Journal of Dynamical and Control Systems (2024) 30:31
https://doi.org/10.1007/s10883-024-09707-y

Null Controllability of an Abstract Riesz-spectral Boundary Control Systems

Abdellah Lourini[1] · **Mohamed El Azzouzi**[2] · **Mohamed Laabissi**[1]

Received: 5 September 2023 / Revised: 12 March 2024 / Accepted: 27 July 2024

Abstract
This paper addresses the null controllability of an abstract boundary control systems in Hilbert spaces where the system operator is of Riesz type. Consequently, this document establishes a criterion for null controllability in such systems based on initial data, utilizing the moment problem. Furthermore, this criterion is formulated by employing a null controllability criterion that is applicable to a corresponding linear system with internal control. Finally, we apply our approach to the heat equation and the Mullins equation, demonstrating the practicality of our methodology.

Keywords Null controllability · Boundary control · Unbounded control operator · Dirichlet operator · Semigroup · Heat equation · Mullins equation

Mathematics Subject Classification (2010) 93A10 · 93B05 · 47D06

1 Introduction

The study of null controllability has been a significant research area in the field of partial differential equations, aiming to manipulate the evolution of systems to reach a desired state through appropriate control inputs. A fundamental aspect of this pursuit involves investigating the controllability of equations subjected to boundary control conditions. Such equations often arise in various scientific and engineering applications, ranging from heat transfer and fluid dynamics to structural mechanics and population dynamics. Refer to [1–5] for further insights on this topic.

In recent years, several researchers have dedicated their efforts to exploring the concept of boundary null controllability in specific systems, often based on the moment problem. Notably, this methodology was also employed by Fattorini and Russell in tackling controllability issues related to second-order parabolic equations, see [6]. For instance, in [7], the authors investigated the null controllability of the linear heat equation with Dirichlet bound-

✉ Abdellah Lourini
louriniabdellah3@gmail.com

1 Department of mathematics, Faculty of Sciences, Chouaib Doukkali University, BP 20, El Jadida 2400, Morocco

2 Department STIN, National School of Applied Sciences, Chouaib Doukkali University, BP 1166, El Jadida 2400, Morocco

ary control, transforming the control problem into a moment problem. Similarly, in [8], the null controllability of one-dimensional compressible Navier-Stokes equations, linearized around a constant steady state with periodic boundary conditions, was examined using the moment problem as a foundation. This approach has also been explored in other works, see for example [9, 10].

The null controllability of infinite-dimensional control systems, in which the evolution of the state variable x is described by a linear equation where the input signal is internal, has been studied by various researchers in the literature, see [11, 12]. In this study, we examine the situation where the force acting on the system only takes place on a certain boundary area. To that purpose we separate the boundary area from the state space. This idea was developed by G. Greiner [13] to examine boundary conditions of differential equations as a domain perturbation of an unbounded linear operator. The class of boundary control systems is considered to be a fundamental and essential class of infinite-dimensional unbounded control systems.

In this contribution, we investigate the null controllability of linear boundary control systems in the abstract framework of the form

$$(BCP)\begin{cases} \dot{x}(t) = A_m x(t),\ t > 0,\\ Qx(t) = Bu(t),\ t \geq 0,\\ x(0) = x_0, \end{cases}$$

where the state $x(\cdot)$ and the boundary control $u(\cdot)$ are in some appropriate Hilbert spaces and A_m, Q and B are linear operators. These systems garnered significant attention in the early seventies, particularly through the contributions of [14, 15]. Significant results have been obtained concerning the existence of solutions, as well as the exact, approximate and positive controllability of boundary control systems (BCP) [16, 17]. In [14], H.O. Fattorini established abstract characterizations of null controllability for boundary control systems (BCP). While these abstract results are theoretically significant, they can be impractical in certain cases. Consequently, many authors in the literature often focus on investigating the controllability of specific systems, particularly parabolic linear equations in one space dimension. In this context, the major contributions of authors F.O. Fattorini and D.L. Russell have considerably enriched our understanding of this type of equation. In their work [6], they extensively examined the exact controllability of linear parabolic partial differential equations, basing the proposed characterization on the moment problem. See also [18].

Recently, null controllability of boundary control systems (BCP) has been discussed in [19]. Their approach involved a transformation of the problem of null controllability, where the control operator is unbounded, into a problem of null controllability, where the control operator is bounded. A central aim of their investigation was to explore the null controllability of flows in networks controlled in a single vertex. In this context, they established a matrix equality that characterizes the conditions for achieving null controllability. For more information on the various controllability concepts associated with (BCP), see [19] and the references therein.

In [19], the authors refrained from providing a comprehensive characterization of the null controllability of (BCP) in the abstract context. Consequently, the primary objective of this paper is to delve into criteria concerning the null controllability of (BCP). To this end, we focus on a class of Riesz operators and use the semigroup theory and moment problem to derive practical characterizations for achieving null controllability of (BCP) in the abstract framework.

To be more precise, we delve into the realm of initial data that can be driven to zero in a finite time frame through an appropriate choice of boundary control in $L^2_{loc}(\mathbb{R}_+, U)$.

The starting point of this paper is the work done in [19], where the resolution of the null controllability problem for (BCP) is transmuted into the resolution of null controllability for an associated infinite-dimensional linear control system. This approach proves to be a fundamental tool in our analysis.

Subsequently, our attention shifts to the characterization of null controllability of the associated system. In this context, the control problem is reduced to a moment problem. This aspect is particularly critical, as it engages the family of real exponentials $\{e^{\lambda_n t}\}_{n\geq 0}$, where the λ_n represent the eigenvalues of a Riesz operator. Consequently, we show that the input leading to null controllability have Fourier coefficients that increase exponentially with increasing frequency. In this context, we develop criteria for assessing the null controllability of the (BCP) system.

The paper is structured as follows: In Section 2, we introduce the necessary hypotheses to formulate the (BCP) problem and outline their corresponding solutions. Proceeding to Section 3, we establish the fundamental notion of null controllability for the system (BCP), along with its associated properties. Subsequently, we prove our main results concerning the null controllability of the system (BCP). In Section 4, we shift our focus to the null controllability analysis of the heat equation. Finally, in Section 5, we delve into the examination of null controllability of the Mullins equation.

2 Problem Setting

The goal of this paper is to explore the null controllability of the system (BCP) with the aim of formulating a more specific criterion. In line with the approach of [19], we propose the following conditions to achieve this goal:

- The state, the boundary, and the control space are three Hilbert spaces X, ∂X, and U, respectively,
- $A_m : D(A_m) \subset X \longrightarrow X$ is a linear closed, densely defined operator,
- $Q \in \mathcal{L}([D(A)], \partial X)$ is a boundary operator, where $[D(A_m)]$ is the subspace $D(A_m)$ equipped with the graph norm,
- $B \in \mathcal{L}(U, \partial X)$ is a bounded operator.
- $u(\cdot)$ is a control function in $L^2_{loc}(\mathbb{R}_+, U)$

In order to establish well-posedness for the abstract Cauchy problem (BCP), the introduction of the following assumption was necessary.

Assumptions 1

(H1) The operator $A \subset A_m$ with domain $D(A) = ker Q$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X.
(H2) The operator $Q : D(A_m) \longrightarrow \partial X$ is surjective.

Building upon these assumptions, it becomes essential to present certain results established in [13, Lemma 1.2, Lemma 1.3], as these findings play a crucial role in our analysis.

Lemma 1 *The following assertions are satisfied for each $\lambda \in \rho(A)$.*

1. *$D(A_m) = D(A) \oplus ker(\lambda - A_m)$;*
2. *$Q|_{ker(\lambda - A_m)}$ is invertible and the operator $Q_\lambda := (Q|_{ker(\lambda-A_m)})^{-1} : \partial X \longrightarrow ker(\lambda - A_m) \subset X$ is bounded;*

For $\lambda \in \rho(A)$, Q_λ is called the Dirichlet operator. Define the operator

$$B_\lambda := Q_\lambda B \in \mathcal{L}(U, \ker(\lambda - A_m)).$$

Under the above assumptions and for $u \in L^2_{loc}(\mathbb{R}_+, U)$, it has been shown that if x is a classical solution of (BCP), i.e. $x(\cdot) \in C^1(\mathbb{R}_+, X)$ with $x(t) \in D(A_m)$ for all $t \geq 0$ satisfying the boundary systems (BCP), then it is given by the variation of constat formula

$$x(t) = S(t)x_0 + (\lambda - A_{-1}) \int_0^t S(t-s)B_\lambda u(s)ds, \quad t \geq 0 \tag{1}$$

for some $\lambda \in \rho(A)$, here A_{-1} denotes the generator of the extrapolated semigroup associated to $(S(t))_{t\geq 0}$, see [17].

In this study, the concept of Riesz spectral operators, as introduced in [11], is employed to establish the admissibility of mild solutions (1) for the system (BCP) and to investigate their null controllability.

Definition 1 *Let $A : D(A) \subset X \to X$ be a linear and closed operator with simple eigenvalues λ_n and corresponding eigenvectors $\phi_n \in D(A)$, $n \in \mathbb{N}$. A is a Riesz-spectral operator if*
(1) $\{\phi_n, n \in \mathbb{N}\}$ is a Riesz basis:

(a)$\{\phi_n, n \in \mathbb{N}\}$ is maximal, i.e., $\overline{\operatorname{span}_{\mathbb{K}} \phi_n} = X$;
(b)there exist constants $m_R, M_R \in \mathbb{R}^*_+$ such that for all $N \in \mathbb{N}$ and all $\alpha_0, \ldots, \alpha_N \in \mathbb{K}$

$$m_R \sum_{n=0}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=0}^{N} \alpha_n \phi_n \right\|_X^2 \leq M_R \sum_{n=0}^{N} |\alpha_n|^2$$

(2) the closure of $\{\lambda_n, n \in \mathbb{N}\}$ is totally disconnected, i.e. for any distinct $a, b \in \overline{\{\lambda_n, n \in \mathbb{N}\}}$, $[a, b] \not\subset \overline{\{\lambda_n, n \in \mathbb{N}\}}$.

From [11], the Riesz spectral operator ensures the following properties:
The eigenvalues of the adjoint operator A^* are provided for $n \in \mathbb{N}$ by $\mu_n \triangleq \overline{\lambda_n}$ and the associated eigenvectors $\psi_n \in D(A^*)$ can be selected such that $\{\phi_n, n \in \mathbb{N}\}$ and $\{\psi_n, n \in \mathbb{N}\}$ are biorthogonal, i.e., for all $n, m \in \mathbb{N}$, $\langle \phi_n, \psi_m \rangle_X = \delta_{n,m}$.
Moreover, the sequence of vectors $\{\psi_n, n \in \mathbb{N}\}$ is a Riesz basis. For all $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$,

$$\sum_{n\in\mathbb{N}} |\alpha_n|^2 < \infty \Leftrightarrow \sum_{n\in\mathbb{N}} \alpha_n \phi_n \in X,$$

and for all $x \in X$,

$$x = \sum_{n\in\mathbb{N}} \langle x, \psi_n \rangle_X \, \phi_n = \sum_{n\in\mathbb{N}} \langle x, \phi_n \rangle_X \, \psi_n.$$

On the other hand, A is the generator of a C_0-semigroup S if and only if $\sup_{n\in\mathbb{N}} \operatorname{Re} \lambda_n < \infty$. In this case, the C_0-semigroup S is given by

$$S(t)x = \sum_{n\in\mathbb{N}} e^{\lambda_n t} \langle x, \psi_n \rangle_X \, \phi_n \text{ for all } t \in \mathbb{R}_+, \ x \in X. \tag{2}$$

Remark 1 *[11] The semigroup $(S(t))_{t\geq 0}$ is analytic if A is a Riesz-spectral operator and for some $\omega \in \mathbb{R}$ and $c < 0$*

$$Re(\lambda_n) \leq \omega \text{ and } Re(\lambda_n) - \omega \leq c|Im(\lambda_n)| \text{ for all } n \in \mathbb{N}. \tag{3}$$

To establish the subsequent corollary, we can make use of the regularity result presented in [20, Theorem 6.2.4, page 70] and [21]. Hence, we introduce the following assumption:

Assumptions 2 *A is a Riesz-spectral operator and the semigroup* $(S(t))_{t\geq 0}$ *is analytic.*

Corollary 1 *Under Assumptions 1 and 2, the system (BCP) has a unique solution x in* X_{-1}*, that satisfies*

$$x \in C([0, \infty[, X) \cap C^1([0, \infty[, X_{-1}).$$

Moreover, this solution it is given by

$$x(t) = S(t)x_0 + (\lambda - A)\int_0^t S(t-s)B_\lambda u(s)ds, \quad t \geq 0. \tag{4}$$

For more detailed information on formula (4), refer to [19] and the references therein.

3 Null Controllability of Boundary Control Problems

3.1 Definitions and Related Properties

Building upon our previous work in [19], we further investigate null controllability by connecting the problem (BCP) with the standard control system (A, B_λ), where λ belongs to $\rho(A)$. This system is described as follows:

$$(A, B_\lambda)\begin{cases} \dot{x}(t) = Ax(t) + B_\lambda u(t), \ t \geq 0, \\ x(0) = x_0. \end{cases} \tag{5}$$

The mild solution of (A, B_λ) is given by the variation of parameters formula

$$x(t) = S(t)x_0 + \int_0^t S(t-s)B_\lambda u(s)ds, \tag{6}$$

where $u \in L^2_{loc}(\mathbb{R}_+, U)$.

Our objective is to investigate the null controllability of the system (BCP) at a given time $T > 0$. To achieve this goal, we introduce the following definition:

Definition 2 *Let Z be a subset of the space X.*

- *The system* (A, B_λ) *(resp. (BCP)) is said to be exactly Z-null controllable at time T if for all* $x_0 \in Z$*, there is a control* $u \in L^2([0, T], U)$ *such that the corresponding solution of the system satisfies* $x(T) = 0$*.*
- *The system* (A, B_λ) *(resp. (BCP)) is said to be exactly Z-null controllable in finite time if for all* $x_0 \in Z$*, there is a time T and a control* $u \in L^2([0, T], U)$ *such that the corresponding solution of the system satisfies* $x(T) = 0$*.*

In the study by [19], the null controllability of the system (BCP) within Banach spaces was examined. The authors considered a scenario where the state space is non-reflexive, and the control function exists in the space $L^1([0, T], U)$). Operating within the bounds of the hypotheses presented in Assumptions 1 and under the assumption: There exist $\gamma > 0$, and $\lambda_0 \in \mathbb{R}$ such that

$$\|Qx\| \geq \gamma\lambda\|x\| \text{ for all } \lambda > \lambda_0, \text{ and } x \in \ker(\lambda - A_m). \tag{7}$$

they successfully established a correlation between the null controllability of (BCP) and that of (A, B_λ).

In this paper, we aim to present analogous results in the context of Hilbert spaces. Here, $L^1([0, T], U)$ will be replaced by $L^2([0, T], U)$, and the assumption (7) will be substituted with the condition that the semigroup $S(t)$ is analytic.

In the absence of an analytic semigroup $S(t)$, while considering the hypothesis $\int_0^t S(t-s)B_\lambda u(s)ds \in D(A)$ for all $t > 0$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$, we deduce a necessary condition for achieving exact X-null controllability of the system (A, B_λ).

Theorem 1 *[19] Let $\lambda \in \rho(A)$ and Assumptions 1 hold. If the system (A, B_λ) is exactly X-null controllable at a time T, then the semigroup $(S(t))_{t\geq 0}$ is eventually differentiable.*

Remark 2 *Clearly, if the semigroup $(S(t))_{t\geq 0}$ is analytic, it is also eventually differentiable. Note that Assumptions 2 is not a necessary condition for studying the null controllability of the system (A, B_λ) or (BCP). It is introduced to ensure the admissibility of the solution of (BCP) as given by (4).*

Concerning the null controllability of (BCP), as discussed in [19], by considering the semigroup $S(t)$ analytic instead of eventually differentiable, we obtain the following result.

Theorem 2 *[19] Under Assumptions 1 and 2, the following assertions hold.*

i) If the system (BCP) is exactly X-null controllable at time T, then (A, B_λ) is X-null controllable at time $T + \varepsilon$ for any $\varepsilon > 0$.
ii) Conversely, if the system (A, B_λ) is exactly X-null controllable at time T, then (BCP) is X-null controllable at the same time T.

From [19], under Assumptions 1 and 2, we derive an important characterization of the null controllability for (BCP).

Corollary 2 *[19] The system (BCP) is exactly X-null controllable at time T, if and only if, the system (A, B_λ) is $D(A)$-null controllable at the time T.*

Remark 3 *As demonstrated in [19], it is not difficult to prove that the system (BCP) is exactly Z-null controllable at time T, if and only if, the system (A, B_λ) is $R(\lambda, A)Z$-null controllable at the time T.*

The result of the previous Corollary remains true for the null controllability in finite time.

Theorem 3 *[19] The system (A, B_λ) is exactly $D(A)$-null controllable in finite time, if and only if, the system (BCP) is exactly X-null controllable in finite time.*

For the control law which steers the initial state of the (BCP) system to the origin in time T, we have the following result.

Theorem 4 *Let $\lambda \in \rho(A)$, $x_0 \in X$ and $u \in L^2([0, T], U)$.*

The control law u can drive the solution $x(., x_0, u)$ of (BCP) to zero if and only if it can drive the solution $x(., R(\lambda, A)x_0, u)$ of (A, B_λ) to zero.

3.2 Main Results

Within the framework of Assumptions 1 and 2, we focus on the class of Riesz-spectral boundary control systems associated with (BCP). For any $\lambda \in \rho(A)$, the system (A, B_λ)

has a unique solution $x(\cdot)$, as characterized by (6). Furthermore, as evidenced by (2), this solution can be explicitly expressed as:

$$x(T) = \sum_{n\in\mathbb{N}} e^{\lambda_n T} \langle x_0, \psi_n\rangle_X \phi_n + \sum_{n\in\mathbb{N}} \int_0^T e^{\lambda_n (T-s)} \langle B_\lambda u(s), \psi_n\rangle_X \, ds \, \phi_n,$$
$$= \sum_{n\in\mathbb{N}} e^{\lambda_n T} \langle x_0, \psi_n\rangle_X \phi_n + \sum_{n\in\mathbb{N}} \int_0^T e^{\lambda_n s} \langle B_\lambda v(s), \psi_n\rangle_X \, ds \, \phi_n,$$

for all $T > 0$, $x_0 \in X$ and $u \in L^2([0, T], U)$ where $v(s) = u(T - s)$.

Using the results presented in Corollary 2 and Theorem 4, establishing a criterion for the X-null controllability of (BCP) can be simplified to establishing a criterion for the $D(A)$-null controllability of (A, B_λ). To this end, we introduce the following theorem:

Theorem 5 *Let $\lambda \in \rho(A)$. The system (BCP) is X-null controllable at time T if and only if for every $x_0 \in D(A)$, there exist $u \in L^2([0, T], U)$ such that*

$$-e^{\lambda_n T} \langle x_0, \psi_n\rangle_X = \int_0^T e^{\lambda_n s} \langle B_\lambda v(s), \psi_n\rangle_X \, ds \text{ for all } n \in \mathbb{N}, \tag{8}$$

where $v(s) = u(T - s)$.

Proof The system (A, B_λ) is $D(A)$-null controllable at time T if and only if for every $x_0 \in D(A)$, there exist $u \in L^2([0, T], U)$ such that the corresponding solution of the system satisfies $x(T) = 0$. i.e.,

$$\sum_{n\in\mathbb{N}} e^{\lambda_n T} \langle x_0, \psi_n\rangle_X \phi_n + \sum_{n\in\mathbb{N}} \int_0^T e^{\lambda_n s} \langle B_\lambda v(s), \psi_n\rangle_X \, ds \, \phi_n = 0.$$

Since $\{\phi_n, n \in \mathbb{N}\}$ is a Riesz basis, then, (A, B_λ) is $D(A)$-null controllable at time T if and only if for every $x_0 \in D(A)$, there exist $u \in L^2([0, T], U)$ such that

$$-e^{\lambda_n T} \langle x_0, \psi_n\rangle_X = \int_0^T e^{\lambda_n s} \langle B_\lambda v(s), \psi_n\rangle_X \, ds \text{ for all } n \in \mathbb{N}.$$

The proof is completed by Corollary 2. □

Solving the (8) problem has allowed us to establish a connection with the following result:

Lemma 2 *[6] Let $c_n \in \mathbb{R}$ and $\mu_n \geq 0$ for all $n \in \mathbb{N}$. The following moment problem*

$$\int_0^T v(s) e^{-\mu_n s} ds = c_n, \quad n \in \mathbb{N}, \tag{9}$$

has an absolutely convergent solution $v \in L^2([0, T], \mathbb{R})$ if, for some $\delta > 1$,

$$\sum_{n=0}^{\infty} |c_n| e^{\delta\sqrt{\mu_n}} < \infty. \tag{10}$$

Moreover, the solution $v \in L^2([0, T], \mathbb{R})$ of (9) is given by

$$v(s) = \sum_{n=0}^{\infty} c_n E_n(s),$$

where the sequence $\{E_n\}_n \subset L^2([0,T],\mathbb{R})$ *are the biorthogonal functions to the set* $\{e^{-\mu_n s}\}_n$ *in* $L^2([0,T],\mathbb{R})$*, i.e.,*

$$\int_0^T E_n(s)e^{-\mu_m s}ds = \delta_{nm}, \quad n,m \in \mathbb{N}.$$

Remark 4 *Since* A *is a Riesz-spectral operator and a generator of a* C_0*-semigroup, the subset* $\sigma^+(A)$ *of* $\sigma(A)$*, which consists of eigenvalues with positive real parts, is limited to a finite number of elements. Hence, there exist* $N_0 \in \mathbb{N}$ *such that*

$$-\lambda_n \geq 0 \textit{ for all } n \geq N_0.$$

For the remainder of this paper, we will employ the change of variable $\mu_n = -\lambda_n$ for any $n \in \mathbb{N}$. This leads to the equation (8) being expressed as

$$-e^{-\mu_n T}\langle x_0, \psi_n\rangle_X = \int_0^T e^{-\mu_n s}\langle B_\lambda v(s), \psi_n\rangle_X\, ds \text{ for all } n \in \mathbb{N}, \tag{11}$$

where $v(s) = u(T-s)$.

It is now evident that addressing the null controllability question for the (BCP) involves investigating the existence of v in $L^2([0,T],\mathbb{R})$, the solution to the equation (11). In order to develop this characterization, we assume that the control law $v(s)$ takes the following form:

$$v(s) = \sum_{n=0}^{\infty} u_n E_n(s), \text{ for all } s \in [0,T], \tag{12}$$

for some $u_n \in U$, $n = 0, 1, \ldots$.

This formulation leads to the following characterization for the null controllability of (BCP).

Corollary 3 *Let* $\lambda \in \rho(A)$ *and* $v(\cdot)$ *is given by* (12).

The system (BCP) *is* X*-null controllable at time* T *if and only if for every* $x_0 \in D(A)$*, there exist* $\{u_n\}_{n\in\mathbb{N}} \subset U$ *such that*

$$-e^{-\mu_n T}\langle x_0, \psi_n\rangle_X = \langle B_\lambda u_n, \psi_n\rangle_X \textit{ for all } n \in \mathbb{N}. \tag{13}$$

Proof According to Theorem 5 and Lemma 2, along with equations (11) and control law (13), the system (BCP) exhibits X-null controllability at time T if and only if, for every initial state $x_0 \in D(A)$, there exists a sequence $u_{n n\in\mathbb{N}} \subset U$ satisfying

$$-e^{-\mu_n T}\langle x_0, \psi_n\rangle_X = \sum_{m=0}^{\infty}\langle B_\lambda u_m, \psi_n\rangle_X \int_0^T e^{-\mu_n s}E_m(s)ds \text{ for all } n \in \mathbb{N}.$$

This implies that

$$-e^{-\mu_n T}\langle x_0, \psi_n\rangle_X = \langle B_\lambda u_n, \psi_n\rangle_X \text{ for all } n \in \mathbb{N},$$

□

In the single-input case, i.e., when $U = \mathbb{R}$ or $U = \mathbb{C}$, we can derive the following corollary:

Corollary 4 *Assume that*

$$\langle B_\lambda 1, \psi_n\rangle_X \neq 0 \textit{ for all } n \in \mathbb{N}. \tag{14}$$

Then, the system (BCP) is X-null controllable at all time T. Additionally, the corresponding control law is given by

$$u(s) = -\sum_{n=0}^{\infty} \frac{e^{-\mu_n T} \langle R(\lambda, A)x_0, \psi_n\rangle_X}{\langle B_\lambda 1, \psi_n\rangle_X} E_n(T-s), \tag{15}$$

i.e., an initial state $x_0 \in X$ of system (BCP) can be transferred to zero state in time T by the control $u(\cdot)$ defined in (15).

Proof Let $x_0 \in X$ and $\lambda \in \rho(A)$. Define $\tilde{x}_0 = R(\lambda, A)x_0$. Following the proof of Theorem 5 and utilizing criterion (13) from Corollary 3, we establish that the initial state $\tilde{x}_0 \in D(A)$ of system (A, B_λ) can be transferred to zero state in time T by the control $u(\cdot) = v(T - \cdot)$, where $v(\cdot)$ is given in (12) with

$$-e^{-\mu_n T} \langle \tilde{x}_0, \psi_n\rangle_X = u_n \langle B_\lambda 1, \psi_n\rangle_X \quad \text{for all } n \in \mathbb{N}.$$

This yields the expression for u_n as:

$$u_n = \frac{-e^{-\mu_n T} \langle R(\lambda, A)x_0, \psi_n\rangle_X}{\langle B_\lambda 1, \psi_n\rangle_X} \quad \text{for all } n \in \mathbb{N}.$$

Consequently, the initial state $R(\lambda, A)x_0$ of system (A, B_λ) can be transferred to zero state in time T by the control in (15). By applying Theorem 4, the proof is concluded. □

Remark 5 *In the case of a single input, the condition (14) is a necessary condition for the null controllability of (BCP). Indeed, if the condition (14) is not satisfied, i.e., there exists a number p in $\mathbb{N}$ such that $\left\langle B_\lambda 1, \psi_p\right\rangle_X = 0$, then when the initial conditions x_0 satisfy $\left\langle x_0, \psi_p\right\rangle_X \neq 0$, the condition (13) is not satisfied. Consequently, by Corollary 3, the system (BCP) is not X-null controllable for any time T.*

4 Null Boundary Controllability of a One-dimensional Heat Equation

As a concrete application to demonstrate the findings presented in this paper, we consider a specific boundary control system. The system is described by the one-dimensional heat equation defined over the spatial domain [0, 1] and involves a Dirichlet boundary control applied at point 0. The system is described as follows:

$$\begin{cases} \dot{x}(t,z) = \frac{\partial^2}{\partial z^2} x(t,z), & t \geq 0, \quad z \in [0,1], \\ x(t,0) = u(t), \quad x(t,1) = 0, & t \geq 0, \\ x(0,z) = x_0(z), & z \in [0,1]. \end{cases} \tag{16}$$

for a given function $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R})$ and $x_0 \in L^2([0,1], \mathbb{R})$.

To illustrate this example into our framework, we introduce the following:

i) the state space $X = L^2([0,1], \mathbb{R})$,
ii) the boundary space $\partial X = \mathbb{R}$,
iii) the control space $U := \mathbb{R}$,
iv) the control operator $B := Id_{\mathbb{R}}$,
v) the operator $A_m := \dfrac{d^2}{dz^2}$ with domain

$$D(A_m) = \{f \in H^2([0,1], \mathbb{R}) / \quad f(1) = 0\},$$

vi) the boundary operator $Q : D(A_m) \longrightarrow \mathbb{R}$, $Q(f) = f(0)$,
vii) the operator $A \subset A_m$ with domain $D(A) = \ker Q$.

As it is shown in [11], the operator A is a Riesz-spectral operator and serves as the infinitesimal generator of an analytic semigroup $(S(t))_{t\geq 0}$ on X defined by

$$S(t)x_0(z) = \sum_{n=1}^{\infty} 2e^{-\mu_n t} \sin(n\pi z) \int_0^1 sin(n\pi\xi)x_0(\xi)d\xi.$$

Here, the spectrum of A is given by $\sigma(A) = \{-\mu_n / \mu_n = n^2\pi^2,\ n \in \mathbb{N}^*\}$, and the associated eigenvectors $\left\{\phi_n(\cdot) := \sqrt{2}\sin(n\pi\cdot),\ n \in \mathbb{N}^*\right\}$ form an orthogonal basis for $L^2([0, 1], \mathbb{R})$.

It is evident that the boundary operator Q is surjective. Additionally, we verify that all the assumptions stated in Assumptions 1 and 2 are satisfied. To compute the associated Dirichlet operator Q_λ, we focus on the null controllability of equation (16), which is independent of the choice of λ see [19]. For simplicity, we set $\lambda = 0$. Therefore, a straightforward calculation yields

$$(Q_0 y)(z) = (1 - z)y := yh(z).$$

We now introduce the associated system (A, B_0) for (16)

$$\begin{cases} \dot{y}(t, z) = \dfrac{\partial^2}{\partial z^2} y(t, z) + h(z)u(t), & t \geq 0, \quad z \in]0, 1[, \\ y(0) = g(\cdot) := R(0, A)x_0 = -A^{-1}x_0. \end{cases} \tag{17}$$

Using integration by parts, we obtain

$$\langle B_\lambda 1, \phi_n\rangle_X = \sqrt{2}\int_0^1 sin(n\pi z)h(z)dz = \frac{\sqrt{2}}{n\pi} \neq 0.$$

Now, by Corollary 4, we deduce the following characterization of the null controllability of (16).

Proposition 1 *The control system (16) is X-null controllable at all time $T > 0$. Furthermore, for any given initial state $x_0 \in X$, the following control law*

$$u(s) = \sum_{n=1}^{\infty} \frac{n\pi e^{-\mu_n T} \left\langle A^{-1}x_0, \phi_n\right\rangle_X}{\sqrt{2}} E_n(T - s) \text{ for all } s \in [0, T], \tag{18}$$

ensures that $x(T, x_0, u) = 0$, where $x(T, x_0, u) = 0$ represents the solution of (16).

The resolvent operator $-A^{-1}$ is defined as follows:

$$-A^{-1}x_0 = \sum_{p=1}^{\infty} \frac{1}{p^2\pi^2} \left\langle x_0, \phi_p\right\rangle \phi_n.$$

Consequently, the control law (18) takes the form:

$$u(s) = -\sum_{n=1}^{\infty} \frac{e^{-\mu_n T} \langle x_0, \phi_n\rangle_X}{n\pi\sqrt{2}} E_n(T - s) \text{ for all } s \in [0, T].$$

5 Null Boundary Controllability for the Mullins Equation

This section explores another concrete application of our abstract results presented in this paper. We will investigate the null boundary controllability of the Mullins equation, described by the following form:

$$\begin{cases} \dot{x}(t,z) = -\frac{\partial^4}{\partial z^4}x(t,z), & 0<z<1, \quad 0<t<T, \\ \frac{\partial}{\partial z}x(t,0) = u(t), \quad \frac{\partial^3}{\partial z^3}x(t,0) = 0, & 0<t<T, \\ x(t,1) = 0, \quad \frac{\partial^2}{\partial z^2}x(t,1) = 0, & 0<t<T, \\ x(0,z) = x_0(z), & 0<z<1, \end{cases} \tag{19}$$

where $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R})$ and $x_0 \in L^2([0,1], \mathbb{R})$.

Here, the equation (19) takes the form of (BCP) with

- $X = L^2([0,1], \mathbb{R})$, $\partial X = \mathbb{R}$, $U := \mathbb{R}$ and $B := Id_{\mathbb{R}}$,
- the operator $A_m := -\dfrac{d^4}{dz^4}$ with domain

$$D(A_m) = \{f \in H^4([0,1], \mathbb{R}) / \ \frac{d^3}{dz^3}f(0) = f(1) = \frac{d^2}{dz^2}f(1) = 0\},$$

- the boundary operator $Q : D(A_m) \longrightarrow \mathbb{R}$, $Q(f) = \frac{d}{dz}f(0)$.

We can show that the eigenvalues and normalized eigenfunctions of operator $A := A_m$ with $D(A) = \ker(Q)$, are as follows

$$\mu_n = -\big(\frac{\pi}{2} + n\pi\big)^4 \text{ and } \phi_n(z) = \sqrt{2}\cos\big((\frac{\pi}{2} + n\pi)z\big) \text{ for all } n \in \mathbb{N}.$$

Therefore, the eigenvectors $\Big\{\phi_n(\cdot) := \sqrt{2}\cos\big((\frac{\pi}{2} + n\pi)\cdot\big),\ n \in \mathbb{N}\Big\}$ form an orthogonal Riesz basis for $L^2([0,1], \mathbb{R})$. Furthermore, the condition (3) is fulfilled. Thus, the Assumptions 2 and $(H1)$ in Assumptions 1 are satisfied. On the other hand, the hypothesis $(H2)$ is evidently valid.

Now, the task at hand is to identify the operator Q_λ for some $\lambda \in \rho(A)$. Given that $0 \notin \sigma(A)$, we choose $\lambda = 0$. Thus, we have for any $f \in \ker(A_m)$ and $y \in \partial X := \mathbb{R}$

$$Qf = y \Longleftrightarrow \begin{cases} \frac{d^4}{dz^4}f(z) = 0, \qquad z \in]0,1[\\ \frac{d^3}{dz^3}f(0) = 0, \\ \frac{d}{dz}f(0) = y, \\ f(1) = \frac{d^2}{dz^2}f(1) = 0. \end{cases}$$

We get $\big(Q_0 y\big)(z) = (z-1)y$, for all $y \in \mathbb{R}$ and $z \in [0,1]$.

Using integration by parts, for every $n = 0, 1, 2, \cdots$ we obtain

$$\langle B_0 1, \phi_n\rangle_X = \sqrt{2}\int_0^1 \cos\big((\frac{\pi}{2} + n\pi)z\big)(z-1)dz = \frac{-\sqrt{2}}{(\frac{\pi}{2} + n\pi)^2} \neq 0,$$

where $B_0 = Q_0$.

Consequently, utilizing Corollary 4, we can affirm that the control system (19) is X-null controllable for all times $T > 0$.

Remark 6 *If we consider the system* (19) *with the Dirichlet condition* $x(t, 1) = 0$ *replaced by the Neumann condition* $\frac{d}{dz}x(t, 1) = 0$ *and* $\frac{d^2}{dz^2}x(t, 0) = 0$ *replaced by* $\frac{d^3}{dz^3}x(t, 0) = 0$*, we can proceed in a similar manner as mentioned earlier to confirm the validity of hypotheses 1 and 2. However, it's important to note that condition* (14) *is not satisfied for all* $n \in \mathbb{N}$*. Consequently, in this scenario, the system* (19) *is not null controllable. Indeed, under these conditions, we have:*

$$\mu_n = -(n\pi)^4,\ n \in \mathbb{N}, \phi_0(z) = 1 \text{ and } \phi_n(z) = \sqrt{2}\cos\left(n\pi z\right) \text{ for all } n \in \mathbb{N}^*.$$

Now, we choose $\lambda = -1 \in \rho(A)$*, we have for any* $f \in \ker(-1 - A_m)$ *and* $y \in \mathbb{R}$

$$Qf = y \Longleftrightarrow \begin{cases} \frac{d^4}{dz^4} f(z) = f(z), \qquad z \in]0, 1[\\ \frac{d^3}{dz^3} f(0) = 0, \\ \frac{d}{dz} f(0) = y, \\ \frac{d}{dz} f(1) = \frac{d^3}{dz^3} f(1) = 0. \end{cases}$$

We get $\left(Q_{-1}y\right)(z) := yh(z) = y\left(\alpha_1 e^z + \alpha_2 e^{-z} + \alpha_3 \cos(z) + \alpha_4 \sin(z)\right)$*, for all* $y \in \mathbb{R}$ *and* $z \in [0, 1]$*, where*

$$\begin{aligned} \alpha_1 &= -\tfrac{1}{2(e^2-1)}, \\ \alpha_2 &= -\tfrac{e^2}{2(e^2-1)}, \\ \alpha_3 &= \tfrac{\cos(1)}{2\sin(1)}, \\ \alpha_4 &= \tfrac{1}{2} \end{aligned}$$

Furthermore, for $n = 0$*, by a simple integral, we obtain:*

$$\begin{aligned} \langle B_{-1}1, \phi_0\rangle_X &= \int_0^1 h(z)dz, \\ &= 0. \end{aligned}$$

This implies that the condition (14) *is not true. Therefore, according to Remark 5, the system* (19) *is not null controllable for all time* T*.*

6 Conclusion

In this contribution, we have delved into the null controllability aspects of an abstract boundary control system, focusing on a class of Riesz-spectral operators. Our study prove that the null controllability can be effectively characterized through the moment problem. However, it is important to note that this characterisation applies to the study of null controllability in certain classes of parabolic equations.

For our future endeavors, we aim to extend our exploration to the null controllability of bilinear boundary control systems. This specific type of system has been investigated in [22]. Nonetheless, it's worth noting that the method of moments proves to be less effective in this particular scenario. We anticipate that alternative approaches will be necessary to tackle the challenges posed by bilinear systems in the context of null controllability.

References

1. Allal B, Fragnelli G, Salhi J. Null controllability for a degenerate population equation with memory. Applied Mathematics & Optimization. 2022;86(3):41.
2. Carreño N, Cerpa E, Mercado A. Boundary controllability of a cascade system coupling fourth-and second-order parabolic equations. Systems & Control Letters. 2019;133: 104542.
3. Chorfi SE, Guermai GE, Khoutaibi A, Maniar L. Boundary null controllability for the heat equation with dynamic boundary conditions. arXiv preprint arXiv:2206.10701. (2022)
4. Guerrero S, Kassab K. Carleman estimate and null controllability of a fourth order parabolic equation in dimension $n \geq 2$. Journal de Mathématiques Pures et Appliquées. 2019;121:135–61.
5. Guo YJL. Null boundary controllability for a fourth order parabolic equation. Taiwan J Math. 2002;6(3):421–31.
6. Fattorini HO, Russell DL. Exact controllability theorems for linear parabolic equations in one space dimension. Arch Ration Mech Anal. 1971;43(4):272–92.
7. Micu S, Zuazua E. On the lack of null-controllability of the heat equation on the half-line. Trans Am Math Soc. 2001;353(4):1635–59.
8. Chowdhury S, Mitra D. Null controllability of the linearized compressible navier-stokes equations using moment method. J Evol Equ. 2015;15:331–60.
9. Ismailov MI, Oner I. Null boundary controllability for some biological and chemical diffusion problems. Evolution Equations and Control Theory. 2023;12(5):1287–99.
10. Oner I. The null boundary controllability for the mullins equation with periodic boundary conditions. An International Journal of Optimization and Control: Theories & Applications (IJOCTA). 2023;13(1):116–22.
11. Curtain RF, Zwart H. An Introduction to Infinite-dimensional Linear Systems Theory vol. 21. Springer, 2012
12. Vieru A. On null controllability of linear systems in banach spaces. Systems & control letters. 2005;54(4):331–7.
13. Greiner G. Perturbing the boundary-conditions of a generator. Houst J Math. 1987;13(2):213–29.
14. Fattorini HO. Boundary control systems. SIAM Journal on Control. 1968;6(3):349–85.
15. Slemrod M. Stabilization of boundary control systems. J Differential Equations. 1976;22(2):402–15.
16. Boulite S, Bouslous H, El Azzouzi M, Maniar L. Approximate positive controllability of positive boundary control systems. Positivity. 2014;18(2):375–93.
17. Engel KJ, Fijavž MK, Klöss B, Nagel R, Sikolya E. Maximal controllability for boundary control problems. Applied Mathematics & Optimization. 2010;62(2):205–27.
18. Russell DL. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. SIAM Rev. 1978;20(4):639–739.
19. El Azzouzi M, Lourini A, Laabissi M. Null controllability of networks systems. Journal of Dynamical and Control Systems, 1–18 (2022)
20. Arendt W. Semigroups and evolution equations: functional calculus, regularity and kernel estimates. In: Handbook of Differential Equations: Evolutionary Equations vol. 1, pp. 1–85. Elsevier, 2002.
21. Tucsnak, M., Weiss, G.: Observation and Control for Operator Semigroups. Springer, 2009.
22. Lourini A, El Azzouzi M, Laabissi M. Exponential stabilization of riesz-spectral bilinear boundary control systems. Systems & Control Letters. 2023;181: 105649.