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Classification of Singular Solutions in a Nonlinear Fourth-Order Parabolic Equation

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Abstract

This paper deals with the fourth-order parabolic equation $u_t + \Delta^2 u = u^{p(x)} \log u$ in a bounded domain, subject to homogeneous Navier boundary conditions. For subcritical and critical initial energy cases, we combine the Galerkin's method with the generalized potential well method to prove the existence of global solutions. By the concavity arguments, we obtain the results about blow-up solutions. For super critical initial energy case, we use some ordinary differential inequalities to study the extinction of solutions. Moreover, extinction rate, blow-up rate and time, and decay estimate of solutions are discussed.

Keywords Fourth-order parabolic equation · Logarithmic source · Blow-up · Extinction · Initial energy

Mathematics Subject Classification (2010) 35K25 · 35A01 · 35B40 · 35B44

1 Introduction

In this paper, we study the following fourth-order parabolic problem involving a variable exponent and logarithmic nonlinearity,

$$\begin{cases} u_t + \Delta^2 u = u^{p(x)} \log u, & (x,t) \in \Omega \times (0,T), \\ u = \Delta u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary $\partial\Omega$; $u_0(x) \in H_0^2(\Omega)$ satisfies the compatibility conditions on the boundary in the trace sense; $p(x)$ is Hölder-continuous in Ω, satisfying the Zhikov-Fan's condition $|p(x) - p(y)| \le Q/\log(1/|x-y|)$ for $x, y \in \Omega$, $|x-y| < \delta$, where constants $Q > 0$ and $0 < \delta < 1$. The problem (1.1) comes from the discussion on the properties of medical magnetic resonance images. $\Delta^2 u$ is the capillarity-driven surface diffusion term. The nonlinear source $u^{p(x)} \log u$ describes some kinds of Gaussian noise, where the function u represents the random noise with respect to (x,t).

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The Navier boundary conditions are also referred to "stree-free" or "slip" boundary conditions. The term Δu is no longer subject to the zero-flux boundary condition whereas the image always satisfies the zero-flux boundary condition. For more information on background, the interested readers may refer to the works [18, 30]. There are little results about the singularity of solutions to the fourth-order parabolic equations with variable exponents. For the high-order equations with constant exponents, we referred the interested readers to the works [8, 12, 13, 20, 23, 25, 27, 31, 32]. For the second-order parabolic problems with variable exponents, the interested readers could find some results in the works [1, 2, 4–7, 10, 11, 14, 16, 17, 19, 22, 26, 28].

Philippin considered the following parabolic problem with coefficient of t in [21],

$$\begin{cases} u_t + \Delta^2 u = k(t)|u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u = 0,\ \frac{\partial u}{\partial \eta} = 0 \text{ or } \Delta u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where the dimension $n \geq 2$ and p is a positive constant. By using the Sobolev type inequalities and constructing auxiliary functions, they obtained some upper and lower bounds of blow-up time of (1.2).

Li and Liu in [15] studied the following parabolic problem involving logarithmic nonlinearity,

$$\begin{cases} u_t + \Delta^2 u = |u|^{q-2}u \log u, & (x,t) \in \Omega \times (0,T), \\ u = \Delta u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \in H_0^2(\Omega) \setminus \{0\}, \end{cases} \tag{1.3}$$

where constant $2 < q < 2 + \frac{4}{n}$. If $\frac{1}{2}\int_\Omega |\Delta u_0|^2 \mathrm{d}x \leq \frac{1}{q}\int_\Omega |u_0|^q \log|u_0| \mathrm{d}x - \frac{1}{q^2}\int_\Omega |u_0|^q \mathrm{d}x$ and $\int_\Omega |\Delta u_0|^2 \mathrm{d}x > \int_\Omega |u_0|^q \log|u_0| \mathrm{d}x$, then there exist global solutions of (1.3). If $\frac{1}{2}\int_\Omega |\Delta u_0|^2 \mathrm{d}x < \frac{1}{q}\int_\Omega |u_0|^q \log|u_0| \mathrm{d}x - \frac{1}{q^2}\int_\Omega |u_0|^q \mathrm{d}x$ and $\int_\Omega |\Delta u_0|^2 \mathrm{d}x < \int_\Omega |u_0|^q \log|u_0| \mathrm{d}x$, then there exist blow-up solutions.

Qu, Zhou, and Liang in [24] employed the concavity method to study blow-up solutions of the fourth-order parabolic equation involving nonstandard growth conditions,

$$\begin{cases} u_t + \Delta^2 u = u^{p(x)}, & (x,t) \in \Omega \times (0,T), \\ u = \Delta u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \in H_0^2(\Omega). \end{cases} \tag{1.4}$$

They proved that if $(\mathcal{H})$ holds and $\frac{1}{2}\|\Delta u_0\|_2^2 \leq \int_\Omega \frac{1}{p(x)+1} u_0^{p(x)+1} \mathrm{d}x$, then the weak solutions of (1.4) blow up in the sense of $\lim_{t\to T^-}\int_0^t \|u\|_2^2 \mathrm{d}\tau = +\infty$. Moreover, the bounds of blow-up time and rate are discussed.

Inspired by the works [9, 10, 15, 16, 21, 24, 28, 29], we would study the singular solutions to problem (1.1). We would use the sign of the difference between the energy functional and the potential depth to classify the initial energy into three subcases. Moreover, the asymptotic estimates are discussed also, which include the bounds of extinction rate, blow-up rate and time, and decay rate of weak solutions to problem (1.1). This paper is arranged as follows. In Section 2, we give the main results of the present paper. Moreover, we add Table 1 to show the optimal classification of the initial energy on the existence and nonexistence of singular solutions. Sections 3, 4, and 5 will be devoted to the subcritical, critical, and super critical cases, respectively.

Table 1 Complete classification of initial energy and Nehari energy

Classification	$J(u_0)-d$	$I(u_0)$	$p(x)$	$\|u_0\|_2$	*Solution*	*Main results*
Subcritical	$-$	$+$	$p(x)p'(x) < 4^*+1$		G.E.	Th. 2.1
Subcritical	$J(u_0) < 0$		$0 < p^- < p^+ < p^* < 1$		N.E.	Th. 2.8
Subcritical	$0 < J(u_0) < d$	$-$	$1 < p^- < p^+ < 4^*$	$\|u_0\|_2 > \tilde{C}$	B.P.	Th. 2.2
Critical	0	$+$ or 0	$p(x)p'(x) < 4^*+1$		G.E.	Th. 2.3
Critical	0	$-$	$1 < p^- < p^+ < 4^*$	$\|u_0\|_2 > \tilde{C}$	B.P.	Th. 2.4
Super critical	$+$	$+$	$1 < p^- < p^+ < 4^*$	$\|u_0\|_2 \le \lambda_{J(u_0)}$	G.E.	Th. 2.5 (i)
Super critical	$+$	0	$0 < p^- < p^+ < p^* < 1$	$\|u_0\|_2 > \overline{C}$	E.	Cor. 2.1
Super critical	$+$	$-$	$1 < p^- < p^+ < 4^*$	$\|u_0\|_2 \ge \Lambda_{J(u_0)}$	B.P.	Th. 2.5 (ii)

2 Main Results

At the beginning, we give some preliminaries. The set

$$L^{p(x)}(\Omega) := \left\{u : u \text{ is measurable in } \Omega,\ \text{modular } \mathcal{A}_{p(x)}(u) := \int_\Omega |u|^{p(x)}\mathrm{d}x < +\infty\right\},$$

with the Luxemburg's norm $\|u\|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}\,\mathrm{d}x < 1\}$, is a separable and uniformly convex Banach space. The following inequalities show the relation between $\|u\|_{p(x)}$ and $\mathcal{A}_{p(x)}(u)$ (see [3]), for all $u \in L^{p(x)}(\Omega)$, $\min\left\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\right\} \le \mathcal{A}_{p(x)}(u) \le \max\left\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\right\}$. We equip the Banach space $H_0^2(\Omega)$ with the norm $\|u\|_{H_0^2} = \|\Delta u\|_2$. There is the Sobolev embedding relationship $H_0^2(\Omega) \hookrightarrow L^{p^++1}(\Omega) \hookrightarrow L^{p(x)+1}(\Omega)$, where

$$\|u\|_{p(x)+1} \le D_1\|\Delta u\|_2 \le D\|\Delta u\|_2 \quad \text{for constant } D := \max\{D_1, 1\}. \tag{2.1}$$

Let $\mu > 0$ be small such that $p(x) + \mu \le p^+ + \mu < 4^*$. We have

$$\|u\|_{p(x)+1+\mu} \le B_1\|\Delta u\|_2 \le B\|\Delta u\|_2 \quad \text{for constant } B := \max\{B_1, 1\}. \tag{2.2}$$

Suppose λ_1 is the first eigenvalue of the clamped plate problem $\Delta^2 u - \lambda u = 0$ in Ω with $\partial_\nu u = \partial_\nu \Delta u = 0$ on $\partial\Omega$, where ν denotes the unit outward normal vector. We have the following inequality $\|u\|_2^2 \le \lambda_1^{-1}\|\Delta u\|_2^2$. For all $u \in H_0^2(\Omega)$, we give the following notations:

$$\text{energy functional } J(u) := \frac{1}{2}\|\Delta u\|_2^2 - \int_\Omega \frac{u^{p(x)+1}\log u}{p(x)+1}\mathrm{d}x + \int_\Omega \frac{u^{p(x)+1}}{(p(x)+1)^2}\mathrm{d}x, \tag{2.3}$$

$$\text{Nehari functional } I(u) := \|\Delta u\|_2^2 - \int_\Omega u^{p(x)+1}\log u\,\mathrm{d}x, \tag{2.4}$$

$$\begin{aligned} \text{Nehari manifold } \mathcal{N} &:= \left\{u \in H_0^2(\Omega) : I(u) = 0,\ \|\Delta u\|_2 \ne 0\right\}, \\ \text{potential depth } d &:= \inf_{u\in\mathcal{N}} J(u), \\ \text{potential well } \mathcal{W} &:= \left\{u \in H_0^2(\Omega) : I(u) > 0,\ J(u) < d\right\} \cup \{0\}. \end{aligned} \tag{2.5}$$

Define $\mathcal{V} := \left\{u \in H_0^2(\Omega) : I(u) < 0,\ J(u) < d\right\}$, $\mathcal{N}_+ := \{u \in H_0^2(\Omega) : I(u) > 0\}$, and $\mathcal{N}_- := \left\{u \in H_0^2(\Omega) : I(u) < 0\right\}$. Denote the sublevels of J as $J^s := \left\{u \in H_0^2(\Omega) : J(u) < s\right\}$ and $\mathcal{N}^s := \mathcal{N} \cap J^s \ne \emptyset$ for $\forall s > d$. For all constant $s > d$, we define $\lambda_s := \inf\{\|u\|_2 : u \in \mathcal{N}^s\}$ and $\Lambda_s := \sup\{\|u\|_2 : u \in \mathcal{N}^s\}$. Obviously, λ_s and Λ_s is non-increasing and non-decreasing with respect to s, respectively.

Now, we give the definition of weak solutions for problem (1.1), which could be proved by the standard procedure in [8].

Definition 2.1 *Denote T be a positive constant. A function u is a weak solution of (1.1) if $u \in L^\infty\left((0,T); H_0^2(\Omega)\right)$ satisfies $u_t \in L^2((0,T); L^2(\Omega))$ and*

$$(u_t, v) + (\Delta u, \Delta v) = \left(u^{p(x)} \log u, v\right), \quad \text{for a.e. } t \in (0,T), \tag{2.6}$$

and for any test function $v \in L^2((0,T); H_0^2(\Omega))$ and $(\cdot,\cdot)$ as the inner product in $L^2(\Omega)$. Moreover,

$$\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + J(u) = J(u_0), \quad \text{for a.e. } t \in (0,T). \tag{2.7}$$

If u is a weak solution to (1.1) for every bounded $T > 0$, then we call it a weak global solution. □

2.1 Subcritical Initial Energy $J(u_0) < d$

Theorem 2.1 *Let $p(x)p'(x) < 4^* + 1$ be in force and $u_0 \in H_0^2(\Omega)$. For $J(u_0) < d$ and $I(u_0) > 0$, problem (1.1) has a global solution $u \in L^\infty\left((0,+\infty); H_0^2(\Omega)\right)$ with $u_t \in L^2((0,+\infty); L^2(\Omega))$ and $u(t) \in \mathcal{W}$ for all $0 \le t < +\infty$. Moreover, the weak solution is unique if it is bounded. Additionally, if $J(u_0) < d_0$ and $I(u_0) > 0$, then $\|u(t)\|_2 \le \|u_0\|_2 \exp\{-\delta^* t\}$ and*

$$\|u(t)\|_{H_0^2} \le \sqrt{\frac{2(p^-+1)}{p^--1}[J(u_0) + \|u_0\|_2^2]} \exp\left[-\frac{\alpha^*\lambda_1(p^--1)}{\lambda_1(p^--1)+2(p^-+1)}t\right], \tag{2.8}$$

where positive constant $d_0 := \frac{p^--1}{2(p^-+1)}\left(\frac{\mathrm{e}\mu}{B^{p^++1+\mu}}\right)^{\frac{2}{p^--1+\mu}}$; B is defined in (2.2); Positive constants $\delta^* := \lambda_1 - \lambda_1\left(J(u_0)/d_0\right)^{\frac{p^--1+\mu}{2}}$ and

$$\alpha^* := \frac{2(p^++1)(p^-+1)\left[1 - \left(J(u_0)/d_0\right)^{\frac{p^--1+\mu}{2}}\right]}{(p^+-1)(p^-+1)^2 + 2M(p^++1) + 2(p^++1)(p^-+1)\left(1 - J(u_0)/d_0\right)^{\frac{p^--1+\mu}{2}}},$$

where M is a positive constant to be determined later.

$$\|u(t)\|_{p(x)+1} \le \sqrt{\frac{2(p^-+1)}{p^--1}[J(u_0) + \|u_0\|_2^2]}B \exp\left[-\frac{\alpha^*\lambda_1(p^--1)}{\lambda_1(p^--1)+2(p^-+1)}t\right]. \tag{2.9}$$

Moreover, the energy functional $J(u)$ decays in the following sense,

$$\sqrt{J(u(t))} \le \sqrt{J(u_0) + \|u_0\|_2^2} \exp\left[-\frac{\alpha^*\lambda_1(p^--1)}{\lambda_1(p^--1)+2(p^-+1)}t\right]. \tag{2.10}$$

It could be tested that the radical functions in (2.8–2.10) make sense. In fact, since $J(u_0) < d_0$ and $I(u_0) > 0$ and by (2.3), one could find out that $J(u_0) > 0$.

Theorem 2.2 *Let the variable exponent $p(x)$ satisfies that*

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < 4^* := \begin{cases} \frac{n+4}{n-4}, & n > 4, \\ +\infty, & 1 \le n \le 4. \end{cases} \tag{$\mathcal{H}$}$$

Assume u is a weak solution of problem (1.1) *with*

$$u_0 \in H_0^2(\Omega), \quad \|u_0\|_2 > \tilde{C} := \sqrt{\frac{4d(p^- + 1)}{\lambda_1(p^- - 1)}}. \tag{2.11}$$

For $0 < J(u_0) < d$ and $I(u_0) < 0$, u blows up at some finite T satisfying $\lim_{t\to T^-} \int_0^t \|u\|_2^2 \mathrm{d}\tau = +\infty$. Furthermore, $T \le \frac{2\int_0^{t^*}\|u\|_2^2\mathrm{d}\tau}{(p^- -1)\|u(t^*)\|_2^2} + t^*$, where $t^* := \sqrt{\frac{2\|u_0\|_2^2}{\lambda_1(p^- -1)(d-J(u_0))}}$.

2.2 Critical Initial Energy $J(u_0) = d$

Theorem 2.3 *Let $p(x)p'(x) < 4^* + 1$ be in force and $u_0 \in H_0^2(\Omega)$. If $J(u_0) = d$ and $I(u_0) \ge 0$, then problem* (1.1) *admits a global weak solution $u \in L^\infty((0, +\infty); H_0^2(\Omega))$ with $u_t \in L^2((0, +\infty); L^2(\Omega))$ and $u(t) \in \overline{\mathcal{W}} := \mathcal{W} \cup \partial\mathcal{W}$ for $0 \le t < +\infty$. In addition, the weak solution is unique if it is bounded.*

Theorem 2.4 *Let ($\mathcal{H}$) be in force. Assume u is a weak solution of problem (1.1) and the initial data satisfy* (2.11)*. If $J(u_0) = d$ and $I(u_0) < 0$, then there exists a finite time T such that u blows up in the sense of $\lim_{t\to T^-} \int_0^t \|u\|_2^2 \mathrm{d}\tau = +\infty$. Furthermore, $T \le \frac{2\int_0^{t^*}\|u\|_2^2\mathrm{d}\tau}{(p^- -1)\|u(t^*)\|_2^2} + t^*$, where t^* is determined by $\lambda_1(p^- - 1)\int_0^t \|u\|_2^2 \mathrm{d}\tau > 2(p^- + 1)\|u_0\|_2^2$.*

2.3 Super Critical Initial Energy $J(u_0) > d$

Theorem 2.5 *Let ($\mathcal{H}$) be in force and $J(u_0) > d$.*

(i) *If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2 \le \lambda_{J(u_0)}$, the weak solution u of problem* (1.1) *in its H_0^2-norm exists globally and $u(t) \to 0$ as $t \to +\infty$;*

(ii) *If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2 \ge \Lambda_{J(u_0)}$, the weak solution u of problem* (1.1) *in its H_0^2-norm blows up in finite time.*

For $p(x) \equiv p$, a positive constant, we give the following Theorem 2.6 to illustrate that there exists u_0 such that $J(u_0)$ is arbitrary large, and the corresponding solution $u(x, t)$ to problem (1.1) with u_0 as initial datum blows up in finite time as well.

Theorem 2.6 *Let $p > 1$ be in force. For any $M > d$, there exists $u_M \in \mathcal{N}_-$ satisfying $J(u_M) = M$ and the weak solution u of problem* (1.1) *blows up in finite time.*

2.4 Extinction or Non-extinction in Finite Time

The following result shows the extinction properties of weak solutions for $0 < p(x) < 1$. Let p^* be a positive constant satisfying $p^+ < p^* < 1$.

Theorem 2.7 *If* $0 < p^- < p^+ < p^* < 1$, *and*

$$\|u_0\|_2 > \left[\frac{|\Omega|^{\frac{1-p^*}{2}}}{\lambda_1 \mathrm{e}(p^* - p^+)} \right]^{\frac{1}{1-p^*}} := \overline{C}, \tag{2.12}$$

then the weak solution u of (1.1) *vanishes at* T_* *and* $\frac{2}{\lambda_1(1-p^*)} \le T_* \le \frac{-2|\Omega|^{\frac{1-p^*}{2}}}{\lambda_1 \mathrm{e}(p^*-p^+)Y(0)}$. *More precisely,*

$$\begin{cases} \|u\|_2 \le \left(\|u_0\|_2^{1-p^*} + Y(0)t/2 \right)^{\frac{1}{1-p^*}}, & 0 < t < T_*, \\ \|u\|_2 = 0, & t \in [T_*, +\infty), \end{cases} \tag{2.13}$$

where $T_* := -2\|u_0\|_2^{1-p^*}/Y(0)$ *and* $Y(0) := (1-p^*)\left[\frac{|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)} - \lambda_1 \|u_0\|_2^{1-p^*} \right] < 0.$

Corollary 2.1 *Let* $0 < p^- < p^+ < p^* < 1$ *and* (2.12) *be in force. If* $J(u_0) > d$ *and* $I(u_0) = 0$, *then the results of Theorem 2.7 hold.*

In the following, we show a result on non-extinction of solutions.

Theorem 2.8 *Let* $0 < p^- < p^+ < p^* < 1$ *and* $J(u_0) < 0$. *The weak solution u of (1.1) does not vanish in finite time.*

2.5 Remarks

We summarize the main results in the above three subsections. For convenience, we define some notations: "G.E." means "Global existence"; "B.P." means "Blow-up"; "N.E." means "Non-extinction"; "E." means "Extinction"; "Th." means "Theorem"; "Cor." means "Corollary". We characterize the singularity of solutions by the help of Table 1.

The classification for the singularity of solutions is optimal with respect to the initial energy. In fact, Nehari energy $I(u_0) \neq 0$ provided $J(u_0) < d$. It was a pity that we have not solved the case where the energy $J(u_0)$ meets the potential depth.

3 Proof of Theorems 2.1–2.2

Lemma 3.1 *The potential depth* $d > 0$ *which is defined in* (2.5).

Proof Fix any $u \in \mathcal{N}$, we have $I(u) = 0$. Let $\mu > 0$ be small that $p(x)+\mu < p^+ + \mu < 4^*$. By $x^{-\mu} \log x \le (\mathrm{e}\mu)^{-1}$ for $x \ge 1$, $\mu > 0$ and $H_0^2(\Omega) \hookrightarrow L^{p(x)+1+\mu}(\Omega)$, one could obtain that

$$\begin{aligned} \|\Delta u\|_2^2 &= \int_{\Omega_1} u^{p(x)+1} \log u \mathrm{d}x + \int_{\Omega_2} u^{p(x)+1} \log u \mathrm{d}x \\ &\le \frac{1}{\mathrm{e}\mu} \int_{\Omega} u^{p(x)+1+\mu} \mathrm{d}x \le \frac{1}{\mathrm{e}\mu} \max\left\{ \|u\|_{p(x)+1+\mu}^{p^-+1+\mu}, \|u\|_{p(x)+1+\mu}^{p^++1+\mu} \right\} \\ &\le \frac{B^{p^++1+\mu}}{\mathrm{e}\mu} \max\left\{ \|\Delta u\|_2^{p^-+1+\mu}, \|\Delta u\|_2^{p^++1+\mu} \right\}, \end{aligned} \tag{3.1}$$

where $\Omega_1 := \{x \in \Omega : u \geq 1\}$, $\Omega_2 := \{x \in \Omega : 0 < u < 1\}$, which implies $\|\Delta u\|_2 \geq \left(\frac{e\mu}{B^{p^+ +1+\mu}}\right)^{\frac{1}{p^- -1+\mu}}$. Noticing that $p^- > 1$, we have

$$J(u) \geq \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p^- +1}\|\Delta u\|_2^2 + \frac{1}{p^- +1}I(u) \geq \frac{p^- -1}{2(p^- +1)}\left(\frac{e\mu}{B^{p^+ +1+\mu}}\right)^{\frac{2}{p^- -1+\mu}}.$$

Therefore, $d \geq \frac{p^- -1}{2(p^- +1)}\left(\frac{e\mu}{B^{p^+ +1+\mu}}\right)^{\frac{2}{p^- -1+\mu}} > 0$. □

Lemma 3.2 *Let $(\mathcal{H})$ be in force and $u \in H_0^2(\Omega)$.*

(i) If $0 \leq \|\Delta u\|_2 \leq r^* := \left(\frac{e\mu}{B^{p^+ +1+\mu}}\right)^{\frac{1}{p^- -1+\mu}}$, then $I(u) \geq 0$.
(ii) If $I(u) < 0$, then $\|\Delta u\|_2 > r^*$.
(iii) If $I(u) = 0$, then $\|\Delta u\|_2 = 0$ or $\|\Delta u\|_2 \geq r^*$.

Proof (i) By using the Sobolev's inequality (3.1), we have

$$\int_\Omega u^{p(x)+1}\log u \mathrm{d}x \leq \frac{B^{p^+ +1+\mu}}{e\mu}\max\left\{(r^*)^{p^- -1+\mu}, (r^*)^{p^+ -1+\mu}\right\}\|\Delta u\|_2^2 = \|\Delta u\|_2^2,$$

we have $I(u) \geq 0$.

(ii) From $I(u) < 0$ and the Sobolev's inequality (3.1), one could check that $\|\Delta u\|_2 > r^*$.

(iii) If $\|\Delta u\|_2 = 0$, then $I(u) = 0$. If $I(u) = 0$ and $\|\Delta u\|_2 \neq 0$, then by the Sobolev's inequality (3.1), we get $\|\Delta u\|_2 \geq r^*$. □

Proof of Theorem 2.1. **Step 1. Global existence.** We use the Galerkin's approximation and a priori estimate. Let $\{\phi_j(x)\}$ be a system of orthogonal basis of $H_0^2(\Omega)$ and define $u^m(x,t) = \sum_{j=1}^m a_j^m(t)\phi_j(x)$, $m = 1, 2, \cdots$, of (1.1) satisfying that

$$(u_t^m, \phi_j) + (\Delta u^m, \Delta\phi_j) = ((u^m)^{p(x)}\log(u^m), \phi_j), \quad j = 1, 2, \cdots, m, \tag{3.2}$$
$$u^m(x,0) = \textstyle\sum_{j=1}^m b_j^m\phi_j(x) \to u_0(x), \text{ strongly in } H_0^2(\Omega) \text{ as } m \to +\infty. \tag{3.3}$$

By the Peano's theorem, (3.2, 3.3) have a local solution. Multiplying (3.2) by $\frac{\mathrm{d}}{\mathrm{d}t}a_j^m(t)$, we have $\int_0^t \|u_\tau^m\|_2^2 \mathrm{d}\tau + J(u^m) = J(u^m(0))$, $0 \leq t < +\infty$. Since $u^m(x,0) \to u_0(x)$ in $H_0^2(\Omega)$, we have $J(u^m(x,0)) \to J(u_0(x)) < d$, $I(u^m(x,0)) \to I(u_0(x)) > 0$. Thus, for large m, we have

$$\int_0^t \|u_\tau^m\|_2^2 \mathrm{d}\tau + J(u^m) = J(u^m(x,0)) < d, \quad 0 \leq t < +\infty, \tag{3.4}$$

and $I(u^m(x,0)) > 0$, which indicates that $u^m(x,0) \in \mathcal{W}$ for sufficiently large m.

Next, we claim $u^m(x,t) \in \mathcal{W}$ for sufficiently large m and any $t \in [0,T]$. By contradiction, there would exist a sufficiently large m and a constant $t_* \in (0,T]$ such that $u^m(x,t_*) \in H_0^2(\Omega)\setminus\{0\}$ and $J(u^m(x,t_*)) = d$ or $I(u^m(x,t_*)) = 0$. In fact, the former case contradicts to (3.4), while the later one happens, we have $J(u^m(x,t_*)) \geq d$, which also contradicts to (3.4). Hence, our claim is valid. Applying (2.3), (2.4), (3.4) and $I(u^m(x,t)) > 0$ for sufficiently large m, one could obtain

$$\int_0^t \|u_\tau^m\|_2^2 \mathrm{d}\tau + \frac{p^- -1}{2(p^- +1)}\|\Delta u^m\|_2^2 < d,$$

for large m and any $0 \le t < +\infty$, which indicates that

$$\int_0^t \|u_\tau^m\|_2^2 \mathrm{d}\tau < d, \quad 0 \le t < +\infty, \tag{3.5}$$

$$\|u^m\|^2_{H_0^2(\Omega)} < \frac{2d(p^- + 1)}{p^- - 1}, \quad 0 \le t < +\infty. \tag{3.6}$$

Since $p(x)p'(x) < 4^* + 1$, we could choose $\mu > 0$ sufficiently small that $(p(x)+\mu)p'(x) < 4^* + 1$. Therefore, by embedding $H_0^2(\Omega) \hookrightarrow L^{(p(x)+\mu)p'(x)}(\Omega)$, we get

$$\begin{aligned}\int_\Omega |\psi^m(x,t)|^{p'(x)}\mathrm{d}x &= \int_{\Omega_1} |\psi^m(x,t)|^{p'(x)}\mathrm{d}x + \int_{\Omega_2} |\psi^m(x,t)|^{p'(x)}\mathrm{d}x \\ &\le \int_{\Omega_1} |\psi^m(x,t)|^{p'(x)}\mathrm{d}x + \int_{\Omega_2} |(u^m)^{p^-}\log(u^m)|^{p'(x)}\mathrm{d}x \\ &\le (\mathrm{e}\mu)^{-\frac{p^+}{p^- - 1}} \max\left\{ \|u^m\|_{(p(x)+\mu)p'(x)}^{(p^+ + \mu)\frac{p^+}{p^- - 1}}, \|u^m\|_{(p(x)+\mu)p'(x)}^{(p^- + \mu)\frac{p^-}{p^+ - 1}} \right\} + (\mathrm{e}p^-)^{-\frac{p^-}{p^+ - 1}}|\Omega| \\ &\le C_1 \max\left\{ \|\Delta u\|_2^{(p^+ + \mu)\frac{p^+}{p^- - 1}}, \|\Delta u\|_2^{(p^- + \mu)\frac{p^-}{p^+ - 1}} \right\} + C_2 \le C, \end{aligned} \tag{3.7}$$

where $\psi^m(x,t) = (u^m)^{p(x)}\log(u^m)$ and $\Omega_1 := \{x \in \Omega : u^m \ge 1\}$, $\Omega_2 := \{x \in \Omega : 0 < u^m < 1\}$ and the inequalities $|x^{p^-}\log x| \le (\mathrm{e}p^-)^{-1}$ for $0 < x < 1$ and $x^{-\mu}\log x \le (\mathrm{e}\mu)^{-1}$ for $x \ge 1$ and $\mu > 0$ are used.

By the uniform estimates (3.5–3.7), it was seen that the local solutions can be extended globally. Thus, there exist some u and a subsequence of $\{u^m\}$ such that, for each $T > 0$, one could obtain that

$$u_t^m \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.8}$$

$$u^m \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^2(\Omega)), \tag{3.9}$$

$$(u^m)^{p(x)}\log(u^m) \rightharpoonup u^{p(x)}\log u \quad \text{weakly star in } L^\infty((0,T);L^{p'(x)}(\Omega)), \tag{3.10}$$

as $m \to +\infty$. Fix $k \in \mathbb{N}$. In order to show the limit u in (3.8–3.10) is a weak solution of (1.1), one could find $v \in C^1([0,T];H_0^2(\Omega))$ of

$$v(x,t) = \sum_{j=1}^k l_j(t)\phi_j(x), \tag{3.11}$$

where $\{l_j(t)\}_{j=1}^k$ are arbitrarily given C^1 functions. Choosing $m \ge k$ in (3.2) and multiplying (3.2) by $l_j(t)$, one could obtain

$$\int_0^T ((u_t^m, v) + (\Delta u^m, \Delta v))\mathrm{d}t = \int_0^T ((u^m)^{p(x)}\log(u^m), v)\mathrm{d}t. \tag{3.12}$$

Let $m \to +\infty$ in (3.12). By the convergence in (3.8–3.10), we deduce that

$$\int_0^T ((u_t, v) + (\Delta u, \Delta v))\mathrm{d}t = \int_0^T (u^{p(x)}\log u, v)\mathrm{d}t. \tag{3.13}$$

Since functions in (3.11) are dense in $L^2((0,T);H_0^2(\Omega))$, (3.13) holds for all $v \in L^2((0,T);H_0^2(\Omega))$. For arbitrariness of $T > 0$, one has $(u_t, v) + (\Delta u, \Delta v) = \left(u^{p(x)}\log u, v\right)$ for a.e. $t > 0$. To prove (2.7), we first assume that u is smooth enough such that $u_t \in L^2((0,T);H_0^2(\Omega))$. Taking $v = u_t$ in (3.13) it is seen that (2.7) is true. By the density of $L^2((0,T); H_0^2(\Omega))$ in $L^2(\Omega \times (0,T))$, we know (2.7) holds for weak solutions to problem (1.1). The global solutions of (1.1) is proved.

Step 2. Uniqueness of the bounded solution. We assume both u and v be bounded weak solutions to (1.1). For any $\varphi \in H_0^2(\Omega)$, we have $(u_t, \varphi) + (\Delta u, \Delta\varphi) = \left(u^{p(x)} \log u, \varphi\right)$ and $(v_t, \varphi) + (\Delta v, \Delta\varphi) = \left(v^{p(x)} \log v, \varphi\right)$. Subtracting the above two equalities, choosing $\varphi = u - v \in H_0^2(\Omega)$ and integrating over $(0, t)$ for any $t > 0$, we deduce that

$$\int_0^t \int_\Omega (\varphi_t \varphi + |\Delta\varphi|^2) \mathrm{d}x\mathrm{d}t = \int_0^t \int_\Omega \left(u^{p(x)} \log u - v^{p(x)} \log v\right)(u - v)\mathrm{d}x\mathrm{d}t.$$

Since $\varphi(x, 0) = 0$, due to the boundedness of u and v, we have $\int_\Omega \varphi^2(x,t)\mathrm{d}x \le C\int_0^t \int_\Omega \varphi^2(x,t)\mathrm{d}x\mathrm{d}t$, where $C > 0$ is a constant depending on p^-, p^+, and the bounds of u, v. By the Gronwall's inequality, $\int_\Omega \varphi^2(x,t)\mathrm{d}x = 0$. Hence, $\varphi = 0$ a.e. in $\Omega \times (0, +\infty)$ and the uniqueness of bounded solution follows.

Next, we consider the exponential decay of $\|u(t)\|_2$. Since $u(t) \in \mathcal{W}$ for $0 \le t < +\infty$, we have $I(u(t)) \ge 0$. Then it follows from (2.7) and $I(u(t)) \ge 0$ that

$$J(u_0) \ge J(u(t)) \ge \frac{p^- - 1}{2(p^- + 1)}\|\Delta u\|_2^2 + \frac{1}{p^+ + 1} I(u(t)) \ge \frac{p^- - 1}{2(p^- + 1)}\|\Delta u\|_2^2, \quad (3.14)$$

which, together with (2.2), implies

$$\|u\|_{p(x)+1+\mu} \le B\|\Delta u\|_2 \le B\sqrt{\frac{2(p^- + 1)}{p^- - 1} J(u_0)}. \quad (3.15)$$

By the definition of d_0 and (2.2, 3.1, 3.15), we obtain

$$\begin{aligned} \int_\Omega u^{p(x)+1} \log u \mathrm{d}x \le \max &\left\{ \left[\frac{2(p^- + 1)}{p^- - 1} J(u_0)\right]^{\frac{p^- - 1 + \mu}{2}} \frac{B^{p^+ + 1 + \mu}}{\mathrm{e}\mu}, \right. \\ &\left. \left[\frac{2(p^- + 1)}{p^- - 1} J(u_0)\right]^{\frac{p^+ - 1 + \mu}{2}} \left(\frac{B^{p^+ + 1 + \mu}}{\mathrm{e}\mu}\right)^{\frac{p^+ - 1 + \mu}{p^- - 1 + \mu}} \right\} \|\Delta u\|_2^2 \\ &= \left[\frac{J(u_0)}{d_0}\right]^{\frac{p^- - 1 + \mu}{2}} \|\Delta u\|_2^2. \end{aligned} \quad (3.16)$$

Choosing $v = u$ in (2.6) and combining (2.4) with (3.16), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 \le -2\lambda_1 \left[1 - (J(u_0)/d_0)^{\frac{p^- - 1 + \mu}{2}}\right] \|u\|_2^2.$$

Consequently, $\|u(t)\|_2^2 \le \|u_0\|_2^2 \exp\{-\delta t\}$, where constant $\delta := 2\lambda_1 \left[1 - (J(u_0)/d_0)^{\frac{p^- - 1 + \mu}{2}}\right]$.

Finally, we consider the exponential decay of $\|\Delta u(t)\|_2$, $\|u(t)\|_{p(x)+1}$, and $J(u(t))$. By (2.4, 3.16), one could obtain

$$I(u(t)) \ge \left[1 - (J(u_0)/d_0)^{\frac{p^- - 1 + \mu}{2}}\right] \|\Delta u\|_2^2. \quad (3.17)$$

Define an auxiliary function $L(t) := J(u(t)) + \|u(t)\|_2^2$. Then by $\|u\|_2^2 \le \lambda_1^{-1}\|\Delta u\|_2^2$ and (3.14), we get

$$L(t) \le J(u(t)) + \lambda_1^{-1}\|\Delta u(t)\|_2^2 \le \left[1 + \frac{2(p^- + 1)}{\lambda_1(p^- - 1)}\right] J(u(t)). \tag{3.18}$$

By (2.1), we have

$$\begin{aligned}\int_\Omega u^{p(x)+1}\mathrm{d}x &\le \max\left\{\|u\|_{p(x)+1}^{p^-+1}, \|u\|_{p(x)+1}^{p^++1}\right\}\\ &\le \max\left\{D^{p^++1}\left[\frac{2(p^-+1)}{p^--1}J(u_0)\right]^{\frac{p^--1}{2}}, D^{p^++1}\left[\frac{2(p^-+1)}{p^--1}J(u_0)\right]^{\frac{p^+-1}{2}}\right\}\|\Delta u\|_2^2.\end{aligned}$$

Thus,

$$J(u) \le \left[\frac{p^+ - 1}{2(p^+ + 1)} + \frac{M}{(p^- + 1)^2}\right]\|\Delta u\|_2^2 + \frac{1}{p^- + 1} I(u), \tag{3.19}$$

where $M := \max\left\{D^{p^++1}\left[\frac{2(p^-+1)}{p^--1}J(u_0)\right]^{\frac{p^--1}{2}}, D^{p^++1}\left[\frac{2(p^-+1)}{p^--1}J(u_0)\right]^{\frac{p^+-1}{2}}\right\}$. Moreover, it follows from (2.7) and $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = -2I(u)$ that $\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -\|u_t\|_2^2 - 2I(u(t))$, which, together with (3.16, 3.17, 3.19), indicates that for any constant $\alpha > 0$, one could obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t) &\le -\|u_t\|_2^2 - 2I(u) - \alpha J(u) + \alpha\left[\frac{p^+-1}{2(p^++1)} + \frac{M}{(p^-+1)^2}\right]\|\Delta u\|_2^2 + \frac{\alpha}{p^-+1}I(u)\\ &\le -\alpha J(u) + \left\{\alpha\left[\frac{p^+-1}{2(p^++1)} + \frac{M}{(p^-+1)^2}\right]\left[1 - \left(\frac{J(u_0)}{d_0}\right)^{\frac{p^--1+\mu}{2}}\right]^{-1} + \frac{\alpha}{p^-+1} - 2\right\} I(u).\end{aligned}$$

Let

$$\alpha := \frac{4(p^++1)(p^-+1)\left[1 - \left(\frac{J(u_0)}{d_0}\right)^{\frac{p^--1+\mu}{2}}\right]}{(p^+-1)(p^-+1)^2 + 2M(p^++1) + 2(p^++1)(p^-+1)\left[1 - \left(\frac{J(u_0)}{d_0}\right)^{\frac{p^--1+\mu}{2}}\right]} > 0.$$

It follows from (3.18) that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) \le -\alpha J(u(t)) \le -\frac{\alpha\lambda_1(p^- - 1)}{\lambda_1(p^- - 1) + 2(p^- + 1)} L(t),$$

which, together with the definition of $L(t)$, implies

$$\begin{aligned}J(u(t)) + \|u(t)\|_2^2 &= L(t) \le L(0)\exp\left[-\frac{\alpha\lambda_1(p^- - 1)}{\lambda_1(p^- - 1) + 2(p^- + 1)}t\right]\\ &\le \left(J(u_0) + \|u_0\|_2^2\right)\exp\left[-\frac{\alpha\lambda_1(p^- - 1)}{\lambda_1(p^- - 1) + 2(p^- + 1)}t\right].\end{aligned} \tag{3.20}$$

By (3.14, 3.20), we obtain

$$\|\Delta u\|_2^2 \le \frac{2(p^- + 1)}{p^- - 1}J(u(t)) \le \frac{2(p^- + 1)}{p^- - 1}\left(J(u_0) + \|u_0\|_2^2\right)\exp\left[-\frac{\alpha\lambda_1(p^- - 1)}{\lambda_1(p^- - 1) + 2(p^- + 1)}t\right]. \tag{3.21}$$

Finally, it follows from (2.1, 3.21) that

$$\|u\|_{p(x)+1}^2 \le B^2\|\Delta u\|_2^2 \le \frac{2B^2(p^- + 1)}{p^- - 1}\left(J(u_0) + \|u_0\|_2^2\right)\exp\left[-\frac{\alpha\lambda_1(p^- - 1)}{\lambda_1(p^- - 1) + 2(p^- + 1)}t\right].$$

□

Proof of Theorem 2.2. **Step 1. Blow-up in finite time**. Assume that u was a global weak solution to problem (1.1) for $0 < J(u_0) < d$, $I(u_0) < 0$ and define $M(t) := \int_0^t \|u\|_2^2 d\tau$ for $t \geq 0$, one has $M'(t) = \|u\|_2^2$ and

$$M''(t) = 2(u_t, u) = -2\|\Delta u\|_2^2 + 2\int_\Omega u^{p(x)+1} \log u dx = -2I(u). \tag{3.22}$$

Next, we claim $I(u) < 0$ for all $t \in [0, T)$. In fact, if it was false, it follows there would exist a constant of first appearance $t_0 > 0$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $t \in [0, t_0)$. From Lemma 3.2 (ii), we have $\|\Delta u(t)\|_2 > r^* > 0$ for $t \in [0, t_0)$. By the continuity of $\|\Delta u(t)\|_2$ to t and Lemma 3.2 (iii), we get $\|\Delta u(t_0)\|_2 \geq r^* > 0$. Hence, $u(t_0) \in \mathcal{N}$. Then it follows from (2.5) that $J(u(t_0)) \geq d$, which contradicts to (2.7). By computations, one has

$$J(u) \geq \frac{p^- - 1}{2(p^- + 1)}\|\Delta u\|_2^2 + \frac{1}{p^- + 1} I(u). \tag{3.23}$$

By (2.7, 3.22, 3.23), one could obtain

$$\begin{aligned} M''(t) &= -2I(u) \geq (p^- - 1)\|\Delta u\|_2^2 - 2(p^- + 1)J(u) \\ &\geq (p^- - 1)\lambda_1\|u\|_2^2 + 2(p^- + 1)\int_0^t \|u_\tau\|_2^2 d\tau - 2(p^- + 1)J(u_0). \end{aligned}$$

Noticing that $(M'(t))^2 = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u_0\|_2^2 M'(t) - \|u_0\|_2^4$, we have

$$\begin{aligned} &M''(t)M(t) - \frac{p^- + 1}{2}(M'(t))^2 \\ &\geq \lambda_1(p^- - 1)M(t)M'(t) + 2(p^- + 1)M(t)\int_0^t \|u_\tau\|_2^2 d\tau - 2(p^- + 1)M(t)J(u_0) \\ &-2(p^- + 1)\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 - (p^- + 1)M'(t)\|u_0\|_2^2 + \frac{p^- + 1}{2}\|u_0\|_2^4 \\ &\geq \lambda_1(p^- - 1)M(t)M'(t) - 2(p^- + 1)M(t)J(u_0) - (p^- + 1)M'(t)\|u_0\|_2^2 + \frac{p^- + 1}{2}\|u_0\|_2^4 \\ &\geq \left[\frac{\lambda_1}{2}(p^- - 1)M'(t) - 2(p^- + 1)J(u_0)\right]M(t) + \left[\frac{\lambda_1}{2}(p^- - 1)M(t) - (p^- + 1)\|u_0\|_2^2\right]M'(t). \end{aligned}$$

From $M''(t) = -2I(u)$, we have $M''(t) > 0$ for $t \geq 0$. Hence,

$$\begin{aligned} &M''(t)M(t) - \frac{p^- + 1}{2}(M'(t))^2 \\ > &\left[\frac{\lambda_1}{2}(p^- - 1)\|u_0\|_2^2 - 2(p^- + 1)J(u_0)\right]M(t) \\ + &\left[\frac{\lambda_1}{2}(p^- - 1)M(t) - (p^- + 1)\|u_0\|_2^2\right]M'(t). \end{aligned}$$

Since $M'(0) > 0$, we have $M(t) \geq M'(0)t$. Therefore, for sufficiently large $t \geq t^*$, one could obtain $\lambda_1(p^- - 1)M(t) > 2(p^- + 1)\|u_0\|_2^2$. Combining the above inequality and $\|u_0\|_2^2 > \frac{4(p^-+1)d}{\lambda_1(p^- -1)} > \frac{4(p^-+1)J(u_0)}{\lambda_1(p^- -1)}$, we have

$$M''(t)M(t) - \frac{p^- + 1}{2}(M'(t))^2 > 0, \tag{3.24}$$

for sufficiently large $t \geq t^*$. From (3.24), it follows with $\alpha = \frac{p^- - 1}{2} > 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M'(t)}{M^{1+\alpha}(t)}\right) > 0, \quad \forall t \geq t^*, \quad \text{or} \quad \frac{M'(t)}{M^{1+\alpha}(t)} > \frac{M'(t^*)}{M^{1+\alpha}(t^*)}, \quad \forall t > t^*. \tag{3.25}$$

Integrating both sides of (3.25) over (t^*, t) with respect to t, $M(t)$ cannot remain finite for all $t > t^*$, and therefore there is a contradiction.

Step 2. Upper bound of blow-up time. Taking $J(\lambda u) := j(\lambda)$, we have

$$j(\lambda) = \frac{1}{2}\|\Delta(\lambda u)\|_2^2 - \int_\Omega \frac{(\lambda u)^{p(x)+1}\log(\lambda u)}{p(x)+1}\mathrm{d}x + \int_\Omega \frac{(\lambda u)^{p(x)+1}}{(p(x)+1)^2}\mathrm{d}x.$$

It is noticed that $j(\lambda) > 0$ for small $\lambda > 0$, $j(\lambda) \to -\infty$ for $\lambda \to +\infty$ and $j(\lambda)$ is continuous on $[0, +\infty)$ and differentiable on $(0, +\infty)$. Combining this facts, we imply that $j(\lambda)$ attains its maximum value at some number $\lambda_* := \lambda_*(u)$.

By Fermat's theorem, one has $j'(\lambda) = \lambda\|\Delta u\|_2^2 - \int_\Omega \lambda^{p(x)} u^{p(x)+1}\log(\lambda u)\mathrm{d}x$. On the other hand, since $j'(\lambda) = \frac{1}{\lambda}I(\lambda u)$, we obtain $I(\lambda_* u) = 0$. Since

$$\begin{aligned} I(u) &= I(u) - \frac{1}{(\lambda_*)^{p^-+1}} I(\lambda_* u) \\ &= \left[1 - (\lambda_*)^{1-p^-}\right]\|\Delta u\|_2^2 + \int_\Omega (\lambda_*)^{p(x)-p^-} u^{p(x)+1}\log(\lambda_* u)\mathrm{d}x - \int_\Omega u^{p(x)+1}\log u\mathrm{d}x, \end{aligned}$$

and $p(x) > 1$ for a.e. $x \in \Omega$, we derive that $\lambda_* \in (0, 1)$ provided that $I(u) < 0$. And this implies

$$\begin{aligned} d &\leq J(\lambda_* u) - \frac{1}{p^- + 1} I(\lambda_* u) \\ &\leq \left(\frac{1}{2} - \frac{1}{p^- + 1}\right)\|\Delta(\lambda_* u)\|_2^2 \\ &\quad - \int_\Omega \left(\frac{1}{p(x)+1} - \frac{1}{p^-+1}\right)(\lambda_* u)^{p(x)+1}\log u\mathrm{d}x + \int_\Omega \frac{(\lambda_* u)^{p(x)+1}}{(p(x)+1)^2}\mathrm{d}x \\ &\leq (\lambda_*)^2\left[J(u) - \frac{1}{p^-+1}I(u)\right] \leq J(u_0) - \frac{1}{p^-+1}I(u), \end{aligned}$$

which in turn implies for all $t \geq 0$ that $M'(t) \geq 2(p^- + 1)(d - J(u_0))t$ and $M(t) \geq (p^- + 1)(d - J(u_0))t^2$. Integrating both sides of (3.25) over (t^*, t) with respect to t, we obtain $M(t) > \left[\frac{M^{1+\alpha}(t^*)}{\alpha M'(t^*)(t^* - t) + M(t^*)}\right]^{\frac{1}{\alpha}}$, hence, $t \leq \frac{2\int_0^{t^*}\|u\|_2^2\mathrm{d}\tau}{(p^- - 1)\|u(t^*)\|_2^2} + t^*$. □

4 Proof of Theorems 2.3 and 2.5

Proof of Theorem 2.3. Let $\lambda_k = 1 - 1/k$, $k = 2, 3, \cdots$. Consider

$$\begin{cases} u_{kt} + \Delta^2 u_k = u_k^{p(x)}\log u_k, & (x,t) \in \Omega \times (0,T), \\ u_k(x,t) = \Delta u_k(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_k(x,0) = u_{0k}(x) := \lambda_k u_0, & x \in \Omega. \end{cases}$$

Noticing that $u_0 \in H_0^2(\Omega)$, $\lambda_k \in (0,1)$, and $I(u_0) \geq 0$, we have

$$I(u_{0k}) > \int_\Omega \lambda_k^2 \left(|\Delta u_0|^2 - u_0{}^{p(x)+1} \log u_0\right) \mathrm{d}x$$
$$= \lambda_k^2 \left(\int_\Omega |\Delta u_0|^2 \mathrm{d}x - \int_{\Omega u_0}^{p(x)+1} \log u_0 \mathrm{d}x\right) = \lambda_k^2 I(u_0) \geq 0.$$

A simply computation shows

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_k} J(\lambda_k u_0) = \lambda_k \int_\Omega |\Delta u_0|^2 \mathrm{d}x - \int_\Omega \lambda_k^{p(x)} u_0{}^{p(x)+1} \log(\lambda_k u_0) \mathrm{d}x$$
$$> \lambda_k \left(\int_\Omega |\Delta u_0|^2 \mathrm{d}x - \int_{\Omega u_0}^{p(x)+1} \log u_0 \mathrm{d}x\right) = \lambda_k I(u_0) \geq 0.$$

This implies that $J(\lambda_k u_0)$ is strictly increasing with respect to λ_k. Hence, $J(u_{0k}) = J(\lambda_k u_0) < J(u_0) = d$. The remainder can be proved similarly to Theorem 2.1. □

Proof of Theorem 2.4. Since $J(u_0) = d$ and $I(u_0) < 0$, there exists a constant $t_0 > 0$ such that $J(u(x,t)) > 0$ and $I(u(x,t)) < 0$ for $0 < t \leq t_0$. Considering $(u_t, u) = -I(u)$, we have $u_t \not\equiv 0$ for $0 < t \leq t_0$. Furthermore,

$$0 < J(u(t_0)) = d_1 := d - \int_0^{t_0} \|u_\tau\|_2^2 \mathrm{d}\tau < d. \tag{4.1}$$

Taking $t = t_0$ as the initial time, we will prove that $I(u) < 0$ for all $t > t_0$. Otherwise, there must be a constant $t_1 > 0$ such that $I(u(t_1)) = 0$ and $I(u) < 0$ for $t_0 \leq t < t_1$. From Lemma 3.2 (ii), we have $\|\Delta u(t)\|_2 > r^* > 0$ for $t \in [t_0, t_1)$. Then by continuity of $\|\Delta u(t)\|_2$ with respect to t and Lemma 3.2 (iii), we get $\|\Delta u(t_1)\|_2 \geq r^* > 0$. Hence, $u(t_1) \in \mathcal{N}$. It follows from the definition of d that $J(u(t_1)) \geq d$, which contradicts to (4.1). Hence, $I(u) < 0$ for all $t \geq 0$.

Similarly to the proof of Theorem 2.2, we get

$$M''(t)M(t) - \frac{p^- + 1}{2}(M'(t))^2$$
$$\geq \left[\frac{\lambda_1}{2}(p^- - 1)M'(t) - 2(p^- + 1)J(u_0)\right] M(t)$$
$$+ \left[\frac{\lambda_1}{2}(p^- - 1)M(t) - (p^- + 1)\|u_0\|_2^2\right] M'(t).$$

Then from $M''(t) = -2I(u)$, we have $M''(t) > 0$ for $t \geq 0$. Since $M'(0) > 0$, we have $M(t) \geq M'(0)t$. Therefore, for sufficiently large time t, we have $\frac{\lambda_1}{2}(p^- - 1)M(t) > (p^- + 1)\|u_0\|_2^2$. By $\|u_0\|_2 > \tilde{C}$, we have $\frac{\lambda_1}{2}(p^- - 1)M'(t) > 2(p^- + 1)J(u_0)$. Consequently, there exists a suitably large constant t^* such that for all $t \geq t^*$, $M''(t)M(t) - \frac{p^- + 1}{2}(M'(t))^2 > 0$. The other could be proved similarly to the ones of Theorem 2.2. □

5 Proof of Theorems 2.5 and 2.6

Lemma 5.1 *Let* $(\mathcal{H})$ *be in force.*

(i) *dist*$(0, \mathcal{N}) > 0$, *dist*$(0, \mathcal{N}_-) > 0$.

(ii) For any constant $s > 0$, the set $J^s \cap \mathcal{N}_+$ is bounded in $H_0^2(\Omega)$.

Proof (i) For any $u \in \mathcal{N}$, by the definition of d, one could obtain

$$\begin{aligned} d \le J(u) &\le \frac{p^+-1}{2(p^++1)}\|\Delta u\|_2^2 + \frac{1}{p^++1}I(u) + \frac{1}{(p^-+1)^2}\max\left\{\|u\|_{p(x)+1}^{p^-+1}, \|u\|_{p(x)+1}^{p^++1}\right\} \\ &\le \frac{p^+-1}{2(p^++1)}\|\Delta u\|_2^2 + \frac{D^{p^++1}}{(p^-+1)^2}\max\left\{\|\Delta u\|_2^{p^-+1}, \|\Delta u\|_2^{p^++1}\right\}, \end{aligned}$$

which yields that there exists a constant $c_0 > 0$ such that

$$\mathrm{dist}(0, \mathcal{N}) = \inf_{u\in\mathcal{N}} \|\Delta u\|_2 \ge c_0 := \min\left\{\left[\frac{d}{\left(\frac{p^+-1}{2(p^++1)} + \frac{D^{p^++1}}{(p^-+1)^2}\right)}\right]^{\frac{1}{p^++1}}, \sqrt{\frac{d}{\frac{p^+-1}{2(p^++1)} + \frac{D^{p^++1}}{(p^-+1)^2}}}\right\} > 0.$$

For any $u \in \mathcal{N}_-$, we have $\|\Delta u\|_2 \ne 0$. Let $\mu > 0$ be small that $p(x) + \mu < p^+ + \mu < 4^*$. By $x^{-\mu}\log x \le (\mathrm{e}\mu)^{-1}$ for $x \ge 1$, $\mu > 0$ and $H_0^2(\Omega) \hookrightarrow L^{p(x)+1+\mu}(\Omega)$, we have

$$\begin{aligned} \|\Delta u\|_2^2 &< \int_\Omega u^{p(x)+1}\log u\,\mathrm{d}x \le \int_{\Omega_1} u^{p(x)+1}\log u\,\mathrm{d}x \\ &\le \frac{1}{\mathrm{e}\mu}\max\left\{\|u\|_{p(x)+1+\mu}^{p^-+1+\mu}, \|u\|_{p(x)+1+\mu}^{p^++1+\mu}\right\} \\ &\le \frac{B^{p^++1+\mu}}{\mathrm{e}\mu}\max\left\{\|\Delta u\|_2^{p^-+1+\mu}, \|\Delta u\|_2^{p^++1+\mu}\right\}, \end{aligned}$$

where $\Omega_1 := \{x \in \Omega : u \ge 1\}$, $\Omega_2 := \{x \in \Omega : 0 < u < 1\}$, which implies $\|\Delta u\|_2 > \left(\frac{\mathrm{e}\mu}{B^{p^++1+\mu}}\right)^{\frac{1}{p^--1+\mu}}$. Therefore, $\mathrm{dist}(0, \mathcal{N}_-) = \inf_{u\in\mathcal{N}_-}\|\Delta u\|_2 \ge \left(\frac{\mathrm{e}\mu}{B^{p^++1+\mu}}\right)^{\frac{1}{p^--1+\mu}} > 0$.

(ii) For any $u \in J^s \cap \mathcal{N}_+$, $J(u) < s$ and $I(u) > 0$. Therefore,

$$s > J(u) \ge \frac{p^--1}{2(p^-+1)}\|\Delta u\|_2^2 + \frac{1}{p^++1}I(u) > \frac{p^--1}{2(p^-+1)}\|\Delta u\|_2^2,$$

which yields $\|\Delta u\|_2 < \sqrt{\frac{2s(p^-+1)}{p^--1}}$. □

Lemma 5.2 *Let $(\mathcal{H})$ be in force. For any $s > d$, $0 < C_1 \le \lambda_s \le \Lambda_s \le C_2 < +\infty$, where*

$$C_1 := \min\left\{\left[\frac{\mathrm{e}\mu\delta_{\min}(d)}{[\beta(1+|\Omega|)]^{p^++1+\mu}\left(\frac{2s(p^-+1)}{p^--1}\right)^{\frac{(1-\theta)(p^++1+\mu)}{2}}}\right]^{\frac{1}{\theta(p^-+1+\mu)}}, \left[\frac{\mathrm{e}\mu\delta_{\min}(d)}{[\beta(1+|\Omega|)]^{p^++1+\mu}\left(\frac{2s(p^-+1)}{p^--1}\right)^{\frac{(1-\theta)(p^++1+\mu)}{2}}}\right]^{\frac{1}{\theta(p^++1+\mu)}}\right\}, \tag{5.1}$$

$\delta_{\min}(d)$ is a positive constant to be determined later; $C_2 := \sqrt{\frac{2s(p^-+1)}{\lambda_1(p^--1)}}$; β is the optimal constant in the Gagliardo-Nirenberg's inequality

$$\|u\|_{p^++1+\mu} \le \beta\|\Delta u\|_2^{1-\theta}\|u\|_2^\theta, \quad \forall u \in H_0^2(\Omega), \quad \theta := 1 + \frac{n}{2(p^++1+\mu)} - \frac{n}{4} \in (0, 1).$$

Proof Firstly, we estimate the upper bound of Λ_s. For any $u \in \mathcal{N}^s$, by the definition of functionals $J(u)$ and $I(u)$, we obtain

$$s > J(u) \geq \frac{p^- - 1}{2(p^- + 1)}\|\Delta u\|_2^2 + \frac{1}{p^- + 1}I(u) = \frac{p^- - 1}{2(p^- + 1)}\|\Delta u\|_2^2 \geq \frac{\lambda_1(p^- - 1)}{2(p^- + 1)}\|u\|_2^2. \tag{5.2}$$

Therefore, (5.2) indicates that $\|u\|_2 < C_2$. Hence, $\Lambda_s \leq C_2 < +\infty$.

Next, we estimate the lower bound of λ_s. Let $u \in \mathcal{N}^s$, then $I(u) = 0$ and $J(u) < s$. By (2.1), we obtain

$$\begin{aligned} d < J(u) &\leq \frac{p^+ - 1}{2(p^+ + 1)}\|\Delta u\|_2^2 + \frac{1}{p^+ + 1}I(u) + \frac{1}{(p^- + 1)^2}\int_\Omega u^{p(x)+1}\mathrm{d}x \\ &\leq \frac{p^+ - 1}{2(p^+ + 1)}\|\Delta u\|_2^2 + \frac{1}{(p^- + 1)^2}\max\left\{\|u\|_{p(x)+1}^{p^-+1}, \|u\|_{p(x)+1}^{p^++1}\right\} \\ &\leq \frac{p^+ - 1}{2(p^+ + 1)}\|\Delta u\|_2^2 + \frac{D^{p^++1}}{(p^- + 1)^2}\max\left\{\|\Delta u\|_2^{p^-+1}, \|\Delta u\|_2^{p^++1}\right\} \end{aligned}$$

Thus,

$$\|\Delta u\|_2 \geq \delta_{\min}(d) := \min\left\{\left[\frac{d}{\frac{p^+-1}{2(p^++1)} + \frac{D^{p^++1}}{(p^-+1)^2}}\right]^{\frac{1}{p^++1}}, \sqrt{\frac{d}{\left[\frac{p^+-1}{2(p^++1)} + \frac{D^{p^++1}}{(p^-+1)^2}\right]}}\right\}.$$

By $x^{-\mu}\log x \leq (\mathrm{e}\mu)^{-1}$ for $x \geq 1$, $\mu > 0$, the continuous embedding $L^{p^++1+\mu}(\Omega) \hookrightarrow L^{p(x)+1+\mu}(\Omega)$ and (5.2), we could obtain

$$\begin{aligned} \delta_{\min}(d) \leq \|\Delta u\|_2^2 &\leq \int_{\Omega_1} u^{p(x)+1}\log u\mathrm{d}x \\ &\leq \frac{1}{\mathrm{e}\mu}\max\left\{\|u\|_{p(x)+1+\mu}^{p^-+1+\mu}, \|u\|_{p(x)+1+\mu}^{p^++1+\mu}\right\} \\ &\leq \frac{1}{\mathrm{e}\mu}\max\Big\{[\beta(1+|\Omega|)]^{p^-+1+\mu}\|\Delta u\|_2^{(1-\theta)(p^-+1+\mu)}\|u\|_2^{\theta(p^-+1+\mu)}, \\ &\quad [\beta(1+|\Omega|)]^{p^++1+\mu}\|\Delta u\|_2^{(1-\theta)(p^++1+\mu)}\|u\|_2^{\theta(p^++1+\mu)}\Big\} \\ &\leq \frac{[\beta(1+|\Omega|)]^{p^++1+\mu}}{\mathrm{e}\mu}\left[\frac{2s(p^- + 1)}{p^- - 1}\right]^{\frac{(1-\theta)(p^++1+\mu)}{2}}\max\Big\{\|u\|_2^{\theta(p^-+1+\mu)}, \\ &\quad \|u\|_2^{\theta(p^++1+\mu)}\Big\}. \end{aligned}$$

where $\theta := 1 + \frac{n}{2(p^++1+\mu)} - \frac{n}{4} \in (0, 1)$, one could obtain $\|u\|_2 \geq C_1$, which indicates $\lambda_s \geq C_1 > 0$. □

Lemma 5.3 *Let* $p > 1$*. If* $J(u_0) > d$ *and* $\|u_0\|_2 > \left[(1+|\Omega|)^{p+1}(p+1)^2 J(u_0)\right]^{\frac{1}{p+1}}$*, then the weak solution* u *of problem* (1.1) *blows up in finite time.*

Proof By the embedding $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$, we have $\|u_0\|_2 \le (1+|\Omega|)\|u_0\|_{p+1}$,

$$\begin{aligned} J(u_0) &= \frac{1}{2}\|\Delta u_0\|_2^2 - \int_\Omega \frac{{u_0}^{p+1}\log u_0}{p+1}\mathrm{d}x + \int_\Omega \frac{{u_0}^{p+1}}{(p+1)^2}\mathrm{d}x \\ &\ge \frac{p-1}{2(p+1)}\|\Delta u_0\|_2^2 + \frac{1}{p+1}I(u_0) + \int_\Omega \frac{{u_0}^{p+1}}{(p+1)^2}\mathrm{d}x \\ &\ge \frac{1}{p+1}I(u_0) + \int_\Omega \frac{{u_0}^{p+1}}{(p+1)^2}\mathrm{d}x \\ &\ge \frac{1}{p+1}I(u_0) + J(u_0). \end{aligned}$$

Therefore, $I(u_0) < 0$, i.e., $u_0 \in \mathcal{N}_-$. Now, we only need to prove $\|u_0\|_2 \ge \Lambda_{J(u_0)}$. For any $u \in \mathcal{N}_{J(u_0)}$, i.e., $I(u) = 0$, $J(u) < J(u_0)$. Since

$$J(u) \ge \frac{p-1}{2(p+1)}\|\Delta u\|_2^2 + \frac{1}{p+1}I(u) + \frac{1}{(p+1)^2}\int_\Omega u^{p+1}\mathrm{d}x \ge \frac{1}{(p+1)^2}\int_\Omega u^{p+1}\mathrm{d}x,$$

we have

$$\int_\Omega u^2\mathrm{d}x < (1+|\Omega|)^2\left(\int_\Omega u^{p+1}\mathrm{d}x\right)^{\frac{2}{p+1}} < (1+|\Omega|)^2[(p+1)^2J(u_0)]^{\frac{2}{p+1}} < \int_\Omega {u_0}^2\mathrm{d}x,$$

i.e., $\|u_0\|_2 > \Lambda_{J(u_0)}$. This completes the proof of this lemma. □

Proof of Theorem 2.5. Denote $T(u_0)$ be the maximal existence time of (1.1) with initial datum u_0. If $T(u_0) = +\infty$, we denote the ω-limit set of u_0 by $\omega(u_0) = \bigcap_{t\ge 0}\overline{\{u(s) : s \ge t\}}^{H_0^2(\Omega)}$.

(i) Assume that $u_0 \in \mathcal{N}_+$ with $\|u_0\|_2 \le \lambda_{J(u_0)}$. We first claim that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T(u_0))$. If not, there would exist a constant $t_0 \in (0, T(u_0))$ such that $u(t) \in \mathcal{N}_+$ for $t \in [0, t_0)$ and $u(t_0) \in \mathcal{N}$. By $I(u(t)) = -\int_\Omega u_t(x,t)u(x,t)\mathrm{d}x$, $u_t(x,t) \not\equiv 0$ for $(x,t) \in \Omega \times (0, t_0)$. Recalling (2.7), $J(u(t_0)) < J(u_0)$. Thus, $u(t_0) \in \mathcal{N}^{J(u_0)}$. By the definition of $\lambda_{J(u_0)}$, we have

$$\|u(t_0)\|_2 \ge \lambda_{J(u_0)}. \tag{5.3}$$

By $I(u(t)) > 0$ for $t \in [0, t_0)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = -2I(u)$, we have $\|u(t_0)\|_2 < \|u_0\|_2 \le \lambda_{J(u_0)}$, which contradicts to (5.3). Therefore, our claim is true and we have $u(t) \in J^{J(u_0)}$ for all $t \in [0, T(u_0))$. Lemma 5.1 (ii) shows that $u(t)$ is bounded in $H_0^2(\Omega)$ for $t \in [0, T(u_0))$, and the boundedness of $\|u\|_{H_0^2(\Omega)}$ is dependent of t. Moreover, $T(u_0) = +\infty$. Let $\omega \in \omega(u_0)$. By (2.7) and $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = -2I(u)$, one has $\|\omega\|_2 < \lambda_{J(u_0)}$ and $J(\omega) < J(u_0)$, which implies $\omega(u_0) \cap \mathcal{N} = \emptyset$. Hence, $\omega(u_0) = \{0\}$. Therefore, the weak solution u of (1.1) in its H_0^2-norm exists globally and $u(t) \to 0$ as $t \to +\infty$.

(ii) Assume that $u_0 \in \mathcal{N}_-$ with $\|u_0\|_2 \ge \Lambda_{J(u_0)}$. We claim that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T(u_0))$. If not, there would exist a constant $t^0 \in (0, T(u_0))$ such that $u(t) \in \mathcal{N}_-$ for $t \in [0, t^0)$ and $u(t^0) \in \mathcal{N}$. Similarly to case (i), one has $J(u(t^0)) < J(u_0)$, which implies that $u(t^0) \in J^{J(u_0)}$. Therefore, $u(t^0) \in \mathcal{N}^{J(u_0)}$. According to the definition of $\Lambda_{J(u_0)}$, we have

$$\|u(t^0)\|_2 \le \Lambda_{J(u_0)}. \tag{5.4}$$

From $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = -2I(u)$ and $I(u(t)) < 0$ for $t \in [0, t^0)$, we get $\|u(t^0)\|_2 > \|u_0\|_2 \ge \Lambda_{J(u_0)}$, a contradiction to (5.4). Suppose $T(u_0) = +\infty$. For every $\omega \in \omega(u_0)$, by (2.7) and $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 = -2I(u)$, one has $\|\omega\|_2 > \Lambda_{J(u_0)}$ and $J(\omega) < J(u_0)$. Recalling $\Lambda_{J(u_0)}$, we obtain $\omega(u_0) \cap \mathcal{N} = \emptyset$. Thus, $\omega(u_0) = \{0\}$, which contradicts to Lemma 5.1 (i). Hence, $T(u_0) < +\infty$. □

Proof of Theorem 2.6. Assume that $M > d$ and Ω_1 and Ω_2 are two arbitrary disjoint open subdomains of Ω. Assume that $v \in Q(\Omega_1) := \{u : u \in {H_0}^2(\Omega_1)\}$ is an arbitrary nonzero function and take $\alpha > 0$ large enough such that

$$J(\alpha v) = \frac{1}{2}\|\Delta(\alpha v)\|_2^2 - \int_\Omega \frac{\alpha^{1+p}\log\alpha}{1+p} v^{1+p}\mathrm{d}x - \int_\Omega \frac{\alpha^{1+p}}{1+p} v^{1+p}\log v\mathrm{d}x + \frac{\alpha^{1+p}}{(1+p)^2} v^{1+p}\mathrm{d}x < 0,$$

$$\|\alpha v\|_2 > \left[(1+|\Omega|)^{1+p}(1+p)^2 M\right]^{\frac{1}{1+p}}.$$

We fix such a number $\alpha > 0$ and choose $\mu \in Q(\Omega_2) := \{u : u \in {H_0}^2(\Omega_2)\}$ satisfying $M = J(\mu) + J(\alpha v)$. Extend v and μ to be 0 in $\Omega \setminus \Omega_1$ and $\Omega \setminus \Omega_2$. Set $u_M = \alpha v + \mu$. Then $M = J(\alpha v + \mu) = J(u_M)$ and $\|u_M\|_2 > \|\alpha v\|_2 > \left[(1+|\Omega|)^{1+p}(1+p)^2 J(u_M)\right]^{\frac{1}{1+p}}$. By Lemma 5.3, $u_M \in \mathcal{N}_-$ and the weak solution u of problem (1.1) blows up in finite time. □

6 Proof of Theorems 2.7 and 2.8

Lemma 6.1 *Assume a nonnegative continuous* $\varphi(t)$ *satisfies* $\varphi'(t) + 2\lambda_1\varphi(t) \leq \frac{2|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)}\varphi^{\frac{p^*+1}{2}}(t)$, *and* $\varphi(0) > \left[\frac{|\Omega|^{\frac{1-p^*}{2}}}{\lambda_1\mathrm{e}(p^*-p^+)}\right]^{\frac{2}{1-p^*}}$, *where* $0 < p^- < p^+ < p^* < 1$. *There exists a constant* $T_1 > 0$ *such that*

$$\begin{cases} \varphi(t) \leq \left[\varphi^{\frac{1-p^*}{2}}(0) + \frac{\psi(0)}{2}t\right]^{\frac{2}{1-p^*}}, & 0 < t < T_1, \\ \varphi(t) = 0, & t \in [T_1, \infty), \end{cases} \tag{6.1}$$

where $T_1 := -2\varphi^{\frac{1-p^*}{2}}(0)/\psi(0)$, $\frac{2}{\lambda_1(1-p^*)} \leq T_1 \leq \frac{-2|\Omega|^{\frac{1-p^*}{2}}}{\lambda_1\mathrm{e}(p^*-p^+)\psi(0)}$, *and*

$$\psi(0) := \left(1-p^*\right)\left[\frac{|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)} - \lambda_1\varphi^{\frac{1-p^*}{2}}(0)\right] < 0.$$

Proof For simplicity, we denote $M_1 := \frac{2|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)}$, $M_2 := 2\lambda_1$. First, it is easy to observe that the following fact remains true. $\varphi'(t) + M_2\varphi(t) \leq M_1\varphi^{\frac{p^*+1}{2}}(t)$. Define $H(t) := \varphi^{\frac{1-p^*}{2}}(t)$. Then

$$\begin{aligned} H'(t) &\leq \frac{1-p^*}{2}\varphi^{-\frac{1+p^*}{2}}(t)\left[M_1\varphi^{\frac{p^*+1}{2}}(t) - M_2\varphi(t)\right] \\ &= \psi(t) := \frac{1-p^*}{2}\left(M_1 - M_2\varphi^{\frac{1-p^*}{2}}(t)\right). \end{aligned} \tag{6.2}$$

Since $\psi(0) < 0$ and recalling the continuity of $\psi(t)$, there exists a sufficiently small $T_0 > 0$ such that

$$\psi(t) \leq \frac{\psi(0)}{2} < 0, \quad 0 < t \leq T_0. \tag{6.3}$$

The estimates (6.2) and (6.3) indicate that $H'(t) \le \psi(0)/2$, which implies that

$$\begin{cases} H(t) \le H(0) + \psi(0)t/2, & 0 < t < T_1 \le T_0, \\ H(t) = 0, & t \ge T_1 := -2H(0)/\psi(0). \end{cases}$$

Obviously, by the definition of $H(t)$, we have (6.1). □

Lemma 6.2 *Suppose constants p, α, $\beta > 0$ and $h(t)$ be a nonnegative and absolutely continuous function satisfying $h'(t) + \alpha h^p(t) \ge \beta$, $t \in (0, +\infty)$. Then there exists an estimate $h(t) \ge \min\left\{h(0), (\beta/\alpha)^{\frac{1}{p}}\right\}$.*

In fact, the proof of this lemma can be similarly by ones for Lemma 3.2 in [17], where we use $h'(t) + \alpha h^p(t) \ge \beta$, $t \in (0, +\infty)$ instead of $h'(t) + \alpha \max\{h^p(t), h^q(t)\} \ge \beta$, $t \in (0, +\infty)$. □

Proof of Theorem 2.7. Multiplying the nonlinear equation in (1.1) by u and integrating it over $\Omega \times (t, t+h)$ with $h > 0$ and then dividing the result by h yields

$$\frac{1}{h}\int_t^{t+h}\int_\Omega u_\tau u \mathrm{d}x\mathrm{d}\tau + \frac{1}{h}\int_t^{t+h}\int_\Omega |\Delta u|^2 \mathrm{d}x\mathrm{d}\tau = \frac{1}{h}\int_t^{t+h}\int_\Omega u^{p(x)+1}\log u \mathrm{d}x\mathrm{d}\tau. \tag{6.4}$$

Let $h \to 0^+$ in (6.4) and use the Lebesgue differentiation theorem. We have

$$G'(t) + 2\|\Delta u\|_2^2 = 2\int_\Omega u^{p(x)+1}\log u \mathrm{d}x, \tag{6.5}$$

where $G(t) := \|u\|_2^2$. By $x^{-\mu}\log x \le (e\mu)^{-1}$ for $x \ge 1$, $\mu \ge 0$ and Hölder inequality, we obtain

$$\begin{aligned} \int_\Omega u^{p(x)+1}\log u \mathrm{d}x &= \int_{\Omega_1} u^{p(x)+1}\log u \mathrm{d}x + \int_{\Omega_2} u^{p(x)+1}\log u \mathrm{d}x \\ &\le \int_{\Omega_1} u^{p^++1}\log u \mathrm{d}x \le \frac{1}{e(p^* - p^+)}\int_\Omega u^{p^*+1}\mathrm{d}x \\ &\le \frac{|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^* - p^+)} G^{\frac{1+p^*}{2}}(t). \end{aligned} \tag{6.6}$$

By using $\|u\|_2^2 \le \lambda_1^{-1}\|\Delta u\|_2^2$ and (6.5, 6.6), we have $G'(t) + 2\lambda_1 G(t) \le \frac{|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)} G^{\frac{p^*+1}{2}}(t)$. By Lemma 6.1 and the definition of $G(t)$ above, we obtain (2.13). □

Proof of Theorem 2.8. Let $G(t) := \int_\Omega u^2 \mathrm{d}x$. According to the definition of $J(u)$ and applying the results in Lemma, we have

$$\begin{aligned} \frac{1}{2}G'(t) &= \int_\Omega u u_t \mathrm{d}x = -2\int_\Omega \frac{|\Delta u|^2}{2}\mathrm{d}x + \int_\Omega u^{p(x)+1}\log u \mathrm{d}x \\ &\ge -2J(u_0) + \int_\Omega \left(1 - \frac{2}{1+p(x)}\right) u^{p(x)+1}\log u \mathrm{d}x. \end{aligned} \tag{6.7}$$

Moreover, by $x^{-\mu}\log x \le (e\mu)^{-1}$ for $x \ge 1$, $\mu > 0$ and Hölder's inequality, we obtain that

$$\begin{aligned}
&\int_{\Omega}\left(1-\frac{2}{1+p(x)}\right)u^{p(x)+1}\log u\mathrm{d}x\\
&=\int_{\Omega_1}\left(1-\frac{2}{1+p(x)}\right)u^{p(x)+1}\log u\mathrm{d}x+\int_{\Omega_2}\left(1-\frac{2}{1+p(x)}\right)u^{p(x)+1}\log u\mathrm{d}x\\
&\ge\int_{\Omega_1}\left(1-\frac{2}{p^-+1}\right)u^{p^++1}\log u\mathrm{d}x\ge\left(1-\frac{2}{1+p^-}\right)\frac{1}{\mathrm{e}(p^*-p^+)}\int_{\Omega}u^{p^*+1}\mathrm{d}x\\
&\ge\left(1-\frac{2}{1+p^-}\right)\frac{|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)}G^{\frac{1+p^*}{2}}(t).
\end{aligned}\tag{6.8}$$

By (6.7, 6.8) and $p^- < 1$, we have

$$G'(t)+2\left(\frac{2}{1+p^-}-1\right)\frac{|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)}G^{\frac{p^*+1}{2}}(t)\ge -4J(u_0).\tag{6.9}$$

By (6.9) and Lemma 6.2, we show $G(t) \ge \min\left\{G(0),\ (\beta/\alpha)^{\frac{2}{p^*+1}}\right\}$ with

$$\alpha := 2\left(\frac{2}{1+p^-}-1\right)\frac{|\Omega|^{\frac{1-p^*}{2}}}{\mathrm{e}(p^*-p^+)},\quad \beta := -4J(u_0).$$

□

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Declarations

Conflict of interest The authors declare no competing interests.

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