Journal of Dynamical and Control Systems (2023) 29:703–719
https://doi.org/10.1007/s10883-022-09609-x

Exact Boundary Controllability of the Linear Biharmonic Schrödinger Equation with Variable Coefficients

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Received: 30 December 2021 / Revised: 20 May 2022 / Accepted: 13 July 2022 /
Published online: 2 August 2022

Abstract
In this paper, we study the exact boundary controllability of the linear fourth-order Schrödinger equation, with variable physical parameters and clamped boundary conditions on a bounded interval. The control acts on the first spatial derivative at the right endpoint. We prove that this control system is exactly controllable at any time $T > 0$. The proofs are based on a detailed spectral analysis and the use of nonharmonic Fourier series.

Keywords Biharmonic Schrödinger · Boundary control · Fourier series · Riesz basis

Mathematics Subject Classification (2010) 93B05 · 93B07 · 93B12 · 93B60

1 Introduction

The fourth-order cubic nonlinear Schrödinger equation or the so-called biharmonic cubic non-linear Schrödinger equation reads as follows

$$i\partial_t y + \partial_x^4 y - \partial_x^2 y - \mu|y|^2 y = 0, \tag{1}$$

where y is a complex-valued function and μ is a real constant. This equation has been modeled by Karpman [23] and Karpman and Shagalov [24] in order to describe the propagation of intense laser beams in a bulk medium with Kerr nonlinearity when small fourth-order dispersion is taken into account. The fourth-order cubic nonlinear Schrödinger Eq. 1 has various applications in several fields of physics, such as nonlinear optics, plasma physics, superconductivity, and quantum mechanics. We refer to the book of Fibich [19], and for more details, see also [11, 15, 30].

The well-posedness and the dynamic properties of the biharmonic Schrödinger Eq. 1 have been extensively studied from the mathematical perspective, see the paper of Pausader

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[31], see the papers of Capistrano-Filho et al. [16, 17], and also [14, 32] and the references therein.

In this work, we are interested in the study of the exact boundary controllability of a quantum mechanical particle that moves in a one-dimensional non-homogeneous box of length $\ell > 0$. Its wavefunction y evolves according to the linear biharmonic Schrödinger Eq. 1 for $\mu = 0$, with variable physical parameters. More precisely, we consider the following quantum mechanical system

$$\begin{cases} i\rho(x)\partial_t y = -\partial_x^2(\sigma(x)\partial_x^2 y) + \partial_x(q(x)\partial_x y), & (t,x) \in (0,T)\times(0,\ell),\\ y(t,0) = \partial_x y(t,0) = y(t,\ell) = 0,\ \partial_x y(t,\ell) = f(t), & t \in (0,T),\\ y(0,x) = y_0(x), & x \in (0,\ell), \end{cases} \tag{2}$$

where f is a control that acts at the right end $x = \ell$, and y_0 is the initial wavefunction of the particle at time $t = 0$. Throughout the paper, we assume the following assumptions on the coefficients:

$$\rho, \sigma \in H^2(0,\ell),\ q \in H^1(0,\ell), \tag{3}$$

and there exist constants $\rho_0,\ \sigma_0 > 0$, such that

$$\rho(x) \geq \rho_0,\quad \sigma(x) \geq \sigma_0,\quad q(x) \geq 0,\quad x \in 0,\ell. \tag{4}$$

For system (2), the appropriate control notion to study is the exact controllability, which is defined as follows: System (2) is said to be exactly controllable at time $T > 0$ if, given any initial state y_0, there exists a control f such that the corresponding wavefunction $y = y(t,x)$ of the particle satisfies $y(T,.) = 0$.

Let us now describe the existing results on stabilization and control of the Biharmonic Schrödinger system (2). In the case where $\sigma \equiv 0$, we recover the classical second-order Schrödinger equation with variable coefficients occupying the interval $(0,\ell)$. In this context, the stabilization of the second-order Schrödinger equation has been thoroughly studied, see for instance [2–4, 9]. We also refer to [1, 5–8] for related results on exact controllability of the second-order Schrödinger equation, see also [18, 21], and the references therein. The first result on exact controllability of the linear biharmonic Schrödinger Eq. 1 for $\mu = 0$ on a bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 1$, is established by Zheng and Zhongcheng [35]. In that paper, the authors proved that the linearized system

$$\begin{cases} i\partial_t y + \Delta^2 y = 0, & (t,x) \in (0,T)\times\Omega,\\ y = 0,\ \dfrac{\partial y}{\partial \nu} = f\chi_{\Gamma_0}, & (t,x) \in (0,T)\times\partial\Omega\\ y(0,x) = y_0, & x \in \Omega, \end{cases} \tag{5}$$

is exactly controllable at any positive time T, where the control $f \in L^2((0,T)\times\Gamma_0)$ and $\Gamma_0 \subset \partial\Omega$. Their proof uses the Hilbert Uniqueness Method “Lions’ HUM” (cf. Lions [26, 27]) and the multiplier techniques [25]. Later on, Wen et al. [32] proved the well-posedness and the exact controllability for the linear fourth-order Schrödinger system (5) with the boundary observation

$$z(t,x) = -i\Delta((\Delta^2)^{-1}y(t,x)),\ (t,x) \in (0,T)\times\Gamma_0.$$

As a consequence, they established the exponential stability of the closed-loop system under the output feedback $f = -kz$ for any $k > 0$. The same authors in [33] extended these results to the case of a linear fourth-order multi-dimensional Schrödinger equation with hinged boundary control and collocated observation.

The inverse problem of retrieving a stationary potential from boundary measurements for the one-dimensional linear system (2) with $\rho \equiv \sigma \equiv 1$ and $f \equiv 0$ was studied by Zheng [34]. To this end, the author proved a global Carleman estimate for the corresponding fourth-order operator. Exact controllability result is established recently by Gao [22] when the linear system (2) where $\rho \equiv \sigma \equiv 1$ and $q \equiv 0$ has a particular structure. Indeed, in [22], the author considers a forward and backward stochastic fourth-order Schrödinger equation and, again, uses Carleman inequalities for the adjoint problem for proving the exact controllability result. More recently, the global stabilization and exact controllability properties were studied by Capistrano-Filho et al. [15] for the biharmonic cubic non-linear Schrödinger Eq. 1 on a periodic domain $\mathbb{T}$ with internal control supported on an arbitrary sub-domain of $\mathbb{T}$. More precisely, by means of some properties of propagation of compactness and regularity in Bourgain spaces, they first showed that the system is globally exponentially stabilizable. Then, they used this with a local controllability result to get the global controllability for the associated control system. In particular, for the proof of the local controllability result, they combined a perturbation argument with the fixed point theorem of Picard.

To our knowledge, the exact controllability of the fourth-order Schrödinger equation with variable coefficients is still unknown. In this paper, we prove that the linear control system (2) is exactly controllable at any time $T > 0$, where the control $f \in L^2(0, T)$ and the initial condition $y_0 \in H^{-2}(0, \ell)$. Our approach is essentially based on the qualitative theory of fourth-order linear differential equations, and on a precise asymptotic analysis of the eigenvalues and eigenfunctions. Firstly, we prove that all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ associated to the control system (2) with $f(t) \equiv 0$ are algebraically simple. We show that the second derivative of each eigenfunction ϕ_n, $n \in \mathbb{N}^*$, associated with the uncontrolled system does not vanish at the end $x = \ell$. Secondly, by a precise computation of the asymptotics of the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$, we establish that the spectral gap satisfies the following asymptotic

$$|\lambda_{n+1} - \lambda_n| = \frac{4\pi^4}{\gamma^4} n^3 + \mathcal{O}(n^2),\ \gamma := \int_0^\ell \left(\frac{\rho(t)}{\sigma(t)}\right)^{\frac{1}{4}} dt.$$

As a result of the theory of non-harmonic Fourier series and a variant of Ingham's inequality due to Beurling (e.g., [18]), we derive the observability inequalities for the adjoint system at any time $T > 0$, i.e.,

$$C_T^{-1} \|z_0\|^2_{H_0^2(0,\ell)} \le \int_0^T |\partial_x^2 z(t,\ell)|^2 dt \le C_T \|z_0\|^2_{H_0^2(0,\ell)} \tag{6}$$

for some positive constant $C_T > 0$, depending on T, where z is the solution of System (2) without control. Finally, we apply the Lions' HUM to deduce the exact controllability result for System (2).

The rest of the paper is divided as follows: In Section 2, we establish the well-posedness of System (2) without control. In Section 3, we prove the simplicity of all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ and we determinate the asymptotics of the associated spectral gap. In Section 4, we prove the observability estimate (6). Finally in Section 5, we prove the exact controllability result for the linear control problem (2).

2 Well-posedness of the Uncontrolled System

In this section, we show how solutions of System (2) without control can be developed in terms of Fourier series. As a consequence, we establish the existence and uniqueness of solutions of the uncontrolled system (2) with $f(t) \equiv 0$. Towards this end, we consider the following system

$$\begin{cases} i\rho(x)\partial_t z = -\partial_x^2(\sigma(x)\partial_x^2 z) + \partial_x(q(x)\partial_x z), & (t,x) \in (0,T) \times (0,\ell), \\ z(t,0) = \partial_x z(t,0) = z(t,\ell) = \partial_x z(t,\ell) = 0, & t \in (0,T), \\ z(0,x) = z_0, & x \in (0,\ell). \end{cases} \tag{7}$$

First of all, let $L^2_\rho(0,\ell)$ be the weighted Lebesgue space of all complex-valued functions defined on $(0,\ell)$, which is equipped with the inner product

$$\langle y, z\rangle_{L^2_\rho(0,\ell)} = \int_0^\ell y(x)\bar{z}(x)\rho(x)dx, \ \forall\, y, z \in L^2_\rho(0,\ell), \tag{8}$$

where $\bar{\bullet}$ denotes the conjugate of $\bullet$. In what follows, we denote by $H^k(0,\ell)$ the $L^2_\rho(0,\ell)-$based Sobolev spaces for $k > 0$. We consider the following Sobolev space

$$H_0^2(0,\ell) := \left\{y \in H^2(0,\ell) \ : \ y(0) = y'(0) = y(\ell) = y'(\ell) = 0\right\}$$

endowed with the norm

$$\|y\|_{H_0^2(0,\ell)} = \|y''\|_{L^2_\rho(0,\ell)}, \ \forall\, y \in H_0^2(0,\ell). \tag{9}$$

It is easy to show by Rellich's theorem (e.g., [25]) that the space $H_0^2(0,\ell)$ is densely and compactly embedded in the space $L^2_\rho(0,\ell)$. In the sequel, we introduce the operator $\mathcal{A}$ defined in $L^2_\rho(0,\ell)$ by setting:

$$\mathcal{A}y = \rho^{-1}((\sigma y'')'' - (qy')'),$$

on the domain

$$\mathcal{D}(\mathcal{A}) = H^4(0,\ell) \cap H_0^2(0,\ell),$$

which is dense in $L^2_\rho(0,\ell)$.

Lemma 2.1 *The linear operator $\mathcal{A}$ is positive and self-adjoint such that $\mathcal{A}^{-1}$ is compact. Moreover, the spectrum of $\mathcal{A}$ is discrete and consists of a sequence of positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ tending to ∞:*

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots \underset{n\to\infty}{\longrightarrow} \infty,$$

where $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$. The corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ can be chosen to form an orthonormal basis in $L^2_\rho(0,\ell)$.

Proof Let $y \in \mathcal{D}(\mathcal{A})$, then using an integration by parts, we have

$$\langle \mathcal{A}y, y\rangle_{L^2_\rho(0,\ell)} = \int_0^\ell ((\sigma(x)y''(x))'' - (q(x)y'(x))')\bar{y}(x)dx$$
$$= \int_0^\ell \sigma(x)|y''(x)|^2dx + q(x)|y'(x)|^2dx.$$

Since $\sigma > 0$ and $q \geq 0$, then

$$\langle \mathcal{A}y, y\rangle_{L^2_\rho(0,\ell)} > 0 \text{ for } y \not\equiv 0,$$

and hence, the quadratic form has positive real values, which implies that the linear operator $\mathcal{A}$ is symmetric. Furthermore, it is easy to show that $Ran(\mathcal{A} - iI) = L^2_\rho(0,\ell)$, and this means that $\mathcal{A}$ is self-adjoint. Since, by Rellich's theorem (e.g., [25]) the space $H^2_0(0,\ell)$ is continuously and compactly embedded in the space $L^2_\rho(0,\ell)$, then $\mathcal{A}^{-1}$ is compact in $L^2_\rho(0,\ell)$. The lemma is proved.

Now, we give a characterization of some fractional powers of the linear operator $\mathcal{A}$ which will be useful to give a description of the solutions of Problem (7) in terms of Fourier series. According to Lemma 2.1, the operator $\mathcal{A}$ is positive and self-adjoint, and hence it generates a scale of interpolation spaces $\mathcal{H}_\theta$, $\theta \in \mathbb{R}$, see [25, Chapter 1]. For $\theta \geq 0$, the space $\mathcal{H}_\theta$ coincides with $\mathcal{D}(\mathcal{A}^\theta)$ and is equipped with the norm $\|u\|^2_\theta = \langle \mathcal{A}^\theta u, \mathcal{A}^\theta u\rangle_{L^2_\rho(0,\ell)}$, and for $\theta < 0$ it is defined as the completion of $L^2_\rho(0,\ell)$ with respect to this norm. Furthermore, we have the following spectral representation of space $\mathcal{H}_\theta$,

$$\mathcal{H}_\theta = \left\{ u(x) = \sum_{n\in\mathbb{N}^*} c_n\Phi_n(x) \ : \ \|u\|^2_\theta = \sum_{n\in\mathbb{N}^*} \lambda_n^{2\theta}|c_n|^2 < \infty \right\}, \tag{10}$$

where $\theta \in$ R, and the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ are defined in Lemma 2.1. In particular,

$$\mathcal{H}_0 = L^2_\rho(0,\ell) \text{ and } \mathcal{H}_{1/2} = H^2_0(0,\ell),$$

where $L^2_\rho(0,\ell)$ is equipped with the inner product (8). Obviously, the solutions of Problem (7) can be written as

$$z(t,x) = \sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n(x),$$

where the Fourier coefficients are given by

$$c_n := \int_0^\ell z_0(x)\bar{\Phi}_n(x)\rho(x)dx, \ n \in \mathbb{N}^*,$$

and $(c_n) \in \ell^2(\mathbb{N}^*)$. Let us denote by $\mathcal{E}_\theta$ the θ–energy associated to the space $\mathcal{H}_\theta$, then

$$\mathcal{E}_\theta(t) = \|z\|^2_\theta = \sum_{n\in\mathbb{N}^*} \lambda_n^{2\theta}|c_n e^{i\lambda_n t}|^2$$
$$= \sum_{n\in\mathbb{N}^*} \lambda_n^{2\theta}|c_n|^2 = \mathcal{E}_\theta(0),$$

which establishes the conservation of energy along time. As a consequence, by [25, Theorem 1.1], we have the following existence and uniqueness result for Problem (7).

Proposition 2.2 *Let $\theta \in \mathrm{R}$ and $z_0 \in \mathcal{H}_\theta$. Then, Problem (7) has a unique solution $z \in C([0,T], \mathcal{H}_\theta)$, which is written in Fourier series as*

$$z(t,x) = \sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n(x), \tag{11}$$

where $z_0 = \sum_{n\in\mathbb{N}^*} c_n\Phi_n$. Moreover, the θ−energy of System (7) is conserved along the time.

3 Spectral Analysis

In this section, we investigate the main properties of all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the operator $\mathcal{A}$. On one hand, we prove that all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ are algebraically simple, and then, the second derivatives of the corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ do not vanish at $x = \ell$. On the other hand, we establish that the spectral gap "$|\lambda_{n+1} - \lambda_n|$" is uniformly positive. To this end, we consider the following spectral problem which arises by applying separation of variables to system (7),

$$\begin{cases} (\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, & x \in (0,\ell), \\ \phi(0) = \phi'(0) = \phi(\ell) = \phi'(\ell) = 0. \end{cases} \tag{12}$$

It is clear that Problem (12) is equivalent to the following spectral problem

$$\mathcal{A}\phi = \lambda\phi, \quad \phi \in \mathcal{D}(\mathcal{A}),$$

i.e., the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the operator $\mathcal{A}$ and Problem (12) coincide together with their multiplicities. One has:

Theorem 3.1 *All the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the spectral problem (12) are simple such that:*

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \underset{n\to\infty}{\longrightarrow} \infty.$$

Moreover, the corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ satisfy

$$\Phi_n''(\ell) \neq 0 \ \ \forall n \in \mathbb{N}^*. \tag{13}$$

Our main tool in proving this is the following result [13, Lemma 3.2].

Lemma 3.2 *Let u be a nontrivial solution to the linear fourth-order differential equation defined on the segment $[0,\ell]$:*

$$(\sigma(x)u'')'' - (q(x)u')' - \rho(x)u = 0,$$

where the functions $\rho(x) > 0, \sigma(x) > 0$, and $q(x) \geq 0$. If u, u', u'' and $\mathcal{T}u = (\sigma(x)u'')' - q(x)u'$ are nonnegative at $x = 0$ (but not all zero), then they are positive for all $x > 0$. If $u, -u', u''$ and $(-\mathcal{T}u)$ are nonnegative at $x = \ell$ (but not all zero), then they are positive for all $x < \ell$.

Proof of Theorem 3.1 First, we prove that the space of solutions, denoted by Λ_λ, of the following boundary value problem

$$\begin{cases} (\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, & x \in (0,\ell), \\ \phi(0) = \phi'(0) = \phi'(\ell) = 0, \end{cases} \tag{14}$$

is a one-dimensional subspace for $\lambda > 0$, i.e., $\dim \Lambda_\lambda = 1$. Suppose that there exist two linearly independent solutions ϕ_1 and ϕ_2 of problem (14). Both $\phi_1''(0)$ and $\phi_2''(0)$ must be different from zero since otherwise, it would follow from the first statement of Lemma 3.2 that $\phi_i'(\ell) > 0$ $(i = 1, 2)$ which contradicts the last boundary condition in (14). In view of the assumptions about ϕ_1 and ϕ_2, the solution

$$\phi(x) = \phi_1''(0)\phi_2(x) - \phi_2''(0)\phi_1(x)$$

satisfies

$$\phi(0) = \phi'(0) = \phi'(0) = 0 \text{ and } \phi'(\ell) = 0.$$

This again contradicts the first statement of Lemma 3.2 unless $\phi \equiv 0$. Therefore,

$$\dim \Lambda_\lambda = 1,$$

and then, all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of problem (12) are geometrically simple. On the other hand, by Lemma 2.1, the operator $\mathcal{A}$ is self-adjoint in $L^2_\rho(0,\ell)$, and this implies that all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ are algebraically simple. Now, we prove (13). Let $\{\lambda_n, \Phi_n\}$ $(n \in \mathbb{N}^*)$ be an eigenpair of Problem (12), and assume that $\Phi_n^{\ell''}(\ell) = 0$, for some $n \in \mathbb{N}^*$. Then, the eigenfunctions Φ_n satisfy the boundary conditions

$$\Phi_n(\ell) = \Phi_n'(\ell) = \Phi_n''(\ell) = 0, \text{ for some } n \in \mathbb{N}^*,$$

and then, by standard theory of differential equations, one gets

$$\mathcal{T}\Phi_n(\ell) = ((\sigma(x)\Phi_n''(x))' - q(x)\Phi_n'(x))\Big|_{x=\ell} \neq 0, \text{ for some } n \in \mathbb{N}^*.$$

Without loss of generality, let $\mathcal{T}\Phi_n(\ell) < 0$ for some $n \in \mathbb{N}^*$. Since $\lambda_n > 0$, it follows from the second statement of Lemma 3.2 that

$$\Phi_n(x) > 0, \ \Phi_n'(x) < 0, \ \Phi_n''(x) > 0 \text{ and } \mathcal{T}\Phi_n(x) < 0, \ \forall\, x \in [0,\ell],$$

but this contradicts the boundary conditions $\Phi_n(0) = \Phi_n'(0) = 0$. Thus,

$$\Phi_n''(\ell) \neq 0 \ \ \forall n \in \mathbb{N}^*,$$

and this finalizes the proof.

Next, we establish the asymptotic behavior of the spectral gap $|\lambda_{n+1} - \lambda_n|$ for large n. Namely, we have the following theorem:

Theorem 3.3 *The eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the associated spectral problem (12) satisfy the following asymptotic*

$$\sqrt[4]{\lambda_n} := \mu_n = \frac{\pi}{\gamma}\left(n - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{\exp(n)}\right), \ \gamma := \int_0^\ell \left(\frac{\rho(t)}{\sigma(t)}\right)^{\frac{1}{4}} dt. \tag{15}$$

Moreover, one has

$$|\lambda_{n+1} - \lambda_n| = \frac{4\pi^4}{\gamma^4} n^3 + \mathcal{O}(n^2). \tag{16}$$

Proof To simplify notation, throughout the proof, we denote by

$$\zeta(x) := ([\rho(x)]^{\frac{3}{4}}[\sigma(x)]^{\frac{1}{4}})^{-\frac{1}{2}} \text{ and } \mathcal{X} := \int_0^x \left(\frac{\rho(t)}{\sigma(t)}\right)^{\frac{1}{4}} dt. \tag{17}$$

It is known (e.g., [20, Chapter 5, pp. 235–239] and [29, Chapter 2]) that for $\lambda \in \mathbb{C}$, the fourth-order linear differential equation

$$(\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, \quad x \in (0, \ell), \tag{18}$$

has four fundamental solutions $\{\phi_i(x,\lambda)\}_{i=1}^{i=4}$ satisfying the asymptotic forms

$$\begin{cases} \phi_i(x,\lambda) = \zeta(x)\exp(\mu w_i \mathcal{X})[1], \\ \phi_i^{(k)}(x,\lambda) = \zeta(x)(\mu w_i)^k (\frac{\rho(x)}{\sigma(x)})^{\frac{k}{4}} \exp(\mu w_i \mathcal{X})[1], \end{cases} \tag{19}$$

where $\mu^4 = \lambda$, $w_i{}^4 = 1$, $\phi^{(k)} := \frac{\partial^k \phi}{\partial x^k}$ for $k \in \{1,2,3\}$, and $[1] = 1 + \mathcal{O}(\mu^{-1})$ uniformly as $\mu \to \infty$ in a sector $\mathcal{S}_\tau = \{\mu \in \mathcal{C}$ such that $0 \le \arg(\mu + \tau) \le \frac{\pi}{4}\}$ where τ is any fixed complex number. It is convenient to rewrite these asymptotics in the form

$$\begin{aligned}
\phi_1(x,\lambda) &= \zeta(x)\cos(\mu\mathcal{X})[1], \\
\phi_2(x,\lambda) &= \zeta(x)\cosh(\mu\mathcal{X})[1], \\
\phi_3(x,\lambda) &= \zeta(x)\sin(\mu\mathcal{X})[1], \\
\phi_4(x,\lambda) &= \zeta(x)\sinh(\mu\mathcal{X})[1].
\end{aligned}$$

Hence, every solution $\phi(x,\lambda)$ of Eq. 18 can be written in the following asymptotic form

$$\phi(x,\lambda) = \zeta(x)\Big(C_1\cos(\mu\mathcal{X}) + C_2\cosh(\mu\mathcal{X}) + C_3\sin(\mu\mathcal{X}) + C_4\sinh(\mu\mathcal{X})\Big)[1], \tag{20}$$

for some constants C_i, $i = 1,2,3,4$. Also, from (19), we have

$$\begin{aligned}
\phi^{(k)}(x,\lambda) =& \mu^k\zeta(x)\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{k}{4}}(C_1\cos^{(k)}(\mu\mathcal{X}) + C_2\cosh^{(k)}(\mu\mathcal{X}))[1] \\
&+ \mu^k\zeta(x)\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{k}{4}}(C_3\sin^{(k)}(\mu\mathcal{X}) + C_4\sinh^{(k)}(\mu\mathcal{X}))[1],
\end{aligned} \tag{21}$$

where $k \in \{1,2,3\}$. If $\phi(x,\lambda)$ satisfies the boundary conditions

$$\phi(0,\lambda) = \phi'(0,\lambda) = 0,$$

then by relations (20) and (21), we obtain the following asymptotic estimate

$$\begin{cases} \zeta(0)(C_1 + C_2)[1] = 0, \\ \mu\zeta(0)\left(\frac{\rho(0)}{\sigma(0)}\right)^{\frac{1}{4}}(C_3 + C_4)[1] = 0. \end{cases}$$

Then, one gets

$$\phi(x,\lambda) = C_1\zeta(x)(\cos(\mu\mathcal{X}) - \cosh(\mu\mathcal{X}))[1] + C_3(\sin(\mu\mathcal{X}) - \sinh(\mu\mathcal{X}))[1] \tag{22}$$

and

$$\begin{aligned}\phi'(x,\lambda) =& C_1\mu\zeta(x)\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{1}{4}}(\sinh(\mu\mathcal{X}) - \sin(\mu\mathcal{X}))[1]\\ &+ C_3\mu\zeta(x)\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{1}{4}}(\cos(\mu\mathcal{X}) - \cosh(\mu\mathcal{X}))[1],\end{aligned} \tag{23}$$

From the boundary conditions $\phi(\ell,\lambda) = \phi'(\ell,\lambda) = 0$, and the above asymptotics, one obtains

$$\begin{cases} C_1(\cos(\mu\gamma) - \cosh(\mu\gamma))[1] + C_3(\sin(\mu\gamma) - \sinh(\mu\gamma))[1] = 0,\\ C_1(-\sin(\mu\gamma) - \sinh(\mu\gamma))[1] + C_3(\cos(\mu\gamma) - \cosh(\mu\gamma))[1] = 0,\end{cases} \tag{24}$$

where the constant γ is defined in relation (15). The latter homogeneous system of equations in the unknowns C_1 and C_2 has a non-trivial solution if and only if the corresponding determinant is zero, i.e.,

$$((\cos(\mu\gamma) - \cosh(\mu\gamma))^2 + \sin^2(\mu\gamma) - \sinh^2(\mu\gamma))[1] = 0.$$

Equivalently,

$$\mu\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac{1}{4}}(\cos(\mu\gamma)\cosh(\mu\gamma) - 1)[1] = 0. \tag{25}$$

Consequently, the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ are solutions of the following asymptotic characteristic equation

$$\mu\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac{1}{4}}\exp(\mu\gamma)\left(\cos(\mu\gamma) - \frac{2}{\exp(\mu\gamma)}\right)[1] = 0,$$

which can also be written as

$$\cos(\mu\gamma) + \mathcal{O}\left(\frac{1}{\exp(\mu\gamma)}\right) = 0. \tag{26}$$

Since the solutions of the equation $\cos(\mu\gamma) = 0$ are given by

$$\widetilde{\mu_n} = \frac{\pi}{\gamma}\left(n - \frac{1}{2}\right),\ n = 0, 1, 2, ...,$$

it follows from Rouché's theorem that the solutions of Eq. 26 satisfy the following asymptotic

$$\begin{aligned}\mu_n =& \widetilde{\mu_n} + \delta_n\\ =& \frac{\pi}{\gamma}(n - \frac{1}{2}) + \mathcal{O}\left(\frac{1}{\exp(n)}\right),\end{aligned}$$

which proves relation (15). Furthermore, one gets

$$\begin{aligned}\sqrt{\lambda_n} =&\frac{\pi^2}{\gamma^2}\left(n-\frac{1}{2}\right)^2+\mathcal{O}\left(\frac{n}{\exp(n)}\right),\\ =&\frac{\pi^2}{\gamma^2}(n^2-n)+\mathcal{O}(1).\end{aligned}$$

and hence,

$$\begin{aligned}\lambda_{n+1}-\lambda_n =&(\sqrt{\lambda_{n+1}}-\sqrt{\lambda_n})(\sqrt{\lambda_{n+1}}+\sqrt{\lambda_n})\\ =&\frac{\pi^4}{\gamma^4}((n+1)^2-n^2+\mathcal{O}(1))((n+1)^2+n^2-2n+\mathcal{O}(1))\\ =&\frac{4\pi^4}{\gamma^4}n^3+\mathcal{O}(n^2).\end{aligned}$$

The theorem is then proved.

We conclude this section with the following result about the asymptotics of the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ of the spectral problem (12).

Proposition 3.4 *Assume that the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ of the spectral problem (12) are normalized in the sense that $\lim_{n\to\infty}\|\Phi_n\|_{L^2_\rho(0,\ell)}=1$. Then, we have the following asymptotic formula*

$$\Phi_n(x)=\gamma^{-\frac{1}{2}}\zeta(x)(\sin(\mu_n\mathcal{X})-\cos(\mu_n\mathcal{X}))+\mathcal{O}\left(\frac{1}{\mu_n}\right),\quad \text{as } n\to\infty, \tag{27}$$

where the quantities μ_n,γ,ζ, and $\mathcal{X}$ are given by relations (15) and (17), respectively. Furthermore, one has

$$\lim_{n\to\infty}\frac{|\Phi_n''(\ell)|}{\sqrt{\lambda_n}}=2\zeta(\ell)\left(\frac{\rho(\ell)}{\gamma\sigma(\ell)}\right)^{\frac{1}{2}}. \tag{28}$$

Proof If μ_n satisfies relation (25), then by solving the homogeneous system of two Eq. 24, one gets

$$\begin{cases} C_1=C(\cos(\mu_n\gamma)-\cosh(\mu_n\gamma))[1],\\ C_3=C(\sin(\mu_n\gamma)+\sinh(\mu_n\gamma))[1],\end{cases} \tag{29}$$

for some constant $C\neq 0$. Due to this fact and by relations (15) and (22), we obtain the following asymptotic estimate for the eigenfunctions $\phi(x,\lambda_n)$ of Problem (12)

$$\begin{aligned}\phi(x,\lambda_n)=&C\zeta(x)\{(\cos(\mu_n\gamma)-\cosh(\mu_n\gamma))(\cos(\mu_n\mathcal{X})-\cosh(\mu_n\mathcal{X}))\}[1]\\ &+C\zeta(x)\{(\sin(\mu_n\gamma)+\sinh(\mu_n\gamma))(\sin(\mu_n\mathcal{X})-\sinh(\mu_n\mathcal{X}))\}[1].\end{aligned} \tag{30}$$

By relations (15) and (29), it follows

$$C_1=\frac{-C\exp(\mu_n\gamma)}{2},\quad C_3=\frac{C\exp(\mu_n\gamma)}{2},\quad \text{as } n\to\infty,$$

and then, by (30), we have

$$\phi(x,\lambda_n) = C\zeta(x)\frac{\exp(\mu_n\gamma)}{2}(\sin(\mu_n\mathcal{X}) - \cos(\mu_n\mathcal{X}) + \cosh(\mu_n\mathcal{X}) - \sinh(\mu_n\mathcal{X})),$$

as $n\to\infty$, where γ is defined by (15). Therefore, one has

$$\phi(x,\lambda_n) = C\zeta(x)\frac{\exp(\mu_n\gamma)}{2}(\sin(\mu_n\mathcal{X}) - \cos(\mu_n\mathcal{X})), \quad \text{as } n\to\infty. \tag{31}$$

By the change of variables $s = \mathcal{X}$, one gets

$$\begin{aligned}\int_0^\ell (\xi(x))^2\sin^2(\mu_n\mathcal{X})\rho(x)dx &= \int_0^\ell \sin^2\left(\mu_n\int_0^x\left(\frac{\rho(t)}{\sigma(t)}\right)^{\frac14}dt\right)\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac14}dx,\\ &= \int_0^\gamma \sin^2(\mu_n s)ds = \frac{\gamma}{2},\end{aligned}$$

where ξ is defined by (17). Similarly, we have

$$\int_0^\ell (\xi(x))^2\cos^2(\mu_n\mathcal{X})\rho(x)dx = \frac{\gamma}{2},$$

and

$$\int_0^\ell (\xi(x))^2\sin(\mu_n\mathcal{X})\cos(\mu_n\mathcal{X})\rho(x)dx = \frac{\sin^2(\mu_n\gamma)}{2\mu_n}, \quad \text{as } n\to\infty.$$

Consequently, one gets

$$\|\phi(x,\lambda_n)\|_{L^2_\rho(0,\ell)} = |C|\frac{\gamma^{\frac12}\exp(\mu_n\gamma)}{2}\left(1+\mathcal{O}\left(\frac{1}{\mu_n}\right)\right), \quad \text{as } n\to\infty. \tag{32}$$

Let us set

$$\Phi_n(x) := \frac{\phi(x,\lambda_n)}{\lim_{n\to\infty}\|\phi(x,\lambda_n)\|_{L^2_\rho(0,\ell)}}. \tag{33}$$

Then, $(\Phi_n(x))_{n\in\mathbb{N}^*}$ are the normalized eigenfunctions of Problem (12) so that $\lim_{n\to\infty}\|\Phi_n\|_{L^2_\rho(0,\ell)} = 1$. Therefore, by (31) and (32)–(33), we get

$$\Phi_n(x) = \gamma^{-\frac12}\zeta(x)(\sin(\mu_n\mathcal{X}) - \cos(\mu_n\mathcal{X}))\left(1+\mathcal{O}\left(\frac{1}{\mu_n}\right)\right), \quad \text{as } n\to\infty,$$

and then, the asymptotic formula (27) is proved. In a similar way, from the asymptotics (21) and (32), a straightforward computation yields

$$\begin{aligned}\phi''(\ell,\lambda_n) =& C\mu^2\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac12}(\cosh^2(\mu_n\gamma) - \cos^2(\mu_n\gamma))[1]\\ &- C\mu^2\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac12}(\sin(\mu_n\gamma) + \sinh(\mu_n\gamma))^2[1].\end{aligned}$$

As a consequence, one has

$$|\phi''(\ell,\lambda_n)| = 2|C|\mu_n^2\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac{1}{2}}\left|\sin(\mu_n\gamma)\sinh(\mu_n\gamma)\right|[1].$$

Therefore, relying on the asymptotics (15) and (32)–(33), we get

$$|\Phi_n''(\ell)| = 2\gamma^{-\frac{1}{2}}\mu_n^2\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{\mu_n}\right)\right), \text{ as } n\to\infty,$$

and then, relation (28) is proved. The proof is complete.

4 Observability

In this part of the paper, we prove some observability results that are consequences of the asymptotic properties of the previous section. The reason to study these properties is that, by means of the Lions' HUM [27], controllability properties can be reduced to suitable observability inequalities for the adjoint system. As System (2) is reversible in time, we are reduced to the same system, without control. Therefore, consider System (7). One has:

Proposition 4.1 *Let $T>0$ and $z_0\in H_0^2(0,\ell)$. Then, there exists a positive constant $C_T>0$, depending on T, such that*

$$C_T^{-1}\|z_0\|^2_{H_0^2(0,\ell)} \le \int_0^T |\partial_x^2 z(t,\ell)|^2 dt \le C_T\|z_0\|^2_{H_0^2(0,\ell)}. \tag{34}$$

where z is the solution of Problem (7). Consequently, we have the following "hidden regularity" result

$$z_0\in H_0^2(0,\ell) \Rightarrow \partial_x^2 z(t,\ell)\in L^2(0,T). \tag{35}$$

In order to prove Proposition 4.1, we need the following variant of Ingham's inequality due to Beurling (e.g., [18]).

Lemma 4.2 *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence satisfying for some $\delta>0$ the condition*

$$\lambda_{n+1}-\lambda_n>\delta,\ \forall\, n\in\mathbb{Z}.$$

Then, for any $T>2\pi D^+(\lambda_n)$, the family $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0,T)$, that is

$$C_T^{-1}\sum_{n\in\mathbb{Z}}|\widehat{c}_n|^2 \le \int_0^T\left|\sum_{n\in\mathbb{N}^*}\widehat{c}_n e^{i\lambda_n t}\right|^2 dt \le C_T\sum_{n\in\mathbb{Z}}|\widehat{c}_n|^2, \tag{36}$$

for some positive constant $C_T>0$, depending on T, where $D^+(\lambda_n):=\lim_{r\to\infty}\frac{n^+(r,\lambda_n)}{r}$ is the Beurling upper density of the sequence $(\lambda_n)_{n\in\mathbb{N}^*}$, and $n^+(r,\lambda_n)$ denotes the maximum number of terms of the sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ contained in an interval of length r.

Proof of Theorem 4.1 Let $C_c^\infty(0,\ell)$ be the space of all smooth functions defined on $(0,\ell)$ which have compact support in $(0,\ell)$. Let $z_0\in C_c^\infty(0,\ell)$, then by the Fourier series representation (11), $\partial_x^2 z(t,\ell)$ is well-defined for every $t>0$. Furthermore, for any $T>0$, we have the following equality

$$\int_0^T |\partial_x^2 z(t,\ell)|^2 dt = \int_0^T |\sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n''(\ell)|^2 dt. \tag{37}$$

By the first statement of Theorem 3.1 and the gap condition (16), Beurling's Lemma 4.2 states that, for any $T>2\pi D^+(\lambda_n)$, the family $(e^{i\lambda_n t})_{n\in\mathbb{N}^*}$ forms a Riesz basis in $L^2(0,T)$. Consequently, from relations (36) and (37), we deduce that for every $T>2\pi D^+(\lambda_n)$, there exists a positive constant $C_T>0$, depending on T, such that

$$C_T^{-1}\sum_{n\in\mathbb{N}^*}\left|c_n\Phi_n''(\ell)\right|^2 \le \int_0^T |\partial_x^2 z(t,\ell)|^2 dt \le C_T\sum_{n\in\mathbb{N}^*}\left|c_n\Phi_n''(\ell)\right|^2. \tag{38}$$

By relation (15) and the characteristic Eq. 26, we find that the Beurling upper density of the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of Problem (12) satisfies

$$D^+(\lambda_n) = \lim_{n\to\infty}\frac{\gamma^4}{\pi^4(n-\frac{1}{2})^3} = 0.$$

From the second statement of Theorem 3.1, we have $\Phi_n'(\ell)\neq 0$ for all $n\in\mathbb{N}^*$. Then, by (28), there exists a positive $C>0$ such that

$$C^{-1}\lambda_n \le \left|\Phi_n''(\ell)\right|^2 \le C\lambda_n, \quad \text{as } n\to\infty.$$

Therefore, by relation (38), for any $T>0$, one gets

$$C_T^{-1}\sum_{n\in\mathbb{N}^*}\lambda_n\left|c_n\right|^2 \le \int_0^T |\partial_x^2 z(t,\ell)|^2 dt \le C_T\sum_{n\in\mathbb{N}^*}\lambda_n\left|c_n\right|^2, \tag{39}$$

for some new constant $C_T>0$. Due to the spectral representation (10) of the space $\mathcal{H}_\theta$ and Proposition 2.2, one has

$$\|z_0\|^2_{\mathcal{H}_{1/2}} = \|z_0\|^2_{\mathcal{D}(\mathcal{A}^{1/2})} = \sum_{n\in\mathbb{N}^*}\lambda_n|c_n|^2,\ \forall\, z_0\in C_c^\infty(0,\ell).$$

Since $C_c^\infty(0,\ell)$ dense in $H_0^2(0,\ell)$ and $\mathcal{D}(\mathcal{A}^{1/2}) = H_0^2(0,\ell)$, then by (39), we get both relations (34) and (35). This completes the proof.

Remark 4.3 The second inequality in (34) is often called a "*direct inequality*." By a density argument, this inequality gives a sense to the second derivative at $x=\ell$ of the solution z of System (7), for initial data $z_0\in H_0^2(0,\ell)$ (see [25, Chapter 2]). Note that, in this case, the notation $\partial_x^2 z(t,\ell)$ in relation (35) makes no sense by the usual trace theorem.

Remark 4.4 The first inequality in (34) is an "observability" result. Since the total energy of System (7) is conserved, the observability inequality implies that the norm of the solution z in $H_0^2(0,\ell)$ is measured continuously by the quantity $\partial_x^2 z(t,\ell)$ in $L^2(0,T)$.

Remark 4.5 Due to both inequalities in (34), we deduce that $H_0^2(0,\ell)$ is the optimal space of observability, i.e., the largest space of initial data for which the solution of Problem (7) can be estimated by means of the L^2-norm of the quantity $\partial_x^2 z(t,\ell)$.

5 Exact Boundary Controllability

In this final part of the paper, we prove the exact boundary controllability of the control problem (2).

5.1 Well-posedness

Since we are dealing with boundary control, we need to introduce the weaker notion of "solution defined by transposition" in the spirit of [28].

We follow the multiplier method from [25, Chapter 2] and [35], see also [32]. By a density argument, we may assume that the solution y of Problem (2) is smooth enough so that all the computations are rigorous. Then, we multiply (7) by $\bar{y}$ and integrate by parts on $[0,T]\times(0,\ell)$ to obtain

$$i\int_0^\ell\int_0^T(\partial_t z)\bar{y}(t,x)dt\rho(x)dx+\int_0^\ell\int_0^T(\partial_x^2(\sigma(x)\partial_x^2 z)-\partial_x(q(x)\partial_x z))\bar{y}(t,x)dtdx=0.$$

Then, integrating by parts and using the boundary conditions in (2) and (7), we get

$$\begin{aligned}i\int_0^\ell\Big[\bar{y}z(t,x)\Big]_0^T\rho(x)dx=\sigma(\ell)\int_0^T\bar{f}(t)\partial_x^2 z(t,\ell)dt+i\int_0^\ell\int_0^T(\partial_t\bar{y})z(t,x)dt\rho(x)dx\\-\int_0^\ell\int_0^T(\partial_x^2(\sigma(x)\partial_x^2\bar{y})-\partial_x(q(x)\partial_x\bar{y}))z(t,x)dtdx.\end{aligned}$$

and then, using the first equation (2) that y satisfies, we deduce that

$$i\int_0^\ell\bar{y}z(T,x)\rho(x)dx=\sigma(\ell)\int_0^T\bar{f}(t)\partial_x^2 z(t,\ell)dt+i\int_0^\ell\bar{y}_0z_0\rho(x)dx. \tag{40}$$

Let us now define the spaces

$$\mathcal{S}:=H_0^2(0,\ell)\text{ and }\mathcal{S}':=H^{-2}(0,\ell),$$

and the linear functional $\mathcal{L}_T$ on S by

$$\mathcal{L}_T(z_0)=i\langle\bar{y}_0,z_0\rangle_{\mathcal{S}',\mathcal{S}}+\sigma(\ell)\int_0^T\bar{f}(t)\partial_x^2 z(t,\ell)dt, \tag{41}$$

where $\langle .,.\rangle_{X',X}$ denotes the usual duality product. Moreover, we have

$$\|\mathcal{L}_T\|\leq C\big(\|y_0\|_{H^{-2}(0,\ell)}+\|f\|_{L^2(0,T)}\big), \tag{42}$$

for some constant $C(T)>0$. Using relation (40), one can write the identity (41) in the following form

$$\mathcal{L}_T(z_0)=i\langle\bar{y}(T,x),z(T,x)\rangle_{\mathcal{S}',\mathcal{S}}, \tag{43}$$

where z is the solution of Problem (7). This motivates the following definition.

Definition 5.1 We say that y is a weak solution to Problem (2) in the sense of transposition if $y\in C([0,T];H^{-2}(0,\ell))$ satisfies (43) for all $T>0$ and for every $z_0\in\mathcal{S}$.

Hence, the following result.

Proposition 5.2 *Let $T > 0$, and $f \in L^2(0,T)$. Then for any $y_0 \in H^{-2}(0,\ell)$, there exists a unique weak solution of System (2) in the sense of transposition, satisfying*

$$y \in C([0,T];H^{-2}(0,\ell)). \tag{44}$$

Moreover, there exists a constant $C(T) > 0$ such that

$$\|y\|_{L^\infty([0,T];H^{-2}(0,\ell))} \leq C(T)(\|y_0\|_{H^{-2}(0,\ell)} + \|f\|_{L^2(0,T)}). \tag{45}$$

Proof It follows from Proposition 2.2 that, for any $T > 0$, the linear map

$$z(T,.) \longmapsto z_0$$

is an isomorphism from $H_0^2(0,\ell)$ into itself. Hence, by Proposition 4.1, we deduce that the linear map

$$z(T,.) \longmapsto \mathcal{L}_T(z_0)$$

is continuous on $H_0^2(0,\ell)$. Therefore, by duality, Equation (43) makes sense and uniquely determines $y \in L^\infty([0,T];H^{-2}(0,\ell))$. Moreover, from relation (42), it follows that the estimate (45) holds. The continuity with respect to time in (44) is proved by a standard density argument (e.g., see [25, Chapter 2]). Indeed, if we call y_n the corresponding solution of (2) associated to $f_n \in C_c^\infty(0,\ell)$ and $y_{0,n} \in H_0^2(0,\ell)$ such that

$$f_n \to f \text{ in } L^2(0,\ell), \;\; y_{0,n} \to y_0 \text{ in } H^{-2}(0,\ell), \text{ as } n \to \infty.$$

Then, the solution $y_n \in C([0,T];H^{-2}(0,\ell))$. Moreover, we apply the estimate (45) to $(y_n - y)$, and we have

$$\|y_n - y\|_{L^\infty([0,T];H^{-2}(0,\ell))} \leq C(T)(\|y_{0,n} - y_0\|_{H^{-2}(0,\ell)} + \|f_n - f\|_{L^2(0,T)})$$

for some positive constant $C(T) > 0$. Since $C([0,T];H^{-2}(0,\ell))$ is a closed subspace of $L^\infty([0,T];H^{-2}(0,\ell))$, then the desired property follows. The proof is complete.

5.2 Exact Controllability

We are now ready to state our main controllability result. Thanks to the reversibility in time of (2), this system is exactly controllable if and only if the system is null controllable. One has:

Theorem 5.3 *Assume that the coefficients ρ, σ, and q satisfy relations (3) and (4). Given $T > 0$ and $y_0 \in H^{-2}(0,\ell)$, there exists a control $f \in L^2(0,T)$ such that the solution y of the control problem (2) satisfies*

$$y(T,x) = 0, \;\; x \in [0,\ell].$$

Proof By the Lions' HUM [27], solving the exact controllability problem is equivalent to proving an observability inequality for the backward problem. The backward problem is as follows

$$\begin{cases} i\rho(x)\partial_t y = -\partial_x^2(\sigma(x)\partial_x^2 y) + \partial_x(q(x)\partial_x y), & (t,x) \in (0,T)\times(0,\ell), \\ y(t,0) = \partial_x y(t,0) = 0, & t \in (0,T), \\ y(t,\ell) = 0,\ \partial_x y(t,\ell) = \partial_x^2 z(t,\ell), & t \in (0,T), \\ y(T,x) = 0, & x \in (0,\ell), \end{cases} \tag{46}$$

where z is the solution of the uncontrolled system (7). By Proposition 5.2, Problem (46) has a unique weak solution y, satisfying $y_0 := y(0,x) \in H^{-2}(0,\ell)$. Hence, the linear map

$$\Lambda\ :\ H_0^2(0,\ell) \longrightarrow H^{-2}(0,\ell),\ \ z_0 \longmapsto -iy_0$$

is continuous from $H_0^2(0,\ell)$ into $H^{-2}(0,\ell)$. Furthermore, if Λ is shown to be surjective then there exists a control of the form $f(t) = \partial_x^2 z(t,\ell)$ that drives System (2) to rest in time $T > 0$. Since $y(T,x) = 0$, then for the choice of $f(t) = \partial_x^2 z(t,\ell)$, by multiplying (40) by $\bar{z}$, one has

$$\langle -iy_0, \bar{z}_0\rangle_{\mathcal{S}',\mathcal{S}} = \sigma(\ell)\int_0^T \left|\partial_x^2 z(t,\ell)\right|^2 dt.$$

Equivalently,

$$\langle \Lambda(z_0), \bar{z}_0\rangle_{\mathcal{S}',\mathcal{S}} = \sigma(\ell)\int_0^T \left|\partial_x^2 z(t,\ell))\right|^2 dt.$$

By Proposition 4.1, for every $T > 0$ and $z_0 \in H_0^2(0,\ell)$, we have

$$\int_0^T |\partial_x^2 z(t,\ell)|^2 dt \geq C_T^{-1}\|z_0\|^2_{H_0^2(0,\ell)},$$

for some constant $C_T > 0$. Consequently, for every $T > 0$, one obtains

$$\langle \Lambda(z_0), \bar{z}_0\rangle_{\mathcal{S}',\mathcal{S}} \geq C_T^{-1}\sigma(\ell)\|z_0\|^2_{H_0^2(0,\ell)}.$$

Therefore, by the Lax-Milgram Theorem, Λ is surjective. This implies that there exists a control of the form $f(t) = \partial_x^2 z(t,\ell)$ that drives the system (2) to rest in time $T > 0$, and this completes the proof of Theorem 5.3.

Acknowledgements The authors would like to thank the referee for valuable suggestions and comments that helped to improve the paper.

References

1. Ali Mehmeti F, Ammari K, Nicaise S. Nicaise, Dispersive effects and high frequency behaviour for the Schrödinger equation in star-shaped networks. Portug Math. 2015;72:309–55.
2. Ali Mehmeti F, Ammari K, Nicaise S. Dispersive effects for the Schrödinger equation on the tadpole graph. J Math Anal Appl. 2017;448:262–80.
3. Ammari K, Assel R. Spectral analysis and stabilization of the dissipative Schrödinger operator on the tadpole graph, 2021. arXiv:2111.13227.
4. Ammari K, Mercier D, Régnier V. Spectral analysis of the Schrödinger operator on binary tree-shaped networks and applications. Journal of Diff Equat. 2015;259:6923–59.
5. Ammari K, Duca A. Controllability of localised quantum states on infinite graphs through bilinear control fields. Inter Jour Cont. 2021;94:1824–37.
6. Ammari K, Duca A. Controllability of periodic bilinear quantum systems on infinite graphs. J Math Phys. 2020;61(10):101507, 15.

7. Ammari K, Sabri M. Dispersion for Schrödinger operators on regular trees. Anal Math Phys; 2022. 12, no. 2, Paper No. 56.
8. Ammari K, Choulli M, Robbiano L. Observability and stabilization of magnetic Schrödinger equations. J Diff Equat. 2019;267:3289–327.
9. Ammari K, Bchatnia A, Mehenaoui N. Exponential stability for the nonlinear Schrödinger equation on a star-shaped network. Z Angew Math Phys. 2021;72:1–19.
10. Aksas B, Rebiai SE. Uniform stabilization of the fourth order Schrödinger equation. J Math Anal Appl. 2017;446:1794–813.
11. Ben-Artzi A, Koch H, Saut JC. Dispersion estimates for fourth order Schrödinger equations. C R Acad Sci Paris Sér I Math. 2000;330:87–92.
12. Ben Amara J, Bouzidi H. Exact boundary controllability for the Boussinesq equation with variable coefficients. Evol Equat & Cont Theo. 2018;7:403–15.
13. Ben Amara J, Bouzidi H. On the simplicity of eigenvalues of two nonhomogeneous Euler-Bernoulli beams connected by a point mass. J Diff Equat. 2019;267:2083–103.
14. Cui S, Guo C. Well-posedness of higher-order nonlinear Schrödinger equations in Sobolev spaces $H^s(\mathbb{R}^n)$ and applications. Nonl Analy. 2007;67:687–707.
15. Capistrano-Filho RA, Cavalcante M. Stabilization and Control for the Biharmonic Schrödinger Equation. Appl Math Optim. 2021;84:103–44.
16. Capistrano-Filho RA, Cavalcante M, Gallego FA. Lower regularity solutions of the biharmonic Schrödinger equation in a quarter plane. Pacif Journal of Math. 2020;309:35–70.
17. Capistrano-Filho RA, Cavalcante M, Gallego FA. Forcing operators on star graphs applied for the cubic fourth order Schrödinger equation. Disc & Contin Dynam Syst -B; 2021. 1–36.
18. Dáger R, Zuazua E. Wave propagation, observation and control in 1−d flexible multi-structures, Vol. 50. Springer Science & Business Media; 2006.
19. Fibich G. The nonlinear Schrödinger equation: singular solutions and optical collapse. Berlin: Springer; 2015.
20. Fedoryuk MV. Asymptotic Analysis. Springer-Verlag; 1983
21. Hansen S. Exact boundary controllability of a Schrödinger equation with an internal point mass. In: Amer. Cont. Conf. May 24–26, Seattle, USA; 2017. p. 3809–3814
22. Gao P. Carleman estimates for forward and backward stochastic fourth order Schrödinger equations and their applications. Evol Equat & Cont Theo. 2018;7:465–99.
23. Karpman VI. Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrödinger-type equations. Phys Rev E. 1996;53:1336–9.
24. Karpman VI, Shagalov AG. Stability of soliton described by nonlinear Schrödinger type equations with higher-order dispersion. Physica D: Nonl Phen. 2000;144:194–210.
25. Komornik V. Exact Controllability and Stabilization, the Multiplier Method, John Wiley-Masson; 1994
26. Lions JL. Exact controllability, stabilization and perturbation for distributed systems. SIAM Rev. 1988;30:1–68.
27. Lions JL. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Tome 1 and 2. RMA, Paris: Masson; 1988.
28. Lions JL, Magenes E. Non-Homogeneous Boundary Value Problems and Applications. Berlin: Springer; 1972.
29. Naimark MA. Linear differential operators, vol. 167. New York: Ungar; 2007.
30. Pausader B. Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case. Dynamics of PDE. 2007;4:197–225.
31. Pausader B. The cubic fourth-order Schrödinger equation. J Funct Anal. 2007;256:2473–517.
32. Wen R, Chai S, Guo BZ. Well-posedness and exact controllability of fourth order Schrödinger equation with boundary control and collocated observation. SIAM J Control Optim. 2014;52:365–96.
33. Wen R, Chai S, Guo BZ. Well-posedness and exact controllability of fourth-order Schrödinger equation with hinged boundary control and collocated observation. Math Cont Signa Syst. 2016;28:1–28.
34. Zheng C. Inverse problems for the fourth order Schrödinger equation on a finite domain. Math Cont & Relat Fields. 2015;5:177–89.
35. Zheng C, Zhongcheng Z. Exact controllability for the fourth order Schrödinger Equation. Chin Ann Math. 2012;33:395–404.