dependent media, and space recurrent media. For the time periodic case, a traveling wave solution is defined to be a solution $u(t, x)$ with a periodically varying profile $\phi: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a constant speed c, that is, $u(t, x) = \phi(x - ct, t)$ and $\phi(x, t)$ is periodic in t (see [1]). In the space periodic case, a traveling wave solution is defined to be of form $u(t, x) = \phi(x - ct, x)$, where $\phi(x, y)$ is periodic in y (see [58,61]). The reader is also referred to [28,38,55,56] for the study of traveling wave solutions in time and/or space periodic media. For definitions and basic properties of tarveling wave solutions in time dependent media, see [46–48]. H. Matano in his talks at the Fifth Mississippi State Conference on Differential Equations and Computational Simulations (2001) and the Fourth International Conference on Dynamical Systems and Differential Equations (2002) introduced a notion of traveling wave solutions in time independent and spatially recurrent media. A solution $u(t,x)$ of (1.2) with $f(t,x,u) = f(x,u)$ being recurrent in x is a traveling wave solution if $u(t,x) = \phi(x - c(t), \theta_{0,c(t)} f)$, where $\phi : \mathbb{R} \times H(f) \to R$ is continuous and $c(\cdot)$ is a real valued function. It should be pointed that the concept of traveling wave solutions of (1.1) introduced in the present paper extends all the above mentioned concepts by taking $\Omega = H(f)$ for $f(t,x,u) = f(u)$ being both time and space independent, or $f(t,x,u) = f(t,u)$ being periodic (almost periodic, recurrent) in t, or $f(t,x,u) = f(x,u)$ being periodic (almost periodic, recurrent) in x.

Since in nature, many systems are subject to irregular influences arisen from various kind noise (see Section 3 for two examples arising from population genetics and phase transition), it is of great importance to study traveling wave solutions in random media, and in particular, to investigate the existence, uniqueness and stability of traveling waves and to understand the influence of the media and/or spatial randomness on the wave profiles and wave speeds of such solutions. There are some works toward to various aspects of propagating solutions to certain special random equations (see for example [28, 53, 54, 61]). However, the understanding to traveling wave solutions in general random media is very little. Up to the authors knowledge, there is no rigorous definition of solutions in general random media which serve as an analog of the classical traveling wave solutions in both time and space homogeneous media. As in the time almost periodic case, most methods and techniques to study traveling wave solutions in time independent equations will not be applicable to the study of traveling wave solutions in random media.

The objective of the current paper is to provide some theoretical and methological foundation for the study of traveling waves in random media and discuss some simple applications. It first introduces the concept of random traveling wave solutions, which is a natural extension of the classical traveling wave solutions. To study the existence of such solutions, such a point of view that traveling wave solutions are limits of certain wave like solutions is adopted. A general existence theorem is then established and some deterministic properties of wave profile and wave speed are derived. When the media Ω is compact, certain continuity of wave profile is also obtained. Applying the general results to a bistable case, the existence of traveling wave solution is proved. The results obtained in the paper generalize many existing ones on traveling waves.

To be more specific, first in Section 2, among others, we introduce the concept of random traveling wave solutions connecting two random equilibria (traveling wave solution for short) as well as the concept of wave-like solutions. Roughly speaking, a random traveling wave solution is a solution with a random propagating profile and a random propagating speed. A wave-like solution is a solution that does not become flat as time increases. Main results of the paper, Theorems A–C, are also stated in this section. Theorem A concerns the existence of random traveling wave solutions and deterministic properties of average propagating profile and average propagating speed of a random traveling wave solution in general random media and is proved in Section 5. It shows that the existence of a wave-like solution implies the existence of a critical traveling wave solution and when $(\Omega, \mathscr{F}, \mathbb{P}), \{\theta_{t,x}\}_{t,x\in\mathbb{R}})(\theta_{t,x} = \theta_{t,0}$ or $\theta_{0,x})$ is ergodic, the average propagating profile and average propagating speed of a regular traveling wave solution (see Definition 2.2) are deterministic. Note that under appropriate conditions, a critical traveling wave solution is the one among all the traveling wave solutions that has minimal average propagating speed (see Remark 2.1.3)). Theorem B considers continuous properties of critical traveling wave solution when Ω is compact and is proved in Section 6. It shows that the propagating profile of a critical traveling wave solution is continuous in certain sense and that when Ω is periodic, traveling wave solutions are also periodic. Theorem C is proved in Section 7. It discusses the applications of Theorem A to a spatially homogeneous bistable equation and shows the existence of a wave-like solution to such equation and then by Theorem A the existence of traveling wave solutions. We present two examples arising from population genetics and phase transition, namely, a random variant of the Fisher, or KPP, equation and a random variant of bistable equations in Section 3. For the use in proving Theorems A–C, we present some preliminary lemmas in Section 4.

We remark that there are numerous works on traveling wave solutions to various evolution problems. See for example [1,6,12–14,23,24,35,37,44, 46–48,50,52,58–61] for the local continuous evolution problems, see [9,15, 18–20,29,34,39,55,63,64] for the discrete problems, see [7,8,10,16,17] for nonlocal convolution problems, see [31,36,49,57,63] for the problems with delays, see [22,32,40,41] for the problems of coupled evolution equations. The contribution of the current paper is that it provides some theoretical and methological foundation for the further study of traveling waves in random media. Uniqueness and stability of traveling waves in random media as well as applications of general existence, uniqueness and stability theories to general random equations of bistable and KPP types will be studied in forthcoming papers.

2. DEFINITIONS AND MAIN RESULTS

In this section we first introduce some definitions and then state our main results. Throughout this paper, we assume that (1.1) satisfies two hypotheses: **(H1)** and **(H2)**.

(H1) $F:\Omega\times\mathbb{R}\to\mathbb{R}$ is $(\mathscr{F}\times\mathscr{B}_{\mathbb{R}},\mathscr{B}_{\mathbb{R}})$ – measurable, where $\mathscr{B}_{\mathbb{R}}$ denotes the Borel σ-algebra of $\mathbb{R}$. For each fixed $\omega\in\Omega$, $F^{\omega}(t,x,u)=F(\theta_{t,x}\omega,u)$ is globally Hölder continuous in $t,x\in\mathbb{R}$ uniformly with respect to u in bounded sets and is locally Lipschitz continuous in $u\in\mathbb{R}$ uniformly with respect to $t,x\in\mathbb{R}$, that is, for each fixed $\omega\in\Omega$ and $M>0$, there are positive real numbers $L(\omega,M)$ and $\delta(\omega,M)$ such that

$$|F^{\omega}(t,x,u)-F^{\omega}(s,y,u)|\leqslant L(\omega,M)(|t-s|^{\delta(\omega,M)}+|x-y|^{\delta(\omega,M)}) \quad (2.1)$$

and

$$|F^{\omega}(t,x,u)-F^{\omega}(t,x,v)|\leqslant L(\omega,M)|u-v| \quad (2.2)$$

for all $t,s,x,y\in\mathbb{R}$ and $-M\leqslant u,v\leqslant M$. $F(\omega,0)$ is bounded, that is, there is $M_0>0$ such that

$$|F(\omega,0)|\leqslant M_0$$

for $\omega\in\Omega$.

Note that in **(H1)**, there are no regularity or measurability assumptions on $L(\omega,M)$ and $\delta(\omega,M)$.

Let

$$X=C^b_{\text{unif}}(\mathbb{R})=\{u:\mathbb{R}\to\mathbb{R}|u \text{ is bounded and uniformly continuous}\} \quad (2.3)$$

with uniform convergence topology (i.e.$\|\cdot\|_{\infty}$-topology). For each $\omega\in\Omega$ and $u_0\in X$, it is known that (1.1) has a unique (local) solution, denote it by $u(t,\cdot;u_0,\omega)$, with $u(0,\cdot;u_0,\omega)=u_0(\cdot)$ (see [27,30]). Moreover, $u(t,x;u_0,\omega)$ is continuous in t,x, and u_0, and is measurable in ω. Therefore (1.1) generates a (local) random dynamical system (see [4] for general theory on random dynamical systems),

$$\Pi_{t,x}:X\times\Omega\to X\times\Omega, \quad (2.4)$$

$$\Pi_{t,x}(u_0,\omega)=(\pi_{t,x}(u_0,\omega),\theta_{t,x}\omega), \quad (2.5)$$

where $t\in I(u_0,\omega)\equiv\{t\in\mathbb{R}^+|u(t,x;u_0,\omega) \text{ exists } at\, t\}, x\in\mathbb{R}$, and $\pi_{t,x}(u_0,\omega)=u(t,\cdot+x;u_0,\omega)$.

In the following, a function $u:\mathbb{R}\to\mathbb{R}$ is said to be piecewise continuous if it is continuous on $\mathbb{R}\backslash E$, where E is a countable isolated subset of $\mathbb{R}$, and for each $x_0\in E$, both $\lim_{x\to x_0-}u(x)$ and $\lim_{x\to x_0+}u(x)$ exist.

$u: \mathbb{R} \to \mathbb{R}$ is said to have finite discontinuous points if u is continuous on $\mathbb{R}\backslash E$, where E is a finite subset of $\mathbb{R}$. Let

$$BPC(\mathbb{R}) = \{u: \mathbb{R} \to \mathbb{R} | u \text{ is bounded, piecewise continuous, and has finite discontinuous points}\}. \tag{2.6}$$

Observe that for each $u_0 \in BPC(\mathbb{R})$ and $\omega \in \Omega$, solution of (1.1) with initial data $u_0(\cdot)$ exists (see [35]) and we may also write it as $u(t, x; u_0, \omega)$. We denote $u(t, x; u_0(\omega), \omega)$ as the solution of (1.1) with initial data $u_0(\omega)$ for each $\omega \in \Omega$.

Definition 2.1. (1) A map $\phi: \Omega \to X$ is called a **random variable** if it is measurable.
(2) A random variable $\phi: \Omega \to X$ is called a **random equilibrium solution** of (1.1) if

$$\pi_{t,x}(\phi(\omega), \omega) = \phi(\theta_{t,x}\omega)$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.
(**H2**) There are two bounded random equilibrium solutions $u^\pm: \Omega \to X$ of (1.1) with

$$u^+(\omega)(x) - u^-(\omega)(x) \geqslant \delta_0$$

for some $\delta_0 > 0$ and all $\omega \in \Omega$ and $x \in \mathbb{R}$.

Definition 2.2. (1) A solution $u(t, x; u_0(\omega), \omega)$ of (1.1) is called a **random traveling wave solution connecting** $u^\pm(\omega)$ (traveling wave solution for short) if $u(t, \cdot; u_0(\omega), \omega)$ exists for $t \in \mathbb{R}$ and there are $U: \mathbb{R} \times \Omega \to \mathbb{R}, U(\cdot, \omega) \in X$ and is measurable in ω in the X-topology (i.e. $U^*: \Omega \to X, U^*(\omega) = U(\cdot, \omega)$ is measurable), and $c: \mathbb{R} \times \omega \to \mathbb{R}, c(t, \omega)$ is measurable in ω, such that for each $\omega \in \Omega$,

$$\begin{aligned} u_0(\omega) &= U(\cdot, \omega), \\ u^-(\omega)(x) &< U(x, \omega) < u^+(\omega)(x) \quad \text{for } x \in \mathbb{R}, \\ U(x, \omega) - u^\pm(\omega)(x) &\to 0 \quad \text{as } x \to \pm\infty, \end{aligned}$$

and

$$u(t, x; u_0(\omega), \omega) = U(\cdot - c(t, \omega), \theta_{t,c(t,\omega)}\omega) \quad \text{for } t \in \mathbb{R}.$$

Such traveling wave solution is also said to be **generated by** $U(\cdot, \cdot)$.

(2) A traveling wave solution generated by $U(\cdot, \cdot): \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be **critical** if for each $V(\cdot, \cdot): \mathbb{R} \times \Omega \to \mathbb{R}$ which generates a traveling wave solution,

$$V(x + \xi_1(\omega), \omega) \begin{cases} \leqslant U(x + \xi_2(\omega), \theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega), & x \geqslant 0, \\ \geqslant U(x + \xi_2(\omega), \theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega), & x < 0 \end{cases}$$

provided that $\omega \in \Omega$ and $\xi_1(\omega), \xi_2(\omega) \in \mathbb{R}$ satisfy the equality,

$$V(\xi_1(\omega), \omega) = U(\xi_2(\omega), \theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega).$$

(3) A traveling wave solution generated by $U(\cdot, \omega)$ is said to be **regular** if for each $\omega \in \Omega, c(t, \omega)$ differentiable in t and $u^\omega(x)$ is differentiable in x, where $u^\omega(x) = U(0, \theta_{0,x}\omega)$, and

$$\inf_{\omega \in \Omega} \big(\partial_x U(0, \omega) - \partial_x u^\omega(0)\big) > 0.$$

Definition 2.3. A solution $u(t, \cdot; v_0(\omega), \omega)$ of (1.1) is a **wave-like solution** if for each $\omega \in \Omega$,

$$u^-(\omega)(x) < v_0(\omega)(x) < u^+(\omega)(x) \quad \text{for } x \in \mathbb{R},$$
$$\lim_{x \to \pm\infty} \big(v_0(\theta_{0,y}\omega)(x) - u^\pm(\theta_{0,y}\omega)(x)\big) = 0$$

uniformly in $y \in \mathbb{R}$, $v_0(\theta_{0,y}\omega) \in X$ and is continuous in y in the X-topology, and for each $\delta > 0$ and $\omega \in \Omega$, there is $m(\delta, \omega) > 0$ such that for all $y \in \mathbb{R}$ and $t \geqslant 0$,

$$x_+(t, \delta, \omega, y) - x_-(t, \delta, \omega, y) \leqslant m(\delta, \omega),$$

where

$$x_+(t, \delta, \omega, y) = \inf\{x_+ | u(t, x; v_0(\theta_{-t,-y}\omega), \theta_{-t,-y}\omega) \geqslant u^+(\theta_{0,-y}\omega)(x) - \delta \text{ for } x \geqslant x_+\}$$

and

$$x_-(t, \delta, \omega, y) = \sup\{x_- | u(t, x; v_0(\theta_{-t,-y}\omega), \theta_{-t,-y}\omega) \leqslant u^-(\theta_{0,-y}\omega)(x) + \delta \text{ for } x \leqslant x_-\}.$$

Such wave-like solution is also said to be **generated by** $v_0(\omega)$.

Remark 2.1.

(1) If $\phi : \Omega \to X$ is a random equilibrium solution of (1.1), then for each $\omega \in \Omega, \phi^\omega(t, x) \equiv \phi(\theta_{t,x}\omega)(0) = \phi(\theta_{t,0}\omega)(x)$ is a solution. Moreover, if $\theta_{t,x}$ is independent of x (i.e., $\theta_{t,x}\omega = \theta_{t,0}\omega$ for all $t, x \in \mathbb{R}$ and $\omega \in \Omega$), then so is ϕ (i.e., $\phi^\omega(t, x) = \phi^\omega(t, 0)$ for all $t, x \in \mathbb{R}$ and $\omega \in \Omega$). Similarly, if $\theta_{t,x}$ is independent of t (i.e., $\theta_{t,x}\omega = \theta_{0,x}\omega$ for all $t, x \in \mathbb{R}$ and $\omega \in \Omega$), then so is ϕ (i.e., $\phi^\omega(t, x) = \phi^\omega(0, x)$ for all $t, x \in \mathbb{R}$ and $\omega \in \Omega$).

(2) The joint measurability of $U(x,\omega)$ and $c(t,\omega)$ is not assumed in Definition 2.2.1). But $U(x,\omega)$ is always jointly measurable since $U(\cdot,\omega)$ is measurable in ω in the X-topology and $U(x,\omega)$ is uniformly continuous in x for each $\omega\in\Omega$. In some cases it can be shown that $c(t,\omega)$ is differentiable in t and hence is jointly measurable (see for example Theorem C).

(3) In the case that the media is spatially homogeneous (that is, $\theta_{t,x}$ is independent of x) and $(\Omega,\{\theta_{t,0}\}_{t\in\mathbb{R}})$ is almost periodic, if one of $u^{\pm}$ is stable, then a critical traveling wave solution is the one among all the traveling wave solutions that has minimal average propagating speed (see [47]).

(4) When $u^{\omega}(x)\equiv U(0,\theta_{0,x}\omega)$ is independent of x (i.e., $u^{\omega}(x)=u^{\omega}(0)$ for all $x\in\mathbb{R}$ and $\omega\in\Omega$), the condition in Definition 2.2.3),

$$\inf_{\omega\in\Omega}\left(U_x(0,\omega)-u_x^{\omega}(0)\right)>0,$$

becomes

$$\inf_{\omega\in\Omega}U_x(0,\omega)>0.$$

(5) Suppose that $V_0(\cdot,\omega)$ generates a traveling wave solution, $V_0(\cdot,\theta_{0,y}\omega)\in X$ is continuous in $y\in\mathbb{R}$ in the X-topology, and

$$\lim_{x\to\pm\infty}V_0(x,\theta_{0,y}\omega)-u^{\pm}(\theta_{0,y}\omega)(x)=0$$

uniformly with respect to $y\in\mathbb{R}$. Then $V_0(\cdot,\omega)$ generates a wave-like solution. In fact, suppose that

$$u(t,x;V_0(\cdot,\omega),\omega)=V_0(x-c_0(t,\omega),\theta_{t,c_0(t,\omega)}\omega).$$

Then

$$\begin{aligned}&u(t,x;V_0(\cdot,\theta_{-t,-y}\omega),\theta_{-t,-y}\omega)\\&\quad=V_0(x-c_0(t,\theta_{-t,-y}\omega),\theta_{0,-y+c_0(t,\theta_{-t,-y}\omega)}\omega).\end{aligned}$$

By the assumption that $\lim_{x\to\pm\infty}V_0(x,\theta_{0,y}\omega)-u^{\pm}(\theta_{0,y}\omega)(x)=0$ uniformly with respect to $y\in\mathbb{R}$,

$$V_0(x,\theta_{0,-y+c_0(t,\theta_{-t,-y}\omega)}\omega)-u^{\pm}(\theta_{0,-y+c_0(t,\theta_{-t,-y}\omega)}\omega)(x)\to 0$$

as $x\to\pm\infty$ uniformly in $y\in\mathbb{R}$ and $t\geqslant 0$. It then follows that

$$\begin{aligned}&u(t,x+c_0(t,\theta_{-t,-y}\omega);V_0(\cdot,\theta_{-t,-y}\omega),\theta_{-t,-y}\omega)\\&\quad-u^{\pm}(\theta_{0,-y}\omega)(x+c_0(t,\theta_{-t,-y}\omega))\to 0\end{aligned}$$

as $x\to\pm\infty$ uniformly in $y\in\mathbb{R}$ and $t\geqslant 0$. Therefore $u(t,\cdot;V_0(\cdot,\omega),\omega)$ is a wave-like solution.

Definition 2.4. ([25,51,62]) Let Y be a metric space and $f:\mathbb{R}\to Y$ a continuous function.

(1) f is said to be **recurrent** if for each sequence $\{\alpha'_n\}\subset\mathbb{R}$, there is a subsequence $\{\alpha_n\}\subset\{\alpha'_n\}$ such that $\lim_{n\to\infty} f(t+\alpha_n)$ exists in the compact open topology, and for each sequence $\{\alpha_n\}\subset\mathbb{R}$ with $\lim_{n\to\infty} f(t+\alpha_n)=g(t)$ in the compact open topology, there is a sequence $\{\beta_n\}\subset\mathbb{R}$ such that $\lim_{n\to\infty} g(t+\beta_n)=f(t)$ in the compact open topology.

(2) f is **almost automorphic** if for each sequence $\{\alpha'_n\}\subset\mathbb{R}$, there is a subsequence $\{\alpha_n\}\subset\{\alpha'_n\}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+\alpha_n-\alpha_m)=f(t)$$

in the compact open topology.

(3) f is **almost periodic** if for each pair of sequences $\{\alpha'_n\}, \{\beta'_n\}\subset\mathbb{R}$, there are subsequences $\{\alpha_n\}\subset\{\alpha'_n\}$ and $\{\beta_n\}\subset\{\beta'_n\}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+\alpha_n+\beta_m)=\lim_{n\to\infty} f(t+\alpha_n+\beta_n)$$

in the compact open topology.

Definition 2.5. ([4,21,51,62])

(1) $((\Omega,\mathscr{F},\mathbb{P}),\{\theta_{t,x}\}_{t,x\in\mathbb{R}})$ is **ergodic** if $\theta_{t,x}E=E(E\in\mathscr{F})$ for all $t,x\in\mathbb{R}$ implies that $\mathbb{P}(E)=0$ or 1.

(2) Suppose that Ω is a compact metric space, $\mathscr{F}=\mathscr{B}_\Omega, \theta_{t,x}\colon \Omega\to\Omega$ is continuous, and $\theta_{t,x}$ is independent of $t(x)$. Let $\sigma_t\omega=\theta_{t,0}\omega(\sigma_t\omega=\theta_{0,t}\omega)$ (note that $(\Omega,\{\sigma_t\}_{t\in\mathbb{R}})$ is a compact dynamical system). Then

   (i) $(\Omega,\{\sigma_t\}_{t\in\mathbb{R}})$ is a **minimal** if for any $\omega\in\Omega, \{\sigma_t\omega | t\in\mathbb{R}\}$ is dense in Ω.

   (ii) $(\Omega,\{\sigma_t\}_{t\in\mathbb{R}})$ is **almost automorphic** if it is minimal and there is $\omega_0\in\Omega$ such that $\sigma_t\omega_0$ is an almost automorophic function from $\mathbb{R}$ *to* Ω.

   (iii) $(\Omega,\{\sigma_t\}_{t\in\mathbb{R}})$ is **almost periodic** if it is minimal and there is $\omega_0\in\Omega$ such that $\sigma_t\omega_0$ is an almost periodic function from $\mathbb{R}$ to Ω.

   (iv) $(\Omega,\{\sigma_t\}_{t\in\mathbb{R}})$ is **periodic of period** T if there is $\omega_0\in\Omega$ such that $\sigma_{t+T}\omega_0=\sigma_t\omega_0$ and $\Omega=\{\sigma_t\omega_0 | 0\leqslant t\leqslant T\}$.

Remark 2.2.

(1) Let Y be as in Definition 2.4. For a given a continuous function $f:\mathbb{R}\to Y$, let

$$H(f)=cl\{f_\tau\,|\,f_\tau(\cdot)=f(\cdot+\tau),\,\tau\in\mathbb{R}\},$$

where the closure is taken under the compact open topology, and denote $(H(f),\{\sigma_t\}_{t\in\mathbb{R}})$ as the time translation flow, $\sigma_t g=g(\cdot+t)$. Then if f is recurrent (almost automorphic, almost periodic, and periodic), so is $(H(f),\{\sigma_t\}_{t\in\mathbb{R}})$.

(2) Let $\Omega,\theta_{t,x}$ be as in Definition 2.5 (2) with $\theta_{t,x}\omega=\theta_{t,0}\omega(\theta_{t,x}\omega=\theta_{0,x}\omega)$ and let $\sigma_t\omega=\sigma_{t,0}\omega(\sigma_t\omega=\theta_{t,0}\omega)$. If $(\Omega,\{\sigma_t\}_{t\in\mathbb{R}})$ is almost automorphic, then for residually many $\omega_0\in\Omega,\sigma_t\omega_0$ is an almost automorhic function from $\mathbb{R}$ to Ω. If $(\Omega,\{\sigma_t\}_{t\in\mathbb{R}})$ is almost periodic, then, for all $\omega_0\in\Omega,\sigma_t\omega_0$ is an almost periodic function from $\mathbb{R}$ to Ω (see [25,51,62]).

We now state our main results.

Theorem A. *Consider* (1.1). *Suppose that* (1.1) *has a wave-like solution generated by* $v_0(\cdot):\Omega\to X$. *Then*

(1) *There is a critical traveling wave solution of* (1.1). *Moreover critical traveling wave solutions are unique in the sense that for each pair* $U(\cdot,\omega)$ *and* $V(\cdot,\omega)$ *which generate critical traveling wave solutions, there is a measurable function* $\xi(\cdot):\Omega\to\mathbb{R}$ *such that*

$$V(\cdot+\xi(\omega),\omega)=U(\cdot,\theta_{0,\xi(\omega)}\omega).$$

(2) *If* $\theta_{t,x}$ *is independent of* x, $((\Omega,\mathscr{F},\mathbb{P}),\{\theta_{t,0}\}_{t\in\mathbb{R}})$ *is ergodic, and* $U(x,\omega)$ *generates a regular traveling wave solution,* $u(t,x;U(\cdot,\omega),\omega)=U(x-c(t,\omega),\theta_{t,0}\omega)$ *with* $\sup_{\omega\in\Omega}|\partial_t\tilde{u}^\omega(0)|<\infty$, *where* $\tilde{u}^\omega(t)=U(0,\theta_{t,0}\omega)$, *then there is* $g:\Omega\to\mathbb{R}$ *integrable such that*

$$\partial_t c(t,\omega)=g(\theta_{t,0}\omega)$$

and there are $c_*\in\mathbb{R},U_*\in X$, *and* $u_*^\pm\in\mathbb{R}$ *such that for a.e.* $\omega\in\Omega$,

$$\lim_{t\to\infty}\frac{c(t,\omega)}{t}=c_*=\int_\Omega g(\omega)\mathrm{d}\mathbb{P}(\omega),$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t U(x,\theta_{s,0}\omega)\mathrm{d}s=U_*(x)\quad\text{for } x\in\mathbb{R},$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u^+(\theta_{s,0}\omega)(0)\mathrm{d}s = u_*^+,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u^-(\theta_{s,0}\omega)(0)\mathrm{d}s = u_*^-,$$

and

$$u_*^- \leqslant U_*(x) \leqslant u_*^+ \quad \text{for } x \in \mathbb{R}.$$

Moreover, if $\lim_{x\to\pm\infty} U(x, \theta_{s,0}\omega) = u^\pm(\theta_{s,0}\omega)(x)(\equiv u^\pm(\theta_{s,0}\omega)(0))$ *uniformly with respect to* $s \in \mathbb{R}$, *then* $\lim_{x\to\pm\infty} U_*(x) = u_*^\pm$.

(3) *If* $\theta_{t,x}$ *is independent of* t, $((\Omega, \mathscr{F}, \mathbb{P}, \{\theta_{0,x}\}_{x\in\mathbb{R}})$ *is ergodic, and* $U(\cdot,\omega)$ *generates a regular traveling wave solution,* $U(t, x; U(\cdot,\omega), \omega) = U(x - c(t,\omega), \theta_{0,c(t,\omega)}\omega)$ *with* $\partial_t c(t,\omega) \geqslant \bar{\delta}_0$ *for all* $t \in \mathbb{R}, \omega \in \Omega$ *and some* $\bar{\delta}_0 > 0$, *then there is* $g\colon \Omega \to \mathbb{R}$ *with both* g *and* $1/g$ *being integrable such that*

$$\partial_t c(t,\omega) = g(\theta_{0,c(t,\omega)}\omega)$$

and there are $c_* \in \mathbb{R}, U_* \in X$, *and* $u_*^\pm \in \mathbb{R}$ *such that for a.e.* $\omega \in \Omega$,

$$\lim_{t\to\infty}\frac{c(t,\omega)}{t} = c_* = \frac{1}{\int_\Omega \frac{1}{g(\omega)}\mathrm{d}\mathbb{P}(\omega)},$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t U(x, \theta_{0,s}\omega)\mathrm{d}s = U_*(x) \quad \text{for } x \in \mathbb{R},$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u^+(\theta_{0,s}\omega)(0)\mathrm{d}s = u_*^+,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u^-(\theta_{0,s}\omega)(0)\mathrm{d}s = u_*^-,$$

and

$$u_*^- \leqslant U_*(x) \leqslant u_*^+ \quad \text{for } x \in \mathbb{R}.$$

Moreover, if $\lim_{x\to\pm\infty}(U(x, \theta_{0,s}\omega) - u^\pm(\theta_{0,s}\omega)(x)) = 0$ *uniformly in* $s \in \mathbb{R}$, *then* $\lim_{x\to\pm\infty} U_*(x) = u_*^\pm$.

Theorem B. *Assume that the conditions of Theorem A are satisfied and* $U(\cdot,\omega)$ *generates a critical traveling wave solution. Moreover, assume that* Ω *is a compact metric space,* $F(\theta_{t,x}\omega, u)$ *is continuous in* $t, x \in \mathbb{R}, \omega \in \Omega$, *and* $u \in X$, *and* $u^\pm(\omega)$ *is continuous in* ω. *Then the folowing hold.*

(1) *Let* $\tilde{X} = C^b_{\mathrm{unif}}(\mathbb{R})$ *be equipped with the compact open topology. Then* $\Omega_0 = \{\omega_0 \in \Omega \,|\, U(\cdot, \omega)$ *is continuous at* ω_0 *in the* $\tilde{X}-$ *topology*$\}$ *is a residual subset of* Ω *(i.e.* Ω_0 *is the intersection of countably many open dense subsets of* Ω*).*

(2) *If* $\theta_{t,x}$ *is independent of* x*, then* $U(\cdot, \omega)$ *is continous at* $\omega_0 \in \Omega_0$ *in the* X*-topology and* $\theta_{t,0}\Omega_0 = \Omega_0$ *for all* $t \in \mathbb{R}$*. Moreover, if* $((\Omega, \mathscr{F}, \mathbb{R}), \{\theta_{t,0}\}_{t\in\mathbb{R}})$ *is minimal, then for each* $\omega \in \Omega_0, U^\omega(t)(\cdot) \equiv U(\cdot, \theta_{t,0}\omega) \in \tilde{X}$ *is a recurrent function from* $\mathbb{R}$ *to* $\tilde{X}$*. If* $((\Omega, \mathscr{F}, \mathbb{P}), \{\theta_{t,0}\}_{t\in\mathbb{R}})$ *is almost periodic, then* $U^\omega(t)(\cdot)$ *is almost automorphic function from* $\mathbb{R}$ *to* $\tilde{X}$ *for each* $\omega \in \Omega_0$*. If* $(\Omega, \mathscr{F}, \mathbb{P}), \{\theta_{t,0}\}_{t\in\mathbb{R}})$ *is periodic of period* T*, then* $\Omega_0 = \Omega = \{\theta_{t,0}\omega_0 | 0 \leqslant t \leqslant T, \theta_{T,0}\omega_0 = \omega_0, \theta_{t,x}\omega_0 = \theta_{t,0}\omega_0\}$ *for some* ω_0 *and* $U(\cdot, t) = U(\cdot, \theta_{t,0}\omega_0)$ *is periodic in* t *with period* T*. In this case, let*

$$c_0 = \frac{c(T, \omega_0) - C(0, \omega_0)}{T}$$

and

$$V(x, t) = U(x + c_0 t - c(t, \omega_0), \theta_{t,0}\omega_0).$$

Then

$$u(t, x; U(\cdot, \omega_0), \omega_0) = V(x - c_0 t, t)$$

and

$$V(x, t + T) = V(x, t).$$

(3) *If* $\theta_{t,x}$ *is independent of* t *and periodic of period* p *in* $x, \Omega = \{\theta_{0,x}\omega_0) | 0 \leqslant x \leqslant p\}$ *for some* ω_0*, and there is* $T > 0$ *such that* $c(T, \omega_0) = p$*, then letting* $c_0 = p/T$ *and*

$$V(x, t) = U(x + c_0 t - c(t, \omega_0), \theta_{0,c(t,\omega_0)}\omega_0),$$

there holds

$$u(t, x; U(\cdot, \omega_0), \omega_0) = V(x - c_0 t, t)$$

and

$$V(x, t + T) = V(x, t).$$

Letting

$$W(x, y) = U\left(y - c\left(\frac{y - x}{c_0}, \omega_0\right), \theta_{0,c\left(\frac{y-x}{c_0}, \omega_0\right)}\omega_0\right),$$

there holds

$$u(t, x; U(\cdot, \omega_0), \omega_0) = W(x - c_0 t, x)$$

and

$$W(x, y + p) = W(x, y).$$

Remark 2.3.

(1) Theorem B(2) shows that if $\theta_{t,x}$ is independent of x and $((\Omega, \mathscr{F}, \mathbb{P}), \{\theta_{t,0\omega}\}_{t\in\mathbb{R}})$ is periodic of periodic T, then a traveling wave solution is of form

$$u(t, x) = V(x - c_0 t, t),$$

where $V(x, t + T) = V(x, t)$, which fits the definition given in [1].

(2) Theorem B(3) shows that if $\theta_{t,x}$ is independent of t and $((\Omega, \mathscr{F}, \mathbb{P}), \{\theta_{0,x}\omega\}_{x\in\mathbb{R}})$ is periodic of period p, then under some proper assumption, a traveling wave solution is of form

$$u(t, x; U(\cdot, \omega_0), \omega_0) = W(x - c_0 t, x),$$

where $W(x, y + p) = W(x, y)$, which fits the definition given in [58].

To state Theorem C on the applications of Theorem A and B to a bistable case, we assume

(H3)

(1) (1.1) has three random equilibrium solutions $u^{\pm}(\omega)$ and $u^0(\omega)$ with

$$\inf_{\omega\in\Omega, x\in\mathbb{R}} \{u^+(\omega)(x) - u^0(\omega)(x), u^0(\omega)(x) - u^-(\omega)(x)\} > 0.$$

(2) $u^{\pm}(\omega)$ are globally stable in the sense that for all $\alpha > 0, \beta < 0$, and $\omega \in \Omega$,

$$\lim_{t\to\infty} \Big(u(t, x; u^0(\theta_{\tau,y}\omega)(\cdot) + \alpha, \theta_{\tau,y}\omega) - u^+(\theta_{t+\tau,y}\omega)(x)\Big) = 0$$
$$\lim_{t\to\infty} \Big(u(t, x; u^0(\theta_{\tau,y}\omega)(\cdot) + \beta, \theta_{\tau,y}\omega) - u^-(\theta_{t+\tau,y}\omega)(x)\Big) = 0$$

uniformly in $\tau, y, x \in \mathbb{R}$.

(3) $u^0(\omega)$ is unstable in the sense that there are bounded integrable functions $g_{\pm}(t)$ with $\lim_{t\to\infty} \int_0^t g_-(s)\mathrm{d}s = \infty$ such that

$$g_-(t) \leqslant \partial_u F(\theta_{t,x}\omega, u^0(\theta_{t,0}\omega)(x)) \leqslant g_+(t)$$

for $t, x \in \mathbb{R}$ and $\omega \in \Omega$.

Theorem C. *Consider* (1.1). *Assume that* **(H1), (H2), and (H3)** *are satisfied and that* $\theta_{t,x}\omega = \theta_{t,0}\omega$ *for* $t, x \in \mathbb{R}$ *and* $\omega \in \Omega$. *Then* (1.1) *has a wave-like and therefore has a critical traveling wave solution. Moreover, there is* $U(\cdot, \omega)$ *such that it generates a critical traveling wave solution,* $u(t, x; U(\cdot, \omega), \omega) = U(x - c(t, \omega), \theta_{t,0}\omega)$, *and* $c(t, \omega)$ *is differentiable in* t.

3. EXAMPLES

In this Section, we present two random reaction-diffusion equations arising from population genetics and phase transition to which Theorems A, B and/or C can be applied, namely, a random variant of the Fisher, or KPP, equation to which Theorems A and B can be applied, and a random variant of bistable equations to which Theorems A, B and C can be applied.

3.1. A Random Variant of the Fisher, or KPP, Equation

Classically, the so called Fisher, or KPP, equation is as follows:

$$\partial_t u = \partial_x^2 u + mu(1 - u), \tag{3.1}$$

where m is a positive constant. (3.1) models the propagation of genetic composition in a population ([26]). In this model, each individual of the population belongs to one of three possible genotypes, aa, aA and AA. The parameter u represents the fraction of alleles of type a or A amongst the total number of alleles in the population and hence is a certain measure of the genetic composition. The reaction term $mu(1 - u)$ is derived from a knowledge of the relative survival fitness of the three genotypes (m is hence called the fitness coefficient) and the diffusion term $\partial_x^2 u$ arises from the effect of random migration of the individuals.

In reality, the propagation of genetic composition in a population is influenced by various variations of the environment (for example, local temperature) which may be known only in certain probability. Also, the habitat in which the population lives is spatially inhomogeneous in general. Taking these facts into account, it is natural to use proper random variants of the classical Fisher, or KPP, equation to describe the propagation of genetic composition in a population. The following is one of such variants,

$$\partial_t u = \partial_x^2 u + F(\theta_{t,x}\omega, u), \tag{3.2}$$

where $\theta_{t,x}$ is a random process on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and for all fixed $t_0, x_0 \in \mathbb{R}$ and $\omega_0 \in \Omega$, $f(u) \equiv F(\theta_{t_0,x_0}\omega_0, u)$ is of **Fisher or KPP**

type, that is, there are $u^-, u^+ \in \mathbb{R}$ with $u^- < u^+$ such that $f(u^\pm) = 0$, $f(u) \neq 0$ for $u \neq u^\pm$, and $f'(u^-) > 0$, $f'(u^+) < 0$.

Clearly, if $F(\theta_{t,x}\omega, u) = m(\theta_{t,x}\omega)u(1-u)$, where $m(\theta_{t,x}\omega) > 0$ for all $t, x \in \mathbb{R}$ and $\omega \in \Omega$, then (3.2) is a random variant of (3.1). It describes the propogation of genetic composition in a population in which the fitness coefficient depends on both time and sapce in a random way. (3.2) with $F(\theta_{t,x}\omega, u) = m(\theta_{t,x}\omega)u(1-u)$ satisfies **(H1)** provided that $m\colon \omega \to \mathbb{R}$ is a bounded $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$-measurable function and for all fixed $\omega \in \Omega$, $m(\theta_{t,x}\omega)$ is globally Hölder continuous in t and x. It is also easy to see that (3.2) has two random equilibria $u^-(\omega) \equiv 0$ and $u^+(\omega) \equiv 1$ and hence satisfies **(H2)**. Therefore Theorems A and B can be appliled to (3.2) with $F(\theta_{t,x}\omega, u) = m(\theta_{t,x}\omega)u(1-u)$. The application of Theorems A and B to more general type variant of the fisher, or KPP, equation will be studied in forthcoming papers.

It should be mentioned that a proper regularization of stochastic variant of the Fisher, or KPP, equation gives rise to a random one and therefore, the present work would have impact on the study of front propagation in certain stochastic parabolic equations. For example, consider the following stochastic variant of (3.1),

$$\mathrm{d}u = (\partial_x^2 u + m(x)u(1-u))\mathrm{d}t + u(1-u) \circ \mathrm{d}W(t, \omega), \tag{3.3}$$

where $m(x)$ is a bounded smooth function, $(W(t,\cdot))_{t\in\mathbb{R}}$ is a two-sided scalar Wiener process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$,

$$\Omega = C_0(\mathbb{R}, \mathbb{R}) \equiv \{\omega \in C(\mathbb{R}, \mathbb{R}) | \omega(0) = 0\}$$

endowed with the compact open topology, $\mathbb{P}$ is the corresponding Wiener measure, $\mathscr{F}$ is the $\mathbb{P}$-completion of $\mathscr{F}_{\mathscr{B}}$ ($\mathscr{F}_{\mathscr{B}}$ is the Borel σ-algebra on Ω), and $\circ dW(t,\omega)$ denotes Stratonovich's differential. Let $\epsilon \in (0, 1)$ and let $j \in C^\infty(\mathbb{R})$ be compactly supported in (0, 1) with $j \geqslant 0$ and $\int_{\mathbb{R}} j(\xi)\mathrm{d}\xi = 1$. Consider the mollifier $j_\epsilon(\xi) = \epsilon^{-1} j(\epsilon^{-1}\xi)$ and define

$$s_\epsilon(\omega) = \int_0^\infty j'_\epsilon(\xi)\mathrm{d}W(\xi, \omega).$$

Obviously,

$$s_\epsilon(\theta_t\omega) = \int_0^\infty j'_\epsilon(\xi - t)\mathrm{d}W(\xi, \omega),$$

where θ_t is the so called canonical dynamical system on $(\Omega, \mathscr{F}, \mathbb{P})$ defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t) \quad \text{for all } t \in \mathbb{R}.$$

The following random equation can be viewed as a regularization of (3.3),

$$\partial_t u_\epsilon = \partial_x^2 u_\epsilon + (m(x) + s_\epsilon(\theta_t \omega)) u_\epsilon (1 - u_\epsilon). \tag{3.4}$$

(3.4) is a random variant of (3.1) though **(H1)** is not satisfied $(m(x) + s_\epsilon(\theta_t \omega)$ is locally but may not be globally Hölder continuous for some $\omega \in \Omega)$. Nevertheless, some concepts and techniques introduced in the present paper could be applied to (3.4). We will explore applications of the present work to certain stochastic parabolic equations somewhere else.

3.2. A Random Variant of Bistable Equations

Equation

$$\partial_t u = \partial_x^2 u + (1 - u^2)(u - a), \tag{3.5}$$

where $a(-1 < a < 1)$ is a constant, is a so called **bistable equation**. Note that (3.5) has three and only three constant solutions $u^- = -1, u^0 = a, u^+ = 1\,(u^- < u^0 < u^+)$ and $u^\pm$ are stable and u^0 is unstable. (3.5) has been used to describe front propagation in many applied problems including phase transition and nerve propagation. Physically $W(u) = -\int^u (1 - \xi^2)(\xi - a)\mathrm{d}\xi$, the so called double well potential, arises from the free energy or entropy and $\partial_x^2 u$ arises from the internal interaction energy. When the two wells of W have equal size (i.e. a = 0), (3.5) is usually called Allen–Cahn equation, which models the grain boundary motion in a solid material ([2]). In this case, $u(t, x)$ is an order parameter representing the state of the material at time t and position x. The minima $u = \pm 1$ of W are the pure phases and the grain or antiphase boundary is the interface between two regions, one with order parameter 1 and the other -1.

Similarly, in reality, it is important to use proper random variants of bistable equations to describe front propagation in those physical and biological problems traditionally modelled by (3.5). Here is an example of such random variants,

$$\partial_t u = \partial_x^2 u + F(\theta_{t,x}\omega, u), \tag{3.6}$$

where $\theta_{t,x}$ is a random process on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and for all fixed $t_0, x_0 \in \mathbb{R}$ and $\omega_0 \in \Omega$, $f(u) \equiv F(\theta_{t_0,x_0}\omega_0, u)$ is of **bistable type**, that is, there are $u^\pm, u^0 \in \mathbb{R}$ with $u^- < u^0 < u^+$ such that $f(u^\pm) = f(u^0) = 0$, $f(u) \neq 0$ for $u \neq u^\pm, u^0$, and $f'(u^\pm) < 0$, $f'(u^0) > 0$.

Clearly, $F(\theta_{t,x}\omega, u) = (1 - u^2)(u - a(\theta_{t,x}\omega))$ with $-1 < a(\theta_{t,x}\omega) < 1$ is of bistable type. (3.6) with $F(\theta_{t,x}\omega, u) = (1 - u^2)(u - a(\theta_{t,x}\omega))\;(-1 < a(\theta_{t,x}\omega) < 1)$ satisfies **(H1)** and **(H2)** provided that for all $\omega \in \Omega, a(\theta_{t,x}\omega)$ is globally Hölder continuous in t and x. Moreover, if $\theta_{t,x}$ depends only on t (i.e.

$\theta_{t,x}\omega = \theta_{t,0}\omega$ for all $t, x \in \mathbb{R}$ and $\omega \in \Omega$) and $-1/4 \leqslant a(\theta_{t,x}\omega) \leqslant 1/4$, then it is not difficult to prove that (3.6) also satisfies **(H3)**. In fact, it is easily seen that $u^{\pm}: \Omega \to \mathbb{R}$, $u^{\pm}(\omega) = \pm 1$, are two random equilibria of (3.6). By monotone random dynamical system theory (see [5]), there is at least one random equilibrium $u^0: \Omega \to \mathbb{R}$ of (3.6) with $-1/4 \leqslant u^0(\omega) \leqslant 1/4$ for all $\omega \in \Omega$. Moreover, since $1 - (u_0(\omega))^2 - 2u^0(\omega)(u^0(\omega) - a(\omega)) \geqslant 11/16$, every random equilibrium lying between $-1/4$ and $1/4$ is unstable and hence is unique. The rest of **(H3)** can be easily verified. Therefore, Theorems A, B, as well as C can all be applied.

4. PRELIMINARY LEMMAS

In this section we present some lemmas to be used in later sections to prove Theorems A–C.

Let $X = C^b_{\text{unif}}(\mathbb{R})$ be as in (2.3) and $\Pi_t: X \times \Omega \to X \times \Omega$ be as in (2.4) and (2.5). Let $u_0^*(\omega)$ be as follows,

$$u_0^*(\omega)(x) = \begin{cases} u^+(\omega)(x) & \text{for } x \geqslant 0, \\ u^-(\omega)(x) & \text{for } x < 0. \end{cases} \tag{4.1}$$

Clearly, $u_0^* \in BPC(\mathbb{R})$, where $BPC(\mathbb{R})$ is as in (2.6).

Lemma 4.1. *Consider*

$$\partial_t u = \partial_x^2 u + q(t,x)u, \quad x \in \mathbb{R}, \ t > 0. \tag{4.2}$$

where q is a bounded and continuous function. Let $u(t,x)$ be a nonzero classical solution of (4.2).

(1) *For each $t > 0$, the zero set of $u(t,x)$,*

$$z(t) = \{x \in \mathbb{R} | u(t,x) = 0\}$$

is a discrete subset of $\mathbb{R}$.

(2) *If at (t_0, x_0) both u and u_x vanish, then there is a neighborhood $N = [t_0 - \delta, t_0 + \delta] \times [x_0 - \epsilon, x_0 + \epsilon]$ of (t_0, x_0) such that*

  (i) $u(t, x_0 \pm \epsilon) \neq 0$ *for* $|t - t_0| \leqslant \delta$.

  (ii) $u(t_0 + \delta, \cdot)$ *has at most one zero in the interval* $[x_0 - \epsilon, x_0 + \epsilon]$.

  (iii) $u(t_0 - \delta, \cdot)$ *has at least two zeros in the interval* $[x_0 - \epsilon, x_0 + \epsilon]$.

Proof. See [3]. □

Lemma 4.2. *For each* $u_0 \in \mathrm{BPC}(\mathbb{R})$ *and each* $\omega \in \Omega$*, if*

$$\lim_{x\to\pm\infty}\big(u_0(x) - u^{\pm}(\omega)(x)\big) = 0,$$

then for all $t > 0$ *at which* $u(t, x; u_0, \omega)$ *exists,*

$$\lim_{x\to\pm\infty}\big(u(t, x; u_0, \omega) - u^{\pm}(\theta_{t,0}w)(x)\big) = 0.$$

Proof. If follows from standard theory for parabolic equations ([27]). □

Lemma 4.3. *Let* $\omega_n \in \Omega$ *and* $u_n, u_0 \in \mathrm{BPC}(\mathbb{R}) (n \in \mathbb{N})$ *with* $sup_{x\in\mathbb{R}}|u_n(x)|$ *being bounded. If* $F(\theta_{t,x}\omega_n, u) \to f^*(t, x, u)$ *and* $u_n(x) \to u_0(x)$ *as* $n \to \infty$ *in the compact open topology, then for all* $t > 0$ *at which* $u^*(t, x; u_0)$ *and* $u(t, x; u_n, \omega_n)$ *exist,*

$$u(t, x; u_n, \omega_n) \to u^*(t, x; u_0)$$

as $n \to \infty$ *in the compact open topology, where* $u^*(t, x; u_0)$ *is the solution of*

$$\partial_t u = \partial_t^2 u + f^*(t, x, u)$$

with $u^*(0, x; u_0) = u_0(x)$.

Proof. Let $v_n(t, x) = u(t, x; u_n, \omega) - u^*(t, x; u_0)$. Then $v_n(t, x)$ satisfies

$$\begin{aligned}\partial_t v_n &= \partial_x^2 v_n + \partial_u F(\theta_{t,x}\omega_n, u_n^*(t, x))v_n \\ &\quad + F(\theta_{t,x}\omega_n, u^*(t, x; u_0)) - f^*(t, x, u^*(t, x; u_0)),\end{aligned}$$

where $u_n^*(t, x)$ lies between $u(t, x; u_n, \omega_n)$ and $u^*(t, x; u_0)$. Let

$$\tilde{v}(t, x) = \frac{v_n(t, x)}{1 + x^2}.$$

Then $\tilde{v}_n$, satisfies

$$\begin{aligned}\partial_t \tilde{v}_n &= \partial_x^2 \tilde{v}_n + \frac{4x}{1 + x^2}\partial_x \tilde{v}_n + \left[\partial_u F(\theta_{t,x}\omega_n, u_n^*(t, x)) + \frac{2}{1 + x^2}\right]\bar{v}_n \\ &\quad + \frac{F(\theta_{t,x}\omega_n, u^*(t, x; u_0)) - f^*(t, x, u^*(t, x; u_0))}{1 + x^2}.\end{aligned}$$

Note that

$$\tilde{v}_n(0, x) = \frac{v_n(0, x)}{1 + x^2} \to 0$$

uniformly in $x \in \mathbb{R}$ and

$$\frac{F(\theta_{s,x}\omega_n, u^*(s,x;u_0)) - f^*(s,x,u^*(s,x;u_0))}{1+x^2} \to 0$$

uniformly in $0 \leqslant s \leqslant t$ and $x \in \mathbb{R}$. It then follows that

$$\tilde{v}_n(t,x) \to 0$$

as $n \to \infty$ uniformly in $x \in \mathbb{R}$. Hence $v_n(t,x) \to 0$ and then $u(t,x;u_0,\omega_n) \to u^*(t,x;u_0)$ as $n \to \infty$ in the compact open topology. □

Lemma 4.4. *Let $u_\epsilon(\cdot), u_0(\cdot) \in BPC(\mathbb{R})$ be such that*

$$\int_{-\infty}^{\infty} |u_\epsilon(x) - u_0(x)|\,\mathrm{d}x \to 0$$

as $\epsilon \to 0$. Then for all $t > 0$ at which $u(t,x;u_\epsilon,\omega)$ and $u(t,x;u_\epsilon,\omega)$ exist,

$$\lim_{\epsilon \to 0} u(t,x;u_\epsilon,\omega) = u(t,x;u_0,\omega)$$

uniformly in $x \in \mathbb{R}$.

Proof. See [35]. □

Lemma 4.5.

(1) *For all $y_1, y_2 \in \mathbb{R}$ with $y_1 \neq y_2$ and all $t > 0$, there holds*

$$u(t,\cdot;u_0^*(\theta_{-t,-y_1}\omega)(\cdot+y_1), \theta_{-t,0}\omega) > (\text{or } <)$$
$$u(t,\cdot;u_0^*(\theta_{-t,-y_2}\omega)(\cdot+y_2), \theta_{-t,0}\omega).$$

(2) *Assume that*

$$U(x) = \lim_{n\to\infty} u(t_n,x;u_0^*(\theta_{-t_n,-y_n}\omega)(\cdot+y_n), \theta_{-t_n,0}\omega)$$

and

$$\tilde{U}(x) = \lim_{n\to\infty} u(t_n,x;u_0^*(\theta_{-t_n,-\tilde{y}_n}\omega)(\cdot+\tilde{y}_n), \theta_{-t_n,0}\omega)$$

in the compact open topology for some $t_n \to \infty$ and $y_n, \tilde{y}_n \in \mathbb{R}$. If $U(0) = \tilde{U}(0)$, then $U(x) \equiv V(x)$.

Proof. (1) Note that for all $y \in \mathbb{R}$,

$$u_0^*(\theta_{-t,-y}\omega)(x+y) = \begin{cases} u^+(\theta_{-t,0}\omega)(x) & \text{for } x \geqslant -y, \\ u^-(\theta_{-t,0}\omega)(x) & \text{for } x < -y. \end{cases}$$

Hence for all $y_1, y_2 \in \mathbb{R}$ with $y_1 \neq y_2$,

$$u_0^*(\theta_{-t,-y_1}\omega)(x+y_1) \geqslant (\text{or } \leqslant) u_0^*(\theta_{-t,-y_2}\omega)(x+y_2)$$

for all $x \in \mathbb{R}$ depending on $y_1 > (\text{ or } <) y_2$, but

$$u_0^*(\theta_{-t,-y_1}\omega)(x+y_1) \not\equiv u_0^*(\theta_{-t,-y_2}\omega)(x+y_2).$$

(1) Then follows from comparison principal for parabolic equations.
(2) Without loss generality, suppose that $y_n > \tilde{y}_n$. Then

$$u_0^*(\theta_{-t_n,-y_n}\omega)(\cdot+y_n) \geqslant (\text{ and } \neq) u_0^*(\theta_{-t_n,-\tilde{y}_n}\omega)(\cdot+\tilde{y}_n).$$

By (1),

$$\begin{aligned} &u(t,x;u_0^*(\theta_{-t_n,-y_n}\omega)(\cdot+y_n),\theta_{-t_n,0}\omega) \\ &> u(t,x;u_0^*(\theta_{-t_n,-\tilde{y}_n}\omega)(\cdot+\tilde{y}_n),\theta_{-t_n,0}\omega) \end{aligned}$$

for all $x \in \mathbb{R}$. Without loss of generality, we may also assume that

$$\lim_{n\to\infty} u(t_n-1,x;u_0^*(\theta_{-t_n,-y_n}\omega)(\cdot+y_n),\theta_{-t_n,0}\omega) = u^*(x)$$

and

$$\lim_{n\to\infty} u(t_n-1,x;u_0^*(\theta_{-t_n,-\tilde{y}_n}\omega)(\cdot+\tilde{y}_n),\theta_{-t_n,0}\omega) = \tilde{u}^*(x)$$

in the compact open topology. Then we have $u^*(x) \geqslant \tilde{u}^*(x)$ and

$$\begin{aligned} U(x,\omega) &= u(1,x;u^*(\cdot),\theta_{-1,0}\omega), \\ \tilde{U}(x,\omega) &= u(1,x;\tilde{u}^*(\cdot),\theta_{-1,0}\omega) \end{aligned}$$

for all $x \in \mathbb{R}$. If $u^*(x) \not\equiv \tilde{u}^*(x)$, then by comparison principle for parabolic equations, $U(x,\omega) > \tilde{U}(x,\omega)$ for all $x \in \mathbb{R}$. But $U(0,\omega) = \tilde{U}(0,\omega) = u^0(\omega)$, a contradiction. Therefore, $u^*(x) \equiv \tilde{u}^*(x)$ and then $U(x,\omega) \equiv \tilde{U}(x,\omega)$. □

Lemma 4.6. (1) *For each $t \geqslant 0, \omega \in \Omega$, and $u_0 \in \mathrm{BPC}(\mathbb{R})$ with*

$$\lim_{x\to\pm\infty} \left(u_0(x) - u^{\pm}(\omega)(x)\right) = 0$$

and

$$u^-(\omega)(x) < u_0(x) < u^+(\omega)(x) \quad \text{for } -\infty < x < \infty,$$

there is unique $\xi(t,\omega) \in [-\infty,\infty]$ such that

$$u(t,x;u_0^*(\omega),\omega) \begin{cases} > u(t,x;u_0,\omega) & \text{for } x > \xi(t,\omega), \\ < u(t,x;u_0,\omega) & \text{for } x < \xi(t,\omega). \end{cases}$$

(2) *Assume that*

$$U(x) = \lim_{n\to\infty} u(t_n, x; u_0^*(\theta_{-t_n,-y_n}\omega)(\cdot + y_n), \theta_{-t_n,0}\omega)$$

and

$$V(x) = \lim_{n\to\infty} u(t_n, x; v_n^*(\cdot), \theta_{-t_n,0}\omega)$$

in the compact open topology for some $\omega \in \Omega$, $t_n \to \infty$, $y_n \in \mathbb{R}$, *and* $v_n^* \in \mathrm{BPC}(\mathbb{R})$ *with*

$$\lim_{x\to\pm\infty} \left(v_n^*(x) - u^{\pm}(\theta_{-t_n,0}\omega)(x)\right) = 0$$

and

$$u^-(\theta_{-t_n,0}\omega)(x) < v_n^*(x) < u^+(\theta_{-t_n,0}\omega)(x) \quad \textit{for } x \in \mathbb{R}.$$

If $U(0) = V(0)$, *the either* $U(x) \equiv V(x)$ *or*

$$U(x) \begin{cases} > V(x) & \text{for } x > 0, \\ < V(x) & \text{for } x < 0. \end{cases}$$

Proof. (1) First, it is not difficult to see that there are $u_\epsilon^*(\omega), u_\epsilon \in X$ such that $u_\epsilon^*(\omega)(x) = u_0^*(\omega)(x)$ and $u_\epsilon(x) = u_0(x)$ for $|x| \gg 1$,

$$\int_{-\infty}^{\infty} |u_\epsilon^*(\omega)(x) - u_0^*(\omega)(x)| \mathrm{d}x \to 0$$

and

$$\int_{-\infty}^{\infty} |u_\epsilon(x) - u_0(x)| \mathrm{d}x \to 0$$

as $\epsilon \to 0$, and for $0 < \epsilon \ll 1, u_\epsilon^*(\omega)(\cdot) - u_\epsilon(\cdot)$ has exactly one simple zero. Then by Lemma 4.1, for each $t > 0$, there is $\xi_\epsilon(t,\omega) \in [-\infty, \infty]$ such that

$$u(t, x; u_\epsilon^*(\omega), \omega) \begin{cases} > u(t, x; u_\epsilon, \omega) & \text{for } x > \xi_\epsilon(t,\omega), \\ < u(t, x; u_\epsilon, \omega) & \text{for } x < \xi_\epsilon(t,\omega). \end{cases}$$

Take a sequence $\epsilon_n \to 0$. Without loss of generality, assume $\xi_{\epsilon_n}(t,\omega) \to \xi(t,\omega) \in [-\infty, \infty]$. Then by Lemma 4.4, we have

$$u(t, x; u_0^*(\omega), \omega) \begin{cases} \geqslant u(t, x; u_0, \omega) & \text{for } x > \xi(t,\omega), \\ \leqslant u(t, x; u_0, \omega) & \text{for } x < \xi(t,\omega). \end{cases}$$

Note that the above holds for all $t > 0$. It then follows from Lemma 4.1 and comparison principle for parabolic equations that

$$u(t, x; u_0^*(\omega), \omega) \begin{cases} > u(t, x; u_0, \omega) & \text{for } x > \xi(t, \omega), \\ < u(t, x; u_0, \omega) & \text{for } x < \xi(t, \omega). \end{cases}$$

(2) Without loss of generality, we may assume that

$$U_*(x) = \lim_{n\to\infty} u(t_n - 1, x; u_0^*(\theta_{-t_n,-y_n}\omega)(\cdot + y_n), \theta_{-t_n,0})$$

and

$$V_*(x) = \lim_{n\to\infty} u(t_n - 1, x; v_n^*(\cdot), \theta_{-t_n,0}\omega)$$

in the compact open topology. By (1), we may assume there is $\xi_*(t) \in [-\infty, \infty]$ such that

$$U(x) = u(t, x; U_*(\cdot), \theta_{-t,0}\omega) \begin{cases} \geqslant u(t, x; V_*(\cdot), \theta_{-1,0}\omega) = V(x) & \text{for } x > \xi_*(t), \\ \leqslant u(t, x; V_*(\cdot), \theta_{-1,0}\omega) = V(x) & \text{for } x < \xi_*(t) \end{cases}$$

for $0 \leqslant t \leqslant 1$. It then follows from Lemma 4.1 and comparison principle for parabolic eqautions again that either $V(x) \equiv U(x)$, or $V(x) > U(x)$ for all $x \in \mathbb{R}$, or $V(x) < U(x)$ for all $x \in \mathbb{R}$, or

$$U(x) \begin{cases} > V(x) & \text{for } x > \xi_*(1) \\ < V(x) & \text{for } x < \xi_*(1). \end{cases}$$

Therefore, if $U(0) = V(0)$, then either $U(x) \equiv V(x)$ or

$$U(x) \begin{cases} > V(x) & \text{for } x > 0, \\ < V(x) & \text{for } x < 0. \end{cases}$$

□

Lemma 4.7. *Let $G : \mathbb{R} \times \Omega \to \mathbb{R}$ be measurable in $\omega \in \Omega$ (i.e. for each $y \in \mathbb{R}, G(y, \cdot) : \Omega \to \mathbb{R}$ is measurable) and continuous hemicompact in $y \in \mathbb{R}$ (i.e. for each $\omega \in \Omega, G(\cdot, \omega) : \mathbb{R} \to \mathbb{R}$ is continuous and any sequence $\{y_n\} \subset \mathbb{R}$ with $|y_n - G(y_n, \omega)| \to 0$ as $n \to \infty$ has a convergent subsequence). Then G has a deterministic fixed point (i.e. there is $\phi : \Omega \to \mathbb{R}$ such that $G(\phi(\omega), \omega) = \phi(\omega)$) iff G has a random fixed point (i.e. there is measurable $\phi : \Omega \to \mathbb{R}$ such that $G(\phi)(\omega), \omega) = \phi(\omega)$).*

Proof. See [42]. □

Let Y and Z be two compact metric spaces and $P : Z \to Y$ be a homomorphism with $P(Z) = Y$.

Lemma 4.8. *There is a residual subset $Y_0 \subset Y$ such that for all $y_0 \in Y_0$ and $y_n \in Y$ with $y_n \to y_0$ as $n \to \infty$, for each $z_0 \in P^{-1}(y_0)$, there are $z_n \in P^{-1}(y_n)$ such that $z_n \to z_0$ as $n \to \infty$.*

Proof. See [51] or [62]. □

Lemma 4.9. *Suppose that $\theta_{t,x}\omega = \theta_{t,0}\omega (\theta_{t,x}\omega = \theta_{0,x}\omega)$ for all $t, x \in \mathbb{R}$ and $\omega \in \Omega$. Let $\sigma_t\omega = \theta_{t,0}\omega (\sigma_t\omega = \theta_{0,t}\omega)$. if $(\Omega, \{\sigma_t\}_{t\in\mathbb{R}})$ is ergodic and $h \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ (h is real-valued), then there is $\Omega_0 \in \mathscr{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t h(\sigma_s\omega)\ \mathrm{d}s = \int_\Omega h(\omega)\mathrm{d}\mathbb{P}(\omega)$$

for all $\omega \in \Omega_0$.

Proof. see [4]. □

5. PROOF OF THEOREM A

In this section, we shall prove Theorem A stated in Section 2. Throughout this section, we assume that (1.1) has a wave-like solution generated by $v_0 : \Omega \to X$. Let

$$u_0^0(\omega) = \frac{u^+(\omega)(0) + u^-(\omega)(0)}{2}.$$

Clearly, $u_0^0(\cdot) : \Omega \to \mathbb{R}$ is measurable, and

$$u^-(\omega)(0) + \tilde{\delta}_0 \leqslant u_0^0(\omega) \leqslant u^+(\omega)(0) - \tilde{\delta}_0$$

for some $\tilde{\delta}_0 > 0$ and all $\omega \in \Omega$.

Proof of Theorem A(1). Let $u_0^*(\omega)$ be as in (4.1). We shall prove that for each $t > 0$, there is $y(t,\omega)$ measurable in ω such that

$$u(t, y(t,\omega); u_0^*(\theta_{-t,-y(t,\omega)}\omega), \theta_{-t,-y(t,\omega)}\omega) = u_0^0(\omega), \tag{5.1}$$

and

$$U(x,\omega) = \lim_{t\to\infty} u(t, x + y(t,\omega); u_0^*(\theta_{-t,-y(t,\omega)}\omega), \theta_{-t,-y(t,\omega)}\omega) \tag{5.2}$$

exists and $U(\cdot,\omega)$ generates a critical traveling wave solution. Moreover, critical traveling solution is unique. We subdivide the proof into five steps.

Step 1. We prove that for each $t > 0$ and $\omega \in \Omega$, there is $y(t,\omega)$ measurable in $\omega \in \Omega$ such that (5.1) holds. Moreover, $y(t,\omega)$ satisfying (5.1) is unique.

In order to do so, first for given $\omega\in\Omega$ and $t>0$, define

$$G(t,y,\omega)=y+u_0^0(\omega)-u(t,y;u_0^*(\theta_{-t,-y}\omega),\theta_{-t,-y}\omega). \tag{5.3}$$

Then by Lemma 4.3, $G(t,y,\omega)$ is continuous in y and measurable in ω. We claim that

$$\lim_{y\to\pm\infty} u(t,y;u_0^*(\theta_{-t,-y}\omega),\theta_{-t,-y}\omega)-u^\pm(\omega)(0)=0. \tag{5.4}$$

In fact, for all $y_n\to\infty$,

$$u_0^*(\theta_{-t,-y_n}\omega)(x+y_n)\to u^+(\theta_{-t,0}\omega)(x)$$

in the compact open topology. Hence by Lemma 4.3 again,

$$\begin{aligned} u(t,y_n;u_0^*(\theta_{-t,-y_n}\omega),\theta_{-t,-y_n}\omega) &= u(t,0;u_0^*(\theta_{-t,-y_n}\omega)(\cdot+y_n),\theta_{-t,0}\omega)\\ &\to u(t,0;u^+(\theta_{-t,0}\omega)(\cdot),\theta_{-t,0}\omega)\\ &= u^+(\omega)(0). \end{aligned}$$

This implies that

$$\lim_{y\to\infty} u(t,y;u_0^*(\theta_{-t,-y}\omega),\theta_{-t,-y}\omega)=u^+(\omega)(0).$$

Similarly, we can prove that

$$\lim_{y\to-\infty} u(t,y;u_0^*(\theta_{-t,-y}\omega),\theta_{-t,-y}\omega)=u^-(\omega)(0).$$

By (5.4), there are $y_\pm(t,\omega)$ with $y_-(t,\omega)<0<y_+(t,\omega)$ such that

$$G(t,y,\omega)\leqslant y-\frac{\tilde\delta_0}{2}\quad\text{for } y\geqslant y_+(t,\omega)$$

and

$$G(t,y,\omega)\geqslant y+\frac{\tilde\delta_0}{2}\quad\text{for } y\leqslant y_-(t,\omega).$$

Let

$$M(\omega)=\max_{y_-(t,\omega)\leqslant y\leqslant y_+(t,\omega)}|G(t,y,\omega)|.$$

We have

$$G(t,y,\omega)\in[y_-(t,\omega)-M(\omega),\,y_+(t,\omega)+M(\omega)]$$

for each $y\in[y_-(t,\omega)-M(\omega),\,y_+(t,\omega)+M(\omega)]$. Hence there is $y(t,\omega)$ such that

$$G(t,y(t,\omega),\omega)=y(t,\omega)$$

and then by (5.3), (5.1) holds. Moreover, for all $\{y_n\}$ with $y_n - G(t, y_n, \omega)| \to 0$ as $n \to \infty$, we must have $y_-(t, \omega) \leqslant y_n \leqslant y_+(t, \omega)$ for $n \gg 1$. Hence there is a convergent subsequence of $\{y_n\}$. It then follows from Lemma 4.7 that there is $y(t, \omega)$ measurable in ω such that (5.1) holds.

Next, we prove that $y(t, \omega)$ such that (5.1) holds is unique. For otherwise, suppose that $y_1 < y_2$ are such that

$$u(t, y_i; u_0^*(\theta_{-t,-y_i}\omega), \theta_{-t,-y_i}\omega) = u_0^0(\omega)$$

for $i = 1, 2$. By Lemma 4.5,

$$u(t, x; u_0^*(\theta_{-t,-y_1}\omega)(\cdot + y_1), \theta_{-t,0}\omega) < u(t, x; u_0^*(\theta_{-t,-y_2}\omega)(\cdot + y_2), \theta_{-t,0}\omega)$$

for all $t > 0$ and $x \in \mathbb{R}$. But

$$\begin{aligned} u(t, 0; u_0^*(\theta_{-t,-y_1}(\cdot + y_1), \theta_{-t,0}\omega) &= u_0^0(\omega) \\ &= u(t, 0; u_0^*(\theta_{-t,-y_2}\omega)(\cdot + y_2), \theta_{-t,0}\omega), \end{aligned}$$

a contradiction. Therefore, $y(t, \omega)$ satisfying (5.1) is unique.

Step 2. Let

$$U(t, x; \omega) = u(t, x + y(t, \omega); u_0^*(\theta_{-t,-y(t,\omega)}\omega), \theta_{-t,-y(t,\omega)}\omega). \tag{5.5}$$

We prove that $\lim_{t\to\infty} U(t, \cdot; \omega)$ exists in the X-norm and $U(\cdot, \omega) = \lim_{t\to\infty} U(t, \cdot; \omega)$ is measurable in ω in the $X-$norm, $\lim_{x\to\pm\infty}(U(x, \omega) - u^{\pm}(\omega)(x)) = 0$.

We first prove that for each $t > 0$ $U(t, \cdot; \omega) \in X$ is measurable in ω in the X-norm. To do so, define

$$G_1 : \Omega \to \mathbb{R} \times \Omega, \qquad G_1(\omega) = (y(t, \omega), \omega),$$

$$G_2 : \mathbb{R} \times \Omega \to \mathbb{R} \times X, \qquad G_2(y, \omega) = (y, u(t, \cdot; u_0^*(\theta_{-t,-y}\omega), \theta_{-t,-y}\omega),$$

and

$$G_3 : \mathbb{R} \times X \to X, \qquad G_3(y, u)(\cdot) = u(\cdot + y).$$

Clearly G_1 is $(\mathscr{F}, \mathscr{B}_{\mathbb{R}} \times \mathscr{F})$-measurable and G_3 is $(\mathscr{B}_{\mathbb{R}} \times \mathscr{B}_X, \mathscr{B}_X)$-measurable . By Lemma 4.4, G_2 is $(\mathscr{B}_{\mathbb{R}}, \times\mathscr{F}, \mathscr{B}_{\mathbb{R}} \times \mathscr{B}_X)$-measurable. Hence $U(t, \cdot, \omega) = G_3 \circ G_2 \circ G_1(\omega)$ is $(\mathscr{F}, \mathscr{B}_X)$-measurable.

Next, we prove that for all t_1, t_2 with $0 < t_1 < t_2$,

$$U(t_1, x; \omega) \begin{cases} > U(t_2, x; \omega) & \text{for } x > 0 \\ < U(t_2, x; \omega) & \text{for } x < 0. \end{cases} \tag{5.6}$$

Notice that

$$\begin{aligned} U(t,x;\omega) &= u(t,x+y(t,\omega);u_0^*(\theta_{-t,-y(t,\omega)}\omega),\theta_{-t,-y(t,\omega)}\omega)\\ &= u(t,x;u_0^*(\theta_{-t,-y(t,\omega)}\omega)(\cdot+y(t,\omega)),\theta_{-t,0}\omega). \end{aligned}$$

Therefore,

$$U(t_1,x;\omega)=u(t_1,x;u_0^*(\theta_{-t_1,-y(t_1,\omega)}\omega)(\cdot+y(t_1,\omega)),\theta_{-t_1,0}\omega),$$

and

$$\begin{aligned} U(t_2,x;\omega)=u(t_1,x;u(t_2-t_1,\cdot;u_0^*(\theta_{-t_2,-y(t_2,\omega)}\omega)(\cdot+y(t_2,\omega)),\\ \theta_{-t_2,0}\omega),\theta_{-t_1,0}\omega). \end{aligned}$$

By Lemma 4.6(1), (5.6) holds. Hence $\lim_{t\to\infty} U(t,x;\omega)$ exists for each $x\in\mathbb{R}$. Let

$$U(x,\omega)=\lim_{t\to\infty} U(t,x;\omega).$$

Clearly $U(\cdot,\omega)\in X$ and $U(0,\omega)=u_0^0(\omega)$.

We also claim that

$$\lim_{t\to\infty} U(t,x;\omega)=U(x,\omega) \tag{5.7}$$

uniformly in $x\in\mathbb{R}$ and

$$\lim_{x\to\pm\infty}(U(x,\omega)-u^{\pm}(\omega)(x))=0. \tag{5.8}$$

In fact, by the arguments similar to those in proving the existence of $y(t,\omega)$ in step 1, there is $\xi(t,\omega)$ such that

$$u(t,\xi(t,\omega);v_0(\theta_{-t,-\xi(t,\omega)}\omega),\theta_{-t,-\xi(t,\omega)}\omega)=u_0^0(\omega).$$

Then by Lemma (4.6.1),

$$U(t,x,\omega)\begin{cases}\geqslant u(t,x+\xi(t,\omega);v_0(\theta_{-t,-\xi(t,\omega)}\omega),\theta_{-t,-\xi(t,\omega)}\omega), & x\geqslant 0,\\ \leqslant u(t,x+\xi(t,\omega);v_0(\theta_{-t,-\xi(t,\omega)}\omega),\theta_{-t,-\xi(t,\omega)}\omega), & x<0.\end{cases}$$

This implies that

$$\lim_{x\to\pm\infty} U(t,x,\omega)-u^{\pm}(\omega)(x)=0$$

uniformly in $t>0$ and then (5.7), (5.8) follows. Moreover, by the measurability of $U(t,x,\omega)$ in ω and (5.7), $U(\cdot,\omega)$ is measurable in ω in the X-norm.

Step 3. We prove that $U(\cdot,\omega)$ generates a traveling wave solution, that is, there is $c(t,\omega)$ measurable in ω such that

$$u(t,x+c(t,\omega);U(\cdot,\omega),\omega)\equiv U(x,\theta_{t,c(t,\omega)}\omega). \tag{5.9}$$

First, by the arguments similar to those in proving the existence of $y(t,\omega)$ in step 1 again, there is $c(t,\omega)$ measurable in ω such that

$$u(t,c(t,\omega);U(\cdot,\omega),\omega)=u_0^0(\theta_{t,c(t,\omega)}\omega). \tag{5.10}$$

Now, we show that (5.9) holds. Note that

$$U(x,\omega)=\lim_{s\to\infty}u(s,x;u_0^*(\theta_{-s,-y(s,\omega)}\omega)(\cdot+y(s,\omega)),\theta_{-s,0}\omega).$$

Hence,

$$\begin{aligned}&u(t,x+c(t,\omega);U(\cdot,\omega),\omega)\\&=\lim_{s\to\infty}u(s+t,x;u_0^*(\theta_{-s,-y(s,\omega)}\omega)(\cdot+y(s,\omega)+c(t,\omega)),\theta_{-s,c(t,\omega)}\omega)\\&=\lim_{s\to\infty}u(s+t,x;u_0^*(\theta_{-s-t,\tilde y(s+t,\theta_{t,c(t,\omega)}\omega)}\theta_{t,c(t,\omega)}\omega)(\cdot+\tilde y(s+t,\theta_{t,c(t,\omega)}\omega)),\\&\qquad\theta_{-(s+t),0}(\theta_{t,c(t,\omega)}\omega)),\end{aligned}$$

where $\tilde y(t+s,\theta_{t,c(t,\omega)}\omega)=y(s,\omega)+c(t,\omega)$,
and

$$u(t,c(t,\omega);U(\cdot,\omega),\omega)=u_0^0(\theta_{t,c(t,\omega)}\omega).$$

Note also that

$$\begin{aligned}&U(x,\theta_{t,c(t,\omega)}\omega)\\&=\lim_{s\to\infty}u(s+t,x;u_0^*(\theta_{-(s+t),-y(s+t,\theta_{t,c(t,\omega)}\omega}\theta_{t,c(t,\omega)}\omega)(\cdot+y(s+t,\theta_{t,c(t,\omega)}\omega),\\&\qquad\qquad\theta_{-(s+t),0}\theta_{t,c(t,\omega)}\omega)\end{aligned}$$

and

$$U(0,\theta_{t,c(t,\omega)}\omega)=u_0^0(\theta_{t,c(t,\omega)}\omega),$$

Then by Lemma 4.5(2),

$$u(t,x+c(t,\omega);U(\cdot,\omega),\omega)\equiv U(x,\theta_{t,c(t,\omega)}\omega),$$

that is , (5.9) holds.

Step 4. We prove that $U(\cdot,\omega)$ generates a critical traveling wave solution.

Suppose that $V(\cdot,\omega)$ also generates a traveling wave solution. Given $\omega\in\Omega$, suppose that $\xi_1(\omega)$ and $\xi_2(\omega)$ are such that

$$V(\xi_1(\omega),\omega)=U(\xi_2(\omega),\theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega).$$

Note that

$$V(x,\omega)=u(t,x;u(-t,\cdot;V(\cdot,\omega),\omega),\theta_{-t,0}\omega).$$

Hence

$$\begin{aligned}V(x+\xi_1(\omega),\omega)&=u(t,x;u(-t,\cdot+\xi_1(\omega);V(\cdot,\omega),\omega),\theta_{-t,\xi_1(\omega)}\omega)\\&=\lim_{t\to\infty}u(t,x;u(-t,\cdot+\xi_1(\omega);V(\cdot,\omega),\omega),\theta_{-t,\xi_1(\omega)}\omega).\end{aligned}$$

Note also that

$$\begin{aligned}&U(x+\xi_2(\omega),\theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega)\\&\quad=\lim_{t\to\infty}u(t,x;u_0^*(\theta_{-t,-y(t,\theta_{0,\xi_1(\omega)-\xi_2(\omega)}}\theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega)\\&\qquad(\cdot+\bar{y}(t,\omega)),\theta_{-t,\xi_1(\omega)}\omega),\end{aligned}$$

where $\bar{y}(t,\omega)=y(t,\theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega)+\xi_2(\omega)$. By Lemma 4.6(2), we have

$$U(x+\xi_2(\omega),\theta_{0,\xi_1(\omega)-\xi_2(\omega)}\omega)\begin{cases}\geqslant V(x+\xi_1(\omega),\omega), & x\geqslant 0,\\ \leqslant V(x+\xi_1(\omega),\omega), & x\leqslant 0.\end{cases}$$

Therefore, U generates a critical traveling wave solution.

Step 5. We prove that critical traveling wave solutions are unique. Suppose that both U and V generate critical wave solutions. First, by the arguments similar to those in proving the existence of $y(t,\omega)$ in step 1 again, there is a measurable $\xi(\omega)$ such that $V(\xi(\omega),\omega)=u_0^0(\theta_{0,\xi(\omega)}\omega)=U(0,\theta_{0,\xi(\omega)}\omega)$. Then by the arguments in step 4,

$$V(x+\xi(\omega),\omega)\begin{cases}\leqslant U(x,\theta_{0,\xi(\omega)}\omega) & x\geqslant 0,\\ \geqslant U(x,\theta_{0,\xi(\omega)}\omega) & x\leqslant 0.\end{cases}$$

On the other hand,

$$U(x,\theta_{0,\xi(\omega)}\omega)\begin{cases}\leqslant V(x+\xi(\omega),\omega) & x\geqslant 0,\\ \geqslant V(x+\xi(\omega),\omega) & x<0.\end{cases}$$

Hence $V(\cdot+\xi(\omega),\omega)=U(\cdot,\theta_{0,\xi(\omega)}\omega)$.

The proof of Theorem (A 1) is then completed. □

Proof of Theorem A(2). Let $\sigma_t\omega = \theta_{t,0}\omega$. Suppose that $U(\cdot,\omega)$ generates a regular travelimg wave solution,

$$u(t,x;U(\cdot,\omega),\omega) = U(x - c(t,\omega),\sigma_t\omega),$$
$$u(t,c(t,\omega);U(\cdot,\omega),\omega) = U(0,\sigma_t\omega) = \tilde{u}^{\omega}(t).$$

First of all, we have that

$$\partial_t u(t,c(t,\omega);U(\cdot,\omega),\omega) + \partial_t c(t,\omega)\partial_x u(t,c(t,\omega);U(\cdot,\omega),\omega)$$
$$= \partial_t \tilde{u}^{\omega}(t) = \partial_t \tilde{u}^{\sigma_t\omega}(0).$$

Therefore,

$$\partial_t c(t,\omega) = \frac{\partial_t \tilde{u}^{\sigma_t\omega}(0) - \partial_x^2 U(0,\sigma_t\omega) - F(\sigma_t\omega, u_0^0(\sigma_t\omega))}{\partial_x U(0,\sigma_t\omega)}.$$

Let

$$g(\omega) = \frac{\partial_t \tilde{u}^{\omega}(0) - \partial_x^2 U(0,\omega) - F(\omega, u_0^0(\omega))}{\partial_x U(0,\omega)}$$

Then $g(\omega)$ is integrable and

$$\partial_t c(t,\omega) = g(\sigma_t\omega).$$

It follows from Lemma 4.9 that there is $\Omega_1 \in \mathscr{F}$ with $\mathbb{P}(\Omega_1) = 1$ and $c_* \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{c(t,\omega)}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t g(\sigma_s\omega)\mathrm{d}s = c_* = \int_{\Omega} g(\omega)\mathrm{d}\mathbb{P}(\omega)$$

for $\omega \in \Omega_1$.

By Lemma 4.9, there is $\Omega_2 \subset \Omega$ with $\mathbb{P}(\Omega_2) = 1$ and $u_*^{\pm} \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u^{\pm}(\sigma_s\omega)(0)\mathrm{d}s = u_*^{\pm}$$

exists for any $\omega \in \Omega_2$.

Now, let $\mathbb{Q}$ be the set of rational numbers. Since $U(\cdot,\omega) \in X$ is measurable in ω and $U(x,\omega)$ is bounded in x and ω, by Lemma 4.9 again, there is $\Omega_3 \in \mathscr{F}$ with $\mathbb{P}(\Omega_3) = 1$ such that $\lim_{t\to\infty} \frac{1}{t}\int_0^t U(x,\sigma_s\omega)\mathrm{d}s$ exists for all $x \in \mathbb{Q}$ and $\omega \in \Omega_3$. Note that $U(x,\omega)$ is uniformly continuous in x. This implies that for all $x \in \mathbb{R}$ and $\omega \in \Omega_3$, the limit $\lim_{t\to\infty} \frac{1}{t}\int_0^t U(x,\sigma_s\omega)\mathrm{d}s$ exists. Let

$$U_*(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t U(x,\sigma_s\omega)\mathrm{d}s$$

for $x \in \mathbb{R}$ and $\omega \in \Omega_3$. We have $U_* \in X$ and U_* is monotone. Note that $u^{\pm}(\omega)(x) = u^{\pm}(\omega)(0)$. Then

$$u^-(\omega)(0) < U(x,\omega) < u^+(\omega)(0)$$

for all $x \in \mathbb{R}$. Hence

$$u_*^- \leqslant U_*(x) \leqslant u_*^+ \quad \text{for } x \in \mathbb{R}.$$

Clearly, if $\lim_{x\to\pm\infty} U(x,\sigma_s\omega) = u^{\pm}(\sigma_s\omega)(0)$ uniformly for $s \in \mathbb{R}$, then $U_*(\pm\infty) = u_*^{\pm}$. Theorem A(2) then folows. □

Proof of Theorem A(3). Let $\sigma_x\omega = \theta_{0,x}\omega$. Observe that

$$u(t, c(t,\omega); U(\cdot,\omega), \omega) = U(0, \sigma_{c(t,\omega)}\omega) = u^{\omega}(c(t,\omega)).$$

Since $U(\cdot,\omega)$ generates a regular traveling wave solution, we have

$$\begin{aligned}&\partial_t u(t, c(t,\omega); U(\cdot,\omega), \omega) + \partial_x u(t, c(t,\omega); U(\cdot,\omega),\omega)\partial_t c(t,\omega)\\ &\quad = \partial_t c(t,\omega)\partial_x u^{\omega}(c(t,\omega)\end{aligned}$$

and then

$$\partial_t c(t,\omega) = -\frac{\partial_x^2 U(0,\sigma_{c(t,\omega)}\omega) + F(\sigma_{c(t,\omega)}\omega, u_0^0(\sigma_c(t,\omega)\omega))}{\partial_x U(0,\sigma_{c(t,\omega)}\omega) - \partial_x u^{\omega}(c(t,\omega))}.$$

Note that $\partial_x u^{\omega}(c(t,\omega)) = \partial_x u^{\sigma_{c(t,\omega)}\omega}(0)$. Let

$$g(\omega) = -\frac{\partial_x^2 U(0,\omega) + F(\omega, u_0^0(\omega))}{\partial_x U(0,\omega) - \partial_x u^{\omega}(0)}.$$

Then

$$\partial_t c(t,\omega) = g(\sigma_{c(t,\omega)}\omega).$$

Since $\partial_t c(t,\omega) \geqslant \tilde{\delta}_0 > 0$ for $t \in \mathbb{R}, \omega \in \Omega$ and some $\tilde{\delta}_0 > 0$, there is $t(\xi,\omega)$ such that $\xi = c(t(\xi,\omega),\omega)$. Hence

$$\partial_\xi t(\xi,\omega) = \frac{1}{g\left(\sigma_\xi\omega\right)}.$$

By Lemma 4.9,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\eta}\int_0^{\eta} \partial_\xi t(\xi,\omega)\mathrm{d}\xi &= \frac{1}{\lim_{t\to\infty}\frac{1}{t}\int_0^t \partial_t c(t,\omega)dt}\\ &= \int_{\Omega} \frac{1}{g(\omega)}\mathrm{d}\mathbb{P}(\omega)\end{aligned}$$

for a.e. $\omega \in \Omega$. This implies that

$$\lim_{t\to\infty} \frac{c(t,\omega)-c(0,\omega)}{t} = \frac{1}{\int_\Omega \frac{1}{g(\omega)} \mathrm{d}\mathbb{P}(\omega)}$$

for a.e. $\omega \in \Omega$. The rest of Theorem A(3) follows from the same arguments as in Theorem A(2). $\square$

6. PROOF OF THEOREM B

In this section, we shall prove Theorem B. Throughout this section, we assume that Ω is compact, $\mathscr{F} = \mathscr{B}_\Omega$, $F(\theta_{t,x}\omega, u)$ is continuous in $t, x \in \mathbb{R}, \omega \in \Omega$, and $u \in X$, $u^\pm(\omega)$ are continuous in ω. We assume that (1.1) has a critical traveling wave solution generated by $U(\cdot,\omega)$,$u(t,x;U(\cdot,\omega),\omega) = U(x-c(t,\omega), \theta_{t,c(t,\omega)}\omega)$, $U(0,\omega) = u_0^0(\omega)$, where

$$u_0^0(\omega) = \frac{u^+(\omega)(0) + u^-(\omega)(0)}{2}.$$

Proof of theorem B(1). First of all, let $\tilde{X} = C^b_{unif}(\mathbb{R})$ be equipped with open compact topology. Let

$$Y = cl\{(U(\cdot,\omega),\omega) | \omega \in \Omega\} \subset \tilde{X} \times \Omega,$$

where the closure is taken in the $\tilde{X} \times \Omega$-topology. Let

$$P: Y \to \Omega, \qquad P(V(\cdot),\omega) = \omega.$$

Then Y is a compact subset of $\tilde{X} \times \Omega$ and P is continuous. Let

$$\bar{h}: \Omega \to 2^Y, \qquad \bar{h}(\omega) = P^{-1}(\omega).$$

By Lemma 4.8, there is a residual subset $\Omega_0 \subset \Omega$ such that $\bar{h}$ is a continuous on Ω_0.

Next, we claim that for each $\omega \in \Omega$ and $(V(\cdot),\omega) \in Y$, there holds

$$U(x,\omega) \begin{cases} \geqslant V(x) & \text{for } x \geqslant 0, \\ \leqslant V(x) & \text{for } x \leqslant 0. \end{cases} \tag{6.1}$$

In fact, for each $\omega \in \Omega$ and $(V(\cdot),\omega) \in Y$, suppose that

$$(V(\cdot),\omega) = \lim_{n\to\infty} (U(\cdot,\omega_n),\omega_n)$$

in $\tilde{X} \times \Omega$-topology. Note that

$$U(x,\omega_n) = \lim_{t\to\infty} u(t, x + y(t,\omega_n); u_0^*(\theta_{-t,-y(t,\omega_n)}\omega_n), \theta_{-t,-y(t,\omega_n)}\omega_n)$$

in the uniform convergence topology. Without loss of generality, we may assume that for some $t_n \to \infty$,

$$V(x) = \lim_{n\to\infty} u(t_n, x + y(t_n, \omega_n); u_0^*(\theta_{-t_n, -y(t_n,\omega_n)}\omega_n), \theta_{-t_n,-y(t_n,\omega_n)}\omega_n)$$

in the compact open topology. Hence by Lemma 4.3,

$$\begin{aligned} &u(t, x; V(\cdot), \omega) \\ &= \lim_{n\to\infty} u(t + t_n, x + y(t_n, \omega_n); u_0^*(\theta_{-t_n,-y(t_n,\omega_n)}\omega_n), \theta_{-t_n,-y(t_n,\omega_n)}\omega_n) \end{aligned}$$

in the compact open topology for all $t \in \mathbb{R}$. Therefore $u(t, x; V(\cdot), \omega)$ exists for $t \in \mathbb{R}$ and

$$\begin{aligned} V(x) &= u(t, x; u(-t, \cdot; V(\cdot), \omega), \theta_{-t,0}\omega) \\ &= \lim_{t\to\infty} u(t, x; u(-t, \cdot; V(\cdot), \omega), \theta_{-t,0}\omega) \end{aligned}$$

in the compact open topology. Note also that

$$U(x, \omega) = \lim_{t\to\infty} u(t, x; u_0^*(\theta_{-t,-y(t,\omega)}\omega)(\cdot + y(t, \omega)), \theta_{-t,0}\omega).$$

By the continuity of $u^{\pm}(\omega)$ in ω and comparison principle for parabolic equations,

$$u^-(\omega)(x) < V(x) < u^+(\omega)(x)$$

for $x \in \mathbb{R}$. Clearly, $V(0) = u_0^0(\omega)$. Hence by Lemma 4.6(2), (6.1) holds.

Now we claim that for each $\omega_0 \in \Omega_0$, $\bar{h}(\omega_0) = \{(U(\cdot, \omega_0), \omega_0)\}$ is a singleton. In fact, if $(V(\cdot), \omega_0) \in \bar{h}(\omega_0)$, then there are $(U(\cdot, \omega_n), \omega_n)$ such that $(U(\cdot, \omega_n), \omega_n) \to (V(\cdot), \omega_0)$ as $n \to \infty$ in $\tilde{X} \times \Omega$ topology. By the continuity of $\bar{h}$ at ω_0, there are $(V_n, \omega_n) \in Y$ such that $(V_n, \omega_n) \to (U(\cdot, \omega_0), \omega_0)$ in $\tilde{X} \times \Omega$ topology. By (6.1),

$$U(x, \omega_n) \begin{cases} \geqslant V_n(x) & \text{for } x \geqslant 0, \\ \leqslant V_n(x) & \text{for } x \leqslant 0. \end{cases}$$

Hence we must have

$$V(x) \begin{cases} \geqslant U(x, \omega_0) & \text{for } x \geqslant 0, \\ \leqslant U(x, \omega_0) & \text{for } x \leqslant 0. \end{cases}$$

By (6.1) again,

$$U(x, \omega_0) \begin{cases} \geqslant V(x) & \text{for } x \geqslant 0, \\ \leqslant V(x) & \text{for } x \leqslant 0. \end{cases}$$

Hence $V(x)=U(x,\omega_0)$ for all $x\in\mathbb{R}$ and $\bar{h}(\omega_0)=\{(U(\cdot,\omega_0),\omega_0)\}$ is a singleton.

Finally, it is not difficult to see that $h:\Omega\to X, h(\omega)=U(\cdot,\omega)$ is continuous at $\omega_0\in\Omega$ in the $\tilde{X}$-topology iff $\bar{h}(\omega_0)$ is a singleton. It then follows that $h(\omega)=U(\cdot,\omega)$ is continuous at each $\omega_0\in\Omega_0$ in the $\tilde{X}$-topology. □

Proof of Theorem B(2). Suppose that $\theta_{t,x}\omega=\theta_{t,0}\omega$ for all $t,x\in\mathbb{R}$ and $\omega\in\Omega$. Let $\sigma_t\omega=\theta_{t,0}\omega$.

First of all, we claim that for each $(V(\cdot),\omega)\in Y$, $V(x)$ is either strictly monotone or a constant function. In fact, for each $(V(\cdot),\omega)\in Y$, assume that

$$V(x)=\lim_{n\to\infty} u(t_n,x+y(t_n,\omega_n);u_0^*(\theta_{-t_n,-y(t_n,\omega_n)}\omega_n),\theta_{-t_n,-y(t_n,\omega_n)}\omega_n)$$

for some $t_n\to\infty$ in the compact open topology. Then for all $t\in\mathbb{R}$,

$$\begin{aligned}&u(t,x;V(\cdot),\omega)\\&=\lim_{n\to\infty} u(t+t_n,x+y(t_n,\omega_n);u_0^*(\theta_{-t_n,-y(t_n,\omega_n)}\omega_n),\theta_{-t_n,-y(t_n,\omega_n)}\omega_n)\end{aligned}$$

in the compact open topology. It then follows from comparison principle for parabolic equations that $V(x)$ is either strictly monotone or is a constant function.

Next, we prove that $U(\cdot,\omega)$ is continuous at each $\omega_0\in\Omega_0$ in the X-topology and $\sigma_t\Omega_0=\Omega_0$ for all $t\in\mathbb{R}$. Note that $\omega\in\Omega_0$ if and only if $P^{-1}(\omega)$ is a singleton.

Assume $\omega_0\in\Omega_0$ and $\omega_n\in\Omega$ with $\omega_n\to\omega_0$. By Theorem B(1), $U(\cdot,\omega_n)\to U(\cdot,\omega_0)$ in the $\tilde{X}$-topology. The continuity of $u^{\pm}(\omega)$ together with the monotonicity of $U(\cdot,\omega)$ then implies that $U(\cdot,\omega_n)\to U(\cdot,\omega_0)$ in the X-topology.

Assume $\omega_0\in\Omega$. For each fixed $t>0$, suppose that $(V(\cdot),\sigma_t\omega_0)\in Y$. Then there is $\omega_n\to\omega_0$ such that $U(\cdot,\sigma_t\omega_n)\to V(\cdot)$ in the compact open topology. Note that $U(\cdot,\omega_n)\to U(\cdot,\omega_0)$ in uniform convergence topology. Hence

$$U(x,\omega_n)-u^{\pm}(\omega_n)(0)\to 0$$

as $x\to\pm\infty$ uniformly in $n\geqslant 1$ and then

$$u(t,x;U(\cdot,\omega_n),\omega_n)-u^{\pm}(\sigma_t\omega_n)(0)\to 0$$

as $x\to\pm\infty$ uniformly in $n\geqslant 1$. Hence $c(t,\omega_n)$ is bounded with respect to n and

$$U(x,\sigma_t\omega_n)-u^{\pm}(\sigma_t\omega_n)(0)=u(t,x+c(t,\omega_n);U(\cdot,\omega_n),\omega_n)-u^{\pm}(\sigma_t\omega_n)(0)\to 0$$

as $x \to \pm\infty$ uniformly in $n \geqslant 1$. It then follows that

$$u(-t, x; U(\cdot, \sigma_t \omega_n), \sigma_t \omega_n) - u^{\pm}(\omega_n)(0) \to 0$$

as $x \to \pm\infty$ uniformly in $n \geqslant 1$ and therefore $c(-t, \sigma_t \omega_n)$ is bounded with respect to n. Without loss of generality, suppose that

$$c(-t, \sigma_t \omega_n) \to c^*$$

as $n \to \infty$. Then

$$\begin{aligned} U(x, \omega_n) &= u(-t, x + c(-t, \sigma_t \omega_n); U(\cdot, \sigma_t \omega_n), \sigma_t \omega_n) \\ &= u(-t, x + c^*; U(\cdot, \sigma_t \omega_n), \sigma_t \omega_n) \\ &\quad + u(-t, x + c(-t, \sigma_t \omega_n); U(\cdot, \sigma_t \omega_n), \sigma_t \omega_n) \\ &\quad - u(-t, x + c^*; U(\cdot, \sigma_t \omega_n), \sigma_t \omega_n) \\ &\to u(-t, x + c^*; V(\cdot), \sigma_t \omega_0) \\ &= u(-t, x; V(\cdot + c^*), \sigma_t \omega_0) \end{aligned}$$

as $n \to \infty$ in the compact open topology. On the other hand,

$$\begin{aligned} U(x, \omega_n) &\to U(x, \omega_0) \\ &= u(-t, x + c(-t, \sigma_t \omega_0); U(\cdot, \sigma_t \omega_0), \sigma_t \omega_0) \\ &= u(-t, x; U(\cdot + c(-t, \sigma_t \omega_0), \sigma_t \omega_0), \sigma_t \omega_0) \end{aligned}$$

as $n \to \infty$ in uniformly convergece topology. We then must have

$$U(\cdot + c(-t, \sigma_t \omega_0), \sigma_t \omega_0) = V(\cdot + c^*)$$

and then $V(x)$ is strictly monotone in x. Since $U(0, \sigma_t \omega_0) = V(0) = u_0^0(\sigma_t \omega_0)$, we must have $c^* = c(-t, \sigma_t \omega_0)$ and then $V(\cdot) = U(\cdot, \sigma_t \omega_0)$. Threfore $P^{-1}(\sigma_t \omega)$ is a singleton and then $\sigma_t \omega_0 \in \Omega_0$. This implies that $\sigma_t \Omega_0 = \Omega_0$ for all $t \in \mathbb{R}$.

Second, if $((\Omega, \mathscr{F}, \mathbb{P}), \{\theta_{t,0}\}_{t \in \mathbb{R}})$ is minimal, then for each $\omega_0 \in \Omega_0$, by the continuity of $\bar{h}$ at ω_0, $cl\{(U(\cdot, \sigma_t \omega_0), \sigma_t \omega_0) | t \in \mathbb{R}\}$ is minimal and hence $U^{\omega_0}(t)(\cdot) = U(\cdot, \sigma_t \omega_0)$ is a recurrent function from $\mathbb{R}$ to $\tilde{X}$.

Now suppose that $((\Omega, \mathscr{F}, \mathbb{P}), \{\sigma_t\}_{t \in \mathbb{R}})$ is almost periodic. Then for each $\omega_0 \in \Omega_0$ and $\{\alpha_n'\} \subset \mathbb{R}$, there is $\{\alpha_n\} \subset \{\alpha_n'\}$ such that

$$\lim_{n \to \infty} \lim_{m \to \infty} \sigma_{-\alpha_n + \alpha_m} \omega_0 = \omega_0.$$

By the continuity of $\bar{h}$ at ω_0, we have

$$\lim_{n \to \infty} \lim_{m \to \infty} U(\cdot, \sigma_{-\alpha_n + \alpha_m} \omega_0) = U(\cdot, \omega_0)$$

in $\tilde{X}$. Therefore, $U(\cdot, \sigma_t\omega_0)$ is an almost automorphic function from $\mathbb{R}$ to $\tilde{X}$.

Finally suppose that $\sigma_T\omega_0 = \omega_0$ and $\Omega = \{\sigma_t\omega_0 | 0 \leqslant t \leqslant T\}$. By $\sigma_t\Omega_0 = \Omega_0$, we have $\Omega_0 = \Omega$ and hence $U(\cdot, t) \equiv U(\cdot, \sigma_t\omega_0)$ is periodic in t with periodic T. Let

$$c_0 = \frac{c(T, \omega_0) - c(0, \omega_0)}{T}$$

and

$$V(x, t) = U(x + c_0 \cdot t - c(t, \omega_0), \sigma_t\omega_0).$$

Note that

$$c(t + T, \omega_0) = c(t, \omega_0) + c(T, \omega_0).$$

Then

$$c_0 \cdot (t + T) - c(t + T, \omega_0) = c_0 \cdot t - c(t, \omega_0).$$

Hence

$$V(x, t + T) = V(x, T)$$

and

$$u(t, x; U(\cdot, \omega_0), \omega_0) = V(x - c_0 t, t). \qquad \square$$

Proof of Theorem B(3). Let $\sigma_x\omega = \theta_{0,x}\omega$. Note $\Omega = \{\sigma_x\omega_0 | 0 \leqslant x \leqslant p\}$ and $\sigma_p\omega_0 = \omega_0$. Suppose that $c(T, \omega_0) = p$. Then

$$c(t + T, \omega_0) = c(t, \omega_0) + p.$$

Let

$$c_0 = \frac{p}{T}$$

and

$$V(x, t) = U(x + c_0 t - c(t, \omega_0), \sigma_{c(t,\omega_0)}\omega_0)$$

for $0 \leqslant t \leqslant T$. Then

$$V(x, t + T) = V(x, t)$$

and

$$
\begin{aligned}
u(t,x;U(\cdot,\omega_0),\omega_0) &= U(x-c(t,\omega_0),\sigma_{c(t,\omega_0)},\omega_0)\\
&= U(x-c_0t+c_0t-c(t,\omega_0),\sigma_{c(t,\omega_0)}\omega_0)\\
&= V(x-c_0t,t).
\end{aligned}
$$

Let c_0 be as above and

$$
W(x,y)=U\left(y-c\left(\frac{y-x}{c_0},\omega_0\right),\sigma_{c(\frac{y-x}{c_0},\omega_0)}\omega_0\right)
$$

for $x,y\in\mathbb{R}$. Then

$$
c\left(\frac{y+p-x}{c_0},\omega_0\right)=c\left(\frac{y-x}{c_0}+T,\omega_0\right)=c\left(\frac{y-x}{c_0},\omega_0\right)+p.
$$

Hence

$$
W(x,y+p)=W(x,y)
$$

for $x,y\in\mathbb{R}$. Clearly,

$$
u(t,x;U(\cdot,\omega_0,\omega_0)=W(x-c_0t,y). \qquad \square
$$

7. PROOF OF THEOREM C

In this section, we shall prove Theorem C. Throughout this section, we assume that (1.1) satisfies **(H1), (H2), (H3)** and that the media is spatially homogeneous, that is, $\theta_{t,x}\omega=\theta_{t,0}\omega$ for all $t,x\in\mathbb{R}$ and $\omega\in\Omega$. Let $u^{\pm}(\omega)$ and $u^0(\omega)$ be the random equilibrium solutions of (1.1) assumed **in (H3)**. Note that $u^{\pm}(\omega)(x)=u^{\pm}(\omega)(0)$ and $u^0(\omega)(x)=u^0(\omega)(0)$ for $x\in\mathbb{R}$. For simplicity in notation we denote $u^{\pm}(\omega)$ and $u^0(\omega)$ as $u^{\pm}(\omega)(0)$ and $u^0(\omega)(0)$, respectively. We also assume that $\zeta(\cdot),\eta(\cdot)$, and $H(\cdot)$ are smooth functions with the following properties:

$$
\begin{aligned}
\zeta(s) &= \tfrac{1}{2}\left(1+\tanh\tfrac{s}{2}\right), \quad s\in\mathbb{R},\\
\eta(s) &= \begin{cases} 0 & \text{if } s<0,\\ 1 & \text{if } s\geqslant 4,\end{cases}\\
\eta'(s) &\geqslant 0 \quad\text{and}\quad |\eta''(s)|\leqslant 2 \quad \text{for } s\in\mathbb{R},\\
H(s) &= \begin{cases} 1 & \text{for } s\geqslant 0,\\ 0 & \text{for } s<0.\end{cases}
\end{aligned}
$$

Note that

$$
\zeta'=\zeta(s)(1-\zeta(s)),\qquad \zeta''(s)=\zeta(s)(1-\zeta(s))(1-2\zeta(s)). \tag{7.1}
$$

Let

$$u_0^{\omega}(x) = u^-(\omega)(1-\zeta(x)) + u^+(\omega)\zeta(x). \tag{7.2}$$

We first prove the following lemmas.

Lemma 7.1. *There is $\tau > 0$ and $\chi : \Omega \to \mathbb{R}$ bounded such that the following hold.*

(1) *Let $v^1(t, x; \omega), v^2(t, x; \omega)$ be the solutions of*

$$\partial_t v = \partial_x^2 v + \partial_u F(\theta_{t,x}\omega, u^0(\theta_{t,0}\omega))v(t, x) \tag{7.3}$$

with $v^1(t, 0; \omega) = H(x), v^2(0, x; \omega) = -1 + 2H(x)$. Then

$$\begin{aligned} v^1(\tau, x; \omega) &\geqslant 3 \quad \text{for } x \geqslant \chi(\omega), \\ v^2(\tau, x; \omega) &\leqslant -3 \quad \text{for } x < \chi(\omega). \end{aligned}$$

(2) *Let $u_\delta^1(t, x; \omega), u_\delta^2(t, x; \omega)$ be solutions of (1.1) with*

$$u_\delta^1(0, x; \omega) = u^0(\omega) + \delta H(x)$$

and

$$u^2(0, x; \omega) = u^0(\omega) + \delta(-1 + 2H(x)).$$

There is $\delta_1 > 0$ such that for $0 < \delta \leqslant \delta_1$,

$$\begin{aligned} u_\delta^1(\tau, x, \omega) &\geqslant u^0(\theta_{\tau,0}\omega) + 2\delta \quad \text{for } x \geqslant \chi(\omega), \\ u_\delta^2(\tau, x; \omega) &\leqslant u^0(\theta_{\tau,0}\omega) - 2\delta \quad \text{for } x \leqslant \chi(\omega). \end{aligned}$$

(3) *Let $u_\delta^3(t, x; \omega), u_\delta^4(t, x; \omega)$ be solutions of (1.1) with*

$$u_\delta^3(0, x; \omega) = u^0(\omega) + \delta H(x) - (u^0(\omega) - u^-(\omega))H(-h - x)$$

and

$$\begin{aligned} u_\delta^4(0, x; \omega) = {} & u^0(\omega) + \delta(-1 + 2H(x)) \\ & + (u^+(\omega) - u^0(\omega) - \delta)H(x - h). \end{aligned}$$

Let δ_1 be as in (2). Then for each $0 < \delta \leqslant \delta_1$, there is $h_1(\delta)$ such that for all $h \geqslant h_1(\delta)$.

$$\begin{aligned} u_\delta^3(\tau, x; \omega) &\geqslant u^0(\theta_{\tau,0}\omega) + \delta \quad \text{for } x \geqslant \chi(\omega), \\ u_\delta^4(\tau, x, \omega, x_0) &\leqslant u^0(\theta_{\tau,0}\omega) - \delta \quad \text{for } x \leqslant \chi(\omega). \end{aligned}$$

Proof. (1) First of all, denote $v(t,x;v_0,\omega)$ as the solution of (7.3) with $v(0,x;,v_0,\omega)=v_0(x)$. Clearly, $v(t,x;0,\omega)=0, 0\leqslant v^1(t,x;\omega)\leqslant v(t,x;1,\omega)$, and

$$v^2(t,x;\omega)=-v(t,x;1,\omega)+2v^1(t,x;\omega).$$

By **(H3)**,

$$g_-(t)\leqslant \partial_u F(\theta_{t},_x\,\omega, u^0(\theta_{t,0}\omega))\leqslant g_+(t)$$

for $t,x\in\mathbb{R}$ and $\omega\in\Omega$, and

$$\int_0^t g_-(s)\mathrm{d}s\to\infty$$

as $t\to\infty$. Without loss of generality, we may assume that

$$\inf_{t\in\mathbb{R}} g_+(t)>0. \tag{7.4}$$

Let

$$\gamma_\pm(t)=\int_0^t g_\pm(s)\mathrm{d}s.$$

We claim that

$$v^1(t,-\infty;\omega)=0,\quad v^1(t,\infty;\omega)\geqslant e^{\gamma_-(t)}.$$

In fact, let $\epsilon>0$ and

$$w^+(t,x)=\rho(\epsilon)e^{2\gamma+(t)}+\eta(\epsilon x)e^{\gamma_+(t)}.$$

Then

$$\begin{aligned}
&\partial_t w^+-\partial_x^2 w^+-\partial_u F(\theta_{t,x}\omega,u^0(\theta_{t,0}\omega))w^+\\
&\geqslant \partial_t w^+-\partial_x^2 w^+-g_+(t)w^+\\
&=(2\rho(\epsilon)e^{2\gamma_+(t)}+\eta(\epsilon x)e^{\gamma_+(t)})g_+(t)-\epsilon^2\eta''(\epsilon x)e^{\gamma_+(t)}\\
&\quad -g_+(t)\rho(\epsilon)e^{2\gamma_+(t)}-\eta(\epsilon x)g_+(t)e^{\gamma_+(t)}\\
&=\rho(\epsilon)e^{2\gamma_+(t)}g_+(t)-\epsilon^2\eta''(\epsilon x)e^{\gamma_+(t)}.
\end{aligned}$$

Let

$$\rho(\epsilon)=\sup_{t,x\in\mathbb{R}}\frac{|\epsilon^2\eta''(\epsilon x)|}{g_+(t)}.$$

Then $w^+(t, x+4/\epsilon)$ is a supersolution of (7.3). Note that $v^1(0, x; \omega) \leqslant w^+(0, x+4/\epsilon)$ for $x \in \mathbb{R}$ and $\rho(\epsilon) \to 0$ as $\epsilon \to 0$. By comparison principle for parabolic equations, we have

$$v^1(t, x; \omega) \leqslant w^+\left(t, x, +\frac{4}{\epsilon}\right) \quad \text{for} \quad x \in \mathbb{R}. \tag{7.5}$$

This implies that

$$\lim \sup_{x \to -\infty} v^1(t, x; \omega) \leqslant \rho(\epsilon) e^{2\gamma_+(t)}$$

for all $\epsilon > 0$. Let $\epsilon \to 0$, we have

$$v^1(t, -\infty; \omega) = 0.$$

Note that $-w^+(t, -x+4/\epsilon)$ is a subsolution of (7.3) and $e^{\gamma_-(t)}$ is also a subsolution of (7.3). Let

$$w^-(t, x) = e^{\gamma_-(t)} - w^+(t, -x).$$

Then $w^-(t, x-4/\epsilon)$ is s subsolution of (7.3). Clearly $v^1(0, x; \omega) \geqslant w^-(0, x-4/\epsilon)$ for $x \in \mathbb{R}$. Hence by comparison principle for parabolic equations again,

$$v^1(t, x; \omega) \geqslant w^-\left(t, x - \frac{4}{\epsilon}\right) \quad \text{for} \quad x \in \mathbb{R}. \tag{7.6}$$

This implies that

$$\lim \inf_{x \to \infty} v^1(t, x; w) \geqslant e^{\gamma_-(t)} - \rho(\epsilon) e^{2\gamma_+(t)}$$

for all $\epsilon > 0$. Let $\epsilon \to 0$, we have

$$\lim \inf_{x \to \infty} v^1(t, x; \omega) \geqslant e^{\gamma_-(t)}$$

for $t > 0$.

Next let τ be such that $e^{\gamma_-(\tau)} = 9$. Then $v(\tau, x; 1, \omega) \geqslant 9$ for any $\omega \in \Omega$. By (7.5) and (7.6), it is not difficult to see that there are $\chi_\pm$ such that $v^1(\tau, x; \omega) < 2$ for $x \leqslant \chi_-$ and $\omega \in \Omega$, and $v^1(\tau, x; \omega) > 5$ for $x \geqslant \chi_+$ and $\omega \in \Omega$. Hence there is $\chi(\omega) \in [\chi_-, \chi_+]$ such that

$$v^1(\tau, x; \omega) \geqslant 3 \quad \text{for} \quad x \geqslant \chi(\omega),$$

and

$$v^2(\tau, x; \omega) \leqslant -3 \quad \text{for} \quad x \leqslant \chi(\omega).$$

(2) Let

$$w^1(t,x) = u^0(\theta_{t,0}\,\omega) + \delta v^1(t,x;\omega) - \delta^2 e^{K_1 t},$$

where v^1 is as in (1), K_1 and δ will be determined later. Then

$$\begin{aligned}
&\partial_t w^1 - \partial_x^2 w^1 - F(\theta_{t,x}\omega, w^1(t,x))\\
&= F(\theta_{t,x}\omega, u^0(\theta_{t,0}\omega)) - F(\theta_{t,x}\omega, u^0(\theta_{t,0}\omega) + \delta v^1(t,x;\omega) - \delta^2 e^{K_1 t}))\\
&\quad + \delta\partial_u F(\theta_{t,x}\omega, u^0(\theta_{t,0}\omega))v^1(t,x;\omega) - \delta^2 K_1 e^{K_1 t}\\
&= \delta^2(\partial_u F(\theta_{t,x}\omega, u^0(\theta_{t,0}\omega))e^{K_1 t} - K_1 e^{K_1 t}\\
&\quad - \tfrac{1}{2}\partial_u^2 F(\theta_{t,x}\omega, u^*(t,x;\omega))(v^1(t,x;\omega) - \delta e^{K_1 t})^2),
\end{aligned}$$

where $u^*(t,x;\omega)$ lies between $u^0(\theta_{t,0}\omega)$ and $w^1(t,x)$. Then for K_1 large enough, δ_1 small enough such that $\delta_1 e^{K_1\tau} \leqslant 1$,

$$\partial_t w^1 - \partial_x^2 w^1 - F(\theta_{t,x}\,\omega, w^1) \leqslant 0$$

for $0 \leqslant t \leqslant \tau$ and $0 < \delta \leqslant \delta_1$. Note that $u_\delta^1(0,x;\omega) \geqslant w^1(0,x)$ for $x \in \mathbb{R}$. Hence, $u_\delta^1(t,x;\omega) \geqslant w^1(t,x)$ for $x \in \mathbb{R}$. For $0 < \delta \leqslant \delta_1$, we have

$$u_\delta^1(\tau,x;\omega) \geqslant u^0(\theta_{\tau,0}\,\omega) + 2\delta \quad \text{for} \quad x \geqslant \chi(\omega).$$

Similarly, we can prove that

$$u_\delta^2(\tau,x;\omega) \leqslant u^0(\theta_{\tau,0}\omega) - 2\delta \quad \text{for} \quad x \leqslant \chi(\omega).$$

(3) Let $0 < \delta \leqslant \delta_1$ and $u_\delta^1(\tau,x;\omega)$ be as in (2). For given positive constants ϵ, K_2, C, and $M \geqslant \sup_{\omega\in\Omega}(u^+(\omega) - u^-(\omega))$, define

$$\begin{aligned}
w^2(t,x;\omega) &= u_\delta^1(t,x;\omega) - \rho(\epsilon)e^{2K_2 t} - M\eta\,(-\epsilon(x - \chi(\omega) - C(t-\tau)))\\
&\qquad \times (1 - \eta(\epsilon(x - \chi(\omega) - C(t-\tau)))) .
\end{aligned}$$

Then

$$\begin{aligned}
&\partial_t w^2 - \partial_x^2 w^2 - F(\theta_{t,x}\omega, w^2(t,x;\omega))\\
&= F(\theta_{t,x}\omega, u_\delta^1(t,x;\omega)) - F(\theta_{t,x}\omega, w^2(t,x;\omega))\\
&\quad -2K_2\rho(\epsilon)e^{2K_2 t}\\
&\quad -\epsilon CM\big[\eta'(-y)(1-\eta(y)) + \eta(-y)\eta'(y)\big]_{y=\epsilon(x-\chi(\omega)-C(t-\tau))}\\
&\quad -\epsilon^2 M\big[\eta''(-y)(1-\eta(y)) + 2\eta'(-y)\eta'(y) - \eta(-y)\eta''(y)\big]_{y=\epsilon(x-\chi(\omega)-C(t-\tau))} .
\end{aligned}$$

Let

$$\rho(\epsilon) = \epsilon^2 \sup_{y\in\mathbb{R}} \left|\eta''(-y)(1-\eta(y)) + 2\eta'(-y)\eta'(y) - \eta(-y)\eta''(y)\right|$$

and

$$K_2 = (1+M)\sup\{1 + |\partial_u F(\theta_{t,x}\,\omega, u)| : \hat{u}^1_\delta(t,x;\omega) \leqslant u \leqslant u^1_\delta(t,x;\omega)\},$$

where $\hat{u}^1_\delta(t,x;\omega) = u^1_\delta(t,x;\omega) - 1 - M$. Then

$$-K_2\rho(\epsilon)e^{2K_2 t} - \epsilon^2 M\left[\eta''(-y)(1-\eta(y)) + 2\eta'(-y)\eta'(y) - \eta(-y)\eta''(y)\right] \leqslant 0 \quad (7.7)$$

for all $0 \leqslant t \leqslant \tau$ and $x \in \mathbb{R}$, where $y = \epsilon(x - \chi(\omega) - C(t-\tau))$
and

$$F(\theta_{t,x}\,\omega, u^1_\delta(t,x;\omega)) - F(\theta_{t,x}\omega, w^2(t,x;\omega)) - K_2\rho(\epsilon)e^{2K_2 t} \leqslant 0 \quad (7.8)$$

for all (t,x) with $0 \leqslant t \leqslant \tau$ and $\eta(-\epsilon(x - \chi(\omega) - C(t-\tau))$ or $1 - \eta(-\epsilon(x - \chi(\omega) - C(t-\tau)) \leqslant \rho(\epsilon)$. Now let

$$\gamma = \min\{\eta'(-y)(1-\eta(y)) + \eta(-y)\eta'(y) | \rho(\epsilon) \leqslant \eta(-y) \leqslant 1 - \rho(\epsilon)\}$$

and

$$C = \frac{K_2}{\gamma\epsilon(1+M)}.$$

Let $\epsilon = \epsilon(\delta)$ be such that

$$\rho(\epsilon(\delta))e^{2K_2\tau} \leqslant \delta.$$

Then

$$\begin{aligned} &F(\theta_{t,x}\omega, u^1_\delta(t,x;\omega)) - F(\theta_{t,x}\omega, w^2(t,x,;\omega)) \\ &-\rho(\epsilon)K_2 e^{2K_2 t} - \epsilon C M[\eta'(-y)(1-\eta(y)) + \eta(-y)\eta'(y)] \leqslant 0, \end{aligned} \quad (7.9)$$

where $y = \epsilon(x - \chi(\omega) - C(t-\tau))$ and (t,x) is such that $0 \leqslant t \leqslant \tau, \rho(\epsilon) \leqslant \eta(-y) \leqslant 1 - \rho(\epsilon)$. By (7.7)–(7.9),

$$\partial_t w^2 - \partial_x^2 w^2 - F(\theta_{t,x}\omega, w^2(t,x;\omega)) \leqslant 0$$

for $0 \leqslant t \leqslant \tau$ and $x \in \mathbb{R}$. Let

$$h_1(\delta) = \frac{4}{\epsilon(\delta)} + \sup_{\omega\in\Omega}|\chi(\omega)| + C\tau.$$

Then

$$u^3_\delta(0,x;\omega) \geqslant w^2(0,x;\omega)$$

for $x \in \mathbb{R}$. By comparison principle for parabolic equations, we have

$$u^3_\delta(t,x;\omega) \geqslant w^2(t,x;\omega)$$

for $0 \leqslant t \leqslant \tau$ and $x \in \mathbb{R}$. In particular,

$$\begin{aligned} u_\delta^3(\tau, x; \omega) &\geqslant w^3(\tau, x; \omega) = u_\delta^1(\tau, x; \omega) - \rho(\epsilon)e^{2K_2\tau} \\ &\geqslant u^0(\theta_{\tau,0}\omega) + 2\delta - \delta = u^0(\theta_{\tau,0}\omega) + \delta \end{aligned}$$

for $x \geqslant \chi(\omega)$.

Similarly, we can prove that

$$u_\delta^4(\tau, x; \omega) \leqslant u^0(\theta_{\tau,0}\omega) - \delta$$

for $x \leqslant \chi(\omega)$. □

Lemma 7.2. *Let τ be as in Lemma 7.1. There is $\tilde{\chi}\colon \Omega \to \mathbb{R}$ bounded such that the following hold.*

(1) *Let $\tilde{v}^1(t, x; \omega), \tilde{v}^2(t, x; \omega)$ be the solutions of* (7.3) *with* $\tilde{v}^1(t, 0; \omega) = -1 + H(x), \tilde{v}^2(0, x; \omega) = -1 + 2H(x)$. *Then*

$$\begin{aligned} &\tilde{v}^1(\tau, x; \omega) \leqslant -3 \quad \text{for } x \leqslant \tilde{\chi}(\omega), \\ &\tilde{v}^2(\tau, x; \omega) \geqslant 3 \qquad \text{for } x \geqslant \tilde{\chi}(\omega). \end{aligned}$$

(2) *Let $\tilde{u}_\delta^1(t, x; \omega), \tilde{u}_\delta^2(t, x; \omega)$ be solutions of* (1.1) *with*

$$\tilde{u}_\delta^1(0, x; \omega) = u^0(\omega) + \delta(-1 + H(x))$$

and

$$\tilde{u}^2(0, x; \omega) = u^0(\omega) + \delta(-1 + 2H(x)).$$

There is $\delta_2 > 0$ such that for each $0 < \delta \leqslant \delta_2$,

$$\begin{aligned} &\tilde{u}_\delta^1(\tau, x; \omega) \leqslant u^0(\theta_{\tau,0}\omega) - 2\delta \quad \text{for } x \leqslant \tilde{\chi}(\omega), \\ &\tilde{u}_\delta^2(\tau, x; \omega) \geqslant u^0(\theta_{\tau,0}\omega) + 2\delta \quad \text{for } x \geqslant \tilde{\chi}(\omega). \end{aligned}$$

(3) *Let $\tilde{u}_\delta^3(t, x; \omega), \tilde{u}_\delta^4(t, x; \omega)$ be the solutions of (1.1) with*

$$\begin{aligned} &\tilde{u}_\delta^3(0, x; \omega) = u^0(\omega) + \delta(-1 + H(x)) + (u^+(\omega) - u^0(\omega))H(x - h), \\ &\tilde{u}_\delta^4(0, x; \omega) = u^0(\omega) + \delta(-1 + 2H(x)) - (u^0(\omega) - u^-(\omega) - \delta)H(-x - h). \end{aligned}$$

Let δ_2 be as in (2). Then for each $0 < \delta \leqslant \delta_2$, there is $h_2(\delta)$ such that for all $h \geqslant h_2(\delta)$,

$$\begin{aligned} &\tilde{u}_\delta^3(\tau, x; \omega) \leqslant u^0(\theta_{\tau,0}\omega) - \delta \quad \text{for } x \leqslant \tilde{\chi}(\omega), \\ &\tilde{u}_\delta^4(\tau, x; \omega) \geqslant u^0(\theta_{\tau,0}\omega) + \delta \quad \text{for } x \geqslant \tilde{\chi}(\omega). \end{aligned}$$

Proof. It can be proved by similar arguments as in Lemma 7.1. □

Lemma 7.3. *For each* $0<\delta\leqslant\delta^*(\delta^*=\min(\delta_1,\delta_2))$*, there is* $\epsilon^*(\delta)$ *such that*

$$0<\xi_+(t,\omega,\delta)-\xi_-(t,\omega,\delta)\leqslant\xi_+(0,\omega,\delta)-\xi_-(0,\omega,\delta)+\epsilon^*(\delta)$$

for all $t\geqslant 0$ *and* $\omega\in\Omega$*, where* $\xi_\pm(t,\omega,\delta)$ *are such that*

$$u(t,\xi_\pm(t,\omega,\delta);u_0^\omega,\omega)=u^0(\theta_{t,0}\omega)\pm\delta.$$

Proof. By Lemma 4.2, $\xi_\pm(t,\omega,\delta)$ are well defined. Let $\xi_0(t,\omega)$ be such that

$$u(t,\xi_0(t,\omega);u_0^\omega,\omega)=u^0(\theta_{t,0}\omega).$$

Note that $\xi_0(t,\omega)$ is also well defined.

First we prove that for any $t_0\geqslant 0$,

$$\xi_+(t_0+\tau,\omega,\delta)-\xi_-(t_0+\tau,\omega,\delta)\leqslant\max\{\xi_+(t_0,\omega,\delta)-\xi_-(t_0,\omega,\delta),2h^*(\delta)\}.\quad(7.10)$$

We prove (7.10) for the case $t_0=0$. The case $t_0\neq 0$ can be proved similarly.

First of all, note that one of the following must hold,

$$\xi_+(0,\omega,\delta)-\xi_0(0,\omega)\geqslant h^*(\delta),\quad(7.11)$$

$$\xi_0(0,\omega)-\xi_-(0,\omega,\delta)\geqslant h^*(\delta),\quad(7.12)$$

and

$$h^*(\delta)\geqslant\max\{\xi_+(0,\omega,\delta)-\xi_0(0,\omega),\xi_0(0,\omega)-\xi_-(0,\omega,\delta)\}.\quad(7.13)$$

Next, suppose that (7.11) holds. Then

$$\begin{aligned}u_0^\omega(x+\xi_+(0,\omega,\delta)-\xi_0(0,\omega))&=u(0,x+\xi_+(0,\omega,\delta)-\xi_0(0,\omega);u_0^\omega,\omega)\\&\geqslant u_\delta^3(0,x-\xi_0(0,\omega);\omega)\end{aligned}$$

and

$$\begin{aligned}u_0^\omega(x+\xi_-(0,\omega,\delta)-\xi_0(0,\omega))&=u(0,x+\xi_-(0,\omega,\delta)-\xi_0(0,\omega),u_0^\omega,\omega)\\&\leqslant u_\delta^4(0,x-\xi_0(0,\omega);\omega)\end{aligned}$$

for $x\in\mathbb{R}$. By comaprison principle for parabolic equations,

$$u(\tau,x+\xi_+(0,\omega,\delta)-\xi_0(0,\omega);u_0^\omega,\omega)\geqslant u_\delta^3(\tau,x-\xi_0(0,\omega);\omega)$$

and

$$u(\tau,x+\xi_-(0,\omega,\delta)-\xi_0(0,\omega);u_0^\omega,\omega)\leqslant u_\delta^4(\tau,x-\xi_0(0,\omega);\omega)$$

for $x\in\mathbb{R}$, and then by Lemma 7.1,

$$\xi_+(\tau,\omega,\delta)\leqslant\chi(\omega)+\xi_+(0,\omega,\delta)$$

and

$$\xi_-(\tau,\omega,\delta) \geqslant \chi(\omega) + \xi_-(0,\omega,\delta).$$

Hence

$$\xi_+(\tau,\omega,\delta) - \xi_-(\tau,\omega,\delta) \leqslant \xi_+(0,\omega,\delta) - \xi_-(0,\omega,\delta) \leqslant \xi_+(0,\omega,\delta) - \xi_-(0,\omega,\delta).$$

Similarly, if (7.12) holds, then by Lemma 7.2 the above inequality holds. If (7.13) holds, then

$$\begin{aligned} u_0^\omega(x+h^*(\delta)) &= u(0,x+h^*(\delta);u_0^\omega,\omega) \\ &\geqslant u_\delta^3(0,x-\xi_0(0,\omega);\omega) \end{aligned}$$

and

$$\begin{aligned} u_0^\omega(x-h^*(\delta)) &= u(0,x-h^*(\delta);u_0^\omega,\omega) \\ &\leqslant u_\delta^4(0,x-\xi_0(0,\omega);\omega) \end{aligned}$$

for $x \in \mathbb{R}$. By comparison principal for parabolic equations again,

$$u(\tau,x+h^*(\delta);u_0^\omega,\omega) \geqslant u_\delta^3(\tau,x-\xi_0(0,\omega);\omega)$$

and

$$u(\tau,x-h^*(\delta);u_0^\omega,\omega) \leqslant u_\delta^4(\tau,x-\xi_0(0,\omega);\omega)$$

for $x \in \mathbb{R}$. It then follows from Lemma 7.1 that

$$\xi_+(\tau,\omega,\delta) \leqslant \chi(\omega) + h^*(\delta) + \xi_0(0,\omega)$$

and

$$\xi_-(\tau,\omega,\delta) \geqslant \chi(\omega) - h^*(\delta) + \xi_0(0,\omega).$$

Hence

$$\xi_+(\tau,\omega,\delta) - \xi_-(\tau,\omega,\delta) \leqslant 2h^*(\delta).$$

Therefore, (7.10) holds.

Now, by (7.10), for each $0<\delta\leqslant\delta^*$, there is $\epsilon_1(\delta)$ such that

$$\xi_+(t+t_0,\omega,\delta) - \xi_-(t+t_0,\omega,\delta) \leqslant \xi_+(t_0,\omega,\delta) - \xi_-(t_0,\omega,\delta) + \epsilon_1(\delta)$$

for $t_0 \geqslant 0, t\in[0,\tau], \omega\in\Omega$. Let $\epsilon^*(\delta)=\epsilon_1(\delta)+2h^*(\delta)$. We have that for all $t\geqslant 0$,

$$\xi_+(t,\omega,\delta) - \xi_-(t,\omega,\delta) \leqslant \xi_+(0,\omega,\delta) - \xi_-(0,\omega,\delta) + \epsilon^*(\delta). \qquad \square$$

Lemma 7.4. *For each $M>0$, there is $C>0$ such that for each pair $\alpha_+,\alpha_-\in X$ with $-M\leqslant\alpha_-(x)<\alpha_+(x)\leqslant M$ for $x\in\mathbb{R}$, each $c\geqslant C$, and each $\omega\in\Omega$, the following hold.*

(1) *Let*

$$v^+(t,x)=u(t,x;\alpha_+,\omega)\zeta(x+ct)+u(t,x;\alpha_-,\omega)(1-\zeta(x+ct)),$$
$$v^-(t,x)=u(t,x;\alpha_-,\omega)\zeta(x+ct)+u(t,x;\alpha_+,\omega)(1-\zeta(x+ct)).$$

Then v^+ *and* v^- *are super- and sub-solutions of* (1.1)*, respectively.*

(2) *Let*

$$w^+(t,x)=u(t,x;\alpha_-,\omega)\zeta(x-ct)+u(t,x;\alpha_+,\omega)(1-\zeta(x-ct)),$$
$$w^-(t,x)=u(t,x;\alpha_+,\omega)\zeta(x-ct)+u(t,x;\alpha_-,\omega)(1-\zeta(x-ct)).$$

Then w^+ *and* w^- *are also super- and sub-solutions of* (1.1)*, respectively.*

Proof. We prove that $v^+(t,x)$ is a super-solution. Other statements can be proved similarly, Denote $u^\pm(t,x)$ as $u^\pm(t,x;\alpha_\pm,\omega)$.

First, a direct computation yields

$$\begin{aligned}
&\partial_t v^+(t,x)-\partial_x^2 v^+(t,x)-F(\theta_{t,x}\omega,v^+(t,x))\\
&\quad=F(\theta_{t,x}\omega,u^+(t,x))\xi(x+ct)+F(\theta_{t,x}\omega,u^-(t,x))(1-\xi(x+ct))\\
&\qquad-F(\theta_{t,x}\omega,u^+(t,x)\xi(x+ct)+u^-(t,x)(1-\xi(x+ct))\\
&\qquad+c\xi'(x+ct)(u^+(t,x)-u^-(t,x))-\xi''(x+ct)(u^+(t,x)-u^-(t,x))\\
&\qquad-2\partial_x u^+(t,x)\xi'(x+ct)+2\partial_x u^-(t,x)\xi'(x+ct).
\end{aligned}$$

Note that

$$\begin{aligned}
&F(\theta_{t,x}\omega,u^+(t,x))\xi(x+ct)+F(\theta_{t,x}\omega,u^-(t,x))(1-\xi(x+ct))\\
&\qquad-F(\theta_{t,x}\omega,u^+(t,x)\xi(x+ct)+u^-(t,x)(1-\xi(x+ct))\\
&\quad=\partial_u^2F(\theta_{t,x}\omega,u^{**}(t,x))\cdot(u^*(t,x)-u^-(t,x))\\
&\qquad\cdot(u^+(t,x)-u^-(t,x))\zeta(x+ct)(1-\zeta(x+ct))
\end{aligned}$$

for some $u^{**}(t,x),u^*(t,x)$ between $u^-(t,x)$ and $u^+(t,x)$. It then follows from (7.1) that

$$\begin{aligned}
&\partial_t v^+(t,x)-\partial_x^2 v^+(t,x)-F(\theta_{t,x}\omega),v^+(t,x))\\
&\quad=\xi(x+ct)(1-\xi(x+ct))(u^+(t,x)-u^-(t,x))\\
&\qquad\times\Bigg(c-(1-2\xi(x+ct))-\partial_u^2F(\theta_{t,x}\omega,u^{**}(t,x))(u^*(t,x)-u^-(t,x))\\
&\qquad-\frac{2\partial_x u^+(t,x)-2\partial_x u^-(t,x)}{u^+(t,x)-u^-(t,x)}\Bigg).
\end{aligned}$$

It is then not difficult to see that for each given $M>0$, there is $C>0$ such that when $c\geqslant C$, v^+ is a super-solution. □

Lemma 7.5. *Let* δ^* *and* $\xi_\pm(t,\omega,\delta)$ *be as in Lemma 7.3. Let* $\tilde{\xi}_\pm(t,\omega,\delta)$ *be such that*

$$u(t,\tilde{\xi}_\pm(t,\omega,\delta);u_0^\omega,\omega)=u^\pm(\theta_{t,0}\omega))\mp\delta.$$

Then for each $0<\delta\leqslant\delta^*$, *there is* $\tilde\epsilon^*(\delta)$ *such that*

$$\tilde\xi_+(t,\omega,\delta)-\tilde\xi_-(t,\omega,\delta)\leqslant\xi_+(0,\omega,\delta)-\xi_-(0,\omega,\delta)+\tilde\epsilon^*(\delta)$$

for all $t>0$ *and* $\omega\in\Omega$.

Proof. First, we claim that there is $c>0,\epsilon_1^*(\delta)>0,\tilde T(\delta)>0$ $(0<\delta\leqslant\delta^*)$ such that for each $\omega\in\Omega$,

$$\tilde\xi_+(t+t_0,\omega,\delta)-\tilde\xi_-(t+t_0,\omega,\delta)\leqslant\xi_+(t_0,\omega,\delta)-\xi_-(t_0,\omega,\delta)+\tilde\epsilon_1^*(\delta)+2ct \quad (7.14)$$

for all $t\geqslant\tilde T(\delta), t_0\geqslant 0$. We prove the case that $t_0=0$. The case $t_0\neq 0$ can be proved similarly.

For each $\omega\in\Omega, 0<\delta\leqslant\delta^*$, define

$$w^+(t,x;\omega)=u(t,x;u^0(\omega)-\delta,\omega)(1-\zeta(x+ct))+u(t,x;u^+(\omega)+\delta,\omega)\zeta(x+ct)$$

and

$$w^-(t,x;\omega)=u(t,x;u^-(\omega)-\delta,\omega)(1-\zeta(x-ct))+u(t,x;u^0(\omega)+\delta,\omega)\zeta(x-ct).$$

Clearly, there is $\chi^*(\delta)$ such that

$$\begin{aligned} &w^+(0,x;\omega)\geqslant u^+(\omega) \quad &&\text{for } x\geqslant\chi^*(\delta),\\ &w^-(0,x,\omega)\leqslant u^-(\omega) &&\text{for } x\leqslant-\chi^*(\delta). \end{aligned}$$

Hence

$$w^-(0,x-\xi_+(0,\omega,\delta)-\chi^*(\delta);\omega)\leqslant u_0^\omega(x)\leqslant w^+(x-\xi_-(0,\omega,\delta)+\chi^*(\delta);\omega)$$

for $x\in\mathbb{R}$. This together with Lemma 7.4 and comparison principle for parabolic equations implies that when $c\gg 1$,

$$w^-(t,x-\xi_+(0,\omega,\delta)-\chi^*(\delta);\omega)\leqslant u(t,x;u_0^\omega,\omega)\leqslant w^+(t,x-\xi_-(0,\omega,\delta)+\chi^*(\delta);\omega)$$

for $t>0$ and $x\in\mathbb{R}$. By the stability of $u^\pm(\omega)$, there is $\tilde T(\delta)>0, \tilde\chi^*(\delta)>0$ such that for $t\geqslant\tilde T(\delta)$,

$$w^+(t,x-\xi_-(0,\omega,\delta)-\chi^*(\delta);\omega)\leqslant u^-(\theta_{t,0}\omega)+\delta$$

for $x\leqslant-\tilde\chi^*(\delta)-ct+\xi_-(0,\omega,\delta)+\chi^*(\delta)$
and

$$w^-(t,x-\xi_+(0,\omega,\delta)-\chi^*(\delta);\omega)\geqslant u^+(\theta_{t,0}\omega)-\delta$$

for $x\geqslant\tilde\chi^*(\delta)+ct+\xi_+(0,\omega,\delta)+\chi^*(\delta)$.
Hence

$$\tilde\xi_+(t,\omega,\delta)\leqslant\xi_+(0,\omega,\delta)+\chi^*(\delta)+\tilde\chi^*(\delta)+ct$$

and

$$\tilde\xi_-(t,\omega,\delta)\geqslant\xi_-(0,\omega,\delta)-\chi^*(\delta)-\tilde\chi^*(\delta)-ct$$

for $t \geqslant \tilde{T}(\delta)$. This implies (7.14) holds with $\epsilon_1^*(\delta) = 2\chi^*(\delta) + 2\tilde{\chi}^*(\delta)$. Let $\tilde{\epsilon}_1^*(\delta) = \epsilon_1^*(\delta) + \epsilon^*(\delta)$. It then follows from Lemma 7.3 that

$$\tilde{\xi}_+(t,\omega,\delta) - \tilde{\xi}_-(t,\omega,\delta) \leqslant \xi_+(0,\omega,\delta) - \xi_-(0,\omega,\delta) + \tilde{\epsilon}_1^*(\delta) + 2c\tilde{T}(\delta)$$

for all $t \geqslant \tilde{T}(\delta)$. Clearly, there is $\tilde{\epsilon}_2(\delta) > 0$ such that

$$\tilde{\xi}_+(t,\omega,\delta) - \tilde{\xi}_-(t,\omega,\delta) \leqslant \tilde{\epsilon}_2^*(\delta)$$

for $0 \leqslant t \leqslant \tilde{T}(\delta)$. This implies that

$$\tilde{\xi}_+(t,\omega,\delta) - \tilde{\xi}_-(t,\omega,\delta) \leqslant \xi_+(0,\omega,\delta) - \xi_-(0,\omega,\delta) + \tilde{\epsilon}^*(\delta)$$

for all $t \geqslant 0, 0 < \delta \leqslant \delta^*$,
where

$$\tilde{\epsilon}^*(\delta) = \max\{\tilde{\epsilon}_2^*(\delta), \tilde{\epsilon}_1^*(\delta) + 2c\tilde{T}(\delta)\}. \qquad \square$$

Proof of Theorem C. Note that for $0 < \delta \ll 1, \xi_+(0,\omega,\delta), \xi_-(0,\omega,\delta)$ is bounded in $\omega \in \Omega$. Then by Lemma 7.5, $v_0(\omega) = u_0^\omega(\cdot)$ generates a wave-like solution. Therefore (1.1) has a wave-like solution. It then follows from Theorem A that (1.1) has a critical traveling wave solution. Moreover, by the arguments in the proof of Theorem A and comparison principal for parabolic equations, there is $U(\cdot,\omega)$ such that $U(0,\omega) = u^0(\omega)$, $\partial_x U(x,\omega) > 0$, and $U(\cdot,\omega)$ generates a critical traveling wave solution, $u(t,x;U(\cdot,\omega),\omega) = U(x - c(t,\omega), \theta_{t,0}\omega)$. Note that

$$u(t, c(t,\omega); U(\cdot,\omega),\omega) = U(0,\theta_{t,0}\omega) = u^0(\theta_{t,0}\omega)$$

and $\partial_x u(t, c(t,\omega); U(\cdot,\omega),\omega) = \partial_x U(0,\theta_{t,0}\omega) > 0$. It then follows fromt the differentiability of $u^0(\theta_{t,0}\omega)$ and the regularity of $u(t,x;U(\cdot,\omega),\omega)$ that $c(t,\omega)$ is differentiable in t and

$$\partial_t c(t,\omega) = \frac{F(\theta_{t,0}\omega, u^0(\theta_{t,0}\omega)) - \partial_x^2 U(0,\theta_{t,0}\omega) - F(\theta_{t,0}\omega, U(0,\theta_{t,0}\omega))}{\partial_x U(0,\theta_{t,0}\omega)}. \qquad \square$$

ACKNOWLEDGEMENTS

The current paper is partially supported by NSF grant DMS-0103381 and the author of the paper would like to thank the referee for the important suggestions and careful reading of the current paper.

REFERENCES

1. Alikakos, N. D., Bates, P. W., and Chen, X. (1999). Periodic traveling waves and locating oscillating patterns in multidimensional domains. *Trans. Amer. Math. Soc.* **351**, 2777–2805.
2. Allen, S. M., and Cahn, J. W. (1979). A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Met.* **27**, 1085–1095.

3. Angenent, S. B. (1988). The zero set of a solution of a parabolic equation. *J. Reine Angew. Math.* **390**, 79–96.
4. Arnold, L. (1998). *Random Dynamical Systems*, Springer Monographs in Mathematics, Springer, Berlin.
5. Arnold, L., and Chueshov, I. (1998). Order-Preserving random dynamical systems: Equilibria, attractors, applications. *Dyn. Stability Syst.* **13**, 265–280.
6. Aronson, D. G., and Weinberger, H. F. (1975). Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, In *Partial Differential equations and Related Topics*, Goldstein, J. (ed.), Lecture Notes in Math. **446**, Springer, New York, pp. 5–49.
7. Bates, P. W., and Chen, F. (2002). Spectral analysis and multidimensional stability of traveling waves for nonlocal allen-cahn Equation. *J. Math. Anal. Appl.* **273**, 45–57.
8. Bates, P. W., and Chen, F. (2002). Traveling wave solutions for a nonlocal phase-field system. *Interfaces Free Bound.* **4**, 227–238.
9. Bates, P. W., and Chmaj, A. (1999). A discrete convolution model for phase transitions. *Arch. Ration. Mech. Anal.* **150**, 281–305.
10. Bates, P. W., Fife, P. C., Ren, X., and Wang, X. -F. (1997). Traveling waves in a convolution model for phase transitions. *Arch. Rational Mech. Anal.* **138** (2), 105–136.
11. Bell, J. (1981). Some threshold results for models of myelinated nerves. *Math. Biosc.* 181–190.
12. Berestycki, H., Larrouturou, B., and Lions, P. L. (1990). Multidimensional traveling wave solutions of a flame propagation model. *Arch. Rational Mech. Anal.* **111**, 33–49.
13. Berestycki, H., Larrouturou, B., and Roquejoffre, J. M. (1992). Stability of traveling fronts in a model for flame propagation, Part I: Linear stability, *Arch. Rational Mech. Anal.* **117**, 97–117.
14. Berestyski, H., and Nirenberg, L. (1992). Traveling fronts in cylinders. *Ann. Inst. H. Poincaré Anal. Non Lineáire* **9**, 497–572.
15. Cahn, J. W., Mallet-Paret, J., and Van Vleck, E. S. (1999). Traveling wave solutions for systems of ODE's on a two-dimensional spatial lattice. *SIAM J. Appl. Math.* **59**, 455–493.
16. Chen, F. (2002). Almost periodic traveling waves of nonlocal evolution equations. *Nonlinear Anal.* **50**, 807–838.
17. Chen, X. F. (1997). Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations. *Adv. Differential Equations* **2**(1), 125–160.
18. Chen, X. F., and Guo, J. -S. (2002). Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations. *J. Diff. Eqs.* **184**, 549–569.
19. Chow, S. -N., Mallet-Paret, J., and Shen, W. (1998). Traveling waves in lattice dynamical systems. *J. Diff. Eqs.* **149**, 248–291.
20. Chow, S. -N., and Shen, W. (1995). Stability and bifurcation of traveling wave solutions in coupled map lattices. *Dyn. Syst. Appl.* **4**, 1–26.
21. Ellis, R. (1969). Lectures on Topological Dynamics, Benjamin, New York.
22. Faria, T., Huang, W., and Wu, J. Traveling Waves for Delayed Reaction- Diffusion Equations with Non-local Response, preprint.
23. Fife, P. C., and Mcleod, J. B. (1977). The approach of solutions of nonlinear diffusion equations to traveling front solutions. *Arch. Rational, Mech. Anal.* **65**, 335–361.
24. Fife, P. C., and Mcleod, J. B. (1981). A phase plane discussion of convergence to traveling fronts for nonlinear diffusions. *Arch. Rational Mech. Anal.* **75**, 281–315.
25. Fink, A. M. (1974). *Almost Periodic Differential Equations*, Lecture Notes in Mathematics **377**, Springer-Verlag, Berlin/Heidelberg/New York.
26. Fisher, R. A. (1937). The advance of advantageous genes. *Ann. of Eugenics* **7**, 355–369.
27. Friedman, A. (1964). *Partial Differential equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ.

28. Gärtner J., and Freidlin, M. I. (1979). On the propagation of concenttration waves in periodic and random media. *Soviet Math. Dokl.* **20**, 1282–1286.
29. Hankerson, D., and Zinner, B. (1993). Wavefronts for a cooperative tridiagonal system of differential equations. *J. Dynamics Diff. Eq.* **5**(2), 359–373.
30. Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. **840**, Springer-Verlag, Berlin.
31. Huang, J., and Zou, X. (2002). Traveling wavefronts in diffusive and cooperative lotka-volterra system with delays. *J. Math. Anal. Appl.* **271**, 455–466.
32. Huang, W. (2001). Uniqueness of bistable traveling wave for mutualist species. *J. Dynam. Diff. Eq.* **13**, 147–183.
33. Kametaka, Y. (1976). On the nonlinear diffusion equation of kolmogorov-petrovskii-piskunov type. *Osaka J. Math.* **13**, 11–66.
34. Keener, J. P., (1987). Propagation and its failure in coupled systems of discrete excitable cells. *SIAM J. Appl. Math.* **47**(3), 556–572.
35. Kolmogorov, A., Petrovckii, I., and Piskunov, N. (1937). A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem. *Bjul. Moskovskogo Gos. Univ.* **1**, 1–26.
36. Lan, K.Q., and Wu, J. H. (2003). Traveling wavefronts of scalar reaction-diffusion equations with and without delays. *Nonlinear Analysis: Real World Applications* **4**, 173–188.
37. Levermore, C. D., and Xin, J. X. (1992). Multidimensional stability of traveling waves in a bistable reaction-diffusion equation. II. *Comm. Partial Diff. Eq.* **17**(11, 12), 1901–1924.
38. Lui, R. (1989). Biological growth and spread modeled by systems of recursions. I. *Mathematical Theory, Math. Biosci.* **93**, 269–295. II. Biological Theory, *Math. Biosci.* **93**, 297–311.
39. Mallet-Paret, J. (1999). The global structure of traveling waves in spatially discrete dynamical systems. *J. Dynam. Diff. Eq.* **11**, 49–127.
40. Mischaikow, K., and Hutson, V. (1993). Traveling waves for mutualist species. *SIAM J. Math. Anal.* **24**, 987–1008.
41. Mischaikow, K., and Reineck, J. F. (1993). Traveling waves in predator-prey systems. *SIAM J. Math. Anal.* **24**, 1179–1214.
42. O'regan, D. (1998). New fixed-points results for 1-set contractive set valued maps. *Computers Math. Appl.* **25**, 27–34.
43. Roquejoffre, J. M. (1992). Stability of traveling fronts in a model for flame propagation, Part II: Nonlinear stability. *Arch. Rational Mech. Anal.* **117**, 119–153.
44. Sattinger, D. H. (1976). On the stability of waves of nonlinear parabolic systems. *Advances in Math.* **22**, 312–355.
45. Scott, A. C. (1970). *Active and Nonlinear Wave Propagation in Electronics*, Wiley- Interscience, NewYork.
46. Shen, W. (1999). Traveling waves in time almost periodic structures governed by bistable nonlinearities, I. Stability and Uniqueness. *J. Diff. Eq.* **159**, 1–54.
47. Shen, W. (1999). Traveling waves in time almost periodic structures governed by bistable nonlinearities. II. Existence. *J. Diff. Eq.* **159**, 55–101.
48. Shen, W. (2001). Dynamical systems and traveling waves in almost periodic structures. *J. Diff. Eq.* **169**, 493–548.
49. Smith, H. L., and Zhao, X. -Q. (2000). Global asymptotic stability of traveling waves in delayed reaction-diffusion equations. *SIAM J. Math. Anal.* **31**, 514–534.
50. Uchiyama, K. (1978). The behavior of solutions of some nonlinear diffusion equations for large time. *J. Math. Kyoto Univ.* **18–3**, 453–508.
51. Veech, W. A. (1965). Almost automorphic functions on groups. *Amer. J. Math.* **87**, 719–751.

52. Volpert, A. I., Volpert, V. A., and Volpert, V. A. (1994). Traveling wave solutions of parabolic systems, translation of mathematical monographs. **140** Amer. Math. Soc., Providence, Rhode Island.
53. Wehr, J., and Xin, J. (1996). White noise perturbation of the viscous shock fronts of the burgers equations. *Commun. Math. Phys.* **181**, 183–203.
54. Wehr, J., and Xin, J. (1997). Front speed in the burgers equation with a random flux. *J. Statist. Phys.* **88**, 843–871.
55. Weinberger, H. F., (1982). Long-time behavior of a class of biological models. *SIAM J. Math. Anal.* **13**(3), 353–396.
56. Weinberger, H. F. (2002). On spreading speeds and traveling waves for growth and migration models in a periodic habitat. *J. Math. Biol.* **45**, 511–548.
57. Wu, J., and Zou, X. (2001). Traveling wave fronts of reaction-diffusion systems with delay. *J. Dynam. Diff. Eq.* **13**, 651–687.
58. Xin, J. X. (1991). Existence and stability of traveling waves in periodic media governed by a bistable nonlinearity. *J. Dynam. Diff. Eq.* **3**(4), 541–573.
59. Xin, J. X. (1991). Existence and uniqueness of traveling waves in a reaction-diffusion equation with combustion nonlinearity. *Indiana Univ. Math. J.* **40** (3), 985–1008.
60. Xin, J. X. (1992). Multidimensional stability of traveling waves in a bistable reaction-diffusion equations. I. *Comm. Partial Diff. Eq.* **17**(11–12), 1889–1899.
61. Xin, J. (2000). Front propagation in heterogeneous media. *SIAM Review* **42**, 161–230.
62. Yi, Y. (1998). Almost automorphic and almost periodic dynamics in skew-product semiflows, part I. almost automorphy and almost periodicity. *Memoirs of A. M. S.* **647**.
63. Zinner, B. (1992). Existence of traveling wavefront solutions for the discrete nagumo equation. *J. Diff. Eq.* **96**(1), 1–27.
64. Zinner, B. (1991). Stability of traveling wavefronts for the discrete nagumo equation. *SIAM J. Math. Anal.* **22**(4), 1016–1020.
65. Zou, X. (2002). Delay induced traveling wave fronts in reaction diffusion equations of KPP-Fisher type. *J. Comput. Appl. Math.* **146**, 309–321.