Journal of Dynamical and Control Systems (2020) 26:45–67
https://doi.org/10.1007/s10883-019-9429-z

Asymptotic Stability for a Viscoelastic Equation with Nonlinear Damping and Very General Type of Relaxation Functions

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Received: 27 June 2018 / Revised: 20 December 2018 / Published online: 18 January 2019

Abstract
In this paper, we consider a viscoelastic equation with a nonlinear frictional damping and a relaxation function satisfying $g'(t) \leq -\xi(t)G(g(t))$. Using the Galaerkin method, we establish the existence of the solution and prove an explicit and general decay rate results, using the multiplier method and some properties of the convex functions. This work generalizes and improves earlier results in the literature. In particular, those of Messaoudi (2016) and Mustafa (Math Methods Appl Sci. 2017;V41:192–204).

Keywords Viscoelasticity · Optimal decay · Relaxation functions · Convexity

Mathematics Subject Classification (2010) 35B35 · 35L55 · 75D05 · 74D10 · 93D20

1 Introduction

In this paper, we consider the following viscoelastic problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m-2}u_t = 0, & \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & \text{in } \Omega \times (0,+\infty), \end{cases} \tag{1.1}$$

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where u denotes the transverse displacement of waves and Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega$, g is positive and decreasing function and $m > 1$.

The study of viscoelastic problems has attracted the attention of many authors and several decay and blow up results have been established. In [6], Cavalcanti et al. considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + a(x)u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

where $a : \Omega \to \mathbb{R}^+$ is a function which may vanish on a part of the domain Ω but satisfies $a(x) \geq a_0$ on $\omega \subset \Omega$ and g satisfies, for two positive constants ξ_1 and ξ_2,

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0.$$

They established an exponential decay result under some restrictions on ω. Berrimi and Messaoudi [4] established the result of [6], under weaker conditions on both a and g, to a problem where a source term is competing with the damping term. Fabrizio and Polidoro [11] studied the following system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = 0, & \text{in } \Omega \times (0,\infty) \\ u = 0, & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

and showed that the exponential decay of the relaxation function is a necessary condition for the exponential decay of the solution energy. Cavalcanti and Oquendo [7] considered the following problem

$$u_{tt} - k_0\Delta u + \int_0^t \operatorname{div}[a(x)g(t-s)\nabla u(x,s)]ds + b(x)h(u_t) + f(u) = 0 \tag{1.3}$$

and established, for $a(x) + b(x) \geq \rho > 0$, an exponential stability result for g decaying exponentially and h linear and a polynomial stability result for g decaying polynomially and h nonlinear. Rivera [26] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^n$, with zero boundary and history data and in the absence of external body forces. In the bounded domain case, an exponential decay result was proved for exponentially decaying memory kernels and for the whole space case a polynomial decay result was established and the rate of the decay was given. This latter result was later pushed to a situation where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [5]. In their paper, the authors showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function and may be improved by the regularity of the initial data. The authors considered both cases, the bounded domains and that of a material occupying the entire space. This result was later improved by Baretto et al. [3], where equations related for linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function. For partially viscoelastic materials, Rivera et al. [27, 28] showed that solutions decay exponentially to zero, provided the relaxation function decays in a similar fashion, regardless to the size of the viscoelastic part of the material.

In 2008, Messaoudi [21, 22] generalized the decay rates allowing an extended class of relaxation functions and gave general decay rates from which the exponential and the polynomial decay rates are only special cases. However, the optimality in the polynomial decay case was not obtained. Precisely, he considered relaxation functions that satisfy

$$g'(t) \leq -\xi(t)g(t), \; t \geq 0, \tag{1.4}$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function and showed that the rate of the decay of the energy is the same rate of decay of g, which is not necessarily of exponential or polynomial decay type. After that, a series of papers using Eq. 1.4 has appeared (see, for instance, [13, 19, 20, 25, 29, 30, 34, 35]).

Inspired by the experience with frictional damping initiated in the work of Lasiecka and Tataru [15], another step forward was done by considering relaxation functions satisfying

$$g'(t) \leq -\chi(g(t)). \tag{1.5}$$

This condition, where χ is a positive function, $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and strictly convex near the origin, with some additional constraints imposed on χ, was used by several authors with different approaches. We refer to previous studies [1, 8, 9, 12, 16, 17, 31] and [36], where general decay results in terms of χ were obtained. Here, it should be mentioned that, in [17], it was the first time where Lasiecka and Wang established not only general but also optimal results in which the decay rates are characterized by an ODE of the same type as the one generated by the inequality (1.5) satisfied by g. Mustafa and Messaoudi [33] established an explicit and general decay rate for relaxation function satisfying

$$g'(t) \leq -H(g(t)), \tag{1.6}$$

where $H \in C^1(\mathbb{R})$, with $H(0) = 0$ and H is linear or strictly increasing and strictly convex function C^2 near the origin. In [10], Cavalcanti et al. considered the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.7}$$

with a relaxation function satisfying (1.6) and the additional requirement:

$$\lim \inf_{x \to 0^+} x^2 H'' - xH' + H(x) \geq 0,$$

and that $y^{1-\alpha_0} \in L^1(1, \infty)$, for some $\alpha_0 \in [0, 1)$, where $y(t)$ is the solution of the problem

$$y'(t) + H(y(t)) = 0, \;\; y(0) = g(0) > 0.$$

They characterized the decay of the energy by the solution of a corresponding ODE as in [15]. Recently, Messaoudi and Al-Khulaifi [24] treated (1.7) with a relaxation function satisfying

$$g'(t) \leq -\xi(t)g^p(t), \;\; \forall t \geq 0, \;\; 1 \leq p < \frac{3}{2}. \tag{1.8}$$

They obtained a more general stability result for which the results of [21, 22] are only special cases. Moreover, the optimal decay rate for the polynomial case is achieved without any extra work and conditions as in [16] and [15]. Very recently, Mustafa [32] answered the question when he studied a viscoelastic equation with relaxation function satisfies (2.2) (below) and established an optimal decay result using the multiplier method and some properties of the convex functions. In this paper, we intend to extend the results of Messaoudi [23] and Mustafa [32] to Eq. 1.1.

This paper is organized as follows. In Section 2, we present some notations and material needed for our work. In Section 3, we establish the global existence of the solution of the problem. Some technical lemmas and the decay results are presented in Sections 4 and 5, respectively.

2 Preliminaries

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. Throughout this paper, c and ε are used to denote generic positive constants.

We consider the following hypotheses:

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = \ell > 0, \tag{2.1}$$

and there exists a C^1 function $G : (0, \infty) \to (0, \infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r]$, $r \leq g(0)$, with $G(0) = G'(0) = 0$, such that

$$g'(t) \leq -\xi(t)G(g(t)), \quad \forall t \geq 0, \tag{2.2}$$

where $\xi(t)$ is a positive nonincreasing differentiable function.

$(A2)$ For the nonlinearity in the damping, we assume that

$$1 < m \leq \frac{2n}{n-2}, \text{ if } n > 2$$

and

$$m > 1, \text{ if } n = 1, 2. \tag{2.3}$$

We introduce the "modified" energy associated to problem (1.1)

$$E(t) = \frac{1}{2}\left(||u_t||_2^2 + \left(1 - \int_0^t g(s)ds\right)||\nabla u||_2^2 + (go\nabla u)(t)\right), \tag{2.4}$$

where

$$(go\nabla u)(t) = \int_0^t g(t-s)||\nabla u(t) - \nabla u(s)||_2^2 ds.$$

Direct differentiation, using Eq. 1.1, leads to

$$E'(t) = \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \int_\Omega |u_t|^m dx \leq 0. \tag{2.5}$$

Remark 2.1 If G is a strictly increasing and strictly convex C^2 function on $(0, r]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is strictly increasing and strictly convex C^2 function on $(0, \infty)$. For instance, if $G(r) = a$, $G'(r) = b$, $G''(r) = c$, we can define $\overline{G}$, for $t > r$, by

$$\overline{G}(t) = \frac{c}{2}t^2 + (b - cr)t + \left(a + \frac{c}{2}r^2 - br\right). \tag{2.6}$$

3 Existence

In this section, we state and prove an existence result of problem (1.1).

Definition 3.1 For any pair $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. A function

$$u \in C([0, T], H_0^1(\Omega)), \; u_t \in C^1([0, T], L^2(\Omega)) \cap L^m(\Omega \times (0, \infty))$$

is called a weak solution of Eq. 1.1 if

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t(x,t)w(x)dx + \int_\Omega \nabla u(x,t).\nabla w(x)dx \\ \qquad -\int_\Omega \left(\int_0^t g(t-s)\nabla u(x,s)ds\right).\nabla w(x)dx \\ \qquad\qquad +\int_\Omega |u_t|^{m-2}u_t w(x)dx = 0, \quad \forall w \in H_0^1(\Omega), \quad \text{for a.e. } t \in [0,T], \\ u(0)=u_0,\ u_t(0)=u_1. \end{cases} \tag{3.1}$$

Proposition 3.2 *Let* $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ *be given. Assume that* $(A1)$ *and* $(A2)$ *hold. Then problem* (1.1) *has a unique weak global solution.*

Proof We use the standard Faedo-Galerkin method to prove our result. Let $\{w_j\}_{j=1}^\infty$ be the eigenfunctions of the Laplacian operator subject to Dirichlet boundary conditions. Then $\{w_j\}_{j=1}^\infty$ is orthogonal basis of $H_0^1(\Omega)$ as well as of $L^2(\Omega)$. Let $V_k = span\{w_1, w_2, ..., w_k\}$ and the projections of and initial data on the finite-dimensional subspace V_k are given by

$$u_0^k = \sum_{j=1}^k a_j w_j, \qquad u_1^k = \sum_{j=1}^k b_j w_j$$

where,

$$\begin{cases} u_0^k \to u_0 & \text{in } H_0^1(\Omega) \\ \text{and} & \\ u_1^k \to u_1 & \text{in } L^2(\Omega). \end{cases} \tag{3.2}$$

We search solutions of the form

$$u^k(x) = \sum_{j=1}^k h^{j,k}(t)w_j(x)$$

for the approximate problem in V_k

$$\begin{cases} \int_\Omega u_{tt}^k w dx + \int_\Omega \nabla u^k.\nabla w dx - \int_\Omega\int_0^t g(t-s)\nabla u^k(s).\nabla w ds dx \\ \qquad +\int_\Omega |u_t^k|^{m-2}u_t^k w dx = 0,\ \forall w \in V_k \\ u^k(0)=u_0^k,\ u_t^k(0)=u_1^k. \end{cases} \tag{3.3}$$

This leads to a system of ODE's for unknown functions $h^{j,k}$. Based on standard existence theory for ODE, the system (3.3) admits a solution u^k on a maximal time interval $[0, t_k)$, $0 < t_k < T$, for each $k \in \mathbb{N}$. In fact $t_k = T = +\infty$ and to show this, let $w = u_t^k$ in Eq. 3.3 and integrate by parts to obtain

$$\frac{d}{dt}E^k(t) = \frac{1}{2}(g' o \nabla u^k)(t) - \frac{1}{2}g(t)\|\nabla u^k(t)\|_2^2 - \int_\Omega |u_t^k(t)|^m dx \le 0, \tag{3.4}$$

where

$$E^k(t) = \frac{1}{2}||u_t^k||_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)||\nabla u^k||_2^2 + \frac{1}{2}(go\nabla u^k)(t)$$

Integrate (3.4) over $(0, t)$ to obtain

$$\frac{1}{2}\left(||u_t^k||_2^2 + \left(1-\int_0^t g(s)ds\right)||\nabla u^k||_2^2 + (go\nabla u^k)(t)\right) + \int_0^t\int_\Omega |u_t^k(s)|^m dxds$$
$$= \frac{1}{2}\left(||\nabla u_0^k||_2^2 + ||u_1^k||_2^2\right) - \frac{1}{2}\int_0^t (g' o\nabla u^k)(s)ds. \tag{3.5}$$

This means, using $(A1)$ and Eq. 3.2, that, for some positive constant C independent of t and k,

$$E^k(t) \leq E^k(0) \leq C.$$

Thus, we can extend t_k to infinity and, in addition, we have

$$\begin{cases} (u^k) \text{ is a bounded sequence in } L^\infty(0,T;H_0^1(\Omega)) \\ (u_t^k) \text{ is a bounded sequence in } L^\infty(0,T;L^2(\Omega)) \cap L^m(\Omega \times (0,T)). \end{cases}$$

Therefore, there exists a subsequence of (u^k), still denoted by (u^k), such that

$$\begin{cases} u^k \rightharpoonup^* u \quad \text{in } L^\infty(0,T;H_0^1(\Omega)) \\ u_t^k \rightharpoonup^* u_t \quad \text{in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{3.6}$$

Since (u_t^k) is bounded in $L^m(\Omega\times(0,T))$, then $(|u_t^k|^{m-2}u_t^k)$ is bounded in $L^{\frac{m}{m-1}}(\Omega\times(0,T))$. Hence, up to a subsequence,

$$|u_t^k|^{m-2}u_t^k \rightharpoonup \psi \quad \text{in } L^{\frac{m}{m-1}}(\Omega \times (0,T)). \tag{3.7}$$

Now, our task to show that $\psi = |u_t|^{m-2}u_t$. For this purpose, integrate (3.3) over $(0,t)$ to obtain

$$\begin{aligned} \int_\Omega u_t^k(t) w dx \; &- \int_\Omega u_1^k w dx + \int_0^t \int_\Omega \nabla u^k(s).\nabla w dx ds \\ &- \int_\Omega \int_0^t \left(\int_0^s g(s-\tau)\nabla u^k(\tau) d\tau \right).\nabla w ds dx \\ &+ \int_\Omega \int_0^t |u_s^k(s)|^{m-2} u_s^k(s) w ds dx = 0, \quad \forall w \in V_j, \; \forall j = 1,2,...,k. \end{aligned} \tag{3.8}$$

Convergences (3.2), Eqs. 3.6 and 3.7 allow us to pass to the limit in Eq. 3.8, as $k \to +\infty$, and get

$$\begin{aligned} \int_\Omega u_t(t) w dx \; &- \int_\Omega u_1 w dx + \int_0^t \int_\Omega \nabla u(s).\nabla w dx ds \\ &- \int_\Omega \int_0^t \left(\int_0^s g(s-\tau)\nabla u(\tau) d\tau \right) \nabla w ds dx \\ &+ \int_\Omega \int_0^t \psi(s) w ds dx = 0, \quad \forall w \in V_k, \; \forall k \geq 1 \end{aligned} \tag{3.9}$$

which implies that Eq. 3.9 is valid for any $w \in H_0^1(\Omega)$. Using the fact that the left hand side of Eq. 3.9 is an absolutely continuous function, hence it is differentiable for a.e $t \in (0,\infty)$, and we get

$$\begin{aligned} \frac{d}{dt} \int_\Omega u_t(x,t) w(x) dx &+ \int_\Omega \nabla u(x,t).\nabla w(x) dx \\ &- \int_\Omega \left(\int_0^t g(t-s)\nabla u(x,s) ds \right) \nabla w(x) dx \\ &+ \int_\Omega \psi(t) w(x) dx = 0, \quad \forall w \in H_0^1(\Omega), \text{ for a.e. } t \in [0,T]. \end{aligned} \tag{3.10}$$

Now, define

$$X^k = \int_0^T \int_\Omega \left(|u_t^k|^{m-2} u_t^k - |v|^{m-2} v \right) (u_t^k - v) dx dt \geq 0, \quad \forall v \in L^m((0,T), H_0^1(\Omega)). \tag{3.11}$$

This is true by the following elementary inequality (see Theorem 6.1, p. 222 [18]):

$$(|a|^{q-2}a - |b|^{q-2}b)(a-b) \geq 0, \quad \text{for } a, b \in \mathbb{R}, \ q \geq 1. \tag{3.12}$$

So, by using Eq. 3.5, we get

$$\begin{aligned}X^k &= \frac{1}{2}\left(||\nabla u_0^k||_2^2 + ||u_1^k||_2^2 + (go\nabla u^k)(0)\right) - \frac{1}{2}\int_0^T (g'o\nabla u^k)(s)ds \\ &-\frac{1}{2}\left(||u_t^k||_2^2 + \left(1 - \int_0^t g(s)ds\right)||\nabla u^k||_2^2 + (go\nabla u^k)(T)\right) - \int_0^T \int_\Omega |u_t^k|^{m-2}u_t^k v dxdt \\ &-\int_0^T \int_\Omega |v|^{m-2}v(u_t - v)dxdt.\end{aligned}$$

Taking $k \to +\infty$, we obtain

$$\begin{aligned}0 \leq \limsup X^k = \frac{1}{2}&\left(||\nabla u_0(t)||_2^2 + ||u_1||_2^2\right) - \frac{1}{2}\int_0^t (g'o\nabla u)(s)ds \\ &-\frac{1}{2}\left(||u_t||_2^2 + \left(1 - \int_0^t g(s)ds\right)||\nabla u||_2^2 + (go\nabla u)(t)\right) \\ &-\int_0^T \int_\Omega \psi(t) v dxdt - \int_0^T \int_\Omega |v|^{m-2}v(u_t - v)dxdt.\end{aligned} \tag{3.13}$$

Replacing w by u_t in Eq. 3.10 and integrating over $(0, T)$,we obtain

$$\begin{aligned}&-\frac{1}{2}\left(||\nabla u_0(t)||_2^2 + ||u_1||_2^2\right) - \frac{1}{2}\int_0^T (g'o\nabla u)(s)ds \\ &\quad+\frac{1}{2}\left(||u_t||_2^2 + \left(1 - \int_0^t g(s)ds\right)||\nabla u||_2^2 + (go\nabla u)(T)\right) + \int_0^T \int_\Omega \psi u_t dxdt = 0.\end{aligned} \tag{3.14}$$

Combining Eqs. 3.13 and 3.14, we arrive at

$$\begin{aligned}0 \leq \limsup X^k &= \int_0^T \int_\Omega \psi u_t dxdt - \int_0^T \int_\Omega \psi v dxdt \\ &\quad-\int_0^T \int_\Omega |v|^{m-2}v(u_t - v)dxdt \\ &\leq \int_0^T \int_\Omega (\psi - |v|^{m-2}v)(u_t - v)dxdt.\end{aligned}$$

Hence,

$$\int_0^T \int_\Omega (\psi - |v|^{m-2}v)(u_t - v)dxdt \geq 0, , \quad \forall v \in L^m(\Omega \times (0, T))$$

by density of $H_0^1(\Omega)$ in $L^m(\Omega)$.
Let $v = \lambda z + u_t$, $z \in L^m(\Omega \times (0, T))$. So, we get, $\forall \lambda \neq 0$,

$$-\lambda \int_0^T \int_\Omega \left(\psi - |\lambda z + u_t|^{m-2}(\lambda z + u_t)\right) z dxdt \leq 0, \quad z \in L^m(\Omega \times (0, T)).$$

Let $\lambda > 0$. So we have

$$\int_0^T \int_\Omega \Big(\psi - |\lambda z + u_t|^{m-2}(\lambda z + u_t)\Big) z dx dt \leq 0, \quad z \in L^m(\Omega \times (0,T)).$$

As $\lambda \to 0$, we get

$$\int_0^T \int_\Omega \Big(\psi - |u_t|^{m-2} u_t\Big) z dx dt \leq 0, \quad z \in L^m(\Omega \times (0,T)). \tag{3.15}$$

Similarly, for $\lambda < 0$, we get

$$\int_0^T \int_\Omega \Big(\psi - |u_t|^{m-2} u_t\Big) z dx dt \geq 0, \quad z \in L^m(\Omega \times (0,T)). \tag{3.16}$$

Thus, Eqs. 3.15 and 3.16 imply that $\psi = |u_t|^{m-2}u_t$. Hence Eq. 3.10 becomes

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_t(x,t) w(x) dx &+ \int_\Omega \nabla u(x,t).\nabla w(x) dx \\ &- \int_\Omega \left(\int_0^t g(t-s)\nabla u(x,s) ds\right).\nabla w(x) dx \\ &+ \int_\Omega |u_t|^{m-2} u_t w(x) dx = 0, \quad \forall w \in H_0^1(\Omega) \end{aligned}$$

To handle the initial conditions, we note that

$$\begin{aligned} u^k &\rightharpoonup u \quad \text{weakly in } L^2(0,T; H_0^1(\Omega)) \\ u_t^k &\rightharpoonup u_t \quad \text{weakly in } L^2(0,T; L^2(\Omega)) \end{aligned} \tag{3.17}$$

Thus, using Lion's Lemma [18] and Eq. 3.2, we easily obtain

$$u(x,0) = u_0(x).$$

As in [14], multiply (3.3) by $\phi \in C_0^\infty(0,T)$ and integrate over $(0,T)$, we obtain for any $w \in V_k$

$$\begin{aligned} -\int_0^T \int_\Omega u_t^k w \phi'(t) dx dt &= -\int_0^T \int_\Omega \nabla u^k.\nabla w \phi dx dt \\ &+ \int_0^T \int_\Omega \int_0^{+\infty} g(s) \nabla u^k(t-s).\nabla w \phi ds dx dt - \int_0^T \int_\Omega |u_t^k|^{m-2} u_t^k w \phi dx dt \end{aligned} \tag{3.18}$$

As $k \to +\infty$, we have for any $w \in H_0^1(\Omega)$ and any $\phi \in C_0^\infty((0,T))$,

$$\begin{aligned} -\int_0^T \int_\Omega u_t w \phi'(t) dx dt &= -\int_0^T \int_\Omega \nabla u.\nabla w \phi dx dt \\ &+ \int_0^T \int_\Omega \int_0^{+\infty} g(s) \nabla u(t-s).\nabla w \phi ds dx dt - \int_0^T \int_\Omega |u_t|^{m-2} u_t w \phi dx dt \end{aligned} \tag{3.19}$$

This means (see [14]),

$$u_{tt} \in L^2([0,T), H^{-1}(\Omega)).$$

Recalling that $u_t \in L^2((0,T), L^2(\Omega))$, we obtain

$$u_t \in C([0,T), H^{-1}(\Omega)).$$

So, $u_t^k(x,0)$ makes sense and

$$u_t^k(x,0) \to u_t(x,0) \text{ in } H^{-1}(\Omega)$$

But

$$u_t^k(x,0) = u_1^k(x) \rightarrow u_1(x) \text{ in} L^2(\Omega)$$

Hence

$$u_t(x,0) = u_1(x)$$

For uniqueness, let us assume that problem (1.1) has two solutions u and v. Then, $w = u - v$ satisfies

$$\begin{cases} w_{tt} - \Delta w + \int_0^t g(t-s)\Delta w(s)ds + \left(|u_t|^{m-2}u_t - |v_t|^{m-2}v_t\right) = 0, & \text{in } \Omega \times (0,T) \\ w = 0, & \text{on } \partial\Omega \times (0,T) \\ w(x,0) = 0, \quad w_t(x,0) = 0, & \text{in } \Omega \times (0,T). \end{cases} \tag{3.20}$$

Now, multiply (3.20) by w_t and integrate over $\Omega \times (0,t)$ to obtain

$$||w_t||_2^2 + ||\nabla w||_2^2 + (go\nabla w)(t) - \int_0^t (g'o\nabla w)(s)ds$$
$$\int_0^t g(s)\|\nabla w(s)\|_2^2 ds + 2\int_0^t \int_\Omega \left(|u_t|^{m-2}u_t - |v_t|^{m-2}v_t\right)(u_t - v_t)dxds = 0.$$

Hence, by using inequality (3.12), we have

$$||w_t||_2^2 + ||\nabla w||_2^2 \le 0$$

which implies that $w = C$. In fact, $C = 0$ since $w = 0$ on $\partial\Omega$. Which completes the proof. □

4 Technical Lemmas

In this section, we establish several lemmas needed for the proof of our main result. We adopt some results from [23] and [32] without proof.

Lemma 4.1 *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds\right)^2 dx \le C_\alpha(ho\nabla u)(t) \tag{4.1}$$

where, for any $0 < \alpha < 1$,

$$C_\alpha = \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)}ds \quad \textit{and} \quad h(t) = \alpha g(t) - g'(t). \tag{4.2}$$

Proof The Use of Eq. 4.2 and the Cauchy Schwarz inequality gives

$$\int_\Omega \left(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds\right)^2 dx$$
$$\le \int_\Omega (\int_0^t \frac{g(t-s)}{\sqrt{\alpha g(t-s) - g'(t-s)}}\sqrt{\alpha g(t-s) - g'(t-s)}|\nabla u(s) - \nabla u(t)|ds)^2 dx$$
$$\le \left(\int_0^t \frac{g^2(s)}{\alpha g(s) - g'(s)}ds\right)\int_0^t \left[\alpha g(t-s) - g'(t-s)\right]||\nabla u(s) - \nabla u(t)||_2^2 ds$$
$$\le C_\alpha(h \circ \nabla u)(t). \tag{4.3}$$

□

Lemma 4.2 [23, 32] *Under the assumptions* $(A1)$ *and* $(A2)$, *the functional*

$$\psi_1(t) := \int_\Omega u u_t dx$$

satisfies, along the solution, the estimate

$$\begin{aligned}\psi_1'(t) \le -\tfrac{\ell}{2}\|\nabla u\|_2^2 + \|u_t\|_2^2 + \tfrac{C_\alpha}{2\ell}(ho\nabla u)(t)\\ +c(\delta)\textstyle\int_\Omega |u_t|^m dx, \qquad \text{if } m \ge 2\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\psi_1'(t) \le -\frac{\ell}{2}\|\nabla u\|_2^2 + \|u_t\|_2^2 + c\frac{C_\alpha}{2\ell}(ho\nabla u)(t)\\ +c(\delta,\Omega)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \qquad \text{if } m < 2.\end{aligned} \tag{4.5}$$

Lemma 4.3 [23, 32] *Under the assumptions* $(A1)$ *and* $(A2)$, *the functional*

$$\psi_2(t) := -\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, along the solution, the estimate

$$\begin{aligned}\psi_2'(t) \le c\delta\|\nabla u\|_2^2 - \left(\int_0^t g(s)ds - \delta\right)\|u_t\|_2^2 + \left((\frac{3c}{\delta}+1)C_\alpha + \frac{c}{\delta}\right)(ho\nabla u)(t)\\ +C(\delta)\int_\Omega |u_t|^m dx, \qquad \text{if } m \ge 2\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\psi_2'(t) \le c\delta\|\nabla u\|_2^2 - \left(\int_0^t g(s)ds - \delta\right)\|u_t\|_2^2 + + \left((\frac{3c}{\delta}+1)C_\alpha + \frac{c}{\delta}\right)(ho\nabla u)(t)\\ +c(\delta,\Omega)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \quad \text{if } m < 2\end{aligned} \tag{4.7}$$

Lemma 4.4 [32] *Under the assumptions* $(A1)$ *and* $(A2)$, *the functional*

$$\psi_3(t) = \int_\Omega \int_0^t r(t-s)|\nabla u(s)|^2 dsdx, \tag{4.8}$$

satisfies, along the solution of Eq. 1.1, *the estimate*

$$\psi_3'(t) \le -\frac{1}{2}(go\nabla u)(t) + 3(1-\ell)\int_\Omega |\nabla u(t)|^2 dx. \tag{4.9}$$

where $r(t) = \int_t^{+\infty} g(s)ds$.

Proof By Young's inequality and the fact that $r'(t) = -g(t)$,we see that

$$\begin{aligned}\psi_3'(t) &= r(0)\int_\Omega |\nabla u(t)|^2 dx - \int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2 dx\\ &= -\int_\Omega\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|^2 dsdx\\ &\quad -2\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))dsdx + r(t)\int_\Omega |\nabla u(t)|^2 dx.\end{aligned}$$

Now,

$$-2\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))dsdx$$
$$\leq 2(1-\ell)\int_\Omega |\nabla u(t)|^2dx+\frac{\int_0^t g(s)ds}{2(1-\ell)}\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2dsdx.$$

Using the facts that $r(t)\leq r(0)=1-\ell$ and $\int_0^t g(s)ds\leq 1-\ell$, Eq. 4.9 is established. □

Lemma 4.5 [32] *There exist positive constants d and t_1 such that*

$$g'(t)\leq -dg(t),\quad \forall t\in[0,t_1]. \tag{4.10}$$

Proof By $(A1)$,we easily deduce that $\lim_{t\to+\infty} g(t)=0$. Hence, there is $t_1>0$ large enough such that

$$g(t_1)=r$$

and

$$g(t)\leq r,\qquad \forall t\geq t_1.$$

As g and ξ are positive nonincreasing continuous and G is a positive continuous function, then, for all $t\in[0,t_1]$,

$$\begin{cases} 0<g(t_1)\leq g(t)\leq g(0)\\ 0<\xi(t_1)\leq \xi(t)\leq \xi(0),\end{cases}$$

which implies that there are two positive constants a and b such that

$$a\leq \xi(t)G(g(t))\leq b.$$

Consequently, for all $t\in[0,t_1]$,

$$g'(t)\leq -\xi(t)G(g(t))\leq -\frac{a}{g(0)}g(0)\leq -\frac{a}{g(0)}g(t). \tag{4.11}$$

□

Remark 4.6 Using the fact that $\frac{\alpha g^2(s)}{\alpha g(s)-g'(s)}<g(s)$ and recalling the Lebesgue dominated convergence theorem, we can easily deduce that

$$\alpha C_\alpha=\int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s)-g'(s)}ds\to 0 \text{ as } \alpha\to 0. \tag{4.12}$$

Lemma 4.7 *Assume that $(A1)$ and $(A2)$. Then there exist strictly positive constants $N,\varepsilon_1,\varepsilon_2,\lambda,c$ such that the functional*

$$L=NE(t)+N_1\psi_1(t)+N_2\psi_2(t)$$

satisfies, for all $t\geq t_1$,

$$L\sim E, \tag{4.13}$$

$$L'(t)\leq -\lambda_0E(t)+\frac{1}{4}(go\nabla u)(t),\qquad \text{if } m\geq 2 \tag{4.14}$$

and

$$L'(t)\leq -\lambda_0E(t)+c(go\nabla u)(t)+c\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}},\quad \text{if } m<2. \tag{4.15}$$

Proof For the proof of Eq. 4.13, we refer the reader to [22]. Now, we prove inequality (4.14). Let $g_1 := \int_0^{t_1} g(s)ds > 0$. By using Eqs. 2.5, 4.4 and 4.6, recalling that $g' = (\alpha g - h)$ and taking $\delta = \frac{\ell}{4N_2}$, we easily see that, for all $t \geq t_1$,

$$L'(t) \leq -\left(\frac{\ell}{2}N_1 - \frac{\ell}{4}\right) || \nabla u||_2^2 - \left(N_2 g_1 - \frac{\ell}{4} - N_1\right) ||u_t||_2^2 + \frac{\alpha}{2} N (g \circ \nabla u)(t)$$
$$- \left(\frac{1}{2}N - \frac{4c}{\ell}N_2^2 - C_\alpha \left(\frac{c}{2\ell}N_1 + \frac{12c}{\ell}N_2^2 + N_2\right)\right) (h \circ \nabla u)(t). \tag{4.16}$$

At this point, we choose N_1 large enough so that

$$\frac{\ell}{2}N_1 - \frac{\ell}{4} > 4(1-\ell)$$

and then N_2 large enough so that

$$N_2 g_1 - \frac{\ell}{4} - N_1 - 1 > 0.$$

Now, using Remark 4.6, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8\left(\frac{cN_1}{2\ell} + \frac{12cN_2^2}{\ell} + N_2\right)}. \tag{4.17}$$

Next, we choose N large enough so that

$$\frac{1}{4}N - \frac{4c}{N_2^2} > 0 \text{ and } \alpha = \frac{1}{2N} < \alpha_0,$$

which gives

$$\frac{1}{2}N - \frac{4c}{\ell}N_2^2 - C_\alpha \left(\frac{c}{2\ell}N_1 + \frac{12c}{\ell}N_2^2 + N_2\right) > 0.$$

Therefore, we arrive at

$$L'(t) \leq -4(1-\ell)|| \nabla u||_2^2 - ||u_t||_2^2 + \frac{1}{4}(g \circ \nabla u)(t). \tag{4.18}$$

Combining Eqs. 2.4 and 4.18, Eq. 4.14 is established. The same calculations hold, for $m < 2$, using Eqs. 2.5, 4.5 and 4.7, give Eq. 4.15. □

Corollary 4.8 *There exists an equivalent functional $L_1 \sim E$ such that,*

$$L_1'(t) \leq -\lambda E(t) + c \int_{t_1}^{t} g(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 dxds, \qquad \text{if } m \geq 2 \tag{4.19}$$

and

$$L_1'(t) \leq -\lambda E(t) + c \int_{t_1}^{t} g(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 dxds + c\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \quad \text{if } 1 < m < 2, \tag{4.20}$$

for some positive constants λ and c.

Proof Using Eqs. 2.5 and 4.10 we conclude that, for any $t \geq t_1$,

$$\int_0^{t_1} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{-1}{d} \int_0^{t_1} g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds$$
$$\leq -cE'(t) \tag{4.21}$$

By letting $L_1(t) = L(t) + cE(t)$ and combining Eqs. 4.14 and 4.21, Eq. 4.19 is established. Similar calculations hold, for $m < 2$, to obtain (4.20). □

5 Stability

In this section we state and prove our main result. We start with the following lemmas.

Lemma 5.1 *Assume that* $(A1)$ *and* $(A2)$ *hold and* $m \geq 2$. *Then, the energy functional satisfies the following estimate*

$$\int_0^{+\infty} E(s)ds < \infty \tag{5.1}$$

Proof Let $F(t) = L(t) + \psi_3(t)$, then using Eq. 4.9, we obtain

$$F'(t) \leq -(1-\ell)\int_\Omega |\nabla u|^2 dx - \int_\Omega u_t^2 dx - \frac{1}{4}(go\nabla u)(t) \tag{5.2}$$

Using Eqs. 2.5 and 5.2, we obtain

$$\begin{aligned} F'(t) &\leq -bE(t) \\ &\leq -bE(t) - cE'(t), \end{aligned}$$

where b is a positive constant. Therefore,

$$b\int_{t_1}^{t} E(s)ds \leq F_1(t_1) - F_1(t) \leq F_1(t_1) < \infty, \tag{5.3}$$

where $F_1(t) = F(t) + cE(t) \sim E$. □

Lemma 5.2 *Assume that* $(A1)$ *and* $(A2)$ *hold and* $1 < m < 2$. *Then, the energy functional satisfies the following estimate*

$$\int_0^{+\infty} E^{\frac{m}{2m-2}}(s)ds < \infty. \tag{5.4}$$

Proof Let $F(t) = L(t) + \psi_3(t)$, then using (4.9) and (4.15), we obtain

$$\begin{aligned} F'(t) &\leq -(1-\ell)\int_\Omega |\nabla u|^2 dx - \int_\Omega u_t^2 dx - \frac{1}{4}(go\nabla u)(t) + c\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}} \\ &\leq -cE(t) + c\left(-E'(t)\right)^{\frac{2m-2}{m}} \end{aligned} \tag{5.5}$$

By multiplying Eq. 5.5 by $E^q(t)$, $q > 0$,and using Young's inequality, we get

$$\begin{aligned} E^q(t)F'(t) &\leq -cE^{q+1}(t) + E^q(t)\left(-cE'(t)\right)^{\frac{2m-2}{m}} \\ &\leq -cE^{q+1}(t) + \varepsilon E^{\frac{qm}{2-m}}(t) + C(\varepsilon)\left(-E'(t)\right) \end{aligned} \tag{5.6}$$

By choosing $q = \frac{2-m}{2m-2}$ and taking ε small, Eq. 5.6 yields

$$E^q(t)F'(t) \leq -cE^{q+1}(t) + C\left(-E'(t)\right) \tag{5.7}$$

Let $F_2(t) = E^q(t)F(t) + CE(t)$ then Eqs. 2.5, 4.13 and 5.7, lead to

$$E^{q+1}(t) \leq -cF_2'(t). \tag{5.8}$$

Therefore,

$$c\int_{t_1}^{t} E^{q+1}(s)ds \leq F_2(t_1) - F_2(t) \leq F_2(t_1) < \infty, \quad \forall t > t_1, \tag{5.9}$$

which gives Eq. 5.4 since $1 + q = \frac{m}{2m-2}$. □

Remark 5.3 Using Hölder's inequality and Eq. 5.4, we obtain, for $1 < m < 2$,

$$\begin{aligned}\int_{t_1}^{t} E(s)ds &\leq (t-t_1)^{\frac{q}{1+q}} \left(\int_{t_1}^{t} E^{q+1}(s)ds\right)^{\frac{1}{1+q}} \\ &\leq c\,(t-t_1)^{\frac{q}{1+q}} = c\,(t-t_1)^{\frac{2-m}{m}}, \quad \forall t > t_1. \end{aligned} \tag{5.10}$$

Let's define

$$I(t) := -\int_{t_1}^{t} g'(s)\int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq -cE'(t), \tag{5.11}$$

Lemma 5.4 *Under the assumptions* $(A1)$ *and* $(A2)$ *, we have the following estimates*

$$\int_{t_1}^{t} g(s)\int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{1}{p}\overline{G}^{-1}\left(\frac{pI(t)}{\xi(t)}\right), \qquad m \geq 2 \tag{5.12}$$

$$\int_{t_1}^{t} g(s)\int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{(t-t_1)^{\frac{2m-2}{m}}}{p}\overline{G}^{-1}\left(\frac{pI(t)}{(t-t_1)^{\frac{2m-2}{m}}\xi(t)}\right), \quad 1 < m < 2. \tag{5.13}$$

where $p \in (0,1)$ *and* $\overline{G}$ *is an extension of* G *such that* $\overline{G}$ *is strictly increasing and strictly convex* C^2 *function on* $(0,\infty)$*; see Remark 2.1.*

Proof First, we define the following quantity

$$\lambda(t) := p\int_{t_1}^{t}\int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds.$$

Using Eqs. 2.4 and 5.1, we obtain

$$\begin{aligned}\lambda(t) &\leq C\int_{t_1}^{t}\left(\|\nabla u(t-s)\|_2^2 + \|\nabla u(t)\|_2^2\right)ds \\ &\leq C\int_{0}^{t}\left(\|\nabla u(t-s)\|_2^2 + \|\nabla u(t)\|_2^2\right)ds \\ &\leq C\int_{0}^{t}[E(t-s) + E(t)]\,ds \\ &\leq 2C\int_{0}^{t} E(t-s)ds \\ &\leq 2C\int_{0}^{t} E(\tau)ds < 2C\int_{0}^{\infty} E(\tau)ds < \infty. \end{aligned} \tag{5.14}$$

Also, we can choose p so small that, for all $t > t_1$,

$$\lambda(t) < 1. \tag{5.15}$$

Since G is strictly convex on $(0, r]$ and $G(0) = 0$,then

$$G(\theta z) \leq \theta G(z),\ 0 \leq \theta \leq 1 \text{ and } z \in (0, r]. \tag{5.16}$$

The use of Eqs. 2.2, 5.15, 5.16 and Jensen's inequality yields

$$\begin{aligned} I(t) &= \frac{1}{p\lambda(t)} \int_{t_1}^{t} \lambda(t)(-g'(s)) \int_\Omega p|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{1}{p\lambda(t)} \int_{t_1}^{t} \lambda(t)\xi(s)G(g(s)) \int_\Omega p|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{\xi(t)}{p\lambda(t)} \int_{t_1}^{t} \overline{G}(\lambda(t)g(s)) \int_\Omega p|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{\xi(t)}{p}\overline{G}(p \int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds) \\ &= \frac{\xi(t)}{p}\overline{G}(p \int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds) \end{aligned} \tag{5.17}$$

This gives Eq. 5.12 when $m \geq 2$. In the case $1 < m < 2$ and for the proof of Eq. 5.13, we define the following

$$\lambda_1(t) := \frac{p}{(t-t_1)^{\frac{2m-2}{m}}} \int_{t_1}^{t} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds,$$

then using Eqs. 2.4 and 5.10, we easily see that

$$\lambda_1(t) \leq c,$$

then choosing $p \in (0, 1)$ small enough so that Eq. 5.15 holds and

$$\lambda_1(t) < 1,\ \text{for all } t > t_1, \tag{5.18}$$

The use of Eqs. 2.2, 5.16, 5.18 and Jensen's inequality leads to

$$\begin{aligned} I(t) &= \frac{1}{p\lambda_1(t)} \int_{t_1}^{t} \lambda_1(t)(-g'(s)) \int_\Omega p|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{1}{p\lambda_1(t)} \int_{t_1}^{t} \lambda_1(t)\xi(s)G(g(s)) \int_\Omega p|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{\xi(t)}{p\lambda_1(t)} \int_{t_1}^{t} \overline{G}(\lambda_1(t)g(s)) \int_\Omega p|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{(t-t_1)^{\frac{2m-2}{m}}\xi(t)}{p}\overline{G}(\frac{p}{(t-t_1)^{\frac{2m-2}{m}}} \int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds) \\ &= \frac{(t-t_1)^{\frac{2m-2}{m}}\xi(t)}{p}\overline{G}\left(\frac{p}{(t-t_1)^{\frac{2m-2}{m}}} \int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds\right). \end{aligned} \tag{5.19}$$

This implies that

$$\int_{t_1}^{t} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dx ds \leq \frac{(t-t_1)^{\frac{2m-2}{m}}}{p} \overline{G}^{-1} \left(\frac{pI(t)}{(t-t_1)^{\frac{2m-2}{m}} \xi(t)} \right).$$

□

Theorem 5.5 *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* $(A1)$ *and* $(A2)$ *hold and* $m \geq 2$*. Then there exist strictly positive constants* c_1*,* c_2*,* k_1 *and* k_2 *such that the solution of* Eq. 1.1 *satisfies, for all* $t \geq t_1$*,*

$$E(t) \leq c_1 e^{-c_2 \int_{t_1}^{t} \xi(s) ds}, \text{ if } G \text{ is linear} \tag{5.20}$$

$$E(t) \leq k_2 G_1^{-1} \left(k_1 \int_{t_1}^{t} \xi(s) ds \right), \text{ if } G \text{ is nonlinear,} \tag{5.21}$$

where $G_1(t) = \int_t^{r_1} \frac{1}{sG'(s)} ds$*.*

Proof **Case 1**: G is linear.

Using Eqs. 2.2 and 2.5, we get

$$\begin{aligned} \xi(t) \int_{t_1}^{t} g(t-s) \int_{\Omega} |\nabla u(t) - \nabla u(s)|^2 dx ds &\leq \int_{t_1}^{t} \xi(s) g(s) \int_{\Omega} |\nabla u(t) - \nabla u(s)|^2 dx ds \\ &\leq - \int_{t_1}^{t} g'(s) \int_{\Omega} |\nabla u(t) - \nabla u(s)|^2 dx ds \\ &\leq -2E'(t) \end{aligned} \tag{5.22}$$

Multiplying (4.19) by $\xi(t)$ and using Eq. 5.22, we obtain

$$\begin{aligned} \xi(t) L_1'(t) &\leq -\lambda \xi(t) E(t) + c \xi(t) (go\nabla u)(t) \\ &\leq -\lambda \xi(t) E(t) - 2cE'(t) \end{aligned}$$

which gives, as $\xi(t)$ is non-increasing,

$$(\xi L_1 + 2cE)' \leq -\lambda \xi(t) E(t), \qquad \forall t \geq t_1. \tag{5.23}$$

Hence, using the fact that $\xi L + 2cE \sim E$, we easily obtain

$$E(t) \leq c_1 e^{-c_2 \int_{t_1}^{t} \xi(s) ds}. \tag{5.24}$$

Case 2: G is non-linear.

Using (4.19) and (5.12), we obtain

$$L_1'(t) \leq -\lambda E(t) + c \left(\overline{G}\right)^{-1} \left(\frac{pI(t)}{\xi(t)} \right). \tag{5.25}$$

Then, the functional $\mathcal{F}_1$, defined by

$$\mathcal{F}_1(t) := \overline{G}' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) L_1(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$.

$$\alpha_1 \mathcal{F}_1(t) \leq E(t) \leq \alpha_2 \mathcal{F}_1(t) \tag{5.26}$$

and

$$\begin{aligned}\mathcal{F}_1'(t) &= \varepsilon_0 \tfrac{E'(t)}{E(0)} \overline{G}'' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) L_1(t) + \overline{G}' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) L_1{}'(t) \\ &\leq -\lambda E(t) \overline{G}' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) + c\overline{G}' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) \overline{G}^{-1} \left(\tfrac{pI(t)}{\xi(t)}\right).\end{aligned} \tag{5.27}$$

Let $\overline{G}^*$ be the convex conjugate of $\overline{G}$ in the sense of Young [2], then

$$\overline{G}^*(s) = s(\overline{G}')^{-1}(s) - \overline{G}\left[(\overline{G}')^{-1}(s)\right], \quad \text{if } s \in (0, \overline{G}'(r)] \tag{5.28}$$

and $\overline{G}^*$ satisfies the following generalized Young inequality

$$AB \leq \overline{G}^*(A) + \overline{G}(B), \quad \text{if } A \in (0, \overline{G}'(r)], \ B \in (0, r]. \tag{5.29}$$

So, with $A = \overline{G}' \left(\varepsilon_0 \frac{E'(t)}{E(0)}\right)$ and $B = \overline{G}^{-1} \left(\frac{pI(t)}{\xi(t)}\right)$ and using Eqs. 2.5 and 5.27–5.29, we arrive at

$$\begin{aligned}\mathcal{F}_1'(t) &\leq -\lambda E(t) \overline{G}' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) + c\overline{G}^* \left(\overline{G}' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right)\right) + c\left(\tfrac{pI(t)}{\xi(t)}\right) \\ &\leq -\lambda E(t) \overline{G}' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) + c\varepsilon_0 \tfrac{E(t)}{E(0)} \overline{G}' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) + c\left(\tfrac{pI(t)}{\xi(t)}\right).\end{aligned} \tag{5.30}$$

So, multiplying (5.30) by $\xi(t)$ and using (5.11) and the fact that $\varepsilon_0 \frac{E(t)}{E(0)} < r, \overline{G}' \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = G' \left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$, we get

$$\begin{aligned}\xi(t)\mathcal{F}_1'(t) &\leq -\lambda \xi(t) E(t) G' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) + c\xi(t)\varepsilon_0 \tfrac{E(t)}{E(0)} G' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) + cpI(t) \\ &\leq -\lambda \xi(t) E(t) G' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) + c\xi(t)\varepsilon_0 \tfrac{E(t)}{E(0)} G' \left(\varepsilon_0 \tfrac{E(t)}{E(0)}\right) - cE'(t)\end{aligned}$$

Consequently, with a suitable choice of ε_0, we obtain, for all $t \geq t_1$,

$$\mathcal{F}_2'(t) \leq -k\xi(t) \left(\frac{E(t)}{E(0)}\right) G' \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -k\xi(t) G_2 \left(\frac{E(t)}{E(0)}\right), \tag{5.31}$$

where $\mathcal{F}_2 = \xi \mathcal{F}_1 + cE \sim E$ and $G_2(t) = tG'(\varepsilon_0 t)$. Since $G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 t G''(\varepsilon_0 t)$,then, using the strict convexity of G on $(0, r]$,we find that $G_2'(t), G_2(t) > 0$ on $(0, 1]$. Thus, taking in account (5.26) and (5.31), we easily see that

$$R(t) = \varepsilon \frac{\alpha_1 \mathcal{F}_2(t)}{E(0)}, \quad 0 < \varepsilon < 1,$$

satisfies

$$R(t) \sim E(t) \tag{5.32}$$

and, for some $k_1 > 0$,we have

$$R'(t) \leq -k_1 \xi(t) G_2(R(t)), \quad \forall t \geq t_1.$$

Then, the integration over (t_1, t) yields

$$\int_{t_1}^t \frac{-R'(s)}{G_2(R(s))} ds \geq k_1 \int_{t_1}^t \xi(s) ds.$$

Hence, by an approprite change of variable, we get

$$\int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_1)} \frac{1}{\tau G'(\tau)} d\tau \geq k_1 \int_{t_1}^t \xi(s) ds$$

Thus, we have

$$R(t) \leq \frac{1}{\varepsilon_0} G_1^{-1} \left(k_1 \int_{t_1}^t \xi(s) ds\right), \tag{5.33}$$

where $G_1(t) = \int_t^{r_1} \frac{1}{sG'(s)} ds$. Here, we have used the fact that G_1 is strictly decreasing on $(0, r]$. Therefore (5.21) is established by virtue of Eq. 5.32. □

Remark 5.6 The decay rate of $E(t)$ given by Eq. 5.21 is optimal because it is consistent with the decay rate of $g(t)$ given by Eq. 2.2. In fact,

$$g(t) \leq G_0^{-1}\left(\int_{g^{-1}(r)}^{t} \xi(s)ds\right), \quad \forall t \geq g^{-1}(r),$$

where $G_0(t) = \int_t^r \frac{1}{G(s)}$.

Using the properties of G, G_0 and G_1, we can see that

$$G_1(t) = \int_t^r \frac{1}{sG'(s)}ds \leq \int_t^r \frac{1}{G(s)}ds = G_0(t).$$

Using the fact that G_1 is decreasing, we have

$$G_1^{-1}(G_1(t)) \geq G_1^{-1}(G_0(t)).$$

By putting $\tau = G_0(t)$, we obtain

$$t = G_0^{-1}(\tau) = G_1^{-1}(G_1(t)) \geq G_1^{-1}(\tau).$$

Therefore,

$$G_1^{-1}(\tau) \leq G_0^{-1}(\tau).$$

This shows that Eq. 5.21 provides the best decay rates expected under the very general assumption (2.2).

Theorem 5.7 *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* $(A1)$ *and* $(A2)$ *hold and* $1 < m < 2$*. Then, there exist postive constants* c, m_5, m_6 *such that*

$$E(t) \leq c\left(1 + \int_{t_1}^{t} \xi(s)ds\right)^{-\frac{2m-2}{2-m}}, \qquad \text{if } G \text{ is linear} \tag{5.34}$$

$$E(t) \leq m_6(t-t_1)^{\frac{2m-2}{m}} W_2^{-1}\left(\frac{m_5}{(t-t_1)^{\frac{2-m}{2m-2}} \int_{t_1}^{t} \xi(s)ds}\right), \text{ if } G \text{ is nonlinear,} \tag{5.35}$$

where $W_2(\tau) = \tau^{\frac{m}{2m-2}} G'(\varepsilon_1 \tau)$.

Proof **Case 1: G is linear**

Multiplying (4.20) by $\xi(t)$ and using Eq. 5.22, we obtain

$$\begin{aligned}\xi(t)L'(t) &\leq -\lambda\xi(t)E(t) + c\xi(t)(go\nabla u)(t) + c\xi(t)\big(\textstyle\int_\Omega |u_t|^m dx\big)^{\frac{2m-2}{m}} \\ &\leq -\lambda\xi(t)E(t) - cE'(t) + c\xi(t)\big(\textstyle\int_\Omega |u_t|^m dx\big)^{\frac{2m-2}{m}}\end{aligned}$$

which gives, as $\xi(t)$ is non-increasing,

$$\mathcal{L}'(t) \leq -\lambda\xi(t)E(t) + c\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \qquad \forall t \geq t_1, \tag{5.36}$$

where $\mathcal{L}(t) = \xi(t)L(t) + cE(t) \sim E$. By multiplying the last inequality by $E^q(t)$, $q > 0$, recalling (2.5), and using Young's inequality, we get

$$\begin{aligned}E^q(t)\mathcal{L}'(t) &\leq -\lambda\xi(t)E^{1+q}(t) + c\xi(t)E^q(t)\big(-E'(t)\big)^{\frac{2m-2}{m}}, \\ &\leq -\lambda\xi(t)E^{1+q}(t) + c\varepsilon\xi(t)E^{\frac{qm}{2-m}}(t) + c(\varepsilon)\xi(t)\big(-E'(t)\big)\end{aligned} \tag{5.37}$$

By choosing $q = \frac{2-m}{2m-2}$, Eq. 5.37 yields

$$\begin{aligned} E^q(t)\mathcal{L}'(t) &\le -\lambda\xi(t)E^{1+q}(t) + c\varepsilon\xi(t)E^{1+q}(t) + c(\varepsilon)\left(-E'(t)\right) \\ &\le -(\lambda - c\varepsilon)\xi(t)E^{1+q}(t) + c(\varepsilon)\left(-E'(t)\right). \end{aligned} \tag{5.38}$$

Let $\mathcal{L}_1(t) = E^q(t)\mathcal{L}(t) + c(\varepsilon)E$,then using Eqs. 2.5, 5.38, the fact that $\mathcal{L}_1 \sim E$, and choosing ε small enough, we get

$$\mathcal{L}_1(t)' \le -c\xi(t)\mathcal{L}_1^{1+q}(t) \tag{5.39}$$

The last inequality together with the equivalence relation ($\mathcal{L}_1 \sim E$) give (5.34).

Case 2: G is nonlinear

Using Eqs. 4.19 and 5.13, we obtain

$$L'(t) \le -\lambda E(t) + +c(t-t_1)^{\frac{2m-2}{m}}\left(\overline{G}\right)^{-1}\left(\frac{pI(t)}{(t-t_1)^{\frac{2m-2}{m}}\xi(t)}\right) + c\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \tag{5.40}$$

we find that the functional L_1,defined by

$$L_1(t) := G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)L(t)$$

satisfies, for some $\beta_1, \beta_2 > 0$.

$$\beta_1 L_1(t) \le E(t) \le \beta_2 L_1(t) \tag{5.41}$$

and

$$\begin{aligned} L_1'(t) =& \left(\frac{-(2-m)\varepsilon_1}{m(t-t_1)^{\frac{m+2}{m}}}\frac{E(t)}{E(0)} + \frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\frac{E'(t)}{E(0)}\right)G''\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)L(t) \\ &+G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)L'(t) \\ \le& -\lambda E(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right) \\ &+cG'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)(t-t_1)^{\frac{2m-2}{m}}\left(\overline{G}\right)^{-1}\left(\frac{pI(t)}{(t-t_1)^{\frac{2m-2}{m}}\xi(t)}\right) \\ &+cG'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}. \end{aligned} \tag{5.42}$$

So, with $A = G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)$ and $B = \left(\overline{G}\right)^{-1}\left(\frac{pI(t)}{(t-t_1)^{\frac{2m-2}{m}}\xi(t)}\right)$ and using Eqs. 2.5, 5.28, 5.29 and 5.42 yields

$$\begin{aligned} L_1'(t) \le& -\lambda E(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right) + c(t-t_1)^{\frac{2m-2}{m}}\overline{G}^*\left(G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)\right) \\ &+(t-t_1)^{\frac{2m-2}{m}}\frac{pI(t)}{(t-t_1)^{\frac{2m-2}{m}}\xi(t)} + cG'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}} \\ \le& -\lambda E(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right) + c(t-t_1)^{\frac{2m-2}{m}}\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right) \\ &+\frac{pI(t)}{\xi(t)} + cG'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}. \end{aligned} \tag{5.43}$$

By multiplying the last inequality by $\xi(t)E^{\frac{2-m}{2m-2}}(t)$, using Eqs. 2.5, 5.4, 5.11 and Young's inequality, we get

$$\begin{aligned}
\xi(t)E^{\frac{2-m}{2m-2}}(t)L_1'(t) \le& -\lambda\xi(t)E^{\frac{m}{2m-2}}(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)\\
&+cpI(t)E^{\frac{2-m}{2m-2}}(t)+c\varepsilon_1\xi(t)\frac{E^{\frac{m}{2m-2}}(t)}{E(0)}G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)\\
&+c\xi(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)E^{\frac{2-m}{2m-2}}(t)\big(-E'(t)\big)^{\frac{2m-2}{m}}\\
\le& -\lambda\xi(t)E^{\frac{m}{2m-2}}(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)+c\varepsilon_1\xi(t)\frac{E^{\frac{m}{2m-2}}(t)}{E(0)}G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)\\
&-cE'(t)+c\varepsilon\xi(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)E^{\frac{m}{2m-2}}(t)\\
&+c(\varepsilon)\xi(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)(-E'(t)).
\end{aligned} \tag{5.44}$$

The last inequality becomes

$$\begin{aligned}
\xi(t)E^{\frac{2-m}{2m-2}}(t)L_1'(t) \le& -(\lambda-c\varepsilon-c\varepsilon_1)\xi(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)E^{\frac{m}{2m-2}}(t)\\
&+(c(\varepsilon)+c)(-E'(t)).
\end{aligned} \tag{5.45}$$

Let $L_2(t)=\xi(t)E^{\frac{2-m}{2m-2}}(t)L_1(t)+(c(\varepsilon)+c)E$, then using Eqs. 2.5, 4.13, 5.45 and choosing ε_1 and ε small enough, we get

$$L_2'(t)\le -m_1\xi(t)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)E^{\frac{m}{2m-2}}(t), \tag{5.46}$$

for some $m_1>0$. Then, we have, for $m_2=m_1E(0)$,

$$m_2\left(\frac{E^{\frac{m}{2m-2}}(t)}{E(0)}\right)G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)\xi(t)\le -L_2'(t),\qquad \forall t\ge t_1 \tag{5.47}$$

An integration of Eq. 5.47 yields

$$\int_{t_1}^{t} m_2\frac{E^{\frac{m}{2m-2}}(s)}{E(0)}G'\left(\frac{\varepsilon_1}{(s-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)\xi(s)ds\le -\int_{t_1}^{t}L_2'(s)ds\le L_2(t_1). \tag{5.48}$$

Using the fact that $G', G''>0$ and the non-increasingness of E, we deduce that the map $t\mapsto \frac{E^{\frac{m}{2m-2}}(t)}{E(0)}G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)$ is non-increasing and consequently, we have

$$\begin{aligned}
&m_2\frac{E^{\frac{m}{2m-2}}(t)}{E(0)}G'\left(\frac{\varepsilon_2}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}\xi(s)ds\\
&\le\int_{t_1}^{t}m_2\frac{E^{\frac{m}{2m-2}}(s)}{E(0)}G'\left(\frac{\varepsilon_2}{(s-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(s)}{E(0)}\right)\xi(s)ds\le L_2(t_1)=m_3.
\end{aligned} \tag{5.49}$$

Multiplying each side of Eq. 5.49 by $\frac{1}{(t-t_1)}$, we have

$$m_4\frac{\varepsilon_1}{(t-t_1)}\cdot\left(\frac{E(t)}{E(0)}\right)^{\frac{m}{2m-2}}G'\left(\frac{\varepsilon_1}{(t-t_1)^{\frac{2m-2}{m}}}\cdot\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}\xi(s)ds\le\frac{m_3}{(t-t_1)}. \tag{5.50}$$

Next, we set $W_2(\tau) = \tau^{\frac{m}{2m-2}} G'(\tau)$ which is strictly increasing, then we obtain,

$$W_2\left(\frac{1}{(t-t_1)^{\frac{2m-2}{m}}} \cdot \frac{E(t)}{E(0)}\right)\int_{t_1}^{t} \xi(s)ds \leq \frac{m_5}{(t-t_1)}. \tag{5.51}$$

Finally, for two positive constants m_5 and m_6, we obtain

$$E(t) \leq m_6(t-t_1)^{\frac{2m-2}{m}} W_2{}^{-1}\left(\frac{m_7}{(t-t_1)\int_{t_1}^{t} \xi(s)ds}\right). \tag{5.52}$$

This finishes the proof. □

Example 5.8 The following examples illustrate our results:

1. G is linear and $m \geq 2$
   Let $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$ is small enough so that Eq. 2.1 is satisfied, then $g'(t) = -\xi(t)G(g(t))$ where $G(t) = t$ and $\xi(t) = b$. Therefore, we can use Eq. 5.20 to deduce
$$E(t) \leq c_1 e^{-c_2 t} \tag{5.53}$$
   which is the exponential decay.
2. G is non-linear and $m \geq 2$
   Let $g(t) = ae^{-t^p}$,where $0 < p < 1$ and $a > 0$ is small enough so that g satisfies (2.1), then $g'(t) = -\xi(t)G\,(g(t))$ where $\xi(t) = 1$ and $G(t) = \frac{p^t}{(ln(a/t))^{1/p-1}}$. Since
$$G'(t) = \frac{(1-p)+pln(a/t)}{(ln(a/t))^{1/p}}$$
   and
$$G''(t) = \frac{(1-p)(ln(a/t)+1/p)}{(ln(a/t))^{\frac{1}{p+1}}}.$$
   then the function G satisfies the condition (A1) on $(0, r]$ for any $0 < r < a$.
$$\begin{aligned}
G_1(t) = \int_t^r \frac{1}{sG'(s)}ds &= \int_t^r \frac{\left[\ln \frac{a}{s}\right]^{\frac{1}{p}}}{s\left[1-p+p\ln \frac{a}{s}\right]}ds \\
&= \int_{\ln \frac{a}{r}}^{\ln \frac{a}{t}} \frac{u^{\frac{1}{p}}}{1-p+pu}du \\
&= \frac{1}{p}\int_{\ln \frac{a}{r}}^{\ln \frac{a}{t}} u^{\frac{1}{p}-1}\left[\frac{u}{\frac{1-p}{p}+u}\right]du \\
&\leq \frac{1}{p}\int_{\ln \frac{a}{r}}^{\ln \frac{a}{t}} u^{\frac{1}{p}-1}du \leq \left(\ln \frac{a}{t}\right)^{\frac{1}{p}}.
\end{aligned}$$
   Then, Eq. 5.21 gives
$$E(t) \leq ke^{-kt^p}. \tag{5.54}$$
3. G is linear and $1 < m < 2$
   Let $g(t) = ae^{-b(1+t)}$,where $b > 0$ and $a > 0$ is small enough so that Eq. 2.1 is satisfied, then $g'(t) = -\xi(t)G(g(t))$ where $G(t) = t$ and $\xi(t) = b$. Therefore, applying (5.34), we obtain
$$E(t) \leq \left[\frac{1}{1+t}\right]^{\frac{2m-2}{2-m}}. \tag{5.55}$$
4. G is non-linear and $1 < m < 2$
   Let $g(t) = \frac{a}{(1+t)^2}$, a is chosen so that hypothesis (2.1) remains valid. Then
$$g'(t) = -bG(g(t)), \qquad \text{with} \qquad G(s) = s^{\frac{3}{2}},$$

where b is a fixed constant. Then, $W_2(t) = ct^{\frac{2m-2}{2m-1}}$. Therefore, applying (5.35), we get

$$E(t) \leq \frac{1}{(t-t_1)^{\frac{-3m^2+6m-2}{m(2m-1)}}}, \tag{5.56}$$

for $1 < m < 1 + \frac{\sqrt{3}}{3}$, $\frac{-3m^2+6m-2}{m(2m-1)} > 0$.

Acknowledgments The authors thank an anonymous referee for his/her very careful reading and valuable suggestions. This work was funded by KFUPM under Project #IN161006.

Funding information This work was funded by KFUPM under Project #IN161006.

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