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Multiplicity of Periodic Solutions for a Class of New Super-quadratic Damped Vibration Problems

Shan Jiang[1] · Huijuan Xu[1] · Guanggang Liu[1]

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Abstract

In this paper, we study the following damped vibration problem

$$\begin{cases} \ddot{x} + (q(t)I_{N\times N} + B)\dot{x} + \left(\frac{1}{2}q(t)B - A(t)\right)x + H_x(t,x) = 0, & t \in \mathbb{R}, \\ x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, & T > 0. \end{cases}$$

Under a new super-quadratic condition, we obtain a sequence of periodic solutions with the corresponding energy tending to infinity by using a fountain theorem.

Keywords Super-quadratic · Damped vibration problems · Fountain theorem · Periodic solutions

Mathematics Subject Classification (2010) 34C25 · 70H05

1 Introduction

Consider the following damped vibration problems

$$\begin{cases} \ddot{x} + (q(t)I_{N\times N} + B)\dot{x} + \left(\frac{1}{2}q(t)B - A(t)\right)x + H_x(t,x) = 0, & t \in \mathbb{R}, \\ x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, & T > 0, \end{cases} \tag{1.1}$$

✉ Guanggang Liu
lgg112@163.com

Shan Jiang
jianhshan0422@163.com

Huijuan Xu
xuhuijuan2737@163.com

1 School of Mathematical Sciences, Liaocheng University, Liaocheng 252000, People's Republic of China

where $x = x(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$, $I_{N\times N}$ is the $N \times N$ identity matrix, $q(t) \in L^1(\mathbb{R}, \mathbb{R})$ is T-periodic and satisfies $\int_0^T q(t)\,dt = 0$, $A(t) = [a_{ij}(t)]$ is a T-periodic symmetric $N \times N$ matrix-valued function with $a_{ij} \in L^\infty(\mathbb{R}, \mathbb{R})$ ($\forall i, j = 1, 2, \ldots, N$), $B = [b_{ij}]$ is an anti-symmetric $N \times N$ constant matrix, $H(t, x) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t, and $H_x(t, x)$ denotes its gradient with respect to the x variable.

The existence of periodic solutions for damped vibration problems has been extensively studied in recent decades. When $B = 0$ and $q(t) \equiv 0$, the problem (1.1) is the familiar second order Hamiltonian system, and there have been many results on the existence and multiplicity of periodic solutions (see [2, 7, 8, 13, 15–18, 20, 23] and references therein). When $B = 0$ and $q(t) \neq 0$, the authors [19] studied problem (1.1) and obtained the existence and multiplicity of periodic solutions by using variational methods. When $B \neq 0$ and $q(t) \neq 0$, many authors (see [3–6, 9–11, 14, 21]) have studied the existence and multiplicity of periodic solutions for Eq. 1.1 under various assumptions. In [10], the authors established the variational framework for problem (1.1) and obtained infinitely many nontrivial periodic solutions under the Ambrosetti-Rabinowitz super-quadratic condition by using a symmetric mountain pass theorem. In [21], the author obtained infinitely many periodic solutions for a class of second-order damped vibration systems under super-quadratic and sub-quadratic conditions by using a symmetric mountain pass theorem and an abstract critical point theorem. In [3–5], the author considered (1.1) with nonlinearities being asymptotically linear at infinity, being suplinear at infinity and being sublinear at both zero and infinity, and he obtained infinitely many nontrivial periodic solutions by using the variant fountain theorem [22].

In this paper, we study the multiplicity of periodic solutions for the problem (1.1) under a new super-quadratic condition (see (H_2) below) introduced by Tang and Wu [17], where the authors obtained the existence of periodic solutions for second-order Hamiltonian systems by using a local linking theorem [12]. Here by using a fountain theorem in [1], we shall obtain infinitely many periodic solutions for the problem (1.1) with the corresponding energy tending to infinity. Furthermore, we make the following assumptions:

(H_1) $\lim_{|x|\to\infty} \frac{H(t,x)}{|x|^2} = +\infty$ uniformly for $t \in [0, T]$;

(H_2) there exists $a > 0$ and $r_\infty > 0$ such that $(H_x(t, x), x) \geq \left(2 + \frac{a}{|x|^2}\right) H(t, x)$ for every $x \in \mathbb{R}^N$ with $|x| \geq r_\infty$ and $t \in [0, T]$;

(H_3) there exist $C_1 > 0$, $C_2 > 0$ and $p > 2$ such that

$$|H_x(t, x)| \leq C_1|x|^{p-1} + C_2$$

for $t \in [0, T]$, $x \in \mathbb{R}^N$.

Now we state our main result.

Theorem 1.1 *Assume that* $(H_1)-(H_3)$ *hold and* $H(t, x)$ *is even in* x, *then the problem* (1.1) *has infinitely many periodic solutions with the corresponding energy tending to infinity.*

Remark 1.2 We consider the following general case of (H_2):

$(H_2)_\theta$ there exists $a > 0$ and $r_\infty > 0$ such that $(H_x(t, x), x) \geq \left(2 + \frac{a}{|x|^\theta}\right) H(t, x)$ for some $\theta > 0$ and every $x \in \mathbb{R}^N$ with $|x| \geq r_\infty$ and $t \in [0, T]$.

Clearly, if $0 < \theta \leq 2$, then $(H_2)_\theta$ implies (H_2) and thus Theorem 1.1 also holds under (H_1), $(H_2)_\theta$, and (H_3). However, we do not know whether the theorem is still true for some $\theta > 2$. We think this is an interesting question.

Remark 1.3 Now we give some comparisons between the super-quadratic condition (H_2) and the super-quadratic conditions in related papers [4, 9, 10]. We will see that the super-quadratic condition (H_2) is weaker than those in [4, 9, 10]. In [10], the authors obtained infinitely many periodic solutions of Eq. 1.1 under the Ambrosetti-Rabinowitz super-quadratic condition

(AR) there exist a constant $\mu > 2$ and $r \geq 0$ such that $0 < \mu H(t,x) \leq (H_x(t,x), x)$ for all $x \in R^n \backslash \{0\}$ and $|x| \geq r$.

In [4], under more general super-quadratic conditions, the author obtained infinitely many periodic solutions for Eq. 1.1. He assumed (H_1) and

(SH_1) there exist constants $d_1 > 0$ and $\alpha_1 > 2$ such that

$$|H_x(t,x)| \leq d_1 \left(1 + |x|^{\alpha_1 - 1}\right);$$

(SH_2) $\frac{1}{2}(H_x(t,x), x) \geq H(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$;
(SH_3) there is a constant $b > 0$ such that

$$\liminf_{|x|\to\infty} \frac{(H_x(t,x),x) - 2H(t,x)}{|x|^{\alpha_1}} > b \quad \text{uniformly for} \quad t \in [0,T].$$

Later, the authors [9] obtained infinitely many periodic solutions for Eq. 1.1 under the super-quadratic conditions used in [4] and some weaker conditions, they assumed (H_1) and

(SH_1') $H(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$;
(SH_2') there exists $\alpha > 2$ such that

$$\lim_{|x|\to\infty} \frac{H_x(t,x)}{|x|^{\alpha-1}} < \infty \quad \text{uniformly for} \quad t \in [0,T];$$

(SH_3') there are constants $b > 0$ and $1 \leq \beta \in (\alpha - 2, \infty)$ such that

$$\liminf_{|x|\to\infty} \frac{(H_x(t,x),x) - 2H(t,x)}{|x|^{\beta}} > b \quad \text{uniformly for} \quad t \in [0,T].$$

Clearly, the conditions (SH_1)-(SH_3) or (SH_1')-(SH_3') imply (H_2) and (H_3). On the other hand, let $H(t,x) = |x|^2 \ln(1+|x|^2) + \sin|x|^2 - \ln(1+|x|^2)$ (see [17]). It is not difficult to see that $H(t,x)$ satisfies the conditions $(H_1) - (H_3)$ in Theorem 1.1. However, as pointed in [17], we have

$$\liminf_{|x|\to\infty} \frac{(H_x(t,x),x) - 2H(t,x)}{|x|^{\lambda}} = 0$$

for any $\lambda > 0$, so $H(t,x)$ does not satisfy the conditions (SH_3) and (SH_3'). Hence, the super-quadratic conditions in Theorems 1.1 are weaker than the above super-quadratic conditions.

The paper is organized as follows. In Section 2, we give the variational framework and some important preliminary lemmas. In Section 3, we prove the main result by using a fountain theorem.

2 Preliminaries

In this section, we will give some preliminaries and prove an important compactness result. We will use $\|\cdot\|_{L^p}$ to denote the norm of $L^p\left([0,T];\mathbb{R}^N\right)$ for any $p \geq 1$. Let $W := H_T^1$ be the usual Hilbert space defined by

$$H_T^1 = \left\{x : [0,T] \to \mathbb{R}^N \mid x \text{ is absolutely continuous, } x(0) = x(T),\ \dot{x} \in L^2\left([0,T], \mathbb{R}^N\right)\right\}$$

with the norm

$$\|x\|_0 = \left(\int_0^T e^{Q(t)}\left(|x|^2 + |\dot{x}|^2\right) dt\right)^{1/2},\ x \in W,$$

where $Q(t) = \int_0^t q(s)\,ds$. Since $q(t) \in L^1(\mathbb{R}, \mathbb{R})$ is T-periodic and satisfies $\int_0^T q(t)dt = 0$, we see that $Q(t)$ is a T-periodic continuous function. Then there exist two positive constants c_1 and c_2 such that

$$c_1 \leq e^{Q(t)} \leq c_2, \qquad \forall t \in [0,T]. \tag{2.1}$$

Hence, the norm $\|\cdot\|_0$ is equivalent to the usual one $\|\cdot\|_W$ on W, i.e.,

$$\|x\|_W = \left(\|\dot{x}\|_{L^2}^2 + \|x\|_{L^2}^2\right)^{\frac{1}{2}},\ x \in W.$$

We denote by $\langle\cdot,\cdot\rangle_0$ the inner product corresponding to $\|\cdot\|_0$ on W.

Define the functional φ on W by

$$\varphi(x) = \frac{1}{2}\int_0^T e^{Q(t)}\left[|\dot{x}|^2 + (Bx, \dot{x}) + (A(t)x, x)\right] dt - \int_0^T e^{Q(t)} H(t,x)dt,\ \ x \in W. \tag{2.2}$$

By Lemma 2.1 in [10], we see that φ is continuously differentiable on W. In addition, we have

$$\langle\varphi'(x), y\rangle = \int_0^T e^{Q(t)}\left[(\dot{x}, \dot{y}) - (B\dot{x}, y) - \frac{1}{2}q(t)(Bx, y) + (Ax, y)\right] dt - \int_0^T e^{Q(t)}(H_x(t,x), y)dt$$

for all $x, y \in W$. Then the solutions of problem (1.1) correspond to the critical points of φ (see Lemma 2.2 in [10]).

Let $L : W \to W$ be the linear operator defined by

$$\langle Lx, y\rangle_0 := \int_0^T e^{Q(t)}\left[(B\dot{x}, y) + \frac{1}{2}q(t)(Bx, y)\right] dt,\ \ \forall x, y \in W.$$

By Lemma 2.3 in [10], we have that L is a bounded self-adjoint linear operator and compact on W. Since B is an antisymmetry $N \times N$ constant matrix, using the integration by parts, we know that

$$\langle Lx, x\rangle_0 := \int_0^T e^{Q(t)}\left[(B\dot{x}, x) + \frac{1}{2}q(t)(Bx, x)\right] dt = \int_0^T e^{Q(t)}(B\dot{x}, x)dt.$$

It is not difficult to see that there exists a constant $b_0 > 0$ such that

$$\int_0^T e^{Q(t)}(B\dot{x}, x)dt \leq b_0\|x\|_0^2. \tag{2.3}$$

Let $K : W \to W$ be the operator defined by

$$\langle Kx, y\rangle_0 = \langle Lx, y\rangle_0 + \int_0^T e^{Q(t)}((I - A(t))x(t), y(t))dt,\ \ \forall x, y \in W.$$

Then K is a bounded self-adjoint linear operator and compact on W (see [10]). According to the classical spectral theory, we have the decomposition

$$W = W^0 \bigoplus W^- \bigoplus W^+,$$

where $W^0 := \ker(I - K)$, W^+ and W^- are the positive and negative spectral subspaces of $I - K$ on W respectively. Clearly, W^- and W^0 are finite dimensional. We can define a new equivalent norm $\|\cdot\|$ on W such that $\langle (I - K)x^\pm, x^\pm \rangle_0 = \pm\|x^\pm\|^2$ for $x^\pm \in W^\pm$ with the corresponding inner product denoted by $\langle \cdot, \cdot \rangle$. Then there exist positive constants c_1' and c_2' such that

$$c_1'\|x\|_0 \le \|x\| \le c_2'\|x\|_0. \tag{2.4}$$

We can rewrite the functional φ by

$$\begin{aligned} \varphi(x) &= \frac{1}{2}\langle (I - K)x, x\rangle_0 - \int_0^T e^{Q(t)} H(t, x)dt \\ &= \frac{1}{2}\left(\|x^+\|^2 - \|x^-\|^2\right) - \int_0^T e^{Q(t)} H(t, x)dt, \quad x \in W. \end{aligned} \tag{2.5}$$

Now we state an important lemma from [1] which will be used to prove the main result. Let X be a Hilbert space with $X = X^1 \oplus X^2$, where X^2 is a finite dimensional subspace of X. Let $X_1^1 \subset X_2^1 \subset \cdots$ be a sequence of finite dimensional subspaces of X^1 such that $\overline{\bigcup_{n=1}^{\infty} X_n^1} = X^1$. Let $X_n = X_n^1 \oplus X^2$. We say that $\Phi \in C^1(X, \mathbb{R})$ satisfies the $(C)_c^*$ condition at level $c \in \mathbb{R}$ with respect to $\{X_n\}$ if each sequence $\{x_j\}$ satisfying

$$x_j \in X_{n_j}, n_j \to \infty,\ \Phi(x_j) \to c,\ (1 + \|x_j\|)\left(\Phi|_{X_{n_j}}\right)'(x_j) \to 0$$

contains a subsequence which converges to a critical point of Φ.

Lemma 2.1 *(see Theorem 2.2 of [1]) Suppose that $\Phi \in C^1(X, \mathbb{R})$ is even, i.e., $\Phi(-x) = \Phi(x)$ and*

(a_1) *Φ satisfies the $(C)_c^*$ condition at level $c > 0$ with respect to $\{X_n\}$;*
(a_2) *there exists a sequence $r_k > 0$, $k \in \mathbb{Z}^+$, such that*

$$b_k := \inf_{x \in X_{k-1}^\perp, \|x\| = r_k} \Phi(x) \to \infty;$$

(a_3) *there exists a sequence $R_k > r_k$ such that, for $x = x^1 + x^2 \in X_k = X_k^1 \bigoplus X^2$ with $x^1 \in X_k^1$, $x^2 \in X^2$, and $\max\left\{\|x^1\|, \|x^2\|\right\} = R_k$, one has $\Phi(x) < 0$ and*

$$d_k := \sup_{x \in X_k, \max\{\|x^1\|, \|x^2\|\} \le R_k} \Phi(x) < \infty.$$

Then Φ has an unbounded sequence of critical values.

Here we set $X = W$, $X^1 = W^+$ and $X^2 = W^- \bigoplus W^0$. Let $\{e_m\}_{m=1}^\infty$ be an orthonormal basis of X^1. And define

$$X_n^1 = \operatorname{span}\{e_1, e_2, \cdots, e_n\}, n \in \mathbb{N},$$

$$X_n = X_n^1 \bigoplus X^2.$$

Lemma 2.2 *Suppose that* (H_1) *and* (H_2) *hold, then* φ *satisfies the* $(C)_c^*$ *condition at level* $c > 0$ *with respect to* $\{X_n\}$.

Proof Let $\{x_j\}$ be a sequence in W such that

$$x_j \in X_{n_j},\ n_j \to \infty,\ \varphi(x_j) \to c,\ (1 + \|x_j\|_0)(\varphi|_{X_{n_j}})'(x_j) \to 0. \tag{2.6}$$

Now we prove that the sequence $\{x_j\}$ is bounded in W. If not, there is a subsequence of $\{x_j\}$ (for simplicity still denoted by $\{x_j\}$) satisfying $\|x_j\|_0 \to \infty$ as $j \to \infty$. Let $\omega_j = \frac{x_j}{\|x_j\|_0}$, then $\{\omega_j\}$ is bounded in W. Hence, we may assume that for some $\omega \in W$, there is a subsequence of $\{\omega_j\}$ (still denoted by $\{\omega_j\}$) such that

$$\omega_j \rightharpoonup \omega \quad \text{weakly in W}, \tag{2.7}$$

$$\omega_j \to \omega \quad \text{in C } \left([0,T], \mathbb{R}^N\right). \tag{2.8}$$

We claim that $\omega(t) \equiv 0$. If not, set $E_1 = \{t \in [0,T] : |\omega(t)| > 0\}$, then $|E_1| > 0$, where $|E_1|$ is the Lebesgue measure of E_1. Since $\|x_j\|_0 \to +\infty$, we have $|x_j(t)| \to \infty$ as $j \to \infty$ for any $t \in E_1$. Then by (H_1), we have that

$$\lim_{j\to\infty} \frac{H(t, x_j(t))}{|x_j(t)|^2} = +\infty \quad \text{on } E_1. \tag{2.9}$$

From (H_1) and $H(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, there exists a constant $C_3 > 0$ such that

$$H(t,x) \geq -C_3 \tag{2.10}$$

for any $x \in \mathbb{R}^N$ and $t \in [0,T]$. Then by Eqs. 2.1 and 2.10, we obtain

$$\begin{aligned} \int_0^T \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2} dt &= \int_{E_1} \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2} dt + \int_{[0,T]\setminus E_1} \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2} dt \\ &\geq \int_{E_1} \frac{e^{Q(t)}H(t,x_j)}{|x_j|^2} |\omega_j|^2 dt - \frac{c_2 C_3 T}{\|x_j\|_0^2}. \end{aligned} \tag{2.11}$$

By Eqs. 2.9, 2.11 and Fatou's Lemma, we have

$$\liminf_{j\to\infty} \int_0^T \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2} dt \geq \liminf_{j\to\infty} \int_{E_1} \frac{e^{Q(t)}H(t,x_j)}{|x_j|^2} |\omega_j|^2 dt = +\infty. \tag{2.12}$$

Let $a_0 > 0$ be a constant such that

$$|(A(t)x, x)| \leq a_0 |x|^2, \qquad \forall x \in \mathbb{R}^N. \tag{2.13}$$

By Eqs. 2.1, 2.2, 2.6–2.8, 2.13 and Hölder inequality, for sufficiently large j we have that

$$\begin{aligned}\left|\int_0^T \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2}dt-\frac{1}{2}\right| &\le \frac{|\varphi(x_j)|}{\|x_j\|_0^2}+\frac{1}{2}\left|\int_0^T e^{Q(t)}\left[|\omega_j|^2-(A(t)\omega_j,\omega_j)\right]dt\right| \\ &\quad +\frac{1}{2}\left|\int_0^T e^{Q(t)}(B\omega_j,\dot{\omega}_j)dt\right| \\ &\le \frac{c+1}{\|x_j\|_0^2}+\frac{1+a_0}{2}c_2T\|\omega_j\|_{L^\infty}^2+\frac{\|B\|\|\omega_j\|_{L^\infty}}{2}\int_0^T e^{Q(t)}|\dot{\omega}_j|dt \\ &\le \frac{c+1}{\|x_j\|_0^2}+\frac{1+a_0}{2}c_2T\|\omega_j\|_{L^\infty}^2 \\ &\quad +\frac{\|B\|\|\omega_j\|_{L^\infty}}{2}\left(\int_0^T e^{Q(t)}dt\right)^{\frac{1}{2}}\left(\int_0^T e^{Q(t)}|\dot{\omega}_j|^2dt\right)^{\frac{1}{2}} \\ &\le \frac{c+1}{\|x_j\|_0^2}+\frac{1+a_0}{2}c_2T\|\omega_j\|_{L^\infty}^2+\frac{\|B\|\sqrt{c_2T}}{2}\|\omega_j\|_{L^\infty}\|\omega_j\|_0 \\ &\le \frac{c+1}{\|x_j\|_0^2}+\frac{1+a_0}{2}c_2T\|\omega_j\|_{L^\infty}^2+\frac{\|B\|\sqrt{c_2T}}{2}\|\omega_j\|_{L^\infty} \qquad (2.14)\end{aligned}$$

Then using $\|x_j\|_0 \to \infty$ and the Sobolev inequality $\|\omega_j\|_\infty \le C\|\omega_j\|_0 = C$, by Eq. 2.14, we have that

$$\liminf_{j\to\infty}\int_0^T \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2}dt \le \frac{1}{2}+\frac{1+a_0}{2}c_2C^2T+\frac{\|B\|\sqrt{c_2T}}{2}C, \qquad (2.15)$$

which contradicts with Eq. 2.12. Therefore, the claim is proved.

By the above claim and Eq. 2.14, we see that

$$\lim_{j\to\infty}\int_0^T \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2}dt = \frac{1}{2}. \qquad (2.16)$$

By (H_1) and (H_2), there exists a constant r_2 with $r_2 > r_\infty$ such that

$$(H_x(t,x),x)-2H(t,x) \ge \frac{a}{|x|^2}H(t,x) \ge 0 \qquad (2.17)$$

for any $x \in \mathbb{R}^N$ with $|x| \ge r_2$ and $t \in [0,T]$. And there exist positive constants C_5 and C_6 such that

$$|H(t,x)| \le C_5, \quad |(H_x(t,x),x)-2H(t,x)| \le C_6 \qquad (2.18)$$

for any $x \in \mathbb{R}^N$ with $|x| \le r_2$ and $t \in [0, T]$. Then by Eqs. 2.6, 2.17, and 2.18, we have that

$$\begin{aligned}
\left|\int_0^T \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2}dt\right| &\le \int_0^T \frac{e^{Q(t)}|H(t,x_j)|}{\|x_j\|_0^2}dt \\
&= \int_{\{t||x_j(t)|\le r_2\}} \frac{e^{Q(t)}|H(t,x_j)|}{\|x_j(t)\|_0^2}dt + \int_{\{t||x_j(t)|>r_2\}} \frac{e^{Q(t)}H(t,x_j)}{\|x_j\|_0^2}dt \\
&\le \frac{c_2C_5T}{\|x_j(t)\|_0^2} + \int_{\{t||x_j(t)|>r_2\}} \frac{e^{Q(t)}H(t,x_j)}{|x_j|^2}|\omega_j|^2dt \\
&\le \frac{c_2C_5T}{\|x_j(t)\|_0^2} + \|\omega_j\|_{L^\infty}^2 \int_{\{t||x_j(t)|>r_2\}} \frac{e^{Q(t)}H(t,x_j)}{|x_j|^2}dt \\
&\le \frac{c_2C_5T}{\|x_j(t)\|_0^2} + \frac{\|\omega_j\|_{L^\infty}^2}{a} \int_{\{t||x_j(t)|>r_2\}} e^{Q(t)}(H_x(t,x_j),x_j) - 2H(t,x_j)dt \\
&\le \frac{c_2C_5T}{\|x_j(t)\|_0^2} + \frac{\|\omega_j\|_{L^\infty}^2}{a} \int_{\{t||x_j(t)|\le r_2\}} e^{Q(t)}|(H_x(t,x_j),x_j) - 2H(t,x_j)|dt \\
&\quad + \frac{\|\omega_j\|_{L^\infty}^2}{a}\int_0^T e^{Q(t)}(H_x(t,x_j),x_j) - 2H(t,x_j)dt \\
&\le \frac{c_2C_5T}{\|x_j(t)\|_0^2} + \frac{c_2C_6T\|\omega_j\|_{L^\infty}^2}{a} + \frac{\|\omega_j\|_{L^\infty}^2}{a}(2\varphi(x_j) - \langle\varphi'(x_j),x_j\rangle) \\
&\le \frac{c_2C_5T}{\|x_j(t)\|_0^2} + \frac{(c_2C_6T + 3c)}{a}\|\omega_j\|_{L^\infty}^2 \to 0,
\end{aligned} \tag{2.19}$$

which contradicts (2.16). Hence, $\{x_j\}$ is bounded in W. Then by an argument similar as Proposition 4.1 in [13], we conclude that the $(C)_c^*$ condition is satisfied. □

3 Proof of Theorem 1.1

In this section, we shall use Lemma 2.1 to prove Theorem 1.1.

Lemma 3.1 *Suppose* $(H_1) - (H_3)$ *hold, then there exists a sequence* $r_k > 0$ *such that*

$$b_k := \inf_{x \in X_{k-1}^\perp, \|x\| = r_k} \varphi(x) \to \infty.$$

Proof Set

$$\mu_k := \sup_{x \in X_{k-1}^\perp, \|x\|=1} \|x\|_{L^p}. \tag{3.1}$$

Similarly as [1], we prove that

$$\mu_k \to 0 \tag{3.2}$$

as $k \to \infty$. In fact, if not, there must exist a constant $\varepsilon_0 > 0$ and a sequence $\{x_j\}$ in X such that $x_j \in X_{k_j-1}^\perp$, $\|x_j\| = 1$, $\|x_j\|_{L^p} \ge \varepsilon_0$, and $k_j \to \infty$ as $j \to \infty$. Thus, for any $y \in W$, we have

$$|\langle y, x_j\rangle| = |\langle P_{k_j-1}^\perp y, x_j\rangle| \le \|P_{k_j-1}^\perp y\| \to 0$$

as $j \to \infty$, where $P^{\perp}_{k_j-1}$ is the projection onto $X^{\perp}_{k_j-1}$. Therefore, we have $x_j \rightharpoonup 0$ in X. Since the embedding $X \subset L^p([0,T])$ is compact, we get $x_j \to 0$ in $L^p([0,T])$, which is contrary to $\|x_j\|_{L^p} \geq \varepsilon_0$.

By (H_3), there exists $C_7 > 0$ and $C_8 > 0$ such that

$$|H(t,x)| \leq C_7|x|^p + C_8 \tag{3.3}$$

for every $x \in \mathbb{R}^N$ and $t \in [0,T]$. Note that $X^{\perp}_{k-1} \subset W^+$ for every $k \in \mathbb{Z}^+$. Then by Eqs. 3.1 and 3.3, for $x \in X^{\perp}_{k-1}$, we have that

$$\begin{aligned}
\varphi(x) &= \frac{1}{2}\|x\|^2 - \int_0^T e^{Q(t)}H(t,x)dt \\
&\geq \frac{1}{2}\|x\|^2 - C_7\int_0^T e^{Q(t)}|x|^p dt - C_8\int_0^T e^{Q(t)}dt \\
&\geq \frac{1}{2}\|x\|^2 - C_9\|x\|^p_{L^p} - C_{10} \\
&\geq \frac{1}{2}\|x\|^2 - C_9\mu_k^p\|x\|^p - C_{10},
\end{aligned} \tag{3.4}$$

where $C_9 = C_7c_2$ and $C_{10} = C_8c_2T$. Let $r_k := \left(C_9p\mu_k^p\right)^{\frac{1}{2-p}}$. For every $x \in X^{\perp}_{k-1}$ with $\|x\| = r_k$, by Eq. 3.4, we can obtain that

$$\varphi(x) \geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(C_9p\mu_k^p\right)^{\frac{2}{2-p}} - C_{10}. \tag{3.5}$$

Hence, by $p > 2$, Eqs. 3.2 and 3.5, we have that

$$b_k := \inf_{x\in X^{\perp}_{k-1},\|x\|=r_k} \varphi(x) \to \infty$$

as $k \to \infty$. □

Lemma 3.2 *Suppose that $(H_1)-(H_3)$ hold, then there exists a sequence $R_k > r_k$ such that for $x = x^1 + x^2 \in X_k = X_k^1 \bigoplus X^2$ with $x^1 \in X_k^1$, $x^2 \in X^2$ and* $\max\{\|x^1\|, \|x^2\|\} = R_k$, *one has $\varphi(x) < 0$ and*

$$d_k := \sup_{x\in X_k,\max\{\|x^1\|,\|x^2\|\}\leq R_k} \varphi(x) < \infty.$$

Proof By (H_1), for every $M > 0$, there exists a constant $C_M > 0$ such that

$$e^{Q(t)}H(t,x) \geq M|x|^2 - C_M \tag{3.6}$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^N$. Recall that $X_k^1 \subset W^+$ and $X^2 = W^0 \bigoplus W^-$. Then for every $k \in \mathbb{Z}^+$ and $x = x^1 + x^2 \in X_k = X_k^1 \bigoplus X^2$ with $x^1 \in X_k^1$, $x^2 \in X^2$, by Eqs. 2.5 and 3.6, we have that

$$\begin{aligned}
\varphi(x) &= \frac{1}{2}\left(\|x^1\|^2 - \|x^{2-}\|^2\right) - \int_0^T e^{Q(t)}H(t,x)dt \\
&\leq \frac{1}{2}\left(\|x^1\|^2 - \|x^{2-}\|^2\right) - \int_0^T \left[M|x|^2 - C_M\right]dt \\
&\leq \frac{1}{2}\|x\|^2 - M\|x\|^2_{L^2} + C'_M,
\end{aligned} \tag{3.7}$$

where $C'_M = C_M T$. Note $X_k \subset X$ is a finite dimensional subspace, it follows that the norms $\|\cdot\|$ and $\|\cdot\|_{L^2}$ are equivalent on X_k. Hence, we can choose a sufficiently large M such that

$$\|x\|^2 \leq M\|x\|_{L^2}^2, \qquad \forall x \in X_k. \tag{3.8}$$

Let $R_k = \max\{\sqrt{2C'_M + 1}, r_k + 1\}$, then $R_k > r_k$. For $x \in X_k$ with $\max\{\|x^1\|, \|x^2\|\} = R_k$, since $\|x\| \geq \max\{\|x^1\|, \|x^2\|\}$, it follows from Eqs. 3.7 and 3.8 that

$$\varphi(x) \leq -\frac{1}{2}\|x\|^2 + C'_M \leq -\frac{1}{2} < 0. \tag{3.9}$$

Recall that $\varphi \in C^1(X, \mathbb{R})$ and X_k is finite dimensional, it is easy to see that

$$d_k := \sup_{x \in X_k, \|x\| \leq R_k} \varphi(x) < \infty.$$

□

Proof of Theorem 1.1 Since $H(t, x)$ is even in x, we have that $\varphi(x)$ is an even functional on X. By Lemma 2.2, we see that the assumption (a_1) in Lemma 2.1 holds. From Lemma 3.1, we see that the assumption (a_3) in Lemma 2.1 holds. By Lemma 3.2, we have that the assumption (a_3) in Lemma 2.1 holds. Therefore, by Lemma 2.1, the problem (1.1) has infinitely many periodic solutions $\{x_j\}$ with the corresponding energy $\varphi(x_j) \to \infty$ as $j \to \infty$. □

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Declarations

Conflict of Interest The authors declare no competing interests.

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