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Shadowing Property of Hyperspace for Free Semigroup Actions

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Abstract

In this paper, we introduce a notion of shadowing property for a free semigroup action on a compact metric space, which is different of the notion of the shadowing property introduced by Bahabadi, called chain shadowing property. We study the relation between the shadowing property of a free semigroup action on a compact metric space X and the shadowing property of the induced free semigroup action on the hyperspace 2^X. Specially, we not only theoretically prove that $(F_m^+, \mathcal{F}) \curvearrowright X$ has the (chain) shadowing property if and only if $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the (chain) shadowing property, but also give examples to illustrate it. Finally, we compare the two notions of shadowing for free semigroup actions and obtain an interesting result that if $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property, then it has the chain shadowing property, but not vice versa.

Keywords Pseudo orbit · Shadowing property · Hyperspace · Free semigroup action

Mathematics Subject Classification (2010) 37A35 · 37B40

1 Introduction

Let (X, T, d) be a topological dynamical system, where X is a compact metric space with metric d and $T : X \to X$ is a continuous map. The map T induces in a natural way a

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map $2^T : 2^X \to 2^X$ defined by $2^T(A) = T(A)$ on the space 2^X, called hyperspace, of all nonempty closed subsets equipped with the Vietoris topology which coincides with the topology induced by the Hausdorff metric d_H. Then 2^T is also a continuous map and the hyperspace 2^X is also a compact metric space with the Hausdorff metric d_H that means $(2^X, 2^T, d_H)$ is also a topological dynamical system. The study of relationship about topological properties between T and 2^T was initiated by Bauer and Sigmund [2]. After that, several results dealing with this new relationship continue to appear. For instance, Banks in [4] showed that the transitivity of $(2^X, 2^T, d_H)$ is coincident to its weakly mixing property. In [8] authors showed that if T is a chain transitive map then 2^T is also chain transitive. And the relationship between the entropy of the map T and the entropy of the induced map 2^T are studied in [9]. Kwietniak and Oprocha in [15] studied the relationship of entropy between the hyperspace and the base space and showed that under some nonrecurrence assumption the induced map 2^T is always topologically chaotic. Li, Oprocha and Ye in [16] systematically studied the question when hyperspace is recurrent. Recently, Ji, Chen and Zhou in [14] introduced the entropy order concept for $(2^X, 2^T, d_H)$ and proved that it is coincides with the topological entropy of (X, T, d). Blank in [5] proposed the ε-trajectory and introduced the average shadowing property while studying chaotic dynamical systems. Motivated by the work of Bauer on hyperspace, Gómez Rueda, Illanes and Mendez in [10] study some dynamical properties (like periodicity, recoverability and shadowing property) in hyperspace. Specially, Femández and Good in [7] proposed that following theorem.

Theorem 1.1 [7, Theorem 3.8] *Let X be a compact metric space and let $T : X \to X$ be a continuous map, then 2^T has the shadowing property if and only if T has the shadowing property.*

Recently, many research works have been devoted to the applications of free semigroup actions. For instance, Carvalho, Rodrigues and Varandas in [6] studied finitely generated free semigroup actions on a compact metric space and obtained quantitative information on Poincare recurrence. Maria, Rodrigues and Paulo in [17] introduced a notion of measure theoretical entropy for free semigroup actions and established a variational principle. In [22], Zhu and Ma introduced the notion of topological r-entropy for free semigroup actions on a compact metric space and provided some properties of it. Therefore, we want to generalize the above theorem to free semigroup actions. In [3], Bahabadi and Zamani first introduced the notions of chain shadowing property and average shadowing property for free semigroup actions (IFSs). Osipov and Tikhomirov in [21] proposed the shadowing property of finite generated group actions on compact metric spaces and studied the related properties. In the present paper, we push this direction further, considering the relationship of shadowing property for a free semigroup action between base space dynamical system and hyperspace dynamical system.

The rest of the paper is organized as follows. In Section 2, some basic concepts and notes are introduced. In Section 3, we note the definition of shadowing property for a free semigroup action from Hui and Ma [11] as the chain shadowing property and obtained $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the chain shadowing property if and only if $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property. In Section 4, the definition of shadowing property for a free semigroup action is introduced. We also have $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the shadowing property if and only if $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property. Moreover, we give examples to illustrate that the two shadowing properties for free semigroup actions are inequivalent.

2 Preliminaries

First of all, we introduce some basic definitions of hyperspace.

2.1 Hyperspace

Let X be a compact metric space with metric d. Denote by 2^X the hyperspace on X,

$$2^X = \{A \subseteq X : A \neq \emptyset, \bar{A} = A\},$$

that is, the collection of all nonempty closed subsets of X. Then 2^X is also a compact metric space, equipped with the Hausdorff metric d_H, defined as

$$d_H(A, C) = \inf\{\varepsilon > 0 : A \subseteq B(C, \varepsilon) \quad \text{and} \quad C \subseteq B(A, \varepsilon)\}$$

where $B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ for a set A, that is, $B(A, \varepsilon)$ is an ε-neighborhood of A in X.

The following family

$$\{\langle U_1, \cdots, U_n\rangle : U_1, \cdots, U_n \text{ are nonempty open subsets of } X, n \in \mathbb{N}\},$$

forms a basis for the topology of 2^X called the *Vietoris topology*, where

$$\langle S_1, \cdots, S_n\rangle \doteq \left\{A \in 2^X : A \subset \bigcup_{i=1}^{n} S_i \text{ and } A \cap S_i \neq \emptyset \text{ for each } i = 1, \cdots, n\right\}$$

is defined for arbitrary nonempty subsets $S_1, \cdots, S_n \subseteq X$. It is not hard to see that the Hausdorff topology (the topology induced by the Hausdorff metric d_H) and the Vietoris topology for 2^X coincide [12 Theorem 3.1].

Let $F_n(X) = \{A \in 2^X : \sharp A \leq n\}$, where $\sharp A$ denotes the cardinality of the set A.

Denote by $F(X) = \bigcup\limits_{n=1}^{\infty} F_n(X)$ the collection of all finite subsets of X. Obviously, $F(X)$ is a dense subset of 2^X.

For more details on hyperspaces please see [19] and [20].

2.2 Free Semigroup Action

The free semigroup, written F_m^+, is the set of all finite words of symbols $0, 1, \cdots, m-1$. In F_m^+, we consider the semigroup operation of concatenation defined as usual: if $w = w_n w_{n-1} \cdots w_1 \in F_m^+$ and $w' = w'_k w'_{k-1} \cdots w'_1 \in F_m^+$, then $ww' = w_n w_{n-1} \cdots w_1 w'_k w'_{k-1} \cdots w'_1 \in F_m^+$. For any $w \in F_m^+$, $|w|$ denotes the length of w, that is, if $w = w_n w_{n-1} \cdots w_1 \in F_m^+$, then $|w| = n$ and the identity $e \in F_m^+$ which length is 0, called empty word. We write $w \leq w'$ if there exists a word w'' such that $w' = w''w$.

Remark 2.1 For any $w \in F_m^+$, $|w| \neq 0$, we have $e < w$.

Let X be a compact metric space and let $f_0, f_1, \ldots, f_{m-1}$ be continuous maps on X. We denote by $(F_m^+, \mathcal{F}) \curvearrowright X$ the free semigroup action generated by $\mathcal{F}$=$\{f_0, f_1, \ldots, f_{m-1}\}$.

One way to interpret this statement is to consider a map $\varphi : F_m^+ \times X \longrightarrow X$ such that, for any $w = w_n w_{n-1} \cdots w_1 \in F_m^+$,

$$\begin{aligned}\varphi : F_m^+ \times X &\longrightarrow X\\ (w, x) &\longmapsto f_w(x),\end{aligned}$$

where $f_w(x) = f_{w_n} \circ f_{w_{n-1}} \circ \cdots f_{w_1}(x)$. Specially, $(e, x) \mapsto f_e(x) \equiv x$. Then for any map $f_i \in \mathcal{F}$, the map 2^{f_i},

$$\begin{aligned}2^{f_i} : 2^X &\longrightarrow 2^X\\ A &\longmapsto f_i(A),\end{aligned}$$

is a continuous map on 2^X. Thus the map φ naturally induces a map 2^φ,

$$\begin{aligned}2^\varphi : F_m^+ \times 2^X &\longrightarrow 2^X\\ (w, A) &\longmapsto 2^{f_w}(A).\end{aligned}$$

Therefore, $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ is also a free semigroup action.

Next, we set up some notations about the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$.

For sets $A \subseteq X$ and $K \subseteq F_m^+$ and a word $w \in F_m^+$, we write

$$wA = \{f_w(x) : x \in A\}, \quad Kx = \{f_w(x) : w \in K\}, \quad KA = \{f_w(x) : w \in K \text{ and } x \in A\}.$$

The F_m^+-*orbit* of a point $x \in X$ is the set $F_m^+ x$. We say a set $A \subseteq X$ is F_m^+-*invariant* if $F_m^+ A \subseteq A$.

Throughout this paper, $(F_m^+, \mathcal{F}) \curvearrowright X$ is a free semigroup action on a compact metric space with the metric denoted by d.

3 The Chain Shadowing Property

In this section, we study the relation of the chain shadowing property between base space dynamical system and hyperspace dynamical system. The definition of chain shadowing property under the free semigroup action refers to [11]. In order to distinguish the definition of shadowing property in the next section, we call shadowing defined in [11] as chain shadowing.

Denote the set of all one-side infinite sequences of symbols $0, 1, \cdots, m-1$ by Σ_m^+, i.e.,

$$\Sigma_m^+ = \{\omega = w_1 w_2 \cdots | w_i \in \{0, 1, \cdots, m-1\}\}.$$

Let $\omega = w_1 w_2 \cdots \in \Sigma_m^+$ and $a \leq b \in \mathbb{Z}^+$. We write $\omega|_{[a,b]} = w_a w_{a+1} \cdots w_{b-1} w_b$ and $\overline{\omega|_{[a,b]}} = w_b \cdots w_a$.

For $\omega = w_1 w_2 \cdots \in \Sigma_m^+$, the ω-orbit of $x \in X$ under the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ is the sequence $\{f_\omega^n(x)\}_{n=0}^\infty$, where

$$f_\omega^n(x) = f_{w_n} \circ f_{w_{n-1}} \circ \cdots f_{w_1}(x) \quad \text{and} \quad f_\omega^0(x) = f_e(x) \equiv x.$$

It is obvious that

$$f_\omega^n(x) = f_{\overline{\omega|_{[1,n]}}}(x) = f_w(x),$$

where $w = w_n w_{n-1} \cdots w_1 = \overline{\omega|_{[1,n]}}$. Specially, we denote $e = \omega|_{[1,0]}$.

For $k \in \mathbb{Z}^+$, let

$$D_k = \{\omega_k = w_1 w_2 \cdots w_k : 0 \leq w_i \leq m-1, 1 \leq i \leq k\}.$$

For $\omega_k = w_1 w_2 \cdots w_k \in D_k$, the ω_k-orbit of $x \in X$ under the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ is the finite sequence $\{f_{\omega_k}^n(x)\}_{n=0}^k$.

Definition 3.1 For $\delta > 0$, a sequence $\tau = \{x_i\}_{i=0}^{\infty} \subseteq X$ is called a *δ-pseudo orbit* for the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ if there exists $\omega = w_1 w_2 \cdots \in \Sigma_m^+$, such that

$$d(f_{w_{i+1}}(x_i), x_{i+1}) < \delta \tag{3.1}$$

for all $i \in \mathbb{N}$, where ω is called a *compatible element* on τ.

Denote by $S(\tau) = \left\{\omega \in \Sigma_m^+ : \omega \text{ is a compatible element on } \tau\right\}$.
It is obvious that τ is called a δ-pseudo orbit, if $S(\tau) \neq \emptyset$.

Definition 3.2 A δ-pseudo orbit $\tau = \{x_i\}_{i=0}^{\infty}$ is *ε-chain shadowed* by some point of X, if for any $\omega \in S(\tau)$, there exists $z_\omega \in X$, such that

$$d\left(f_w^i(z_\omega), x_i\right) < \varepsilon \quad \text{for all} \quad i \in \mathbb{N}.$$

Definition 3.3 The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the *chain shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$, such that every δ-pseudo orbit can be ε-chain shadowed by some point of X.

Definition 3.4 For $\delta > 0$, a finite sequence $\tau_r = \{x_i\}_{i=0}^{r}$ is called a *finite δ-pseudo orbit* for the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ provided that there exists $\omega_r = w_1 w_2 \cdots w_r \in D_r$, such that

$$d(f_{w_{i+1}}(x_i), x_{i+1}) < \delta \tag{3.2}$$

for all $0 \leq i \leq r - 1$.

Similarly, denote by $S(\tau_r) = \{\omega_r \in D_r : \omega_r \text{ is a compatible element on } \tau_r\}$.

Definition 3.5 A finite δ-pseudo orbit $\tau_r = \{x_i\}_{i=0}^{r}$ is *ε-chain shadowed* by some point of X, if for any $\omega_r \in S(\tau_r)$ there exists $z_{\omega_r} \in X$, such that

$$d\left(f_{\omega_r}^i(z_{\omega_r}), x_i\right) < \varepsilon \quad \text{for all} \quad 0 \leq i \leq r.$$

Definition 3.6 The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the *finite chain shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $r \in \mathbb{N}$ the finite δ-pseudo orbit τ_r can be ε-chain shadowed by some point of X.

First we give a classic theorem about compact space for the proof of our first result.

Definition 3.7 A collection $\mathcal{C}$ of subsets of X is said to have the *finite intersection property* if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \ldots \cap C_n$ is nonempty.

Lemma 3.8 [18, Theorem 26.9] The topological space X is compact space if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, then the intersection $\bigcap\limits_{C \in \mathcal{C}} C$ is nonempty.

Next, we give some results about the chain shadowing property for a free semigroup action.

Theorem 3.9 *The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property if and only if it has the finite chain shadowing property.*

Proof Suppose the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property. Let $\varepsilon > 0$ and $\delta > 0$ given by the chain shadowing property for $(F_m^+, \mathcal{F}) \curvearrowright X$. For any $r \in \mathbb{N}$, let the finite sequence $\tau_r = \{x_i\}_{i=0}^r$ be a finite δ-pseudo orbit, that means $S(\tau_r) \neq \emptyset$. For any $\omega_r = w_1 w_2 \cdots w_r \in S(\tau_r)$, we can extend τ_r into an infinite sequence $\tau = \{x_i\}_{i=0}^\infty$ by the following way:

$$f_0(x_i) = x_{i+1} \quad \text{for all} \quad i \geq r.$$

Then $\tau = \{x_i\}_{i=0}^\infty$ is a δ-pseudo orbit. Since $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property, for the $\omega = w_1 w_2 \cdots w_r 00 \cdots \in S(\tau)$, there exists $z_\omega \in X$, such that

$$d\left(f_\omega^i(z_\omega), x_i\right) < \varepsilon \quad \text{for all} \quad i \in \mathbb{N}.$$

And $\omega|_{[1,r]} = \omega_r$, then

$$d\left(f_\omega^i(z_\omega)\right., x_i) < \varepsilon \quad \text{for all} \quad 0 \leq i \leq r.$$

Therefore, the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property.

In the converse direction, since $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property. For any $\varepsilon > 0$ there exists $\delta > 0$ such that every finite δ-pseudo orbit is a $\frac{\varepsilon}{2}$-chain shadowed by some point in X. Assume that an infinite sequence $\tau = \{x_i\}_{i=0}^\infty$ is δ-pseudo orbit. For any $\omega \in S(\tau)$ and any $k \in \mathbb{N}$, note $\omega_k = \omega|_{[1,k]}$. Then ω_k is a compatible element on τ_k, where $\tau_k = \{x_i\}_{i=0}^k$. By the finite chain shadowing property, we know there exists z_{ω_k}, such that

$$d\left(f_{\omega_k}^i(z_{\omega_k}), x_i\right) < \frac{\varepsilon}{2} \quad \text{for all} \quad 0 \leq i \leq k.$$

It is obvious that $f_{\omega_k}^i(z_{\omega_k}) \in B\left(x_i, \frac{\varepsilon}{2}\right)$, then

$$z_{\omega_k} \in \left(f_{\omega_k}^i\right)^{-1}\left(B\left(x_i, \frac{\varepsilon}{2}\right)\right),$$

that means

$$z_{\omega_k} \in \overline{\left(f_{\omega_k}^i\right)^{-1}\left(B\left(x_i, \frac{\varepsilon}{2}\right)\right)} \quad \text{for all} \quad 0 \leq i \leq k.$$

And $\omega_k = \omega|_{[1,k]}$, we have

$$\bigcap_{i=0}^{k} \overline{\left(f_\omega^i\right)^{-1}\left(B\left(x_i, \frac{\varepsilon}{2}\right)\right)} \neq \emptyset.$$

By Lemma 3.8 and the arbitrary of k, we have

$$\bigcap_{i \in \mathbb{N}} \overline{\left(f_\omega^i\right)^{-1}\left(B\left(x_i, \frac{\varepsilon}{2}\right)\right)} \neq \emptyset.$$

Hence, there exists $z \in X$,

$$z \in \bigcap_{i \in \mathbb{N}} \overline{\left(f_\omega^i\right)^{-1}\left(B\left(x_i, \frac{\varepsilon}{2}\right)\right)},$$

then

$$z \in \left(f_\omega^i\right)^{-1}\left(B\left(x_i, \frac{\varepsilon}{2}\right)\right), \quad \text{i.e.,} \quad f_\omega^i(z) \in B(x_i, \varepsilon) \quad \text{for all} \quad i \in \mathbb{N}.$$

Thus for any δ-pseudo orbit $\tau = \{x_i\}_{i=0}^\infty$ in X and any $\omega \in S(\tau)$. There exists $z \in X$, such that

$$d(f_\omega^i(z), x_i) < \varepsilon$$

for all $i \in \mathbb{N}$, which implies $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property. □

Theorem 3.10 *Let $(F_m^+, \mathcal{F}) \curvearrowright X$ be the free semigroup action and Y be a dense invariant subset of X, Then the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property if and only if the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the finite chain shadowing property.*

Proof Assume $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property. Let $\varepsilon > 0$ and choose $\delta > 0$ such that every δ-pseudo orbit in X is $\frac{\varepsilon}{2}$-chain shadowed. Let $\tau_r = \{y_i\}_{i=0}^r$ be a finite δ-pseudo orbit in Y. Then τ_r is a finite δ-pseudo orbit in X. Since $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property, then for any $\omega_r = w_1 w_2 \cdots w_r \in S(\tau_r)$ there exists $z_{\omega_r} \in X$, such that

$$d\left(f_{\omega_r}^i(z_{\omega_r}), y_i\right) < \frac{\varepsilon}{2}$$

for all $0 \leq i \leq r$, which implies

$$f_{\omega_r}^i(z_{\omega_r}) \in B\left(y_i, \frac{\varepsilon}{2}\right).$$

Then

$$z_{\omega_r} \in \left(f_{\omega_r}^i\right)^{-1}\left(B\left(y_i, \frac{\varepsilon}{2}\right)\right)$$

for all $0 \leq i \leq r$, that means,

$$\bigcap_{0 \leq i \leq r} \left(f_{\omega_r}^i\right)^{-1}\left(B\left(y_i, \frac{\varepsilon}{2}\right)\right) \neq \emptyset.$$

For $(F_m^+, \mathcal{F}) \curvearrowright X$ is continuous action, thus

$$\bigcap_{0 \leq i \leq r} \left(f_{\omega_r}^i\right)^{-1}\left(B\left(y_i, \frac{\varepsilon}{2}\right)\right) \cap Y \neq \emptyset.$$

That means there exists $z \in Y$ and

$$z \in \bigcap_{0 \leq i \leq r} \left(f_{\omega_r}^i\right)^{-1}\left(B\left(y_i, \frac{\varepsilon}{2}\right)\right) \quad \text{i.e.,} \quad f_{\omega_r}^i(z) \in B(y_i, \varepsilon).$$

Thus, for any finite δ-pseudo orbit τ_r in Y, and any $\omega_r \in S(\tau_r)$ there exists $z \in Y$, such that

$$d\left(f_{\omega_r}^i(z), y_i\right) < \varepsilon$$

for all $0 \leq i \leq r$, which means $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the finite chain shadowing property.

In the converse direction, we assume $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the finite chain shadowing property. Let $\varepsilon > 0$ and $\tau_r' = \{x_i\}_{i=0}^r$ be a finite $\frac{\delta}{3}$-pseudo orbit in X, where $\delta > 0$ is given by the finite chain shadowing property in $(F_m^+, \mathcal{F}) \curvearrowright Y$ for $\frac{\varepsilon}{2}$. Since $f_i \in \mathcal{F}$ is a continuous map and X is compact, then $f_i \in \mathcal{F}$ is a uniformly continuous map for any $i \in \{0, 1 \cdots m-1\}$. That means for $\frac{\delta}{3} > 0$ there exists $\eta > 0$ with $\eta < \frac{\delta}{3}$ and $\eta < \frac{\varepsilon}{2}$ for any $y \in B(x, \eta)$, we have

$$d(f_i(x), f_i(y)) < \frac{\delta}{3} \quad \text{for all} \quad 0 \leq i \leq m-1.$$

For any $\omega_r = w_1 w_2 \cdots w_r \in S(\tau_r')$, we can construct a finite sequence $\tau_r^* = \{y_i\}_{i=0}^r$ in Y such that ω_r is compatible on τ_r^*.

For any $x_i \in \tau_r'$, we take $y_i \in B(x_i, \eta) \cap Y$, then

$$d(f_{w_{i+1}}(x_i), f_{w_{i+1}}(y_i)) < \frac{\delta}{3}.$$

Therefore

$$d(f_{w_{i+1}}(y_i), y_{i+1}) < d(f_{w_{i+1}}(y_i), f_{w_{i+1}}(x_i)) + d(f_{w_{i+1}}(x_i), x_{i+1}) + d(x_{i+1}, y_{i+1})$$
$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

Hence τ_r^* is δ-pseudo orbit in Y and $\omega_r \in S(\tau_r^*)$. Since $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the finite chain shadowing property, there exists $y_{\omega_r} \in Y$, such that

$$d\left(f_{\omega_r}^i(y_{\omega_r}), y_i\right) < \frac{\varepsilon}{2} \quad \text{for all} \quad 0 \le i \le r.$$

Then

$$\left(f_{\omega_r}^i(y_{\omega_r}), x_i\right) < d\left(f_{\omega_r}^i(y_{\omega_r}), y_i\right) + d(y_i, x_i)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for any $0 \le i \le r$. Therefore, the $\frac{\delta}{3}$-pseudo orbit $\tau = \{x_i\}_{i=0}^r$ can be ε-chain shadowed by some point in X.

Hence $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property. □

Theorem 3.11 *If the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the chain shadowing property, then $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property.*

Proof Suppose $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the chain shadowing property. Let $\varepsilon > 0$ and choose $\delta > 0$ be given by the chain shadowing property for $(F_m^+, \mathcal{F}) \curvearrowright 2^X$. Let a sequence $\tau = \{x_i\}_{i=0}^\infty$ be a δ-pseudo orbit for $(F_m^+, \mathcal{F}) \curvearrowright X$, that means, $S(\tau) \neq \emptyset$. For any $\omega = w_1 w_2 \cdots \in S(\tau)$, we have

$$d(f_{w_{i+1}}(x_i), x_{i+1}) < \delta,$$

for all $i \in \mathbb{N}$, that means,

$$d_H(f_{w_{i+1}}(\{x_i\}), \{x_{i+1}\}) < \delta \quad \text{for all} \quad i \in \mathbb{N}. \tag{3.3}$$

Then $\tau^* = \{\{x_i\}\}_{i=0}^\infty$ is a δ-pseudo orbit for $(F_m^+, \mathcal{F}) \curvearrowright 2^X$. Since $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the chain shadowing property, and $\omega \in S(\tau^*)$ there exists $A_\omega \in 2^X$, we have

$$d_H\left(f_\omega^i(A_\omega), \{x_i\}\right) < \varepsilon \quad \text{for all} \quad i \in \mathbb{N}.$$

Then

$$f_\omega^i(A_\omega) \in B(\{x_i\}, \varepsilon) \quad \text{for all} \quad i \in \mathbb{N}.$$

That is,

$$d\left(f_\omega^i(z), x_i\right) < \varepsilon \quad \text{for all} \quad z \in A_\omega, i \in \mathbb{N}.$$

Thus $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property. □

Theorem 3.12 *If the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property, then $(F_m^+, \mathcal{F}) \curvearrowright F(X)$ has the finite chain shadowing property.*

Proof Let $\varepsilon > 0$ and choose $\delta > 0$ be given by the finite chain shadowing property for $(F_m^+, \mathcal{F}) \curvearrowright X$. Let $\tau = \{A_i\}_{i=0}^r$ be a finite δ-pseudo orbit in $F(X)$. Suppose $A_i = \left\{x_i^1, x_i^2, \cdots, x_i^{n_i}\right\}$ and $\sharp A_i = n_i$, for each $1 \le i \le r$.

Claim. There exist a family of δ-pseudo orbits in X, denoted by $\{\tau_j : j \leq n\}$, where $\tau_j = \left(x_0^j, x_1^j, \cdots, x_r^j\right)$, for some $n \in \mathbb{N}$, such that,

$$A_r = \left\{x_r^j : j \leq n\right\} \quad \text{for each} \quad 0 \leq i \leq r.$$

We first construct a δ-pseudo orbit in X. Since $\tau = \{A_i\}_{i=0}^r$ is a finite δ-pseudo orbit, for any $\omega_r = (w_1 w_2 \cdots w_r) \in D_r$, we have

$$d_H(f_{w_i}(A_{i-1}), A_i) < \delta \quad \text{for any} \quad 1 \leq i \leq r.$$

For any $x_r^j \in A_r$, we can chose $x_{r-1}^j \in A_{r-1}$, such that

$$d\left(f_{w_r}\left(x_{r-1}^j\right), x_r^j\right) < \delta.$$

Again, there exists $x_{r-2}^j \in A_{r-2}$, such that

$$d\left(f_{w_{r-1}}\left(x_{r-2}^j\right), x_{r-1}^j\right) < \delta.$$

By induction, we have a family of δ-pseudo orbits $\tau_j = \left(x_0^j, x_1^j, \cdots, x_r^j\right)$ for each $1 \leq j \leq n_r$ such that $A_r = \left\{x_r^j : j \leq n_r\right\}$ and $\left\{x_i^j : j \leq n_r\right\} \subseteq A_i$ for each $i \leq r-1$.

Let $k = max\left\{i < r : \left\{x_i^j : j \leq n_r\right\} \neq A_i\right\}$ (if no such k exists, then we are done), and write $A_k - \left\{x_k^j : j \leq n_r\right\} = \left\{x_k^j : n_r < j \leq n_k'\right\}$. Continuing in above way, there exists $x_{k-1}^j \in A_{k-1}$, such that

$$d\left(f_{w_k}\left(x_{k-1}^j\right), x_k^j\right) < \delta \quad \text{for each} \quad n_r < j \leq n_k'.$$

Thus, we can construct a δ-pseudo orbit $\tau_j' = \left(x_0^j, x_1^j, \cdots, x_k^j\right)$ where $x_i^j \in A_i$, $i \leq k$. Now, since $f_{w_{k+1}}\left(x_k^j\right) \in f_{w_{k+1}}(A_k)$ and $d_H(f_{w_{k+1}}(A_k), A_{k+1}) < \delta$, then there exists $x_{k+1}^j \in A_{k+1}$ such that

$$d\left(f_{w_{k+1}}\left(x_k^j\right), x_{k+1}^j\right) < \delta.$$

Similarly, for each $n_r < j \leq n_k'$, and $k \leq i < r$, there exists $x_{i+1}^j \in A_{i+1}$ such that

$$d\left(f_{w_{i+1}}\left(x_i^j\right), x_{i+1}^j\right) < \delta.$$

Thus, we can extend τ_j' to a δ-pseudo orbit $\tau_j = \left(x_0^j, x_1^j, \cdots, x_r^j\right)$ for each $n_r < j \leq n_k'$.

Repeating this process, we can construct the collection $\left\{\tau_j = \left\{x_i^j\right\}_{i=0}^r : j \leq n\right\}$ of δ-pseudo orbits in X, satisfying $\bigcup_j \left\{x_i^j\right\} = A_i$, for any $0 \leq i \leq r$. We complete the claim.

Since $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite chain shadowing property, for each $\tau_j = \left\{x_i^j\right\}_{i=0}^r$ and $\omega_r = (w_1 w_2 \cdots) \in S(\tau_j)$, there exists a point $z_j \in X$ such that

$$d\left(f_{\omega_r}^i(z_j), x_i^j\right) < \varepsilon \quad \text{for any} \quad 0 \leq i \leq r.$$

Let $B = \{z_1, z_2, \cdots, z_j : j \leq n\} \in F(X)$. By construction we have

$$d_H\left(f_{\omega_r}^i(B), A_i\right) < \varepsilon \quad \text{for any} \quad 0 \leq i \leq r.$$

Consequently, $(F_m^+, \mathcal{F}) \curvearrowright F(X)$ has the finite chain shadowing property. □

Theorem 3.13 *The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property if and only if $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the chain shadowing property.*

Proof By Theorem 3.11, we have if the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the chain shadowing property, then $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property. Conversely, if the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the chain shadowing property, then $(F_m^+, \mathcal{F}) \curvearrowright F(X)$ has the finite chain shadowing property by Theorem 3.12, and $F(x)$ is an invariant dense subset of 2^X, so by Theorem 3.10, $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the chain shadowing property. □

Next, we give an example of a free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ with the chain shadowing property.

Example 3.14 Define the two continuous maps f_0, f_1 on $\Sigma(2) = \{0, 1\}^{\mathbb{N}}$, as follows:

$$f_0(x) = (x_2 x_3 x_4 \cdots); \quad f_1(x) = (x_3 x_4 x_5 \cdots),$$

where $x = (x_0 x_1 x_2 \cdots) \in \Sigma(2)$, the metric

$$d(x, y) = 1/2^{\delta(x,y)} \quad \text{where} \quad \delta(x, y) = \min\{s : s \in \mathbb{N} \quad \text{and} \quad x_s \neq y_s\}.$$

The free semigroup action $(F_2^+, \mathcal{F}) \curvearrowright \Sigma(2)$ generated by $\mathcal{F} = \{f_0, f_1\}$.

Next we show $(F_2^+, \mathcal{F}) \curvearrowright \Sigma(2)$ has the chain shadowing property.

For any $\varepsilon > 0$, choose $\delta = \frac{1}{2^N} < \varepsilon$, where N is a positive integer. Let $\tau = \{x^i\}_{i=0}^{\infty}$ be a δ-pseudo orbit for $(F_2^+, \mathcal{F}) \curvearrowright \Sigma(2)$, and $\omega = w_1 w_2 \cdots \in S(\tau)$. We take $z_\omega = (x^0|_{[0,1+w_1]} x^1|_{[0,1+w_2]} x^2|_{[0,1+w_3]} \cdots)$. We show that z_ω satisfies

$$d\left(f_\omega^i(z_\omega), x^i\right) < \varepsilon \quad \text{for all} \quad i \in \mathbb{N}.$$

By the definition of compatible element ω on τ, we have

$$d(f_{w_{i+1}}(x_i), x_{i+1}) < \delta \quad i \in \mathbb{N},$$

that means

$$x^i|_{[2+w_{i+1}, N+2+w_{i+1}]} = x^{i+1}|_{[0,N]}. \tag{3.4}$$

For any $i \in \mathbb{N}$, we have

$$f_\omega^i(z_\omega) = \left(x^i_{[0,1+w_{i+1}]} x^{i+1}_{[0,1+w_{i+2}]} \cdots\right).$$

Without loss of generality, we suppose there exists $r_i \in \mathbb{N}$ such that

$$N = 2 + w_{i+1} + (1 + w_{i+2}) + \cdots + (1 + w_{i+r_i}). \tag{3.5}$$

Then

$$x^i|_{[0,N]} = x^i|_{[0,1+w_{i+1}]} x^i|_{[2+w_{i+1},N]}. \tag{3.6}$$

By (3.4) and (3.5), we have

$$x^i|_{[2+w_{i+1},N]} = x^{i+1}|_{[0,(1+w_{i+2})+\cdots+(1+w_{i+r_i})]}.$$

Again

$$x^{i+1}|_{[0,(1+w_{i+2})+\cdots+(1+w_{i+r_i})]} = x^{i+1}|_{[0,1+w_{i+2}]} x^{i+1}|_{[2+w_{i+2},(1+w_{i+2})+\cdots+(1+w_{i+r_i})]}.$$

By (3.4), we have

$$x^{i+1}|_{[2+w_{i+2},(1+w_{i+2})+\cdots+(1+w_{i+r_i})]} = x^{i+2}|_{[0,(1+w_{i+3})+\cdots+(1+w_{i+r_i})]}.$$

Again and again, until

$$x^{i+r_i-2}|_{\left[0,\left(1+w_{i+r_i-1}\right)+(1+w_{i+r_i})\right]} = x^{i+r_i-2}|_{\left[0,1+w_{i+r_i-1}\right]}x^{i+r_i-2}|_{\left[2+w_{i+r_i-1},(1+w_{i+r_i-1})+(1+w_{i+r_i})\right]}.$$

By (3.4), we have

$$x^{i+r_i-2}|_{\left[2+w_{i+r_i-1},(1+w_{i+r_i-1})+(1+w_{i+r_i})\right]} = x^{i+r_i-1}|_{[0,1+w_{i+r_i}]}.$$

Hence

$$x^i|_{[0,N]} = x^i|_{[0,1+w_{i+1}]}x^{i+1}|_{[0,1+w_{i+2}]}\cdots x^{i+r_i-1}|_{[0,1+w_{i+r_i}]}.$$

Therefore, for any $i \in \mathbb{N}$,

$$d\left(f_\omega^i(z_\omega), x^i\right) \le \delta < \varepsilon,$$

which implies $(F_2^+, \mathcal{F}) \curvearrowright \Sigma(2)$ has the chain shadowing property.

4 The Shadowing Property

In this section, we introduce a notation of shadowing property for a free semigroup action, which is consistent with the definition of shadowing property of finitely generated group action in [21].

Definition 4.1 For $\delta > 0$, a sequence $\{x_w\}_{w\in F_m^+} \subseteq X$ is called a *δ-pseudo orbit* for the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ provided that

$$d(f_j(x_w), x_{jw}) < \delta \quad \text{for any} \quad f_j \in \mathcal{F},\ w \in F_m^+.$$

Definition 4.2 For $\delta > 0$, a finite sequence $\{x_w\}_{|w|\le n} \subseteq X$ where $w \in F_m^+$ is called a *finite δ-pseudo orbit* for the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ provided that

$$d(f_j(x_w), x_{jw}) < \delta \quad \text{for any} \quad f_j \in \mathcal{F},\ w \in F_m^+ \quad \text{with} \quad |w| \le n-1.$$

Definition 4.3 The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the *shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any δ-pseudo orbit $\{x_w\}_{w\in F_m^+}$ there exists a point $z_e \in X$ such that

$$d(f_w(z_e), x_w) < \varepsilon \quad \text{for any} \quad w \in F_m^+.$$

Definition 4.4 The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the *finite shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any finite δ-pseudo orbit $\{x_w\}_{|w|\le n}$ there exists a point $z_e \in X$ such that

$$d(f_w(z_e), x_w) < \varepsilon \quad \text{for any} \quad |w| \le n.$$

Theorem 4.5 *The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property if and only if it has the finite shadowing property.*

Proof Suppose the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property. Let $\varepsilon > 0$ and $\delta > 0$ be given by the shadowing property for $(F_m^+, \mathcal{F}) \curvearrowright X$. For any $k \in \mathbb{N}$ let $\tau = \{x_w\}_{|w| \leq k}$ be a finite δ-pseudo orbit. We extend it into a sequence $\{x_w\}_{w \in F_m^+}$ by the following way

$$f_{w'}(x_w) = x_{w'w} \quad \text{for any} \quad |w| = k \quad \text{and} \quad w' \in F_m^+.$$

Then $\{x_w\}_{w \in F_m^+}$ is a δ-pseudo orbit for $(F_m^+, \mathcal{F}) \curvearrowright X$, and $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property, that means, there exists $z_e \in X$, satisfying

$$d(f_w(z_e), x_w) < \varepsilon \quad \text{for every} \quad w \in F_m^+.$$

Thus,

$$d(f_w(z_e), x_w) < \varepsilon \quad \text{for any} \quad |w| \leq k.$$

In the converse direction, suppose $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite shadowing. For any $\varepsilon > 0$, we take $\delta > 0$ depend on $\frac{\varepsilon}{2}$. For any δ-pseudo orbit $\{x_w\}_{w \in F_m^+}$ and any $n \in \mathbb{N}$, we take the subsequence $\{x_w\}_{|w| \leq n} \subseteq \{x_w\}_{w \in F_m^+}$. By the definition of finite shadowing property, we know there exists $z_e^n \in X$, such that

$$d\left(f_w\left(z_e^n\right), x_w\right) < \frac{\varepsilon}{2} \quad \text{for any} \quad |w| \leq n.$$

It is obvious that $f_w(z_e^n) \in B(x_w, \varepsilon)$, then

$$z_e^n \in f_w^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right) \quad \text{for any} \quad |w| \leq n.$$

That means

$$z_e^n \in \overline{f_w^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right)}.$$

Therefore, for any finite positive integer $n \in \mathbb{N}$

$$\bigcap_{j=0}^{n} \overline{f_w^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right)} \neq \emptyset, \quad |w| \leq n$$

By Lemma 3.8, we have

$$\bigcap_{w \in F_m^+} \overline{f_\omega^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right)} \neq \emptyset.$$

That is, there exists $z_e \in X$, such that,

$$z_e \in \bigcap_{w \in F_m^+} \overline{f_\omega^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right)} \quad \text{i.e.,} \quad f_\omega(z_e) \in B(x_w, \varepsilon).$$

Consequently, for any δ-pseudo orbit $\{x_w\}_{w \in F_m^+}$, we can find some z_e such that

$$d(f_w(z_e), x_w) < \varepsilon \quad \text{for any} \quad w \in F_m^+.$$

That means $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property. □

Next to prove a general result about shadowing.

Theorem 4.6 *Let $(F_m^+, \mathcal{F}) \curvearrowright X$ be the free semigroup action, and Y is a dense invariant subset of X. Then the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite shadowing property if and only if the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the finite shadowing property.*

Proof Assume $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property. Let $\varepsilon > 0$ and choose $\delta > 0$ such that for every finite δ-pseudo orbit in X is $\frac{\varepsilon}{2}$-shadowed. Let $\{x_w\}_{|w| \le n}$ be a finite δ-pseudo orbit in Y. Then $\{x_w\}_{|w| \le n}$ is also a finite δ-pseudo orbit in X. Since $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite shadowing property, there exists $z'_e \in X$ satisfying

$$d(f_w(z'_e), x_w) < \frac{\varepsilon}{2} \quad \text{for any} \quad |\, w \,| \le n.$$

It is obvious that $f_w(z'_e) \in B\left(x_w, \frac{\varepsilon}{2}\right)$, then

$$z'_e \in f_w^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right) \quad \text{for any} \quad |\, w \,| \le n.$$

That means

$$\bigcap_{|w| \le n} f_w^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right) \ne \emptyset.$$

For F_m^+ is finite generated free semigroup then the members with $|\, w \,| \le n$ is finite. Thus

$$\bigcap_{|w| \le n} f_w^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right)$$

is nonempty open subset of X. That means

$$\bigcap_{|w| \le n} f_w^{-1}\left(B\left(x_w, \frac{\varepsilon}{2}\right)\right) \bigcap Y \ne \emptyset.$$

Therefore, there exists $z_e \in Y$, such that,

$$f_w(z_e) \in B(x_w, \varepsilon) \quad \text{for any} \quad |\, w \,| \le n.$$

By combing the process above, we know for any δ-pseudo orbit $\{x_w\}_{|w| \le n}$ in Y, we can find some $z_e \in Y$ such that

$$d(f_w(z_e), x_w) < \varepsilon \quad \text{for any} \quad |\, w \,| \le n.$$

That means $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the finite shadowing property.

In the converse direction, suppose $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the finite shadowing. Let $\varepsilon > 0$ and $\delta > 0$ given by $\frac{\varepsilon}{2}$-shadowing property for $(F_m^+, \mathcal{F}) \curvearrowright Y$ and $\{x_w\}_{|w| \le n}$ is $\frac{\delta}{3}$-pseudo orbit in X. Since $(F_m^+, \mathcal{F}) \curvearrowright X$ is a continuous action and X is compact, then $(F_m^+, \mathcal{F}) \curvearrowright X$ is a uniformly continuous action that means for any $\frac{\delta}{3} > 0$ there exists a $\eta > 0$ with $\eta < \min\{\frac{\delta}{3}, \frac{\varepsilon}{2}\}$, for each $y_w \in B(x_w, \eta) \cap Y$ we have

$$d(f_j(y_w), f_j(x_w)) < \frac{\delta}{3} \quad \text{for any} \quad f_j \in \mathcal{F}.$$

Hence $\{y_w\}_{|w| \le n}$ is a δ-pseudo orbit in Y, for

$$\begin{aligned} d(f_j(y_w), y_{jw}) &< d(f_j(y_w), f_j(x_w)) + d(f_j(x_w), x_{jw}) + d(x_{jw}, y_{jw}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \end{aligned}$$

Since $(F_m^+, \mathcal{F}) \curvearrowright Y$ has the shadowing property, there is a point $z_e \in Y$ such that

$$d(f_w(z_e), y_w) < \frac{\varepsilon}{2} \quad \text{for any} \quad |\, w \,| \le n.$$

Then

$$d(f_w(z_e), x_w) < d(f_w(z_e), y_w) + d(y_w, x_w)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for any $\frac{\delta}{3}$-pseudo orbit $\{x_w\}_{|w|\leq n}$ in X, there is a point $z_e \in Y \subseteq X$ satisfying

$$d(f_w(z_e), x_w) < \varepsilon \quad \text{for any} \quad |w| \leq n.$$

Hence the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite shadowing property. □

Theorem 4.7 *If the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the finite shadowing property, then the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright F(X)$ has the finite shadowing property.*

Proof For any $\varepsilon > 0$ and choose $\delta > 0$ be given by the shadowing property for $(F_m^+, \mathcal{F}) \curvearrowright X$. For any finite δ-pseudo orbit $\{A_w\}_{|w|\leq n}$ in $F(X)$, we have

$$d_H(f_j(A_w), A_{jw}) < \delta \quad \text{for all} \quad f_j \in \mathcal{F},\ w \in F_m^+ \quad \text{with} \quad |w| \leq n-1.$$

That means, for any $x_w \in A_w$ there exists $x_{jw} \in A_{jw}$ such that,

$$d(f_j(x_w), x_{jw}) < \delta \quad \text{for all} \quad f_j \in \mathcal{F},\ w \in F_m^+ \quad \text{with} \quad |w| \leq n-1. \tag{4.1}$$

Similarly, for any $y_{jw} \in A_{jw}$ there exists $y_w \in A_w$ such that,

$$d(f_j(y_w), y_{jw}) < \delta \quad \text{for all} \quad f_j \in \mathcal{F},\ w \in F_m^+ \quad \text{with} \quad |w| \leq n-1. \tag{4.2}$$

We claim that for any given $|w'| \leq n$ and any $x' \in A_{w'}$, we can construct a finite δ-pseudo orbit $\{x_w\}_{|w|\leq n}$ in X with $x_w \in A_w$ and $x_{w'} = x'$, by the following way.

Case 1. $|w'| = 0$

That means $x' \in A_e$, by (4.1) we can find $x_i \in A_i$ such that

$$d(f_i(x'), x_i) < \delta \quad \text{for all} \quad f_i \in \mathcal{F}.$$

Again, by (4.1) we can find $x_{ji} \in A_{ji}$ such that

$$d(f_j(x_i), x_{ji}) < \delta \quad \text{for all} \quad f_j \in \mathcal{F}.$$

Hence by induction, for any $w \in F_m^+$ with $|w| \leq n-1$ where $x_w \in A_w$, we can find $x_{iw} \in A_{iw}$ such that

$$d(f_i(x_w), x_{iw}) < \delta \quad \text{for all} \quad f_i \in \mathcal{F}.$$

Case 2. $|w'| > 0$.

We can assume $w' = w'_r w'_{r-1} \cdots w'_1$ where $0 < r \leq n$. Then for any $x' \in A_{w'}$ by (4.2), we can find $x_{w^{r-1}} \in A_{w^{r-1}}$, where $w^{r-1} = w'_{r-1} w'_{r-2} \cdots w'_1$, satisfying

$$d(f_{w'_r}(x_{w^{r-1}}), x_{w'_r w^{r-1}}) = d(f_{w'_r}(x_{w^{r-1}}), x_{w'}) < \delta.$$

For $x_{w^{r-1}}$, by (4.2) again we can find $x_{w^{r-2}} \in A_{w^{r-2}}$, where $w^{r-2} = w'_{r-2} w'_{r-3} \cdots w'_1$, such that

$$d(f_{w'_{r-1}}(x_{w^{r-2}}), x_{w'_{r-1} w^{r-2}}) = d(f_{w'_{r-1}}(x_{w^{r-2}}), x_{w^{r-1}}) < \delta.$$

Now applying (4.2) again and again until there exists $x_e \in A_e$ such that

$$d(f_{w^1}(x_e), x_{w^1}) < \delta.$$

For x_e, we repeat the process of *Case 1*. In particular, for the process of taking x_{w^i}, we take the values obtained in the above process, for every $0 \leq i \leq r$. Thus the above method establish our claim.

Then, for each $x_w \in A_w$, we can find a finite δ-pseudo orbit and $A_w \in F(X)$, that means the number of all $x_w \in A_w$ with $\mid w \mid \leq n$ is finite. Without loss of generality, we suppose there exists $m \in \mathbb{N}$ such that each sequence $\left\{x_w^j\right\}_{\mid w \mid \leq n}$ is finite δ-pseudo orbit satisfying $A_w = \left\{x_w^j : j = 1, 2, \cdots, m\right\}$ for every $\mid w \mid \leq n$. For each finite δ-pseudo orbit $\left\{x_w^j\right\}_{\mid w \mid \leq n}$, by the shadowing property of the action $(F_m^+, \mathcal{F}) \curvearrowright X$, we can find $z_e^j \in X$, such that

$$d\left(f_w\left(z_e^j\right), x_w^j\right) < \varepsilon \quad \text{for any} \quad \mid w \mid \leq n.$$

Let $Z = \left\{z_e^j : j = 1, 2, \cdots, m\right\}$, we say Z is ε-shadowing $\{A_w\}_{\mid w \mid \leq n}$.

For any $x_w \in A_w$, there exists positive integer $1 \leq j \leq m$, such that $x_w = x_w^j$ and z_e^j satisfying

$$d\left(f_w\left(z_e^j\right), x_w^j\right) < \varepsilon.$$

That means

$$A_w \subseteq B(f_w(Z), \varepsilon).$$

Similarly,

$$f_w(Z) \subseteq B(A_w, \varepsilon).$$

Hence, we completed the whole proof. □

Theorem 4.8 *The free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property if and only if the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the shadowing property.*

Proof Suppose $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the shadowing property. Let $\varepsilon > 0$, and choose δ be given by shadowing for $(F_m^+, \mathcal{F}) \curvearrowright 2^X$. For any δ-pseudo orbit $\{x_w\}_{w \in F_m^+}$ in X, we can turn it into $\{\{x_w\}\}_{w \in F_m^+}$, which is a δ-pseudo orbit in 2^X. Since $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ has the shadowing property, then there exists a point $Z \in 2^X$, such that

$$d_H(f_w(Z), \{x_w\}) < \varepsilon \quad \text{for any} \quad w \in F_m^+.$$

Then

$$f_w(Z) \subseteq B(x_w, \varepsilon) \quad \text{for any} \quad w \in F_m^+.$$

That is, every point $z \in Z$ is ε-shadowing $\{x_w\}_{w \in F_m^+}$.

Conversely, if $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property, then $(F_m^+, \mathcal{F}) \curvearrowright F(X)$ has the finite shadowing by Theorem 4.7. And $F(X)$ is an invariant dense subset of 2^X, by Theorem 4.5 and Theorem 4.6, thus $(F_m^+, \mathcal{F}) \curvearrowright 2^X$ also has the shadowing property. □

Naturally, we want to know the direct relationship between these two shadowing property definitions.

Corollary 4.9 If the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property, then it has the chain shadowing property.

The following example shows that the converse to Corollary 4.9 is not true.

Example 4.10 By Example 3.14, we know the free semigroup action $(F_2^+, \mathcal{F}) \curvearrowright \Sigma(2)$ has the chain shadowing property. We take $\varepsilon_0 = \frac{1}{4}$ for any $\delta > 0$. Without loss of generality, we suppose $\delta = \frac{1}{2^n}$, where n is a positive integer and $n \geq 2$. Let $\tau' = \{x_i^1\}_{i=0}^{\infty}$ where $x_0^1 = (00 \cdots 00 \cdots) \in \Sigma(2)$.

$$x_i^1 = (\underbrace{00\cdots0}_{n+3-2i}\underbrace{1\cdots1}_{2i}00\cdots) \in \Sigma_(2), \quad 1 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor$$

and

$$x_i^1 = x_{i+1}^1 = (\underbrace{11\cdots1}_{n+3}00\cdots) \quad \text{for all} \quad i \geq \left\lceil \frac{n+3}{2} \right\rceil. \tag{4.3}$$

Therefore τ' is a δ-pseudo orbit for $\omega' = 00\cdots \in S(\tau')$.

Let $\tau'' = \left\{x_i^2\right\}_{i=0}^{\infty}$ where $x_0^2 = (00\cdots00\cdots) \in \Sigma(2)$

$$x_i^2 = (\underbrace{00\cdots0}_{n+4-3i}\underbrace{100100\cdots100}_{3i}00\cdots) \in \Sigma_2, \quad 1 \leq i \leq \left\lfloor \frac{n+4}{3} \right\rfloor$$

and

$$x_i^2 = x_{i+1}^2 = (\underbrace{s100\cdots100}_{n+4}00\cdots) \quad \text{for all} \quad i \geq \lfloor \frac{n+4}{3} \rfloor. \tag{4.4}$$

Where $s = 0$ when $n + 4 \equiv 1(mod\ 3)$, $s = 00$ when $n + 4 \equiv 2(mod\ 3)$, specially, if $n + 4 \equiv 0(mod\ 3)$ we note $s = 100$.

Therefore τ'' is a δ-pseudo orbit for $\omega'' = 11\cdots \in S(\tau'')$.

Next, we construct a δ-pseudo orbit $\tau = \{x_w\}_{w \in F_2^+}$ for F_2^+ by the following way.

If $w = 00\cdots00 \in F_2^+$, then we take $x_w = x_{|w|}^1$; if $w = 11\cdots11 \in F_2^+$ then we take $x_w = x_{|w|}^2$. For others $w \in F_2^+$ there exists a positive integer m such that $w = w_m w_{m-1}\cdots w_1$. We consider the smallest i such that $w_i \neq w_{i+1}$, then note $w' = w_i w_{i-1}\cdots w_1$.

If $w' = 00\cdots0$, then we take

$$x_w = f_{w_m} \circ f_{w_{m-1}} \circ \cdots \circ f_{w_{i+1}} \left(x_i^1\right).$$

If $w' = 11\cdots1$ then we take

$$x_w = f_{w_m} \circ f_{w_{m-1}} \circ \cdots \circ f_{w_{i+1}} \left(x_i^2\right).$$

Suppose there exists $z \in \Sigma(2)$ satisfying z ε-shadows τ, that means

$$d(f_w(z), x_w) < \varepsilon \quad \text{for any} \quad w \in F_2^+.$$

Therefore, for any $\omega \in \Sigma_m^+$, if there exists a sequence $\overline{\tau} = \{y_{w^i}\}_{i=0}^{\infty} \subseteq \tau$ where $w^i = \overline{\omega|_{[1,i]}}$ such that $\omega \in S(\overline{\tau})$, we have

$$d(f_\omega^i(z), y_{w^i}) < \varepsilon \quad \text{for any} \quad i \in \mathbb{N}.$$

For ω' and ω'', we have

$$d\left(f^n_{\omega'}(z), x^1_n\right) < \varepsilon_0 \quad \text{and} \quad d\left(f^m_{\omega''}(z), x^2_m\right) < \varepsilon_0 \quad \text{for any} \quad n, m \in \mathbb{N}. \tag{4.5}$$

By the construction of x^1_n, x^2_m and the values of $\varepsilon_0 = \frac{1}{4}$, we can express (4.5) into

$$\left(f^n_{\omega'}(z)\right)|_{[0,2]} = \left(x^1_n\right)|_{[0,2]} \quad \text{and} \quad \left(f^m_{\omega''}(z)\right)|_{[0,2]} = \left(x^2_m\right)|_{[0,2]} \quad \text{for any} \quad n, m \in \mathbb{N}. \tag{4.6}$$

Let $N = \max\left\{\left\lceil \frac{n+3}{2} \right\rceil, \left\lfloor \frac{n+4}{3} \right\rfloor\right\}$. By (4.3) and (4.6), we have

$$\left(f^{3N}_{\omega'}(z)\right)|_{[0,2]} = \left(x^1_{3N}\right)|_{[0,2]} = 111 \quad \text{and} \quad \left(f^{2N}_{\omega''}(z)\right)|_{[0,2]} = \left(x^2_{2N}\right)|_{[0,2]}.$$

By the definitions of f_0 and f_1, we know $f^{3N}_{\omega'}(z) = f^{3N}_0(z) = f^{2N}_1(z) = f^{2N}_{\omega''}(z)$. But, by (4.3) and (4.4) we realize that $\left(x^1_{3N}\right)|_{[0,2]} \neq \left(x^2_{2N}\right)|_{[0,2]}$. That is a contradiction.

Take a look at the process. There exists $\varepsilon_0 = \frac{1}{4}$. For any $\delta > 0$, we can construct a δ-pseudo orbit $\tau = \{x_w\}_{w \in F^+_2}$. For any $z \in \Sigma(2)$, z can't ε-shadows τ, that means $(F^+_2, \mathcal{F}) \curvearrowright \Sigma(2)$ doesn't have the shadowing property.

Next, we give examples with both the chain shadowing property and the shadowing property.

Example 4.11 Define two continuous maps g_0, g_1 on $\Sigma(2)$ as follows:

$$g_0(x) = (0x_0x_1x_2\cdots); \quad g_1(x) = (1x_0x_1x_2\cdots),$$

where $x = (x_0x_1x_2\cdots) \in \Sigma(2)$.

The free semigroup action $(G^+_2, \mathcal{G}) \curvearrowright \Sigma(2)$ generated by $\mathcal{G} = \{g_0, g_1\}$. Then the free semigroup action $(G^+_2, \mathcal{G}) \curvearrowright \Sigma(2)$ has the chain shadowing property [3, Example 1.2].

Next we show $(G^+_2, \mathcal{G}) \curvearrowright \Sigma(2)$ has the shadowing property. For any $\varepsilon = \frac{1}{2^n} > 0$, choose $\delta = \frac{1}{2^{n+1}} < \varepsilon$. For any δ-pseudo orbit $\{x_w\}_{w \in G^+_2}$ for $(G^+_2, \mathcal{G}) \curvearrowright \Sigma(2)$, i.e.,

$$d(g_{w_i}x_w, x_{w_iw}) < \delta \quad \text{for any} \quad g_{w_i} \in \mathcal{G},\ w \in G^+_2. \tag{4.7}$$

By the definitions of g_0, g_1, for any $x_1, x_2 \in \Sigma(2)$, if $d(x_1, x_2) < \delta$ then

$$d(g_{w_i}(x_1), g_{w_i}(x_2)) < \frac{\delta}{2} \quad \text{for all} \quad g_{w_i} \in \mathcal{G}_2. \tag{4.8}$$

Consider $z = x_e$. For any $w \in G_2^+$, write $w = w_m w_{m-1} \cdots w_1$, by (4.7) and (4.8) we have

$$\begin{aligned}
d(g_w(z), x_w) &= d\left(g_{w_m}\left(g_{w'_{m-1}}(z)\right), x_w\right) \\
&< d\left(g_{w_m}\left(g_{w'_{m-1}}(z)\right), g_{w_m}\left(x_{w'_{m-1}}\right)\right) + d\left(g_{w_m}\left(x_{w'_{m-1}}\right), x_w\right) \\
&< \frac{d\left(g_{w'_{m-1}}(z), x_{w'_{m-1}}\right)}{2} + \delta \\
&< \frac{1}{2}\left(d\left(g_{w_{m-1}}\left(g_{w'_{m-2}}(z)\right), g_{w_{m-1}}\left(x_{w'_{m-2}}\right)\right) + d\left(g_{w_{m-1}}\left(x_{w'_{m-2}}\right), x_{w'_{m-1}}\right)\right) + \delta \\
&< \frac{1}{2}\left(\frac{d\left(g_{w'_{m-2}}(z), x_{w'_{m-2}}\right)}{2} + \delta\right) + \delta \\
&= \frac{d\left(g_{w'_{m-2}}(z), x_{w'_{m-2}}\right)}{4} + \frac{\delta}{2} + \delta \\
&\quad \vdots \\
&< \frac{d\left(g_{w'_1}(z), x_{w'_1}\right)}{2^{m-1}} + \frac{\delta}{2^{m-2}} + \cdots + \frac{\delta}{2} + \delta \\
&< \frac{\delta}{2^{m-1}} + \frac{\delta}{2^{m-2}} + \cdots + \frac{\delta}{2} + \delta \\
&< 2\delta < \varepsilon,
\end{aligned}$$

where $w'_i < w$ and $| w'_i |= i$. Thus $\{x_w\}_{w \in G_2^+}$ is ε-shadowed by z.

Consequently, we shown $(G_2^+, \mathcal{G}) \curvearrowright \Sigma(2)$ has the shadowing property.

In [13], authors constructed an action in S^1 whose minimal set K is a Cantor set and has the shadowing property. And Barzanouni in [1] presented an example to illustrate the shadowing property depends on the metric on non-compact spaces.

Remark 4.12 For any free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ with the discrete metric, we have that the free semigroup action $(F_m^+, \mathcal{F}) \curvearrowright X$ has the shadowing property.

Example 4.13 The free semigroup action $(G_2^+, \mathcal{G}) \curvearrowright 2^{\Sigma(2)}$, which inducted by Example 4.11, has the chain shadowing property and the shadowing property.

The proof process is similar to the above example, so we will only briefly describe it. For any $A_1, A_2 \in 2^{\Sigma(2)}$, $\delta > 0$ if $d_H(A_1, A_2) < \delta$, then

$$d_H(g_{w_i}(A_1), g_{w_i}(A_2)) < \frac{\delta}{2} \quad \text{for all} \quad g_{w_i} \in \mathcal{G}.$$

Similarly, let $Z = A_e$, that is what we want.

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