Journal of Dynamics and Differential Equations, Vol. 16, No. 2, April 2004 (© 2004)

Global Bifurcation and Structure of Turing Patterns in the 1-D Lengyel–Epstein Model*

Jaeduck Jang[1], Wei-Ming Ni[2,4] and Moxun Tang[3]

Received November 12, 2003

This work continues the mathematical study started in ([13], to appear) on the analytic aspects of the Lengyel–Epstein reaction diffusion system. This system captures the crucial feature of the CIMA reaction in an open unstirred gel reactor which gave the first experimental evidence of Turing pattern in 1990. In the one dimensional case, we make a detailed description for the global bifurcation structure of the set of the non-constant steady states. The limiting behavior of the steady states is further clarified using a shadow system approach.

KEY WORDS: Turing patterns; global bifurcation; CIMA reaction; open chemical systems; Lengye–Epstein model; shadow systems.

1. INTRODUCTION

Understanding the mechanisms by which patterns are created in the living system poses one of the most challenging problems in developmental biology. In 1952, Alan Turing suggested in his celebrated paper "chemical basis for morphogenesis" [21] that chemical reactions, with appropriate nonlinear kinetics coupled to diffusion, could lead to the formation of stationary patterns of the type appeared in living organisms. He also argued that the creation of such patterns, which we now call *Turing patterns*,

---

* Dedicated to Professor Shui-Nee Chow on the occasion of his 60th birthday.

[1] Department of Mathematics, Hankuk University of Foreign Studies, Yongin-si, Kyungki-do 449791, South Korea.

[2] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA. E-mail: ni@math.umn.edu

[3] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA. E-mail: mtang@math.umn.edu

[4] To whom corrrespondence should be addressed.

1040-7294/04/0400-0297/0

could play a major role in biological pattern formation. Turing's mechanism is considered to be a central source for the occurrences of coherent patterns in far-from-equilibrium systems, and has been applied in a variety fields in biology and chemistry [10], such as tissue regeneration in *hydra* [4,11], pattern formation in the Belousov–Zhabotinsky (BZ) reaction, and most recently in electrochemical systems [9].

The first experimental evidence of Turing pattern was observed in 1990, nearly 40 years after Turing's prediction, by the Bordeaux group in France, on the chlorite-iodide-malonic acid-starch (CIMA) reaction in an open unstirred gel reactor [2]. In their scheme, the two sides of the gel strip loaded with starch indicator are, respectively, in contact with solutions of chlorite (ClO_2^-) and iodide (I^-) ions on one side, and malonic acid (MA) on the other side, of which are fed through two continuously flow stirred tank reactors. These reactants diffuse into the gel, encountering each other at significant concentrations in a region near the middle of the gel, where the Turing patterns of lines of periodic spots can be observed. This observation represents a significant breakthrough for one of the most fundamental ideas in morphogenesis and biological pattern formation.

The Brandeis group later found that, after a relatively brief initial period, it is really the simpler chlorine dioxide ClO_2–I_2–MA (CDIMA) reaction that governs the formation of the patterns [7,8]. The CDIMA reaction can be described in a five-variable model consists of three component processes. However, observing that three of the five concentrations remain nearly constants in the reaction, Lengyel and Epstein [7, 8] simplified the model to a 2×2 system: Let $u = u(x,t)$ and $v = v(x,t)$ denote the rescaled chemical concentrations of iodide (I^-) and chlorite (ClO_2^-), respectively, where $x \in \Omega$, a smooth, bounded domain in R^n. Then the Lengyel and Epstein model takes the form

$$\frac{\partial u}{\partial t} = \Delta u + a - u - \frac{4uv}{1+u^2}, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \sigma\left[c\Delta v + b\left(u - \frac{uv}{1+u^2}\right)\right], \tag{1.2}$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator, carrying the spatial dependence of the reaction, a and b are the parameters related to the feed concentrations, c the ratio of the diffusion coefficients, $\sigma > 1$, a rescaling parameter depending on the concentration of the starch, enlarging the effective diffusion ratio to σc. We shall assume accordingly that all constants a, b, c, and σ are positive.

Recently we started the mathematical study on the analytic aspects of this system [13]; for various important experimental and numerical studies see [1, 5] and the references therein. We considered positive solutions to the system subject to the initial condition

$$u(x,0)=u_0(x)>0, \qquad v(x,0)=v_0(x)>0, \quad x\in\Omega, \tag{1.3}$$

where $u_0,\ v_0\in C^2(\Omega)\cap C^0(\bar{\Omega})$, and the Neumann boundary condition

$$\partial u/\partial\nu=\partial u/\partial\nu=0, \quad x\in\partial\Omega,\ t>0, \tag{1.4}$$

where ν is the unit outer normal to $\partial\Omega$. We proved that the problem (1.1)–(1.4) admits a unique solution (u,v), which is defined for all $x\in\Omega$ and $t>0$, and is bounded by some positive constants depending only on a, u_0 and v_0. Furthermore, this unique solution enters the "attracting" region

$$\Re_a=(0,a)\times(0,1+a^2)$$

for all t large, regardless of the initial values u_0 and v_0.

We also studied in [13] the existence and non-existence of steady states of (1.1)–(1.4), as a Turing pattern is defined to be the non-constant steady state at the onset of diffusion-driven instability. Our theorems show that, roughly speaking, if the parameter a (related to the feed concentrations), the size of the reactor Ω (reflected by its first eigenvalue), or the "effective" diffusion rate $d=c/b$, is not large enough, then the system (1.1)–(1.4) has no non-constant steady states. On the other hand, we proved that if a lies in a suitable range, then (1.1)–(1.4) possesses non-constant steady states for large d. The proof of the existence uses a degree-theoretical approach combined with the *a priori* bounds. As we noticed, however, such an approach does not provide much information about the shape of the solution.

It is our purpose in this paper to make a better description for the structure of the set of the non-constant steady states, but focusing on the one dimensional case only. We will prove in Section 3 a global bifurcation theorem which gives the existence of non-constant steady states for all d suitably large under a rather natural condition. In Section 4, we shall describe the solution set for all d sufficiently large by solving the corresponding shadow system.

2. PRELIMINARY

In this section we shall recall several results proved in [13], and introduce the basic assumption on the system parameters, see condition (H) below. Clearly, Turing patterns, or more generally non-constant steady states of the Lengyel–Epstein reaction–diffusion system, are necessarily positive non-constant solutions to the elliptic system

$$\Delta u + a - u - \frac{4uv}{1+u^2} = 0, \tag{2.1}$$

$$\mathrm{d}\Delta v + u - \frac{uv}{1+u^2} = 0 \tag{2.2}$$

($d = c/b$), subject to the homogeneous Neumann boundary condition

$$\partial u/\partial \nu = \partial v/\partial \nu = 0, \quad x \in \partial\Omega. \tag{2.3}$$

If $(u, v) = (u(x), v(x))$ is a positive solution to the boundary value problem (2.1)–(2.3), then it is proved in [13] that

$$\frac{a}{5+4a^2} < u < a \quad \text{and} \quad 1 + \left(\frac{a}{5+4a^2}\right)^2 < v < 1 + a^2, \quad x \in \Omega. \tag{2.4}$$

With the help of this *a priori* estimate, we also proved the following non-existence theorem.

Theorem 1. [13]. *There is a constant $d_0 = d_0(a, \lambda_1) > 0$ such that the problem (2.1)–(2.3) does not admit a nonconstant solution for $0 < d < d_0$.*

If $a^2 \leqslant 75$, then the boundary value problem (2.1)–(2.3) does not admit any non-constant solution if

$$\frac{1}{d} > \frac{8a}{5} - \frac{25}{a}. \tag{2.5}$$

In particular, there is no non-constant solution for all $d > 0$ if $a^2 \leqslant 125/8$.

There is a constant $\Lambda = \Lambda(a) > 0$ such that the problem (2.1)–(2.3) does not admit any non-constant solution if $\lambda_1(\Omega) > \Lambda$.

As in [13], we shall maintain the basic hypothesis

$$\text{(H)} \quad 0 < 3\alpha^2 - 5 < \sigma\alpha b, \quad \alpha = a/5$$

in the rest of this paper. This condition is important because it is sufficient and necessary for

(i) the system (1.1) and (1.2) is an *activator–inhibitor* system,
(ii) the unique constant steady state

$$(u^*, v^*) = (\alpha, 1 + \alpha^2), \quad \alpha = a/5$$

of (1.1) and (1.2) subject to the boundary condition (1.4) is diffusion-free stable.

To see this, we let

$$f(u,v)=a-u-\frac{4uv}{1+u^2},\quad g(u,v)=u-\frac{uv}{1+u^2}. \tag{2.6}$$

Then we call the species u an activator, v an inhibitor, and so (1.1) and (1.2) an *activator-inhibitor* system, if

$$f_u(u^*,v^*)>0,\quad g_v(u^*,v^*)<0.$$

This holds if and only if $3\alpha^2-5>0$, which is ensured by (H). Furthermore, we say that (u^*,v^*) is diffusion-free stable if it is stable as an equilibrium of the system of ordinary differential equations

$$\frac{du}{dt}=f(u,v),\quad \frac{du}{dt}=\sigma bg(u,v).$$

Notice that the Jacobian of this system is

$$J=\begin{bmatrix} f_0 & f_1 \\ \sigma bg_0 & \sigma bg_1 \end{bmatrix},$$

where

$$f_0=\frac{3\alpha^2-5}{1+\alpha^2},\quad f_1=-\frac{4\alpha}{1+\alpha^2},\quad g_0=\frac{2\alpha^2}{1+\alpha^2},\quad g_1=-\frac{\alpha}{1+\alpha^2}. \tag{2.7}$$

Since $\det J>0$, the equilibrium is stable if trace $J=f_0+\sigma bg_1<0$, which is again ensured by (H).

Under condition (H), (u^*,v^*) is *Turing unstable* if it is unstable as a steady state of the system of reaction–diffusion equations (1.1) and (1.2) subject to the homogeneous boundary condition. Let $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ be the sequence of eigenvalues for the elliptic operator $-\Delta$ subject to the Neumann boundary condition on Ω, where each λ_i has multiplicity $m_i\geqslant 1$. If

$$\lambda_1<f_0=(3\alpha^2-5)/(1+\alpha^2), \tag{2.8}$$

then we define $i_\alpha=i_\alpha(\alpha,\Omega)$ to be the largest positive integer such that

$$\lambda_i<f_0\quad\text{for}\quad i\leqslant i_\alpha. \tag{2.9}$$

Clearly, if (2.8) is satisfied, then $1\leqslant i_\alpha<\infty$. In this case, we let

$$\tilde{d}=\tilde{d}(\alpha,\Omega)=\min_{1\leqslant i\leqslant i_\alpha} d_i,\quad d_i=\frac{\alpha}{1+\alpha^2}\frac{\lambda_i+5}{\lambda_i(f_0-\lambda_i)}. \tag{2.10}$$

Then the local stability of (u^*,v^*) can be summarized as follows.

Lemma 2.1. [13]. *Assume* (H) *hold. If* $\lambda_1 \geqslant f_0$*, or* $\lambda_1 < f_0$ *and* $0 < d < \tilde{d}$*, then the constant steady state* (u^*, v^*) *is asymptotically stable. If* $\lambda_1 < f_0$*, and* $d > \tilde{d}$*, then* (u^*, v^*) *is unstable, and hence Turing unstable.*

Naturally, one may expect the existence of non-constant steady states as the constant solution is unstable. Our next result gives a partial result.

Theorem 2. [13]. *Assume* (H) *holds. Suppose* $\lambda_1 < f_0 < \lambda_2$ *and* λ_1 *has an odd multiplicity. Then for any* $d > d_1$ *the problem (2.1)–(2. 3) possesses at least one non-constant positive solution.*

We note that the assumption $f_0 > \lambda_1$ in this theorem is a natural one in view of Lemma 2.1. However, it is for the technical reason that we assumed $f_0 < \lambda_2$. As our result in Section 3 shows, it is not needed at least in the one dimensional case.

3. GLOBAL BIFURCATION

In the one-dimensional interval

$$\Omega = (0, \ell), \quad \ell > 0,$$

a steady state of (1.1) and (1.2) is a positive solution $U = (u(x), v(x))$, $0 < x < \ell$, to the elliptic problem

$$u'' + a - u - \frac{4uv}{1+u^2} = 0, \tag{3.1}$$

$$\mathrm{d}v'' + u - \frac{uv}{1+u^2} = 0, \tag{3.2}$$

subject to the boundary condition

$$u' = v' = 0 \quad \text{at } x = 0, \ell. \tag{3.3}$$

When applying the bifurcation theory to the study of the existence of such solutions, we shall fix constants a and ℓ, and treat d as a bifurcation parameter. The local bifurcation theory will be used to give a precise description for the structure of positive solutions near the bifurcation points. The global bifurcation theory is then used to show that these bifurcation curves can be prolonged as long as d is larger than certain critical values.

The eigenvalue problem

$$-\phi'' = \lambda\phi, \quad \phi' = 0 \quad \text{at } 0, \ell$$

possesses a sequence of simple eigenvalues

$$\lambda_j = (\pi j/\ell)^2, \quad j = 0, 1, 2, \ldots,$$

whose corresponding normalized eigenfunctions are given by

$$\phi_j(x) = \begin{cases} 1/\sqrt{\ell}, & j = 0, \\ \sqrt{2/\ell}\cos(\pi j x/\ell), & j > 0. \end{cases} \tag{3.4}$$

This set of eigenfunctions forms an orthonormal basis in $L^2(0,\ell)$. Let $Y = L^2(0,\ell) \times L^2(0,\ell)$ be the Hilbert space with the inner product

$$(U_1, U_2)_Y = (u_1, u_2)_{L^2(0,\ell)} + (v_1, v_2)_{L^2(0,\ell)}$$

for $U_1 = (u_1, v_1)$, $U_2 = (u_2, v_2) \in Y$, and

$$E = \{(u, v) : u, v \in C^2([0,\ell]), \quad u' = v' = 0 \quad \text{at } x = 0, \ell\}.$$

We regard E as a Banach space with usual C^2 norm. Define the map $F : (0,\infty) \times E \to Y$ by

$$F(d, U) = \begin{pmatrix} u'' + f(u, v) \\ \mathrm{d}v'' + g(u, v) \end{pmatrix}, \quad U = (u, v).$$

Then the solutions of the boundary value problem (3.1)–(3.3) are exactly zeros of this map. With $U^* = (u^*, v^*)$, we have

$$F(d, U^*) = 0 \quad \text{for all } d > 0.$$

If there is a number $\tau > 0$ such that every neighborhood of (τ, U^*) contains zeros of F in $(0,\infty) \times E$ not lying on the curve (d, U^*), $d > 0$, then we say that (τ, U^*) is a bifurcation point of the equation $F = 0$ with respect to this curve. By Theorem 1.7 of [3], (τ, U^*) is a bifurcation point provided that:

(a) the partial derivatives F_d, F_U, and F_{dU} exist and are continuous,
(b) ker $F_U(\tau, U^*)$ and $Y/R(F_U(\tau, U^*))$ are one-dimensional,
(c) let ker $F_U(\tau, U^*) = \mathrm{span}\{\Phi\}$, then $F_{dU}(\tau, U^*)\Phi \notin R(F_U(\tau, U^*))$.

Theorem 3. *Suppose j is a positive integer such that $\lambda_j < f_0$ and $d_j \neq d_k$ for any integer $k \neq j$. Then (d_j, U^*) is a bifurcation point of $F = 0$ with respect to the curve (d, U^*), $d > 0$.*

There is a one-parameter family of non-trivial solutions $\Gamma(s)=(d(s), u(s), v(s))$ *of the problem (3.1)–(3.3) for* $|s|$ *sufficiently small, where* $d(s), u(s), v(s)$ *are continuous functions,* $d(0)=d_j$ *and*

$$u(s)=u^*+s\phi_j+o(s), \quad v(s)=v^*+sb_j\phi_j+o(s), \quad b_j=(\lambda_j-f_0)/f_1>0.$$

The zero set of F *consists of two curves* (d, U^*) *and* $\Gamma(s)$ *in a neighborhood of the bifurcation point* (d_j, U^*).

Proof. It suffices to verify conditions (a)–(c) above. Note that

$$L=F_U(d,U^*)=\begin{pmatrix} \Delta+f_0 & f_1 \\ g_0 & d\Delta+g_1 \end{pmatrix}, \quad \Delta=\frac{\partial^2}{\partial^2 x},$$

where f_0, f_1, g_0 and g_1 are given in (2.7). It is clear that the linear operators F_U, F_{dU} and F_d are continuous. Condition (a) is verified. Suppose $\Phi=(\phi,\psi)\in\ker L$, and write $\phi=\Sigma a_i\phi_i$, $\psi=\Sigma b_i\phi_i$. Then

$$\sum_{i=0}^{\infty} B_i \begin{pmatrix} a_i \\ b_i \end{pmatrix}\phi_i=0, \quad B_i=\begin{pmatrix} f_0-\lambda_i & f_1 \\ g_0 & g_1-d\lambda_i \end{pmatrix}. \tag{3.5}$$

Since

$$\det B_i=0 \;\Leftrightarrow\; d=d_i=\frac{\alpha}{1+\alpha^2}\frac{\lambda_i+5}{\lambda_i(f_0-\lambda_i)},$$

taking $d=d_j$ implies that

$$\ker L=\mathrm{span}\{\Phi\}, \quad \Phi=\begin{pmatrix} 1 \\ b_j \end{pmatrix}\phi_j, \tag{3.6}$$

where

$$b_j=\frac{\lambda_j-f_0}{f_1}=\frac{3\alpha^2-5-(1+\alpha^2)\lambda_j}{4\alpha}>0.$$

Consider the adjoint operator

$$L^*=\begin{pmatrix} \Delta+f_0 & g_0 \\ f_1 & d\Delta+g_1 \end{pmatrix}.$$

In the same way as above we obtain

$$\ker L^*=\mathrm{span}\{\Phi^*\}, \quad \Phi^*=\begin{pmatrix} 1 \\ b_j^* \end{pmatrix}\phi_j, \tag{3.7}$$

where $b_j^* = (\lambda_j - f_0)/g_0 < 0$. Since $R(L) = (\ker L^*)^\perp$, the codimension of $R(L)$ is the same as dim ker $L^* = 1$. Condition (b) is thus verified.
Finally, since

$$F_{dU}(d_j, U^*)\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \Phi = \begin{pmatrix} 0 \\ -\lambda_j b_j \phi_j \end{pmatrix},$$

and

$$(F_{dU}(d_j, U^*)\Phi, \Phi^*)_Y = (-\lambda_j b_j \phi_j, b_j^* \phi_j)_{L^2} = -\lambda_j b_j b_j^* > 0,$$

we find $F_{dU}(d_j, U^*)\Phi \notin R(L)$, and so condition (c) is satisfied. The proof is completed. □

We remark that if $d_j = d_k$ for some $j \neq k$, then dim ker $L > 1$ at $d = d_j$. This can be seen from the proof above. By (2.10) one may verify that $d_j = d_k$ for $j \neq k$ if and only if

$$\frac{5\ell^2(f_0 - \lambda_j)}{\pi^2(\lambda_j + 5)} = k^2. \tag{3.8}$$

This theorem shows that if $d_j > 0$, then $(d_j, 0)$ is a bifurcation point with respect to the trivial branch $(d, 0)$. The number of such bifurcation points is thus equal to the number of j for which $d_j > 0$, namely, i_α for given α and ℓ, see (2.9). Let J denote the closure of the non-trivial solution set of $F = 0$, and Γ_j the connected component of $J \cup \{(d_j, 0)\}$ to which $\{(d_j, 0)\}$ belongs. In a neighborhood of the bifurcation point the curve Γ_j is characterized by the eigenfunction ϕ_j. Since ϕ_j has exactly j zeros in the open interval $(0, \ell)$, we call the non-constant solutions in Γ_j *mode* j steady states.

Theorem 3 provides no information on the bifurcating curve Γ_j far from the equilibrium. A further study is therefore necessary in order to understand its global structure. We shall prove that Γ_j *is unbounded*, using the global bifurcation theory of Rabinowitz and the Leray-Schauder degree for compact operators.

Theorem 4. *Under the same assumption of Theorem 3, the projection of the bifurcation curve Γ_j on the d-axis contains (d_j, ∞).*

If $d > \tilde{d}$ and $d \neq d_k$ for any integer $k > 0$, then the problem (3.1)–(3.3) possesses at least one non-constant positive solution.

Proof. As in [13], we first rewrite the system (3.1) and (3.2) in a form that the standard global bifurcation theory can be more conveniently applied: Let $\tilde{u} = u - \alpha, \tilde{v} = v - 1 - \alpha^2$. Then (3.1) and (3.2) is transformed into

$$-\tilde{u}'' = f_0\tilde{u} + f_1\tilde{v} + \tilde{f}(\tilde{u}, \tilde{v}),$$
$$-\mathrm{d}\tilde{v}'' = g_0\tilde{u} + g_1\tilde{v} + \tilde{g}(\tilde{u}, \tilde{v}),$$

where $\tilde{f}$ and $\tilde{g}$ are higher-order terms of $\tilde{u}$ and $\tilde{v}$. The constant steady state $(\alpha, 1+\alpha^2)$ of (3.1–3.2) is shifted to (0, 0) of this new system.

Let G: $h \to w$ denote the Green operator for the boundary value problem

$$f_0 w - w'' = h \quad \text{in } (0, \ell), \quad w' = 0 \quad \text{at } x = 0, \ell$$

and G_d the Green operator for

$$-g_1 w - dw'' = h \quad \text{in } (0, \ell), \quad w' = 0 \quad \text{at } x = 0, \ell,$$

where $f_0 = (3\alpha^2 - 5)/(1+\alpha^2) > 0$ and $g_1 = -\alpha/(1+\alpha^2) < 0$. Put $\tilde{U} = (\tilde{u}, \tilde{v})$,

$$K(d)\tilde{U} = (2f_0G(\tilde{u}) + f_1G(\tilde{v}), g_0G_d(\tilde{u}))$$

and

$$H(\tilde{U}) = (G(\tilde{f}(\tilde{u}, \tilde{v})), G_d(\tilde{g}(\tilde{u}, \tilde{v}))).$$

Recall that

$$E = \{(u, v) : u, v \in C^2([0, \ell]), \ u' = v' = 0 \text{ at } x = 0, \ell\}.$$

Then the boundary value problem (3.1) and (3.3) can be interpreted as the equation

$$\tilde{U} = K(d)\tilde{U} + H(\tilde{U}) \tag{3.9}$$

in E. Note that $K(d)$ is a compact linear operator on E for any given $d > 0$. $H(\tilde{U}) = o(|\tilde{U}|)$ for $\tilde{U}$ near zero uniformly on closed d sub-intervals of $(0, \infty)$, and is a compact operator on E as well.

In order to apply Rabinowitz's global bifurcation theorem [17, Theorem 1.3], we first verify that 1 is an eigenvalue of $K(d_j)$ of algebraic multiplicity one. From the argument in the proof of Theorem 3 it is seen that $\ker(K(d_j) - I) = \ker \ L = \mathrm{span}\{\Phi\}$, so 1 is indeed an eigenvalue of $K = K(d_j)$, and $\dim \ker(K - I) = 1$. As the algebraic multiplicity of the eigenvalue 1 is the dimension of the generalized null space $\cup_{i=1}^{\infty} \ker(K - I)^i$, we need to verify that $\ker(K - I) = \ker(K - I)^2$, or $\ker(K - I) \cap R(K - I) = \{0\}$.

We now compute $\ker(K^* - I)$ following the calculation in [13], where K^* is the adjoint of K. Let $(\phi, \psi) \in \ker(K^* - I)$. Then

$$2f_0G(\phi) + g_0G_d(\psi) = \phi, \quad f_1G(\phi) = \psi.$$

By the definition of G and G_d we obtain

$$-d_j f_1\phi'' = f_\phi\phi + f_\psi\psi, \quad -\psi'' = f_1\phi - f_0\psi,$$

where

$$f_\phi = 2d_j f_0 f_1 + f_1 g_1, \quad f_\psi = f_1 g_0 - 2(f_0 g_1 + d_j f_0^2).$$

Write $\phi = \Sigma a_i\phi_i$, $\psi = \Sigma b_i\phi_i$. Then

$$\sum_{i=0}^{\infty} B_i^* \begin{pmatrix} a_i \\ b_i \end{pmatrix} \phi_i = 0, \quad B_i^* = \begin{pmatrix} f_\phi - d_j f_1\lambda_i & f_\psi \\ f_1 & -f_0 - \lambda_i \end{pmatrix}.$$

By a straightforward calculation one can check that $\det B_i^* = f_1 \ \det B_i$, where B_i is given in (3.5) (replacing d there by d_j). Thus $\det B_i = 0$ only for $i = j$, and $\ker(K^* - I)$ is spanned by $(f_0 + \lambda_j, f_1)\phi_j$. This shows that $\Phi \notin (\ker(K^* - I))^\perp = R(K - I)$, so $\ker(K - I) \cap R(K - I) = \{0\}$ and the eigenvalue 1 has algebraic multiplicity one.

If $0 < d \neq d_j$ is in a small neighborhood of d_j, then the linear operator $I - K(d) : E \to E$ is a bijection and 0 is an isolated solution of (3.9) for this fixed d. The index of this isolated zero of $I - K(d) - H$ is given by

$$i(I - K(d) - H, (d, 0)) = \text{ deg } (I - K(d), B, 0) = (-1)^p,$$

where B is a sufficiently small ball center at 0, and p is the sum of the algebraic multiplicities of the eigenvalues of $K(d)$ that are > 1. For our bifurcation analysis, it is also necessary to verify that this index changes as d crosses d_j, that is, for $\epsilon > 0$ sufficiently small,

$$i(I - K(d_j - \epsilon) - H, (d_j - \epsilon, 0)) \neq i(I - K(d_j + \epsilon) - H, (d_j + \epsilon, 0)). \tag{3.10}$$

Indeed, if μ is an eigenvalue of $K(d)$ with an eigenfunction (ϕ, ψ), then

$$-\mu\phi'' = (2 - \mu) f_0\phi + f_1\psi,$$

$$-d\mu\psi'' = g_0\phi + g_1\mu\psi.$$

Using the Fourier cosine series $\phi = \Sigma a_i \phi_i$ and $\psi = \Sigma b_i \phi_i$ leads to

$$\sum_{i=0}^{\infty} \begin{pmatrix} (2-\mu) f_0 - \mu\lambda_i & f_1 \\ g_0 & \mu g_1 - d\mu\lambda_i \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} \phi_i = 0.$$

Thus the set of eigenvalues of $K(d)$ consists of all μ's that solve the characteristic equation

$$(f_0 + \lambda_i)\mu^2 - 2 f_0 \mu - \frac{f_1 g_0}{d\lambda_i - g_1} = 0, \tag{3.11}$$

where the integer i runs from zero to ∞. For $d = d_j$ in particular, if $\mu = 1$ is a root of (3.11), then a simple calculation leads to $d_j = d_i$, and so $j = i$ by the assumption. Therefore, *without* counting the eigenvalues corresponding to $i = j$ in (3.11), $K(d)$ has the *same* number of eigenvalues > 1 for all d close to d_j, and they have the same multiplicities. For $i = j$ in (3.11), we let $\mu(d)$, $\tilde{\mu}(d)$ denote the two roots. First we find that

$$\mu(d_j) = 1 \quad \text{and} \quad \tilde{\mu}(d_j) = \frac{f_0 - \lambda_j}{f_0 + \lambda_j} < 1.$$

Now for d close to d_j, $\tilde{\mu}(d) < 1$. As the constant term $-f_1 g_0/(d\lambda_i - g_1)$ in (3.11) is a decreasing function of d, there results

$$\mu(d_j + \epsilon) > 1, \quad \text{and} \quad \mu(d_j - \epsilon) < 1.$$

Consequently, $K(d_j + \epsilon)$ has exactly one more eigenvalues that are larger than 1 than $K(d_j - \epsilon)$ does, and by a similar argument above we can show that this eigenvalue has algebraic multiplicity one. This verifies (3.10).

With the help of (3.10), we can use the argument in the proof of the [Theorem 1.3, 17] to conclude that Γ_j either meets infinity in $R \times E$ or meets $(d_k, 0)$ for some $k \neq j$, $d_k > 0$. We now show that the first alternative must occur, following the idea of Nishiura [11] and Takagi [20]: Indeed, if Γ_j is bounded, then it is compact, and Γ_j meets some other bifurcation points. Let k be such that Γ_j meets $(d_k, 0)$, but not $(d_i, 0)$ for any $i > k$. Consider the system (3.1) and (3.2) on the interval $(0, \ell/k)$ subject to the boundary condition

$$u' = v' \quad \text{at } x = 0, \ell/k. \tag{3.12}$$

We first note that if $\bar{U}$ solves (3.1) and (3.2) and (3.12), then one can construct a solution $\bar{U}$ to (3.1) and (3.3) by a reflective and periodic extension: Let $x_n = n\ell/k$, $n = 0, 1, \ldots, k$, and define

$$U(x) = \begin{cases} \bar{U}(x - x_{2n}) & \text{if } x_{2n} \leqslant x \leqslant x_{2n+1}, \\ \bar{U}(x_{2n+2} - x) & \text{if } x_{2n+1} \leqslant x \leqslant x_{2n+2}. \end{cases}$$

It is easy to see that $(d_k, 0)$ is also a bifurcation point of the problem (3.1) and (3.2) and (3.12). Let Λ_k denote the bifurcation branch of this new problem that meets $(d_k, 0)$, then using the same argument above it is clear that it either meets infinity or meets $(d'_k, 0)$ for some $k' > k$. If the second case occurs, then by the above extension one sees that Γ_j meets $(d'_k, 0)$, which violates the definition of k; hence Λ_k meets infinity, and then by the extension again Γ_j meets infinity too. It then follows that the projection of Γ_j on the d interval must be unbounded, since the solutions u, v are bounded by constants independent of d. It also follows from the *a priori* estimates that any solution on the curve Γ_j must be positive. The theorem is thus proved. □

We remark that we do *not* know if it is possible that Γ_j meets some bifurcation points and then reaches infinity; note that our argument above *only* rule out the possibility that Γ_j meets some bifurcation points without finally reaching infinity. If this case occurs, then some bifurcating branches "collide" each other and the solutions undergo a symmetry breaking. Notice further that our theorem *does not* provide the existence of non-constant solutions for $d = d_k$. Indeed, from our proof it can be seen that if (3.1)–(3.3) admits no non-constant positive solution for some $d_k > \tilde{d}$, then the "collision" must occur somewhere; i.e., the non-existence at some $d_k > \tilde{d}$ implies the "collision" of some bifurcation curves. Understanding this phenomenon is very important in studying the pattern formation in living organisms.

4. THE SHADOW SYSTEM

Although Theorem 4 gives existence of the non-constant solutions for large d, and Theorem 3 provides a detailed description on the solution shape near the bifurcation points, little is known for the qualitative properties of the solutions far from the equilibrium. For a better understanding of the solution properties, we consider the problem (3.1)–(3.3) as $d \to \infty$. The limit system is called a *shadow system*, as was first introduced by Keener [6]. For other important progress on the study of shadow systems, see for example [12, 14, 15, 20]. If we rewrite Eq. (3.2) as

$$v'' + d^{-1}\left(u - \frac{uv}{1+u^2}\right) = 0.$$

Then by the *a priori* estimate (2.4) we see that, as $d \to \infty$, $v'' = 0$. It follows from the boundary condition (3.3) that v must be a constant, say $v = \tau$, with

$$1 + \left(\frac{a}{5+4a^2}\right)^2 < \tau < 1 + a^2. \tag{4.1}$$

Integrating (3.2) on $(0,\ell)$ gives

$$\int_0^\ell u\,\mathrm{d}x = \int_0^\ell \frac{u\tau}{1+u^2}\mathrm{d}x.$$

Substituting this into the integration of (3.1) yields

$$\int_0^\ell u\,\mathrm{d}x = \frac{a\ell}{5}.$$

We thus obtain the shadow system

$$u'' + a - u - \frac{4u\tau}{1+u^2} = 0, \tag{4.2}$$

$$\frac{1}{\ell}\int_0^\ell u\,\mathrm{d}x = \frac{a}{5}, \tag{4.3}$$

$$u'(0) = u'(\ell) = 0. \tag{4.4}$$

By (2.4) any positive solution of (4.2)–(4.4) satisfies

$$\frac{a}{5+4a^2} < u < a.$$

Observe that if $u = u(x)$ is a decreasing solution of (4.2)–(4.4), then

$$u_-(x) = u(\ell - x)$$

gives an increasing solution of the same problem. Furthermore, if u is a solution of (4.2)–(4.4) with $m (m \geqslant 1)$ interior critical points, then these points must occur at $k\ell/(m+1)$, $1 \leqslant k \leqslant m$, and all the $m+1$ pieces defined in the subintervals $((k-1)\ell/(m+1),\ k\ell/(m+1))$, $1 \leqslant k \leqslant m+1$, are monotone and are identical up to translation and/or reflection. Hence the set of increasing solutions characterizes all solutions of (4.2)–(4.4). We shall therefore focus on *finding increasing solutions of* (4.2)–(4.4) only.

As in [15,20], our strategy of solving (4.2)–(4.4) is to solve the regular boundary value problem (4.2) and (4.4) first, and then to look for the solutions that satisfy the extra condition (4.3).

Let

$$f(u) = a - u - \frac{4\tau u}{1+u^2} \quad \text{and} \quad F(u) = \int_0^u f(s)\mathrm{d}s.$$

Then we have:

Lemma 4.1. *Let u be an increasing solution of (4.2) and (4.4) with $u(0) = \beta$, $u(\ell) = \theta$. Then $F(\beta) = F(\theta)$ and $F(u(x)) < F(\beta)$ for all $0 < x < \ell$.*

Proof. Let $E(x)=(u'(x))^2/2+F(u(x))$. Then $E'(x)\equiv 0$ and so

$$E(x)=F(\beta)=F(\theta), \quad 0\leqslant x\leqslant \ell \tag{4.5}$$

by the boundary condition (4.4). Since $u'(x)>0$ in $(0,\ell)$, it holds that $F(u(x))<F(\beta)$ for all $0<x<\ell$. □

Consequently, if (4.2) and (4.4) admit a solution, then there must be some $z\in(0,a)$ at which $F(u)$ takes a minimum value, so $f(z)=0$ and f changes sign in a neighborhood of z. To find the range for the parameter τ for which this happens, we need some detailed information for the curve

$$h(u)=\frac{(a-u)(1+u^2)}{4u}, \quad 0<u<a. \tag{4.6}$$

Clearly $f(z)=0$ if and only if $\tau=h(z)$. Since

$$h'(u)=-\frac{2u^3-au^2+a}{4u^2}, \tag{4.7}$$

$h(u)$ is decreasing for $u>0$ close to $u=0$ or $u=a$. As our basic assumption (H) gives

$$3a^2>125, \tag{4.8}$$

we find that

$$h'(\alpha)=\frac{3a^2-125}{20a}>0.$$

Therefore $h(u)$ has exactly two critical points in $(0,a)$.

Lemma 4.2. *Assume (4.8) holds. Let α_- and α_+ denote respectively the minimum and the maximum points of $h(u)$ in $(0,a)$. Then*

$$\max\{1,8/a\}<\alpha_-<\alpha, \quad 2\alpha<\alpha_+<a/2, \tag{4.9}$$

where $\alpha=a/5$. Furthermore, let

$$\tau_-=h(\alpha_-) \quad \textit{and} \quad \tau_+=h(\alpha_+).$$

Then

$$\max\left\{2,1+\frac{64}{a^2}\right\}<\tau_-<1+\frac{a^2}{25}<\tau_+<\frac{a^2}{12}-\frac{1}{4}. \tag{4.10}$$

Proof. By (4.7), α_- and α_+ are the two zeros of $\overline{h}(u) = 2u^3 - au^2 + a$ in $(0, a)$, so (4.9) follows from the simple fact that $\overline{h} > 0$ at $u = 0, 1, 8/a, a/2, a$, and $\overline{h} < 0$ at $u = \alpha, 2\alpha$.

Since $\alpha_- < a/5$, we have $4\alpha_- < a - \alpha_-$ and by (4.9)

$$\tau_- = \frac{(a-\alpha_-)(1+\alpha_-^2)}{4\alpha_-} > 1 + \alpha_-^2 > \max\left\{2, 1 + \frac{64}{a^2}\right\}.$$

As $h(\alpha) = 1 + \alpha^2$, we get

$$\tau_- < 1 + \alpha^2 < \tau_+$$

by the monotonicity of $h(u)$ over (α_-, α_+). Finally, since α_+ is a zero of $\overline{h}(u)$, it holds that $a = a\alpha_+^2 - 2\alpha_+^3$. We have the estimate

$$\begin{aligned}\tau_+ &= \frac{a - \alpha_+ + a\alpha_+^2 - \alpha_+^3}{4\alpha_+} \\ &= \frac{1}{4}(-3\alpha_+^2 + 2a\alpha_+ - 1) \leqslant \frac{a^2}{12} - \frac{1}{4}.\end{aligned}$$

□

We remark that although the estimates in (4.9) and (4.10) are not optimal, they are nevertheless very sharp. For instance, the lower bound $8/a$ in (4.9) cannot be replaced by numbers larger than $9/a$.

Lemma 4.3. *The problem (4.2) and (4.4) admits solutions for some* $\ell > 0$ *if and only if* $\tau \in (\tau_-, \tau_+)$.

Proof. We first prove that there exists no solution to (4.2) and (4.4) for any $\ell > 0$ if $\tau \notin (\tau_-, \tau_+)$. Indeed, if $\tau < \tau_-$, then $f(u)$ has exactly one zero in $(0, a)$. Denote this unique zero by z_+. Then $z_+ > \alpha_+$. Clearly $f(u) > 0$ for $0 < u < z_+$, and $f(u) < 0$ for $z_+ < u < a$. Hence the function $F(u)$ is concave in $(0, a)$. If $\tau = \tau_-$, then α_- is a zero of $f(u)$, which has another zero $z_+ > \alpha_+$ in $(0, a)$. Since $f(u)$ does not change sign near α_-, $F(u)$ is still concave in $(0, a)$. Similarly, we can derive the concavity of $F(u)$ for the case $\tau \geqslant \tau_+$. Hence the non-existence follows from Lemma 4.1.

On the other hand, if $\tau \in (\tau_-, \tau_+)$, then $f(u)$ has exactly three (simple) zeros, denoted by $z_- < z < z_+$, in $(0, a)$, with the interlacing relation

$$z_- < \alpha_- < z < \alpha_+ < z_+. \tag{4.11}$$

It is easily seen that $f<0$ for $z_-<u<z$, $f>0$ for $z<u<z_+$ and $f'(z)>0$. Thus $F(u)$ is a convex function in (z_-,z_+), taking a strict minimum at $u=z$. For a given $\beta\in(z_-,z)$, there are two cases: either $F(\beta)\geqslant F(z_+)$ or

$$f(\beta)<F(z_+). \tag{4.12}$$

Clearly (4.12) holds if β is sufficiently close to z.

Choose $\beta\in(z_-,z)$ such that (4.12) holds. Let $\theta>\beta$ be the unique number in (z,z_+) such that $F(\beta)=F(\theta)$. Let $u=u(x,\beta)$ be the unique solution to the initial value problem

$$u''+f(u)=0,\quad u(0)=\beta\in(z_-,z),\quad u'(0)=0. \tag{4.13}$$

Then $u''(0)=-f(\beta)>0$, so u is initially increasing. We claim that there is some finite $\ell=\ell(\beta)>0$ such that $u'>0$ in $(0,\ell)$ and $u'(\ell)=0$. Indeed, if this is not true, then $u'>0$, and by (4.5), $u<\theta$ for all $x>0$. Let $u_\infty=\lim_{x\to\infty}u(x)$. Then $\beta<u_\infty\leqslant\theta$. But as $x\to\infty$ one has $u''\to0$, so by (4.2) $f(u_\infty)=0$. Since z is the only zero of f in (β,θ) we get $u_\infty=z$, and $\lim_{x\to\infty}E(x)=F(z)<F(\beta)$, which contradicts (4.5). The claim is proved and the existence of solutions to (4.2) and (4.4) then follows. $\square$

Remark 4.1. If $F(z_-)>F(z_+)$, then we define β_0 to be the unique number in (z_-,z) such that $F(\beta_0)=F(z_+)$; if $F(z_-)\leqslant F(z_+)$, then we let $\beta_0=z_-$. For a given $\beta\in(z_-,z)$, condition (4.12) holds if and only if $\beta>\beta_0$. From the argument in the proof of Lemma 4.3 it follows that the initial value problem (4.13) gives rise to an increasing solution of (4.2) and (4.4) if and only if $\beta_0<\beta<z$.

To obtain more precise information for the existence, we shall study the function $\ell(\beta)$, $\beta_0<\beta<z$, in more details. By (4.5) we find that

$$u'(x)=\sqrt{2(F(\beta)-F(u))}>0,\quad x\in(0,\ell)$$

and $u(\ell)=\theta$. It follows that

$$\ell=\int_\beta^\theta\frac{\mathrm{d}u}{\sqrt{2(F(\beta)-F(u))}}. \tag{4.14}$$

This is a singular integral. For a simple evaluation of this integral we shall apply several change of variables to transform it into a regular one, following the original idea of Opial [16] (see also [18, 19] for advanced approach).

For a given number $u\in(\beta,\theta)$, define $u=g(s)$ by the relation

$$F(g(s))-F(z)=s^2/2,\quad \operatorname{sign}s=\operatorname{sign}(u-z)=\operatorname{sign}(f(u)). \tag{4.15}$$

Then $s = g^{-1}(u)$ is well defined and is strictly increasing in (β, θ), since in this interval $F(u)$ is convex and takes a strict minimum at $u = z$. Let $p > 0$ be given by

$$\frac{1}{2}p^2 = F(\beta) - F(z) > 0. \tag{4.16}$$

Then we have

$$\ell(\beta) = \int_{-p}^{p} \frac{g'(s)\mathrm{d}s}{\sqrt{p^2 - s^2}}.$$

Making another change of variable $s = -p\cos t, 0 \leqslant t \leqslant \pi$, we arrive at

$$\ell(\beta) = \int_0^{\pi} g'(-p\cos t)\mathrm{d}t. \tag{4.17}$$

For later purpose, we first express $g'(s), g''(s)$ and $g'''(s)$ as functions of u, following the calculation similar to that in [18, pp. 4–6]. Differentiating the identity (4.15) with respect to s we obtain $f(u)g'(s) = s$. Write

$$\bar{F}(U) = F(u) - F(z).$$

Then by (4.15) we get

$$g'(s) = \frac{\sqrt{2\bar{F}(u)}}{|f(u)|} > 0, \tag{4.18}$$

as long as $s \neq 0$ or $u \neq z$. For $s = 0$, using the L'Hopital's rule we obtain

$$g'(0) \equiv \lim_{u \to z} g'(s) = 1/\sqrt{f'(z)}, \tag{4.19}$$

which implies in particular

$$\lim_{\beta \to z} \ell(\beta) = \pi/\sqrt{f'(z)} = \ell_0. \tag{4.20}$$

Differentiating the identity $f(u)g'(s) = s$ with respect to s further gives

$$f'(u)g'^2(s) + f(u)g''(s) = 1$$

and

$$f''(u)g'^3(s) + 3f'(u)g'(s)g''(s) + f(u)g'''(s) = 0,$$

from which it is straightforward to verify

$$g''(s) = -\frac{f^2 - 2f'\bar{F}}{f^3}(u), \quad g''(0) = -\frac{f''}{3f'^2}(z), \tag{4.21}$$

and

$$g'''(s) = -\frac{g'(s)}{f^4(u)} H(u),$$
$$g'''(0) = \frac{1}{12(f'(z))^{7/2}} [5(f'')^2 - 3f'f'''](z), \tag{4.22}$$

where

$$H(u) = 2f(u)f''(u)\bar{F}(u) + 3f'(u)[f^2(u) - 2f'(u)\bar{F}(u)]. \tag{4.23}$$

The following lemma is technically very useful.

Lemma 4.4. *$H(z) = 0$, and $H(u) < 0$ for $u \in (z_-, z_+)$ and $u \neq z$.*

Proof. Obviously $H(z) = 0$ since $f(z) = 0$ and $\bar{F}(z) = 0$.
For our function $f(u) = a - u - 4\tau u/(1+u^2)$, we have

$$f'(u) = -1 - \frac{4\tau(1-u^2)}{(1+u^2)^2}, \quad f''(u) = \frac{8\tau u(3-u^2)}{(1+u^2)^3},$$

and

$$f'''(u) = \frac{24\tau(1-6u^2+u^4)}{(1+u^2)^4}.$$

In (z_-, z_+), $f(u)$ has one point of inflection $u = \sqrt{3}$, and two critical points c_- and c_+ with $c_- < \sqrt{3} < c_+$. For $u \in (z_-, c_-]$, we have

$$f < 0, \quad f' \leqslant 0, \qquad f'' > 0, \quad \text{and} \quad \bar{F} > 0,$$

verifying at once $H(u) < 0$. The same reasoning works for $u \in [c_+, z_+)$.

For $u \in (c_-, c_+)$, we need a different approach since in this interval f and f'' change sign. Noticing further that we do not know explicitly the function $\bar{F}$, it seems not easy to verify $H(u) < 0$ directly. We shall use an argument based on some clever ideas of Schaaf [18]. As

$$H'(u) = 2f(u)f'''(u)\bar{F}(u) + 5f''(u)[f^2(u) - 2f'(u)\bar{F}(u)],$$

we find that

$$5f''H(u) - 3f'H'(u) = 2f\bar{F}G(u), \quad G(u) = 5f''^2(u) - 3f'(u)f'''(u). \tag{4.24}$$

It is remarkable that G also appears in (4.22).

We claim that $G(u)>0$ for $u\in(c_-,c_+)$. In fact, if $f'''\leqslant 0$, then this is trivial as $f'>0$ in this interval. If $f'''>0$, then $u^4-6u^2+1>0$ and so

$$\begin{aligned} G(u) &= \frac{320\tau^2u^2(3-u^2)^2}{(1+u^2)^6}+72\tau\left(1+\frac{4\tau(1-u^2)}{(1+u^2)^2}\right)\cdot\frac{(1-6u^2+u^4)}{(1+u^2)^4} \\ &> \frac{320\tau^2u^2(3-u^2)^2}{(1+u^2)^6}+\frac{288\tau^2(1-u^2)(1-6u^2+u^4)}{(1+u^2)^6} \\ &= \frac{32\tau^2}{(1+u^2)^6}[10u^2(3-u^2)^2+9(1-u^2)(1-6u^2+u^4)] \\ &= \frac{32\tau^2}{(1+u^2)^6}(9+27u^2+3u^4+u^6)>0. \end{aligned}$$

By (4.22) we have

$$g'''(0)=\frac{1}{12(f'(z))^{7/2}}G(z)>0,$$

and so $g'''(s)>0$ for s close to zero, implying that $H(u)<0$ for all u close to z but not equal to z. Suppose for contradiction that there is some $\xi\in(z,c_+)$ at which $H(\xi)=0$ and $H<0$ for $u\in(z,\xi)$, then $H'(\xi)\geqslant 0$, $f'(\xi)>0$ and $G(\xi)>0$, which contradict (4.24). Thus $H(u)<0$ for all $z<u<z_+$. Similarly, we can prove that $H(u)<0$ for all $z_-<u<z_+$. □

Lemma 4.5. *Let $\beta_0<\beta<z$. Then $\ell'(\beta)<0$.*

Proof. In stead of showing that $\ell'(\beta)<0$ directly, we shall prove an equivalent inequality $\mathrm{d}\ell/\mathrm{d}p>0$. (Note that $\mathrm{d}p/\mathrm{d}\beta<0$, see (4.16)). By (4.17) we have

$$\ell'(p)=-\int_0^\pi \cos t g''(s)\mathrm{d}t \quad \text{and} \quad \ell''(p)=\int_0^\pi \cos^2 t g'''(s)\mathrm{d}t, \quad s=-p\cos t.$$

Since

$$\ell'(0)=\frac{\mathrm{d}\ell}{\mathrm{d}p}(0)=-g''(0)\int_0^\pi \cos t\,\mathrm{d}t=0,$$

and from (4.18), (4.22) and Lemma 4.4 it follows that $\ell''(p)>0$, we conclude that $\mathrm{d}\ell/\mathrm{d}p>0$, completing the proof. □

Now we can present our main result on the problem (4.2) and (4.4).

Theorem 5. *The problem (4.2) and (4.4) has non-trivial solutions if and only if*

$$\tau\in(\tau_-,\tau_+) \quad \text{and} \quad \ell>\ell_0\equiv\pi/\sqrt{f'(z)}. \tag{4.25}$$

Furthermore, if τ and ℓ are given numbers such that (4.25) holds, then the problem admits exactly one increasing solution u, and u must be non-degenerate.

Let $u(0)=\beta$ and $u(\ell)=\theta$. Then $\beta\in(\beta_0,z), \theta\in(z,z_+), \partial\beta/\partial\ell<0$ and $\partial\theta/\partial\ell>0$.

Proof. When $\tau\in(\tau_-,\tau_+)$, it is easy to show that $\ell(\beta)\to\infty$ as $\beta\downarrow\beta_0$, where β_0 is defined in Remark 4.1. Indeed, if $F(z_-)>F(z_+)$, then $\beta_0>z_-$ and $F(\beta_0)=F(z_+)$, in which case $u=u(x,\beta_0)$ is increasing for all $x>0$ and $\lim_{x\to\infty}u=z_+$. If $F(z_-)\leqslant F(z_+)$, then $\beta_0=z_-$ and $u(x,\beta_0)\equiv\beta_0$, for which case one still has $\ell(\beta)\to\infty$ as $\beta\downarrow\beta_0$ by the continuous dependence on initial conditions.

Therefore, if $\tau\in(\tau_-,\tau_+)$, and $\beta\in(\beta_0,z)$, the function $\ell(\beta)$ is decreasing by Lemma 4.5, with the range (ℓ_0,∞) as $\lim_{\beta\uparrow z}=\ell_0$ by (4.20). Combining with Lemma 4.3, we obtain the existence and uniqueness results as stated. Furthermore, in view of Remark 4.1, Lemmas 4.1 and 4.5 we obtain the properties for β and θ.

It remains to show that u must be non-degenerate: Differentiating the relation $u'(\ell,\beta)=0$ with respect to β we have

$$u''(\ell,\beta)\frac{\partial\ell}{\partial\beta}+w'(\ell,\beta)=0, \quad w=w(\ell,\beta)=\frac{\partial u(\ell,\beta)}{\partial\beta},$$

leading to $w'(\ell)=f(\theta)\partial\ell/\partial\beta<0$. As w also solves the problem

$$w''+f'(u)w=0, \quad w(0)=1, \quad w'(0)=0, \tag{4.26}$$

it follows that the only solution to the eigenvalue problem

$$\rho''+f'(u)\rho=0, \quad \rho'(0)=\rho'(\ell)=0, \tag{4.27}$$

is the trivial solution, establishing the non-degeneracy of u. □

Treating the second zero z of $f(u)$ on $(0,a)$ as a function of τ, then $z=z(\tau)$ is well-defined for $\tau\in(\tau_-,\tau_+)$. $z(\tau)$ is an increasing function, with $z\downarrow\alpha_-$ as $\tau\downarrow\tau_-$, and $z\uparrow\alpha_+$ as $\tau\uparrow\tau_+$.

It is easy to check that $z=\alpha$ if and only if $\tau=1+\alpha^2$. Since

$$f'(z)=-\frac{2z^3-az^2+a}{z(z^2+1)},$$

we find that, when $z=\alpha$,

$$f'(z)=-\frac{3\alpha^2-5}{\alpha^2+1}=f_0,$$

in which case $\ell > \ell_0$ if and only if $f_0 > \lambda_1$. Recall that $f_0 > \lambda_1$ is necessary for the instability of (u^*, v^*) as well as the existence of bifurcating solutions of (3.1)–(3.3), see Lemma 2.1, and Theorems 3, 4. However, $f'(z)$ may not take the maximum value at $z = \alpha$. Hence the problem (4.2) and (4.4) could have solutions even if (3.1)–(3.3) possesses no bifurcating solutions.

Next, we try to understand the existence of solutions to (4.2) and (4.4) *when the interval length ℓ is fixed.* To state the result, we introduce the polynomial

$$\zeta(z) = az^4 + 4z^3 - 4az^2 - a. \tag{4.28}$$

One can verify that $\zeta(z)$ has a unique zero, say $z = \alpha_*$, in $(0, a)$. We have

$$\partial f'(z)/\partial z = -\zeta(z)/[z^2(z^2+1)^2].$$

Thus $f'(z)$ takes the maximum value at α_*.

Theorem 6. *Let α_* be the unique zero of $\zeta(z)$. Let $\ell_* = \pi/\sqrt{f'(\alpha_*)}$. If $\ell \leqslant \ell_*$, then the problem (4.2) and (4.4) has no nontrivial solutions for any $\tau > 0$.*

On the other hand, if $\ell > \ell_*$, then there are two numbers $\tau_{\ell-} < \tau_{\ell+}$ such that $(\tau_{\ell-}, \tau_{\ell+}) \subset (\tau_-, \tau_+)$, and the problem (4.2) and (4.4) has non-trivial solutions if and only if $\tau \in (\tau_{\ell-}, \tau_{\ell+})$. Moreover, for each $\tau \in (\tau_{\ell-}, \tau_{\ell+})$, the problem admits exactly one increasing solution u, and u must be non-degenerate.

Proof. By the remark above it is obvious that ℓ_* is the absolute minimum value of $\pi/\sqrt{f'(z(\tau))}$ over $\tau \in (\tau_-, \tau_+)$. It follows that (4.2) and (4.4) admit no nontrivial solutions when $\ell \leqslant \ell_*$ for any $\tau > 0$.

Observe that $f'(z)$ has a unique maximum value at $z = \alpha_*$ in $z \in (\alpha_-, \alpha_+)$, with $f'(\alpha_*) = \pi^2/\ell_*^2$, and f' vanishes at α_- and α_+. Now, if $\ell > \ell_*$, then there must be two numbers $\alpha_{\ell-} < \alpha_{\ell+}$, such that $\alpha_* \in (\alpha_{\ell-}, \alpha_{\ell+}) \subset (\alpha_-, \alpha_+)$, and $f'(\alpha_{\ell-}) = f'(\alpha_{\ell+}) = \pi^2/\ell^2$. Let $\tau_{\ell-} = h(\alpha_{\ell-})$ and $\tau_{\ell+} = h(\alpha_{\ell+})$, see (4.6) for the definition of $h(u)$. We see that whenever $\tau \in (\tau_{\ell-}, \tau_{\ell+})$, it holds that $f'(z(\tau)) > \pi^2/\ell^2$, i.e., $\ell_0 < \ell$; thus there must be a unique, increasing, and non-degenerate solution to the problem (4.2) and (4.4) by Theorem 5. The proof is completed. $\square$

It is easy to see that as $\ell \downarrow \ell_*$, the set $(\tau_{\ell-}, \tau_{\ell+})$ shrinks to an empty set; whereas as $\ell \to \infty$, $(\tau_{\ell-}, \tau_{\ell+})$ expands to the intervel (τ_-, τ_+).

Theorem 7. *Assume $f_0 > \lambda_1$. Then the shadow system (4.2)–(4.4) admits at least one (strictly) increasing solution, and the same number* *of (strictly) decreasing solutions. The corresponding τ must satisfy $\tau \in (\tau_{\ell-}, \tau_{\ell+})$.*

Consequently, if $f_0 > \lambda_k, k > 1$, then the shadow system admits at least $2k$ non-trivial solutions, and for each $1 < j \leqslant k$ there are at least two solutions with exactly $j-1$ interior critical points in $(0, \ell)$.

Proof. We first prove that $f_0 > \lambda_1$ implies $\ell > \ell_*$: Indeed, as we noticed before that for $\tau = 1+\alpha^2$, one has $z(\tau) = \alpha$ and $f'(z) = f_0$. Hence $f_0 < f'(\alpha_*)$, yielding $f'(\alpha_*) > \lambda_1$, and

$$\ell = \frac{\pi}{\sqrt{\lambda_1}} > \frac{\pi}{\sqrt{f'(\alpha_*)}} = \ell_*.$$

Since $f'(\alpha_{\ell-}) = f'(\alpha_{\ell+}) = \lambda_1$, and $f'(\alpha) = f_0 > \lambda_1$, we also find that

$$\alpha_{\ell-} < \alpha < \alpha_{\ell+}. \tag{4.29}$$

For fixed $\ell > \ell_*$ and a given $\tau \in (\tau_{\ell-}, \tau_{\ell+})$, we shall denote the unique solution obtained in Theorem 6 by $u_\ell(x)$, and write

$$\beta_\ell(\tau) = u_\ell(0), \quad A(\tau) = \frac{1}{\ell}\int_0^\ell u_\ell(x)\mathrm{d}x.$$

It follows by the non-degeneracy of u_ℓ that both $\beta_\ell(\tau)$ and $A(\tau)$ must be continuous functions of τ.

If $\tau \in (\tau_{\ell-}, \tau_{\ell+})$ is close to $\tau_{\ell-}$, then $z(\tau)$ is close to $\alpha_{\ell-}$. Since $\ell = \pi/\sqrt{f'(\alpha_{\ell-})}$ we find that β_ℓ is close to $z(\tau)$, and thus close to $\alpha_{\ell-}$ too. Furthermore, the solution $u_\ell(x)$ is *nearly* a constant. Hence $A(\tau)$ is close to $\alpha_{\ell-}$ and by (4.29) we conclude that $A(\tau) < \alpha$. Similarly, for those τ's that are close to $\tau_{\ell+}$ we can establish $A(\tau) > \alpha$. By the continuity of $A(\tau)$ we find that there must be some $\tau \in (\tau_{\ell-}, \tau_{\ell+})$ such that $A(\tau) = \alpha$. This gives the existence of increasing solutions to the shadow system (4.2)–(4.4). The existence of decreasing solutions follows by a reflection.

Finally, if $f_0 > \lambda_k, k > 1$, then for each $1 < j \leqslant k$ it follows that $\ell/j > \pi/\sqrt{f_0} > \ell_*$. Hence we can find a pair of solutions of (4.2)–(4.4) with ℓ replaced with ℓ/j. The existence of solutions with interior critical points in $(0, \ell)$ follows by reflections and/or translations. $\square$

We are in the process of trying to show that $A(\tau)$ is a strictly *increasing* function of τ for *all* $\tau \in (\tau_{\ell-}, \tau_{\ell+})$, which needs a lengthy and highly non-trivial argument. Notice however that it is easy to prove the monotonicity of $A(\tau)$ in some subintervals of $(\tau_{\ell-}, \tau_{\ell+})$. This monotonicity is sufficient to imply the non-degeneracy of the shadow system, and so by a standard argument in [20] it is clear that the solutions obtained in Theorem 6 can be "projected" back to the original system (3.1)–(3.3).

REFERENCES

1. Callahan, T. K., and Knobloch, E. (1999). Pattern formation in three-dimensional reaction-diffusion systems. *Phys. D* **132**, 339–362.
2. Castets, V., Dulos, E., Boissonade, J., and De Kepper, P. (1990). Experimental evidence of a sustained Turing-type equilibrium chemical pattern. *Phys. Rev. Lett.* **64**, 2953–2956.
3. Crandall, M., and Rabinowitz, P. (1971). Bifurcation from simple eigenvalues. *J. Functional Anal.* **8**, 321–340.
4. Gierer, A., and Meinhardt, H. (1972). A theory of biological pattern formation. *Kybernetik* **12**, 30–39.
5. Judd, S. L., and Silber, M. (2000). Simple and superlattice Turing patterns in reaction-diffusion systems: bifurcation, bistability, and parameter collapse. *Phys. D* **136**, 45–65.
6. Keener, J. P. (1978). Activators and inhibitors in pattern formation. *Stud. Appl. Math.* **59**, 1–23.
7. Lengyel, I., and Epstein, I. R. (1991). Modeling of Turing structures in the chlorite-iodide-malonic acid-starch reaction system. *Science* **251**, 650–652.
8. Lengyel, I., and Epstein, I. R. (1992). A chemical approach to designing Turing patterns in reaction-diffusion system. *Proc. Natl. Acad. Sci. USA* **89**, 3977–3979.
9. Li, Y., Oslonovitch, J., Mazouz, N., Plenge, F., Krischer, K., and Ertl, G. (2001). Turing-type patterns on electrode surfaces. *Science* **291**, 2395–2398.
10. Murray, J. D. (1989). *Mathematical Biology,* Springer-Verlag, Berlin.
11. Ni, W.-M. (1998). Diffusion, cross-diffusion, and their spike-layer steady states. *Notices Amer. Math. Soc.* **45**, 9–18.
12. Ni, W.-M., Polčik, P., and Yanagida, E. (2001). Monotonicity of stable solutions in shadow systems. *Trans. Amer. Math. Soc*. **353**, 5057–5069.
13. Ni, W.-M., and Tang, M. Turing patterns in Lengyel–Epstein system for the CIMA reaction, submitted.
14. Ni, W.-M., Takagi, I., and Yanagida, E. (2001). Stability of least energy patterns of the shadow system for an activator-inhibitor model. *Japan J. Ind. Appl. Math.* **18**, 259–272.
15. Nishiura, Y. (1982). Global structure of bifurcating solutions of some reaction-diffusion systems. *SIAM J. Math. Anal.* **13**, 555–593.
16. Opial, Z. (1961). Sur les périodes des solutions de l'équation différentielle $x'' + g(x) = 0$. *Ann. Polon. Math.* **10,** 49–72.
17. Rabinowitz, P. H. (1971). Some global results for nonlinear eigenvalue problems. *J. Functional Anal.* **7**, 487–513.
18. Schaaf, R. (1990). *Global Solution Branches of Two-point Boundary Value Problems*, Lecture Notes in Mathematics, Vol. 1458, Springer-Verlag, Berlin.
19. Shi, J. (2002). Semilinear Neumann boundary value problems on a rectangle. *Trans. Amer. Math. Soc.* **354**, 3117–3154.
20. Takagi, I. (1986). Point-condensation for a reaction-diffusion system. *J. Diff. Eqns* **61**, 208–249.
21. Turing, A. M. (1952). The chemical basis of morphogenesis. *Phil. Trans. Roy. Soc. London Ser. B* **237**, 37–72.