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Four-dimensional Zero-Hopf Bifurcation of Quadratic Polynomial Differential System, via Averaging Theory of Third Order

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Abstract

This article concerns the zero-Hopf bifurcation of a quadratic polynomial differential system in $\mathbb{R}^4$. By using the averaging theory of third order, we provide that at most 25 limit cycles can bifurcate from one singularity with eigenvalues of the form $\pm bi, \quad 0$ and 0.

Keywords Zero-Hopf bifurcation · Averaging theory · Quadratic polynomial differential system

Mathematics Subject Classification (2010) Primary: 34C07 · 34C05 · 34C40

1 Introduction

A Hopf bifurcation takes place at a singular point of a differential system when this changes its stability. More precisely, it is a local bifurcation which can appears when a singular point of a differential system having a pair of complex conjugate eigenvalues crosses the imaginary axis of the complex plane when we move the parameters of the differential system. At this crossing under convenient assumptions on the differential system, one or several small-amplitude limit cycles bifurcate from the singular point.

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When the pair of complex eigenvalues are on the imaginary axis, i.e., they are of the form $\pm bi$, if the other eigenvalues are non-zero, we talk about a Hopf bifurcation, but if some of the other eigenvalues are zero, we say that we have a zero-Hopf bifurcation. Here we are interested in the study of the zero-Hopf bifurcations when all the eigenvalues different from the $\pm bi$ are zero, we denote such kind of zero-Hopf bifurcation a complete zero-Hopf bifurcation. While there is a well developed theory for studying the Hopf bifurcations (see for instance [6, 9]), such theory does not exist for the zero-Hopf bifurcations. For the zero-Hopf bifurcations there are only partial results.

The goal of this paper is to study how many small-amplitude limit cycles can bifurcate in a complete zero-Hopf bifurcation at a singular point of a quadratic polynomial differential system in function of the dimension of the system.

Bautin [1] in 1954 proved that at most 3 small-amplitude limit cycles can bifurcate in a Hopf bifurcation at a singular point of a quadratic polynomial differential system in $\mathbb{R}^2$. Note that in $\mathbb{R}^2$ the notions of Hopf bifurcation, zero-Hopf bifurcation and complete zero-Hopf bifurcation coincide.

Also using Bautin's result it is easy to show that at least 3 small-amplitude limit cycles can bifurcate in a zero-Hopf bifurcation at a singular point of a quadratic polynomial differential system in $\mathbb{R}^3$, for a proof of this last result using averaging theory see the paper [5]. Some other results related with the zero-Hopf bifurcation of quadratic polynomial differential system in $\mathbb{R}^3$ can be found for instance in [8, 12]. Note that in $\mathbb{R}^3$ the notions of zero-Hopf bifurcation and complete zero-Hopf bifurcation coincide.

In 2017 Bendib et al. [2] studied the Hopf bifurcation occurring in vector fields in $\mathbb{R}^3$ via the averaging theory of third order. They obtained at most 10 limit cycles and they provided an example for which exactly 10 limit cycles bifurcate from the origin.

In [4], the authors studied the zero-Hopf bifurcation of a polynomial differential system in $\mathbb{R}^4$ with cubic homogeneous nonlinearities. They provided that for a sufficient condition the system can exhibit at least nine periodic solutions bifurcating from the origin when $\varepsilon = 0$, using the averaging theory of second order.

The aim of this paper is to prove that at least 9 and 25 limit cycles can be bifurcate in a complete zero-Hopf bifurcation of a quadratic polynomial differential system in $\mathbb{R}^4$, by using respectively the averaging theory of second and third order.

Here we are interested in studying the zero-Hopf bifurcation of a quadratic polynomial differential system in $\mathbb{R}^4$ with a singular point at the origin $(0, 0, 0, 0)$ whose linear part has eigenvalues $(a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3) \pm i(b + b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3)$, $c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3$ and $d_1\varepsilon + d_2\varepsilon^2 + d_3\varepsilon^3$, where ε is a small parameter. Such system can be described by the following equations

$$\begin{cases} \dot{x} &= (a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3)x - (b + b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3)y + \displaystyle\sum_{j=0}^{2} \varepsilon^j X_j(x, y, z, w), \\ \dot{y} &= (b + b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3)x + (a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3)y + \displaystyle\sum_{j=0}^{2} \varepsilon^j Y_j(x, y, z, w), \\ \dot{z} &= (c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3)z + \displaystyle\sum_{j=0}^{2} \varepsilon^j Z_j(x, y, z, w), \\ \dot{w} &= (d_1\varepsilon + d_2\varepsilon^2 + d_3\varepsilon^3)w + \displaystyle\sum_{j=0}^{2} \varepsilon^j W_j(x, y, z, w), \end{cases} \tag{1}$$

where

$$\begin{aligned} X_j(x,y,z,w) = {} & a_{j0}x^2 + a_{j1}xy + a_{j2}xz + a_{j3}xw + a_{j4}y^2 \\ & + a_{j5}yz + a_{j6}yw + a_{j7}z^2 + a_{j8}zw + a_{j9}w^2, \end{aligned}$$

$Y_j(x,y,z,w)$, $Z_j(x,y,z,w)$ and $W_j(x,y,z,w)$ have the same expression as $X_j(x,y,z,w)$ by replacing a_{ji} respectively by b_{ji}, c_{ji} and d_{ji} for $j = 0,1,2$ and $i = 0,1,\ldots,9$. The coefficients $a_{ij}, b_{ij}, c_{ij}, d_{ij}, a_1, a_2, a_3, b, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$ are real parameters with $b \neq 0$.

The following Theorem shows our main result on the zero-Hopf bifurcation of the system (1).

Theorem 1 *The following statements hold.*

(a) *At most 2 limit cycles bifurcate from the origin of system* (1) *when* $\varepsilon = 0$ *by applying the averaging theory of first order, and this upper bound is reached.*
(b) *At most 9 limit cycles bifurcate from the origin of system* (1) *when* $\varepsilon = 0$ *by applying the averaging theory of second order, and this upper bound is reached.*
(c) *At most 25 limit cycles bifurcate from the origin of system* (1) *when* $\varepsilon = 0$ *by applying the averaging theory of third order.*

Theorem 1 will be proved using the averaging theory for computing limit cycles. Then, statement (a) of Theorem 1 is proved in Section 3, statement (b) is proved in Section 4 and statement (c) is proved in Section 5. In Sections 3, 4 and 5, we will use Bezout's theorem. This theorem gives the maximum number of zeros of a system of polynomial functions.

Theorem 2 *(Bezout's theorem). Let* P_i *be polynomials in the variables* $(x_1, \cdots, x_n) \in \mathbb{R}^n$ *of degree* d_i *for* $i = 1, \cdots, n$. *Consider the following polynomial system*

$$P_i(x_1, \cdots, x_n) = 0, \quad i = 1, \cdots, n,$$

If the number of solutions of this system is finite, then it is bounded by $d_1 \cdots d_n$.

See [11] for more details on Bezout's theorem.

2 The Averaging Theory of First, Second and Third Order

In this section we recall the averaging theory of first, second, and third order as it was developed in [3] and [7]. This will be the main tool for proving Theorem 1.

Theorem 3 *Consider the differential system*

$$x'(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \tag{2}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* T*-periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$. *Assume that the following hypotheses* (i) *and* (ii) *hold.*

(i) $F_1(t,.) \in C^2(D)$, $F_2(t,.) \in C^1(D)$ *for all* $t \in \mathbb{R}$, $F_1, F_2, F_3, R, D_x^2F_1, D_xF_2$ *are locally lipschitz with respect to x, and R is twice differentiable with respect to* ε. *We define* $F_{k0} : D \longrightarrow \mathbb{R}^n$ *for* $k = 1, 2, 3$ *as*

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_zF_1(s,z).y_1(s,z) + F_2(s,z)\right]ds,$$

$$F_{30}(z) = \frac{1}{T}\int_0^T [\frac{1}{2}y_1(s,z)^T\frac{\partial^2 F_1}{\partial z^2}(s,z)y_1(s,z) + \frac{1}{2}\frac{\partial F_1}{\partial z}(s,z)y_2(s,z)+$$

$$\frac{\partial F_2}{\partial z}(s,z)(y_1(s,z)) + F_3(s,z)]ds,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)dt,$$

$$y_2(s,z) = \int_0^s \left[\frac{\partial F_1}{\partial z}(t,z)\int_0^t F_1(r,z)dr + F_2(t,z)\right]dt.$$

(ii) *For* $V \subset D$ *an open and bounded set and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\setminus\{0\}$, *there exists* $a_\epsilon \in V$ *such that* $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ *and* $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$.

Then, for $|\varepsilon| > 0$ *sufficiently small there exists a T-periodic solution* $\varphi(\cdot,\varepsilon)$ *of the system* (2) *such that* $\varphi(0,\varepsilon) = a_\varepsilon$.

The expression $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order*.

If F_{10} and F_{20} is identically zero and F_{30} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{30} for ε sufficiently small. In this case the previous result provides the *averaging theory of third order*.

For more information about the averaging theory see [10] and [13].

3 Proof of Statement (*a*) of Theorem 1

For proving statement (a) of Theorem 1, we should write system (1) into the normal form for applying the averaging theory of Section 2. First, we rescale the variables, setting $(x, y,$

$z, w) = (\varepsilon X, \varepsilon Y, \varepsilon Z, \varepsilon W)$. Second, changing to cylindrical coordinates $(X, Y, Z, W) = (\rho\cos\theta, \rho\sin\theta, \eta, \xi)$. Finally, we take the angle θ as the new independent variable. Thus in the variables (ρ, η, ξ) system (1) writes

$$\begin{cases} \dfrac{d\rho}{d\theta} = \varepsilon F_{11}(\theta, \rho, \eta, \xi) + \varepsilon^2 F_{21}(\theta, \rho, \eta, \xi) + \varepsilon^3 F_{31}(\theta, \rho, \eta, \xi) + O(\varepsilon^4), \\ \dfrac{d\eta}{d\theta} = \varepsilon F_{12}(\theta, \rho, \eta, \xi) + \varepsilon^2 F_{22}(\theta, \rho, \eta, \xi) + \varepsilon^3 F_{32}(\theta, \rho, \eta, \xi) + O(\varepsilon^4), \\ \dfrac{d\xi}{d\theta} = \varepsilon F_{13}(\theta, \rho, \eta, \xi) + \varepsilon^2 F_{23}(\theta, \rho, \eta, \xi) + \varepsilon^3 F_{33}(\theta, \rho, \eta, \xi) + O(\varepsilon^4). \end{cases} \tag{3}$$

Taking

$$\begin{aligned} x &= (\rho, \eta, \xi), \\ t &= \theta, \\ F_1(t, x) &= (F_{11}(\theta, \rho, \eta, \xi), F_{12}(\theta, \rho, \eta, \xi), F_{13}(\theta, \rho, \eta, \xi)), \\ F_2(t, x) &= (F_{21}(\theta, \rho, \eta, \xi), F_{22}(\theta, \rho, \eta, \xi), F_{23}(\theta, \rho, \eta, \xi)), \\ F_3(t, x) &= (F_{31}(\theta, \rho, \eta, \xi), F_{32}(\theta, \rho, \eta, \xi), F_{33}(\theta, \rho, \eta, \xi)), \end{aligned}$$

and $T = 2\pi$, system (3) is equivalent to system (2). Note that we do not provide the functions F_1, F_2 and F_3 because some of them are huge and they need several pages for writing one of such huge functions. Applying the averaging theory of first order to the system (3). We have that $f_1 = (f_{11}, f_{12}, f_{13})$, where for $i = 1, 2, 3$

$$f_{1i}(\rho, \eta, \xi) = \frac{1}{2\pi}\int_0^{2\pi} F_{1i}(\theta, \rho, \eta, \xi)d\theta.$$

Doing these computations we get that

$$\begin{cases} f_{11}(\rho, \eta, \xi) = \dfrac{1}{b}(\rho(2a_1 + (a_{03} + b_{06})\xi + (a_{02} + b_{05})\eta)) = 0, \\ f_{12}(\rho, \eta, \xi) = \dfrac{1}{b}((c_{00} + c_{04})\rho^2 + 2(c_{09}\xi^2 + \eta(c_1 + c_{08}\xi + c_{07}\eta))) = 0, \\ f_{13}(\rho, \eta, \xi) = \dfrac{1}{b}((d_{00} + d_{04})\rho^2 + 2(\xi(d_1 + d_{09}\xi) + d_{08}\xi\eta + d_{07}\eta^2)) = 0. \end{cases} \tag{4}$$

In order for looking for the limit cycles of system (1) by the averaging theory we need to compute the isolated real roots of the averaged system (4) with $\rho > 0$.

We solve the first equation f_{11} of (4), we obtain the following unique solution

$$\xi = -\frac{2a_1 + (a_{02} + b_{05})\eta}{a_{03} + b_{06}}.$$

Then the second and the third equations become

$$\begin{cases} g_{11} = \dfrac{8a_1^2 c_{09}}{(a_{03}+b_{06})^2} + (c_{00}+c_{04})\rho^2 - \dfrac{2}{(a_{03}+b_{06})^2}(2a_{03}a_1c_{08}+ \\ \quad 2a_1b_{06}c_{08} - 4a_{02}a_1c_{09} - 4a_1b_{05}c_{09} - a_{03}^2c_1 - 2a_{03}b_{06}c_1 - b_{06}^2c_1)\eta + \\ \quad \dfrac{2}{(a_{03}+b_{06})^2}(a_{03}^2c_{07} + 2a_{03}b_{06}c_{07} + b_{06}^2c_{07} - a_{02}a_{03}c_{08} - a_{03}b_{05}c_{08} \\ \quad -a_{02}b_{06}c_{08} - b_{05}b_{06}c_{08} + a_{02}^2c_{09} + 2a_{02}b_{05}c_{09} + b_{05}^2c_{09})\eta^2 = 0, \\ g_{12} = \dfrac{4a_1(2a_1d_{09} - a_{03}d_1 - b_{06}d_1)}{(a_{03}+b_{06})^2} + (d_{00}+d_{04})\rho^2 - \dfrac{2}{(a_{03}+b_{06})^2} \\ \quad (2a_{03}a_1d_{08} + 2a_1b_{06}d_{08} - 4a_{02}a_1d_{09} - 4a_1b_{05}d_{09} + a_{02}a_{03}d_1 + \\ \quad a_{03}b_{05}d_1 + a_{02}b_{06}d_1 + b_{05}b_{06}d_1)\eta + \dfrac{2}{(a_{03}+b_{06})^2}(a_{03}^2d_{07}+ \\ \quad 2a_{03}b_{06}d_{07} + b_{06}^2d_{07} - a_{02}a_{03}d_{08} - a_{03}b_{05}d_{08} - a_{02}b_{06}d_{08} \\ \quad -b_{05}b_{06}d_{08} + a_{02}^2d_{09} + 2a_{02}b_{05}d_{09} + b_{05}^2d_{09})\eta^2 = 0. \end{cases}$$

This system has four real zeros. We eliminate ρ^2 between the two equations $g_{11} = 0$ and $g_{12} = 0$; we obtain a quadratic equation in η which has at most two real zeros. Now, we substitute one of these two zeros in one of the two equations, since there appears only ρ^2, we get two possible real zeros one of them is negative. Since ρ must be positive, system (4) has at most two real zeros with $\rho > 0$.

Let $\left(\bar\rho, \bar\eta, \bar\xi\right)$ be a solution of system (4). In order to have a limit cycle according to the averaging theory in Section 2, we must have

$$D(\bar\rho, \bar\eta, \bar\xi) = det \begin{pmatrix} \dfrac{\partial f_{11}}{\partial \rho} & \dfrac{\partial f_{11}}{\partial \eta} & \dfrac{\partial f_{11}}{\partial \xi} \\ \dfrac{\partial f_{12}}{\partial \rho} & \dfrac{\partial f_{12}}{\partial \eta} & \dfrac{\partial f_{12}}{\partial \xi} \\ \dfrac{\partial f_{13}}{\partial \rho} & \dfrac{\partial f_{13}}{\partial \eta} & \dfrac{\partial f_{13}}{\partial \xi} \end{pmatrix} \Bigg|_{(\rho,\eta,\xi)=(\bar\rho,\bar\eta,\bar\xi)} \neq 0.$$

Therefore by applying the averaging theory of first order, we deduce that system (1) has at most two limit cycles bifurcating from the origin. This case has been studied in [8].

Giving an example shows that system (1) has exactly 2 limit cycles bifurcating from a zero-Hopf bifurcation.

Example 4 We consider the following system

$$\begin{cases} \dfrac{dx}{dt} = -y + 2xw + (x^2 + x)\varepsilon - xy\varepsilon^2, \\ \dfrac{dy}{dt} = x + 2yw + (x - xy)\varepsilon - z^2\varepsilon^2, \\ \dfrac{dz}{dt} = x^2 - 2w^2 + (-x^2 + 3z^2)\varepsilon^2, \\ \dfrac{dw}{dt} = x^2 + \dfrac{2}{3}z^2 - 6w^2 + 3yz\varepsilon. \end{cases} \tag{5}$$

The eigenvalues of the singular point (0, 0, 0, 0) of system (5) are $\dfrac{\varepsilon}{2} \pm \dfrac{1}{2}\sqrt{\varepsilon^2 - 4\varepsilon - 4}$ and 0 of multiplicity 2.

For finding the limit cycles we must solve the averaged system

$$\begin{cases} f_{11}(\rho,\eta,\xi) &= \dfrac{1}{2}\rho(1+4\xi)=0,\\ f_{12}(\rho,\eta,\xi) &= -2\xi^2+\dfrac{1}{2}\rho^2=0,\\ f_{13}(\rho,\eta,\xi) &= \dfrac{2}{3}\eta^2+\dfrac{1}{2}\rho^2-6\xi^2=0. \end{cases} \tag{6}$$

System (6) has the following two roots $(\bar{\rho},\bar{\eta},\bar{\xi})$, with $\rho>0$

$$P_{12}=\left(\frac{1}{2},\pm\frac{\sqrt{6}}{4},\frac{-1}{4}\right).$$

We shall verify that the determinant at these two roots is different from zero , where

$$D(\bar{\rho},\bar{\eta},\bar{\xi})=det\begin{pmatrix} \dfrac{1}{2}(1+4\bar{\xi}) & 0 & 2\bar{\rho}\\ \bar{\rho} & 0 & -4\bar{\xi}\\ \bar{\rho} & \dfrac{4}{3}\bar{\eta} & -12\bar{\xi} \end{pmatrix}.$$

We get that

$$\det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Bigg|_{(\bar{\rho},\bar{\eta},\bar{\xi})=P_{12}}=\pm\frac{\sqrt{6}}{6}\neq 0.$$

Then, this proves that system (5) has exactly two limit cycles bifurcating from the origin for $\varepsilon\neq 0$ sufficiently small.

4 Proof of Statement (*b*) of Theorem 1

For proving statement (b) of Theorem 1 we use the averaging theory of second order. Then, we must annul the averaged system of first order $(f_{11}(\rho,\eta,\xi), f_{12}(\rho,\eta,\xi), f_{13}(\rho,\eta,\xi))$. So we take

$$a_1=d_1=c_1=0,\quad b_{06}=-a_{03},\quad b_{05}=-a_{02},\quad c_{09}=c_{08}=c_{07}=0,$$

$$d_{04}=-d_{00},\quad d_{09}=d_{08}=d_{07}=0,\quad c_{04}=-c_{00}.$$

Considering this conditions to apply the averaging theory of second order. Then from Section 2, we have $f_2=(f_{21}(\rho,\eta,\xi), f_{22}(\rho,\eta,\xi), f_{23}(\rho,\eta,\xi))$, where

$$\begin{cases} f_{21}(\rho,\eta,\xi) &= \dfrac{\rho}{8b^2}(U_0+U_1\rho^2+U_2\xi+U_3\xi^2+U_4\eta+U_5\xi\eta+U_6\eta^2),\\ f_{22}(\rho,\eta,\xi) &= \dfrac{1}{2b^2}(V_0\rho^2+V_1\rho^2\xi+V_2\xi^2+V_3\xi^3+V_4\eta+V_5\rho^2\eta\\ &\quad + V_6\xi\eta+V_7\xi^2\eta+V_8\eta^2+V_9\xi\eta^2+V_{10}\eta^3),\\ f_{23}(\rho,\eta,\xi) &= \dfrac{1}{2b^2}(W_0\rho^2+W_1\xi+W_2\rho^2\xi+W_3\xi^2+W_4\xi^3+W_5\rho^2\eta\\ &\quad + W_6\xi\eta+W_7\xi^2\eta+W_8\eta^2+W_9\xi\eta^2+W_{10}\eta^3), \end{cases} \tag{7}$$

where

$$\begin{aligned}
U_0 &= 8a_2b,\\
U_1 &= a_{00}a_{01} + a_{01}a_{04} - 2a_{00}b_{00} - b_{00}b_{01} + 2a_{04}b_{04} - b_{01}b_{04} + a_{05}c_{00} + b_{02}c_{00}\\
&\quad - a_{02}c_{01} + a_{06}d_{00} + b_{03}d_{00} - a_{03}d_{01},\\
U_2 &= 4b(a_{13} + b_{16}),\\
U_3 &= 4(a_{01}a_{09} + 2a_{09}b_{04} - 2a_{00}b_{09} - b_{01}b_{09} + b_{08}c_{03} - a_{08}c_{06} + 2b_{09}d_{03} - 2a_{09}d_{06}),\\
U_4 &= 4b(a_{12} + b_{15}),\\
U_5 &= 4(a_{01}a_{08} + 2a_{08}b_{04} - 2a_{00}b_{08} - b_{01}b_{08} + b_{08}c_{02} + 2b_{07}c_{03} - a_{08}c_{05} - 2a_{07}c_{06}\\
&\quad + 2b_{09}d_{02} + b_{08}d_{03} - 2a_{09}d_{05} - a_{08}d_{06}),\\
U_6 &= 4(a_{01}a_{07} + 2a_{07}b_{04} - 2a_{00}b_{07} - b_{01}b_{07} + 2b_{07}c_{02} - 2a_{07}c_{05} + b_{08}d_{02} - a_{08}d_{05}),
\end{aligned}$$

and

$$\begin{aligned}
V_0 &= b(c_{10} + c_{14}),\\
V_1 &= -(a_{06}c_{00} + b_{03}c_{00} - a_{03}c_{01} + b_{00}c_{03} + b_{04}c_{03} - c_{03}c_{05} - a_{00}c_{06} - a_{04}c_{06}\\
&\quad + c_{02}c_{06} - c_{06}d_{03} + c_{03}d_{06}),\\
V_2 &= 2bc_{19},\\
V_3 &= -2(b_{09}c_{03} - a_{09}c_{06}),\\
V_4 &= 2bc_2,\\
V_5 &= -(a_{05}c_{00} + b_{02}c_{00} - a_{02}c_{01} + b_{00}c_{02} + b_{04}c_{02} - a_{00}c_{05} - a_{04}c_{05} - c_{06}d_{02} + c_{03}d_{05}),\\
V_6 &= bc_{18},\\
V_7 &= -2(b_{09}c_{02} + b_{08}c_{03} - a_{09}c_{05} - a_{08}c_{06}),\\
V_8 &= 2bc_{17},\\
V_9 &= -2(b_{08}c_{02} + b_{07}c_{03} - a_{08}c_{05} - a_{07}c_{06}),\\
V_{10} &= -2(b_{07}c_{02} - a_{07}c_{05}),
\end{aligned}$$

and

$$\begin{aligned}
W_0 &= b(d_{10} + d_{14}),\\
W_1 &= 2bd_2,\\
W_2 &= -(a_{06}d_{00} + b_{03}d_{00} - a_{03}d_{01} + c_{06}d_{02} + b_{00}d_{03} + b_{04}d_{03} - c_{03}d_{05} - a_{00}d_{06} - a_{04}d_{06}),\\
W_3 &= 2bd_{19},\\
W_4 &= -2(b_{09}d_{03} - a_{09}d_{06}),\\
W_5 &= -(a_{05}d_{00} + b_{02}d_{00} - a_{02}d_{01} + b_{00}d_{02} + b_{04}d_{02} + c_{05}d_{02} - a_{00}d_{05} - a_{04}d_{05}\\
&\quad - c_{02}d_{05} + d_{03}d_{05} - d_{02}d_{06}),\\
W_6 &= 2bd_{18},\\
W_7 &= -2(b_{09}d_{02} + b_{08}d_{03} - a_{09}d_{05} - a_{08}d_{06}),\\
W_8 &= 2bd_{17},\\
W_9 &= -2(b_{08}d_{02} + b_{07}d_{03} - a_{08}d_{05} - a_{07}d_{06}),\\
W_{10} &= -2(b_{07}d_{02} - a_{07}d_{05})\eta^3.
\end{aligned}$$

Hence, from the first equation of system (7) and avoiding the solutions with $\rho = 0$, we isolate ρ^2 and we substitute it in $f_{2i}(\rho, \eta, \xi) = 0$ for $i = 2, 3$. We obtain the following two equations

$$\begin{cases} g_{21} = C_0 + C_1\eta + C_2\xi + C_3\eta^2 + C_4\eta\xi + C_5\xi^2 + C_6\eta^3 + C_7\eta^2\xi + C_8\eta\xi^2 + C_9\xi^3,\\ g_{22} = D_0 + D_1\eta + D_2\xi + D_3\eta^2 + D_4\eta\xi + D_5\xi^2 + D_6\eta^3 + D_7\eta^2\xi + D_8\eta\xi^2 + D_9\xi^3, \end{cases}$$

where C_i and D_i for $i = 0, \cdots, 9$ are real coefficients.

$$
\begin{aligned}
C_0 &= \frac{-V_0U_0}{U_1},\\
C_1 &= \frac{-V_0U_4 - V_5U_0 + V_4U_1}{U_1},\\
C_2 &= \frac{-V_0U_2 - V_1U_0}{U_1},\\
C_3 &= \frac{-V_0U_6 - V_5U_4 + V_8U_1}{U_1},\\
C_4 &= \frac{V_6U_1 - V_0U_5 - V_1U_4 - V_5U_2}{U_1},\\
C_5 &= \frac{-V_0U_3 - V_1U_2 + V_2U_1}{U_1},\\
C_6 &= \frac{-V_5U_6 + V_{10}U_1}{U_1},\\
C_7 &= \frac{-V_1U_6 - V_5U_5 + V_9U_1}{U_1},\\
C_8 &= \frac{-V_1U_5 - V_5U_3 + V_7U_1}{U_1},\\
C_9 &= \frac{-V_1U_3 + V_3U_1}{U_1},
\end{aligned}
$$

and

$$
\begin{aligned}
D_0 &= \frac{-W_0U_0}{U_1},\\
D_1 &= \frac{-W_0U_4 - W_5U_0}{U_1},\\
D_2 &= \frac{-W_0U_2 - W_2U_0 + W_1U_1}{U_1},\\
D_3 &= \frac{-W_0U_6 - W_5U_4 + W_8U_1}{U_1},\\
D_4 &= \frac{W_6U_1 - W_0U_5 - W_2U_4 - W_5U_2}{U_1},\\
D_5 &= \frac{-W_0U_3 - W_2U_2 + W_3U_1}{U_1},\\
D_6 &= \frac{-W_5U_6 + W_{10}U_1}{U_1},\\
D_7 &= \frac{-W_2U_6 - W_5U_5 + W_9U_1}{U_1},\\
D_8 &= \frac{-W_2U_5 - W_5U_3 + W_7U_1}{U_1},\\
D_9 &= \frac{-W_2U_3 + W_4U_1}{U_1}.
\end{aligned}
$$

Looking only at the coefficients of system (1) which appear in C_i and D_i for $i = 0, \cdots, 9$ we see that the coefficients C_i and D_i are all independent because pairwise contain different coefficients of system (1), with the exceptions of the coefficients C_7 and C_8, and D_7 and D_8 that share the same coefficients of system (1). But now looking directly at the explicit expressions of C_7 and C_8, and of D_7 and D_8 we observe that they are also independent.

Since all the coefficients of the two equations $g_{21}(\eta, \xi) = 0$ and $g_{22}(\eta, \xi) = 0$ are independent they can be chosen arbitrary. By Bezout's theorem, system $g_{21}(\eta, \xi) = 0$, $g_{22}(\eta, \xi) = 0$ has nine real roots, and system (7) has at most nine real roots with $\rho > 0$.

Let $\left(\bar{\rho}, \bar{\eta}, \bar{\xi}\right)$ be a solution of system (7). In order to have a limit cycle according to the averaging theory in Section 2, we must have

$$D(\bar{\rho}, \bar{\eta}, \bar{\xi}) = det \begin{pmatrix} \frac{\partial f_{21}}{\partial \rho} & \frac{\partial f_{21}}{\partial \eta} & \frac{\partial f_{21}}{\partial \xi} \\ \frac{\partial f_{22}}{\partial \rho} & \frac{\partial f_{22}}{\partial \eta} & \frac{\partial f_{22}}{\partial \xi} \\ \frac{\partial f_{23}}{\partial \rho} & \frac{\partial f_{23}}{\partial \eta} & \frac{\partial f_{23}}{\partial \xi} \end{pmatrix} \Bigg|_{(\rho,\eta,\xi)=(\bar{\rho},\bar{\eta},\bar{\xi})} \neq 0.$$

Then, we conclude that by the averaging theory of second order system (1) has at most nine limit cycles in a zero-Hopf bifurcation at the origin. This completes the proof of statement (b) of Theorem 1.

Now we give an example which proves that system (1) has exactly 9 limit cycles bifurcating from the origin by the averaging theory of second order.

Example 5 Consider the following quadratic polynomial differential system

$$\begin{cases} \frac{dx}{dt} = -2\varepsilon^2 x - y - x^2 + yz + 2xz - 2xw - yw + \frac{1}{2}zw \\ \quad + \varepsilon(z^2 - xz + xw) + \varepsilon^2(x^2 - yz), \\ \frac{dy}{dt} = -2\varepsilon^2 y + x - 2yz + xz + xw + 2yw + zw + \varepsilon(xy - yz + yw) \\ \quad + \varepsilon^2(xz + yw), \\ \frac{dz}{dt} = 12\varepsilon^2 z - 2x^2 - 2y^2 - 2xz - 4xw + \varepsilon(x^2 - y^2 + 8z^2 - 2zw) + \varepsilon^2 z^2, \\ \frac{dw}{dt} = 24\varepsilon^2 w + 2x^2 - 2y^2 + 2xy - 2xz - 2xw + \varepsilon(-2w^2 - 2zw) \\ \quad + \varepsilon^2(x^2 + xw). \end{cases} \tag{8}$$

The eigenvalues of the singular point (0, 0, 0, 0) of system (8) are $-2\varepsilon^2 \pm i$, $12\varepsilon^2$ and $24\varepsilon^2$. The averaged system associated to system (8) is

$$\begin{cases} f_{21}(\rho, \eta, \xi) = \rho(-2 - \eta + \xi - \eta^2 - 2\xi^2 + \rho^2 - \eta\xi) = 0, \\ f_{22}(\rho, \eta, \xi) = 12\eta + 8\eta^2 - 2\eta\xi + 4\xi^2\eta + 2\xi\eta^2 - 2\rho^2\eta = 0, \\ f_{23}(\rho, \eta, \xi) = 24\xi - 2\xi^2 - 2\eta\xi + 2\xi^2\eta + 2\xi\eta^2 - 2\rho^2\xi = 0. \end{cases} \tag{9}$$

Solving system (9), there are only nine roots $(\bar{\rho}, \bar{\eta}, \bar{\xi})$ with $\rho > 0$, namely

$$P_1 = \left(\sqrt{2}, 0, 0\right), \quad P_2 = \left(\sqrt{12 - \sqrt{5}}, 0, \sqrt{5}\right), \quad P_3 = \left(\sqrt{12 + \sqrt{5}}, 0, -\sqrt{5}\right),$$

$$P_4 = \left(\sqrt{22}, 4, 0\right), \quad P_5 = \left(\sqrt{27}, 4, 1\right), \quad P_6 = \left(\sqrt{14 - 2\sqrt{6}}, -1, \sqrt{6}\right),$$

$$P_7 = \left(\sqrt{14 + 2\sqrt{6}}, -1, -\sqrt{6}\right), \quad P_8 = \left(\sqrt{2}, -1, 0\right), \quad P_9 = \left(\sqrt{21}, 4, -1\right).$$

Since, we must verify that the determinant is different from zero at these roots where

$$D(\bar\rho,\bar\eta,\bar\xi) = det\begin{pmatrix} \begin{array}{c}-2-\bar\eta+\bar\xi-\bar\eta^2\\ -2\bar\xi^2+3\bar\rho^2-\bar\eta\bar\xi\end{array} & \bar\rho(-1-2\bar\eta-\bar\xi) & \bar\rho(1-4\bar\xi-\bar\eta) \\ -4\bar\rho\bar\eta & 12+16\bar\eta-2\bar\xi+4\bar\xi^2+4\bar\xi\bar\eta-2\bar\rho^2 & -2\bar\eta+8\bar\xi\bar\eta+2\bar\eta^2 \\ -4\bar\rho\bar\xi & -2\bar\xi+2\bar\xi^2+4\bar\xi\bar\eta & \begin{array}{c}24-4\bar\xi-2\bar\eta+4\bar\xi\bar\eta\\ +2\bar\eta^2-2\bar\rho^2\end{array} \end{pmatrix},$$

we get

$$\det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Big|_{P_1}=640\neq 0,\ \det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Big|_{P_{23}}=-7680\pm640\sqrt{5}\neq 0,$$

$$\det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Big|_{P_4}=-7040\neq 0,\ \det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Big|_{P_5}=17280\neq 0,$$

$$\det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Big|_{P_{67}}=13440\pm1920\sqrt{6}\neq 0,\ \det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Big|_{P_8}=-960\neq 0,$$

$$\det\left(\frac{\partial(f_{11},f_{12},f_{13})}{\partial(\rho,\eta,\xi)}\right)\Big|_{P_9}=13440\neq 0.$$

Hence, system (8) has exactly nine limit cycles bifurcating from the origin for $\varepsilon\neq 0$ sufficiently small.

5 Proof of Statement (*c*) of Theorem 1

To prove the main result of this work we will use the third order averaging theory. According to the theorem of Section 2, we must annul the averaged system of second order $(f_{21}(\rho,\eta,\xi), f_{22}(\rho,\eta,\xi), f_{23}(\rho,\eta,\xi))$. For this, we take

$$a_2=c_2=d_2=0,\ \ a_{00}=a_{04}=a_{07}=a_{08}=a_{09}=0,\ \ a_{05}=-b_{02},\ \ a_{13}=-b_{16},$$
$$b_{07}=b_{08}=b_{09}=0,\ \ a_{06}=-b_{03},\ \ a_{06}=-b_{03},\ \ b_{00}=-b_{04},\ \ a_{12}=-b_{15},$$
$$c_{01}=c_{03}=c_{06}=0,\ \ d_{01}=d_{02}=d_{05}=0, c_{14}=-c_{10},\ \ d_{14}=-d_{10}.$$
$$c_{17}=c_{18}=c_{19}=0,\ \ d_{17}=d_{18}=d_{19}=0.$$

Applying the averaging theory of third order, we must compute the following expression

$$\begin{aligned} f_3(\rho,\eta,\xi) = \frac{1}{2\pi}\int_0^{2\pi} &[\frac{1}{2}y_1^T(\theta,\rho,\eta,\xi)\frac{\partial^2F_1}{\partial(\rho,\eta,\xi)^2}(\theta,\rho,\eta,\xi)y_1(\theta,\rho,\eta,\xi) \\ &+\frac{1}{2}\frac{\partial F_1}{\partial(\rho,\eta,\xi)}(\theta,\rho,\eta,\xi)y_2(\theta,\rho,\eta,\xi) \\ &+\frac{\partial F_2}{\partial(\rho,\eta,\xi)}(\theta,\rho,\eta,\xi)y_1(\theta,\rho,\eta,\xi)+F_3(\theta,\rho,\eta,\xi)]d\theta, \end{aligned} \tag{10}$$

where

$$y_1(\theta,\rho,\eta,\xi)=\int_0^\theta F_1(t,\rho,\eta,\xi)dt,$$

$$y_2(\theta,\rho,\eta,\xi)=\int_0^\theta\left[\frac{\partial F_1}{\partial(\rho,\eta,\xi)}(t,\rho,\eta,\xi)\int_0^t F_1(s,\rho,\eta,\xi)ds+F_2(t,\rho,\eta,\xi)\right]dt,$$

and

$$f_3(\rho,\eta,\xi) = (f_{31}, f_{32}, f_{33}),$$

$$F_1(\theta,\rho,\eta,\xi) = (F_{11}, F_{12}, F_{13}),$$

$$F_2(\theta,\rho,\eta,\xi) = (F_{21}, F_{22}, F_{23}),$$

$$F_3(\theta,\rho,\eta,\xi) = (F_{31}, F_{32}, F_{33}),$$

$$y_1(\theta,\rho,\eta,\xi) = (y_{11}, y_{12}, y_{13}),$$

$$y_2(\theta,\rho,\eta,\xi) = (y_{21}, y_{22}, y_{23}).$$

So, first we compute the following integral, and we get that

$$\begin{aligned}
&\frac{1}{4\pi}\int_0^{2\pi}\left[y_1^T(\theta,\rho,\eta,\xi)\frac{\partial^2 F_1}{\partial(\rho,\eta,\xi)^2}(\theta,\rho,\eta,\xi)y_1(\theta,\rho,\eta,\xi)\right]d\theta\\
&= \frac{\partial^2 F_1}{\partial\rho^2}y_{11}^2 + \frac{\partial^2 F_1}{\partial\rho\partial\theta}y_{11}y_{12}\\
&+\frac{\partial^2 F_1}{\partial\rho\partial\xi}y_{11}y_{13} + \frac{\partial^2 F_1}{\partial\eta\partial\xi}y_{11}y_{12} + \frac{\partial^2 F_1}{\partial\eta^2}y_{12}^2 + \frac{\partial^2 F_1}{\partial\eta\partial\xi}y_{12}^{y_{13}} + \frac{\partial^2 F_1}{\partial\rho\partial\xi}y_{11}y_{13} + \frac{\partial^2 F_1}{\partial\xi\partial\eta}y_{12}y_{13}\\
&+\frac{\partial^2 F_1}{\partial\xi^2}y_{13}^2\\
&= (G_1(\rho,\eta,\xi), G_2(\rho,\eta,\xi), G_3(\rho,\eta,\xi)).
\end{aligned}$$

Secondly, we compute the second part of the expression, and we get

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2}\frac{\partial F_1}{\partial(\rho,\eta,\xi)}(\theta,\rho,\eta,\xi)y_2(\theta,\rho,\eta,\xi) + \frac{\partial F_2}{\partial(\rho,\eta,\xi)}(\theta,\rho,\eta,\xi)y_1(\theta,\rho,\eta,\xi)\right.\\
&\left.+F_3(\theta,\rho,\eta,\xi)\right]d\theta = (H_1(\rho,\eta,\xi), H_2(\rho,\eta,\xi), H_3(\rho,\eta,\xi)).
\end{aligned}$$

Finally, we obtain the following averaged system of third order ($f_{31}(\rho,\eta,\xi) = G_1 + H_1$, $f_{32}(\rho,\eta,\xi) = G_2 + H_2$, $f_{33}(\rho,\eta,\xi) = G_3 + H_3$).

$$\left\{\begin{array}{rcl}
f_{31}(\rho,\eta,\xi) &=& \frac{1}{8b^3}\rho(8a_3b^2 + A_1\eta + A_2\xi + A_3\xi^2 + A_4\eta^2 + A_5\eta\xi\\
&+& A_6\rho^2 + A_7\rho^2\eta + A_8\rho^2\xi),\\
\\
f_{32}(\rho,\eta,\xi) &=& \frac{1}{8b^3}(8c_3\eta b^2 + B_1\rho^2\xi + B_2\rho^2\eta + B_3\eta^2 + B_4\xi^2 + B_5\eta^4\\
&+& B_6\eta\xi + B_7\rho^2\eta\xi + B_8\rho^2 + B_9\eta^3 + B_{10}\rho^2\eta^2 + B_{11}\eta\xi^2 + B_{12}\eta^2\xi),\\
\\
f_{33}(\rho,\eta,\xi) &=& \frac{1}{8b^3}(8d_3\xi b^2 + K_1\eta\xi + K_2\rho^4 + K_3\rho^2\eta + K_4\rho^2\eta\xi + K_5\rho^2\xi^2\\
&+& K_6\eta^2\xi + K_7\eta\xi^2 + K_8\xi^2 + K_9\eta^2 + K_{10}\rho^2\xi + K_{11}\xi^3 + K_{12}\rho^2),
\end{array}\right. \tag{11}$$

where

$$
\begin{aligned}
A_1 &= 4(b_{25}b^2 + a_{22}b^2),\\
A_2 &= 4(b_{26}b^2 + a_{23}b^2),\\
A_3 &= -4b_{19}b_{01}b + 4a_{19}a_{01}b + 8d_{03}b_{19}b - 8d_{06}a_{19}b + 8b_{04}a_{19}b,\\
A_4 &= -4b_{17}b_{01}b + 4a_{17}a_{01}b + 8c_{02}b_{17}b - 8c_{05}a_{17}b + 8b_{04}a_{17}b,\\
A_5 &= -4a_{18}c_{05}b + 4b_{18}c_{02}b + 4d_{03}b_{18}b - 4d_{06}a_{18}b + 8b_{04}a_{18}b + 4a_{18}a_{01}b - 4b_{18}b_{01}b,\\
A_6 &= d_{00}a_{16}b + d_{00}b_{13}b - a_{02}c_{11}b + c_{00}b_{12}b + a_{14}a_{01}b + 2a_{14}b_{04}b + 2b_{04}a_{10}b\\
&\quad + a_{10}a_{01}b\\
&- b_{14}b_{01}b - b_{10}b_{01}b - a_{03}d_{11}b + c_{00}a_{15},\\
A_7 &= a_{02}c_{05}^2 - a_{02}c_{02}^2 + a_{02}a_{01}^2 - a_{02}b_{01}^2 + c_{05}b_{01}b_{02} + 2a_{02}a_{01}b_{04} + 2a_{02}c_{02}b_{01}\\
&- 2a_{02}c_{05}b_{04} - 2a_{02}c_{05}a_{01} - a_{01}c_{02}b_{02} - 2b_{02}c_{02}b_{04},\\
A_8 &= -a_{03}d_{03}^2 + a_{03}d_{06}^2 + a_{03}a_{01}^2 - a_{03}b_{01}^2 + d_{06}b_{01}b_{03} + 2a_{03}a_{01}b_{04} + 2a_{03}d_{03}b_{01}\\
&- 2a_{03}d_{06}b_{04} - 2a_{03}d_{06}a_{01} - a_{01}d_{03}b_{03} - 2b_{03}d_{03}b_{04},
\end{aligned}
$$

and

$$
\begin{aligned}
B_1 &= -4b_{13}c_{00}b - 4a_{16}c_{00}b - 4c_{02}c_{16}b + 4c_{13}c_{05}b + 4a_{03}c_{11}b + 4d_{03}c_{16}b - 4d_{06}c_{13}b,\\
B_2 &= -4b_{12}c_{00}b - 4a_{15}c_{00}b + 4c_{05}a_{10}b - 4b_{14}c_{02}b - 4b_{10}c_{02}b + 4a_{02}c_{11}b + 4c_{05}a_{14}b,\\
B_3 &= 8c_{27}b^2,\\
B_4 &= 8c_{29}b^2,\\
B_5 &= -c_{00}c_{02}^2 + c_{00}c_{05}^2 + b_{01}c_{00}c_{02} - a_{01}c_{00}c_{05},\\
B_6 &= 8c_{28}b^2,\\
B_7 &= 4d_{06}b_{03}c_{02} - 4a_{03}b_{01}c_{02} + 4a_{03}c_{02}d_{03} + 4a_{03}a_{01}c_{05} - 4a_{03}d_{06}c_{05} - 4b_{03}c_{05}d_{03},\\
B_8 &= 4c_{20}b^2 + 4c_{24}b^2,\\
B_9 &= 8a_{17}c_{05}b - 8b_{17}c_{02}b,\\
B_{10} &= 4a_{02}c_{02}^2 - 4a_{02}c_{05}^2 - 4b_{01}c_{02}a_{02} + 4a_{01}a_{02}c_{05},\\
B_{11} &= 8a_{19}c_{05}b - 8b_{19}c_{02}b,\\
B_{12} &= 8a_{18}c_{05}b - 8b_{18}c_{02}b,
\end{aligned}
$$

and

$$
\begin{aligned}
K_1 &= 8d_{28}b^2,\\
K_2 &= -d_{00}d_{03}^2 + d_{00}d_{06}^2 + b_{01}d_{00}d_{03} - a_{01}d_{00}d_{06},\\
K_3 &= -4b_{12}d_{00}b - 4a_{15}d_{00}b - 4d_{03}d_{15}b + 4d_{12}d_{06}b + 4a_{02}d_{11}b + 4c_{02}d_{15}b - 4c_{05}d_{12}b,\\
K_4 &= -4d_{06}b_{02}c_{02} + 4a_{02}d_{03}c_{02} + 4a_{01}a_{02}d_{06} + 4b_{02}d_{03}c_{05} - 4a_{02}d_{06}c_{05} - 4b_{01}a_{02}d_{03},\\
K_5 &= 4a_{03}d_{03}^2 - 4a_{03}d_{06}^2 - 4b_{01}d_{03}a_{03} + 4a_{01}a_{03}d_{06},\\
K_6 &= 8a_{17}d_{06}b - 8b_{17}d_{03}b,\\
K_7 &= 8a_{18}d_{06}b - 8b_{18}d_{03}b,\\
K_8 &= 8d_{29}b^2,\\
K_9 &= 8d_{27}b^2,\\
K_{10} &= -4b_{13}d_{00}b - 4a_{16}d_{00}b + 4d_{06}a_{10}b - 4b_{14}d_{03}b - 4b_{10}d_{03}b + 4a_{03}d_{11}b + 4d_{06}a_{14}b,\\
K_{11} &= 8a_{19}d_{06}b - 8b_{19}d_{03}b,\\
K_{12} &= 4d_{20}b^2 + 4d_{24}b^2.
\end{aligned}
$$

We solve the first equation f_{31} with respect to ρ and avoiding the solutions with $\rho = 0$, we obtain

$$
\rho^2 = -\frac{A_4\eta^2 + A_3\xi^2 + A_1\eta + A_2\xi + A_5\eta\xi + 8a_3b2}{A_6 + A_8\xi + A_7\eta}.
$$

Substituting ρ^2 in $f_{3i}(\rho,\eta,\xi)=0$ for $i=2,3$, we get the following system

$$
\begin{cases}
g_{31} = \dfrac{1}{8(A_6+A_7\eta+A_8\xi)^2b^3}(I_0 + I_1\xi\eta + I_2\eta^2\xi + I_3\eta\xi^2 + I_4\eta^2+I_5\eta^4+I_6\xi^2 + I_7\xi^4 \\
\quad + I_8\eta^3 + I_9\eta^5 + I_{10}\eta + I_{11}\xi^3 + I_{12}\eta^2\xi^2 + I_{13}\eta^3\xi + I_{14}\eta\xi^3 + I_{15}\xi + I_{16}\eta^3\xi^2 \\
\quad + I_{17}\eta\xi^4 + I_{18}\eta^4\xi + I_{19}\eta^2\xi^3), \\
g_{32} = \dfrac{1}{8(A_6+A_7\eta+A_8\xi)^2b^3}(J_0 + J_1\xi\eta + J_2\eta\xi^2 + J_3\eta^2\xi + J_4\eta^3 + J_5\xi^3 + J_6\eta^2\xi^2 \\
\quad + J_7\eta^2 + J_8\eta^4 + J_9\xi^4 + J_{10}\xi^5 + J_{11}\eta^4\xi + J_{12}\eta^2\xi^3 + J_{13}\eta^3\xi^2 + J_{14}\eta\xi^4 \\
\quad + J_{15}\xi^2 + J_{16}\eta + J_{17}\xi + J_{18}\eta^3\xi + J_{19}\eta\xi^3),
\end{cases}
$$

where

$$
\begin{aligned}
I_0 &= 64B_5a_3^2b^4 - 8B_8a_3b^2A_6,\\
I_1 &= 16c_3b^2A_6A_8 - 8B_1a_3b^2A_7 - 8B_2a_3b^2A_8 + 16B_5a_3b^2A_5 - 8B_7a_3b^2A_6 + B_6A_6^2\\
&- B_1A_1A_6 + 2B_5A_1A_2 - B_2A_2A_6 - B_8A_1A_8 - B_8A_2A_7 - B_8A_5A_6,\\
I_2 &= 16c_3b^2A_7A_8 - 8B_7a_3b^2A_7 - 8B_{10}a_3b^2A_8 + B_{12}A_6^2 - B_{10}A_2A_6 - B_1A_1A_7\\
&- B_1A_4A_6 - B_2A_1A_8 - B_2A_2A_7 - B_2A_5A_6 + 2B_3A_6A_8 + 2B_5A_1A_5 + 2B_5A_2A_4\\
&+ 2B_6A_6A_7 - B_7A_1A_6 - B_8A_4A_8 - B_8A_5A_7,\\
I_3 &= 8c_3b^2A_8^2 - 8B_7a_3b^2A_8 + B_{11}A_6^2 - B_1A_1A_8 - B_1A_2A_7 - B_1A_5A_6 - B_2A_2A_8\\
&- B_2A_3A_6 + 2B_4A_6A_7 + 2B_5A_1A_3 + 2B_5A_2A_5 + 2B_6A_6A_8 - B_7A_2A_6 - B_8A_3A_7\\
&- B_8A_5A_8,\\
I_4 &= B_3A_6^2 + B_5A_1^2 - B_2A_1A_6 - B_8A_1A_7 - B_8A_4A_6 + 16c_3b^2A_6A_7 - 8B_2a_3b^2A_7\\
&+ 16B_5a_3b^2A_4 - 8B_{10}a_3b^2A_6,\\
I_5 &= B_3A_7^2 + B_5A_4^2 - B_2A_4A_7 + 2B_9A_6A_7 - B_{10}A_1A_7 - B_{10}A_4A_6,\\
I_6 &= B_4A_6^2 + B_5A_2^2 - B_1A_2A_6 - B_8A_2A_8 - B_8A_3A_6 - 8B_1a_3A_8b^2 + 16B_5a_3b^2A_3,\\
I_7 &= B_4A_8^2 + B_5A_3^2 - B_1A_3A_8,\\
I_8 &= B_9A_6^2 + 8c_3b^2A_7^2 - B_2A_1A_7 - B_2A_4A_6 + 2B_3A_6A_7 + 2B_5A_1A_4 - B_8A_4A_7\\
&- B_{10}A_1A_6 - 8B_{10}a_3A_7b^2,\\
I_9 &= B_9A_7^2 - B_{10}A_4A_7,\\
I_{10} &= 8c_3b^2A_6^2 - B_8A_1A_6 - 8B_2a_3A_6b^2 + 16B_5a_3A_1b^2 - 8B_8a_3A_7b^2,\\
I_{11} &= -B_1A_2A_8 - B_1A_3A_6 + 2B_4A_6A_8 + 2B_5A_2A_3 - B_8A_3A_8,\\
I_{12} &= B_3A_8^2 + B_4A_7^2 + B_5A_5^2 - B_1A_4A_8 - B_1A_5A_7 - B_2A_3A_7 - B_2A_5A_8 + 2B_5A_3A_4\\
&+ 2B_6A_7A_8 - B_7A_1A_8 - B_7A_2A_7 - B_7A_5A_6 - B_{10}A_2A_8 - B_{10}A_3A_6 + 2B_{11}A_6A_7\\
&+ 2B_{12}A_6A_8,\\
I_{13} &= B_6A_7^2 - B_1A_4A_7 - B_2A_4A_8 - B_2A_5A_7 + 2B_3A_7A_8 + 2B_5A_4A_5 - B_7A_1A_7\\
&+ 2B_9A_6A_8 - B_7A_4A_6 - B_{10}A_1A_8 - B_{10}A_2A_7 - B_{10}A_5A_6 + 2B_{12}A_6A_7,\\
I_{14} &= B_6A_8^2 - B_1A_3A_7 - B_1A_5A_8 - B_2A_3A_8 + 2B_4A_7A_8 + 2B_5A_3A_5 - B_7A_2A_8\\
&- B_7A_3A_6 + 2B_{11}A_6A_8,
\end{aligned}
$$

$$
\begin{aligned}
I_{15} &= -B_8A_2A_6 - 8B_1a_3A_6b^2 + 16B_5a_3A_2b^2 - 8B_8a_3A_8b^2,\\
I_{16} &= B_9A_8^2 + B_{11}A_7^2 - B_7A_4A_8 - B_7A_5A_7 - B_{10}A_3A_7 - B_{10}A_5A_8 + 2B_{12}A_7A_8,\\
I_{17} &= B_{11}A_8^2 - B_7A_3A_8,\\
I_{18} &= B_{12}A_7^2 - B_7A_4A_7 - B_{10}A_4A_8 - B_{10}A_5A_7 + 2B_9A_7A_8,\\
I_{19} &= B_{12}A_8^2 - B_7A_3A_7 - B_7A_5A_8 - B_{10}A_3A_8 + 2B_{11}A_7A_8,
\end{aligned}
$$

and

$$
\begin{aligned}
J_0 &= 64K_2a_3^2b^4 - 8K_{12}a_3b^2A_6,\\
J_1 &= 16d_3b^2A_6A_7 - 8K_3a_3b^2A_8 - 8K_4a_3b^2A_6 + 16K_2a_3b^2A_5 - 8K_{10}a_3b^2A_7\\
&+ K_1A_6^2 - K_3A_2A_6 + 2K_2A_1A_2 - K_{10}A_1A_6 - K_{12}A_1A_8 - K_{12}A_2A_7 - K_{12}A_5A_6,\\
J_2 &= 16d_3b^2A_7A_8 - 8K_4a_3b^2A_8 - 8K_5a_3b^2A_7 + K_7A_6^2 - K_3A_2A_8 - K_3A_3A_6\\
&- K_4A_2A_6 - K_5A_1A_6 - K_{10}A_1A_8 - K_{10}A_2A_7 + 2K_1A_6A_8 + 2K_2A_1A_3\\
&+ 2K_2A_2A_5 + 2K_8A_6A_7 - K_{10}A_5A_6 - K_{12}A_3A_7 - K_{12}A_5A_8,\\
J_3 &= 8d_3b^2A_7^2 - 8K_4a_3b^2A_7 + K_6A_6^2 - K_3A_1A_8 - K_3A_2A_7 - K_3A_5A_6 - K_4A_1A_6\\
&- K_{10}A_1A_7 + 2K_1A_6A_7 + 2K_2A_1A_5 + 2K_2A_2A_4 + 2K_9A_6A_8 - K_{10}A_4A_6\\
&- K_{12}A_4A_8 - K_{12}A_5A_7,\\
J_4 &= -K_3A_1A_7 - K_3A_4A_6 + 2K_9A_6A_7 + 2K_2A_1A_4 - K_{12}A_4A_7,\\
J_5 &= K_{11}A_6^2 + 8d_3b^2A_8^2 - K_5A_2A_6 - K_{10}A_2A_8 + 2K_2A_2A_3 + 2K_8A_6A_8 - K_{10}A_3A_6\\
&- K_{12}A_3A_8 - 8K_5a_3A_8b^2,\\
J_6 &= K_2A_5^2 + K_8A_7^2 + K_9A_8^2 - K_3A_3A_7 - K_3A_5A_8 - K_4A_1A_8 - K_4A_2A_7\\
&+ 2K_1A_7A_8 + 2K_2A_3A_4 - K_4A_5A_6 - K_5A_1A_7 - K_5A_4A_6 - K_{10}A_4A_8\\
&- K_{10}A_5A_7 + 2K_6A_6A_8 + 2K_7A_6A_7,\\
J_7 &= K_2A_1^2 + K_9A_6^2 - K_3A_1A_6 - K_{12}A_1A_7 - K_{12}A_4A_6 - 8K_3a_3A_7b^2 + 16K_2a_3b^2A_4,\\
J_8 &= K_2A_4^2 + K_9A_7^2 - K_3A_4A_7,\\
J_9 &= K_8A_8^2 + K_2A_3^2 - K_5A_2A_8 + 2K_{11}A_6A_8 - K_5A_3A_6 - K_{10}A_3A_8,\\
J_{10} &= K_{11}A_8^2 - K_5A_3A_8,\\
J_{11} &= K_6A_7^2 - K_4A_4A_7,\\
J_{12} &= K_6A_8^2 + K_{11}A_7^2 - K_4A_3A_7 - K_4A_5A_8 - K_5A_4A_8 - K_5A_5A_7 + 2K_7A_7A_8,\\
J_{13} &= K_7A_7^2 - K_4A_4A_8 - K_4A_5A_7 - K_5A_4A_7 + 2K_6A_7A_8,\\
J_{14} &= K_7A_8^2 - K_4A_3A_8 - K_5A_3A_7 - K_5A_5A_8 + 2K_{11}A_7A_8,\\
J_{15} &= K_2A_2^2 + K_8A_6^2 - K_{10}A_2A_6 - K_{12}A_2A_8 - K_{12}A_3A_6 + 16d_3b^2A_6A_8 - 8K_5a_3b^2A_6\\
&+ 16K_2a_3b^2A_3 - 8K_{10}a_3b^2A_8,\\
J_{16} &= -K_{12}A_1A_6 - 8K_3a_3A_6b^2 + 16K_2a_3A_1b^2 - 8K_{12}a_3A_7b^2,\\
J_{17} &= 8d_3b^2A_6^2 - K_{12}A_2A_6 - 8K_{10}a_3A_6b^2 + 16K_2a_3A_2b^2 - 8K_{12}a_3A_8b^2,\\
J_{18} &= K_1A_7^2 - K_3A_4A_8 - K_3A_5A_7 - K_4A_1A_7 + 2K_6A_6A_7 + 2K_9A_7A_8 - K_{10}A_4A_7\\
&- K_4A_4A_6 + 2K_2A_4A_5,\\
J_{19} &= K_1A_8^2 - K_3A_3A_8 - K_4A_2A_8 - K_4A_3A_6 + 2K_2A_3A_5 + 2K_7A_6A_8 - K_5A_1A_8\\
&+ 2K_{11}A_6A_7 - K_5A_2A_7 - K_5A_5A_6 - K_{10}A_3A_7 - K_{10}A_5A_8 + 2K_8A_7A_8.
\end{aligned}
$$

Hence, it is easy to verify that this system has 25 real solutions by Bezout's theorem. So, the coefficients of system (11) can be taken in such a way that this system has at most 25 real solutions different from zero for $\rho > 0$.

Let $(\bar{\rho}, \bar{\eta}, \bar{\xi})$ be a solution of system (11). In order to have a limit cycle according to the averaging theory in Section 2, we must have

$$
D(\bar{\rho}, \bar{\eta}, \bar{\xi}) = det \begin{pmatrix} \dfrac{\partial f_{31}}{\partial \rho} & \dfrac{\partial f_{31}}{\partial \eta} & \dfrac{\partial f_{31}}{\partial \xi} \\ \dfrac{\partial f_{32}}{\partial \rho} & \dfrac{\partial f_{32}}{\partial \eta} & \dfrac{\partial f_{32}}{\partial \xi} \\ \dfrac{\partial f_{33}}{\partial \rho} & \dfrac{\partial f_{33}}{\partial \eta} & \dfrac{\partial f_{33}}{\partial \xi} \end{pmatrix} \Bigg|_{(\rho,\eta,\xi)=(\bar{\rho},\bar{\eta},\bar{\xi})} \neq 0.
$$

In short, we deduce that system (1) has at most 25 limit cycles in a zero-Hopf bifurcation at the origin, using the averaging theory of third order. This completes the proof of statement (c) of Theorem 1.

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