Journal of Dynamical and Control Systems (2020) 26:611–627
https://doi.org/10.1007/s10883-019-09475-0

Approximate Controllability of Second-order Non-autonomous System with Finite Delay

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Received: 14 May 2019 / Revised: 2 December 2019 / Published online: 3 January 2020

Abstract
In this article, we shall study the approximate controllability of certain non-autonomous second-order nonlinear differential problems with finite delay in the infinite dimensional space. Sufficient conditions are proposed and proved for the controllability of such systems. Further, we briefly discuss the approximate controllability of impulsive as well as integro-differential problem. We establish these results by utilizing Schauder's fixed-point analysis approach. Finally, the application of the proposed results is presented by giving an example.

Keywords Second-order control system · Delay · Approximate controllability · Schauder's fixed-point theorem · Evolution operator

Mathematics Subject Classification (2010) 34K35 · 34K30 · 93B05 · 93C25

1 Introduction

In 1960, Kalman had given a very basic concept of mathematical control theory called controllability. Generally, controllability is described as qualitative property of dynamical control systems. It has great importance in several fields of research such as ecology, economics and biology. In crude sense, controllability means to check whether a dynamical

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control system can be steer from one state to another state by using some suitable control. However, approximate controllability means that the system can be steered to an arbitrary small neighbourhood of a final state. Hence, approximate controllability is a weaker concept of controllability. So, the study of approximate controllability results for the nonlinear systems involving control parameter in the infinite dimensional spaces is of concern. Controllability results of nonlinear systems have been established by several authors in the past few decades. Many authors [1, 4, 5, 11, 13, 14, 17, 20] have further investigated the theory of controllability to infinite dimensional systems and formulated sufficient conditions for the various types of controllability.

In [19], Sakthivel et al. established the approximate controllability results for the second-order systems with state-dependent delay by utilizing Schauder's fixed-point theorem. Mahmudov et al. [16] studied approximate controllability of second-order neutral stochastic evolution system using semi-group theory and Banach fixed-point theorem. In [8], Henriquez investigated the existence of solutions of non-autonomous second-order functional differential equation with infinite delay using Leray-Sachauder's Alternative fixed-point theorem. Vijaykumar et al. [26] studied the approximate controllability for a class of fractional neutral integro-differential inclusion with state-dependent delay using Dhage's fixed-point theorem. In [18], Sakthivel et al. studied the approximate controllability of fractional nonlinear differential inclusion with initial and non-local conditions by using Bohnenblust-Karlin's fixed-point theorem. Moreover, in recent, a few survey papers of good quality and dealing with various types of controllability are published by Babiarz [2, 3, 9]. In these papers, controllability results of various types of dynamical systems with and without integer-order have been presented. However, to the study of the controllability of dynamical systems, fixed-point technique has been effectively utilized. Motivated by this fact, we use Schauder's fixed-point theorem to study the existence and uniqueness of the mild solution and approximate controllability of the non-autonomous control problem. The proposed control problem in a Banach space X is considered as follows:

$$\begin{aligned} v''(t) &= A(t)v(t) + A_1 v_+ \mathcal{C}u(t) + g(t, v_, v(t)), \quad t \in I = [0, T], \\ v(t) &= \phi(t), \quad v'(0) = y_0, \quad t \in [-\tau, 0] \end{aligned} \tag{1.1}$$

where $v : I \to X$ is the state function. Let $u(\cdot) \in L^2(I, U)$ be the control function and U is a Hilbert space. The closed linear operator A generates continuous cosine family $\Phi(t)$. The operator $A_1 : C([-\tau, T], X) \to L_2([0, T], X)$ is a bounded linear operator. Also, $\mathcal{C}$ is a bounded linear operator from Y to W. Let $W = L_2([0, T], X)$, $W_\tau = L_2([-\tau, T], X)$, $0 < \tau < T$ and $Y = L_2([0, T], U)$. $v_(\theta) = v(t + \theta)$, $\theta \in [-\tau, 0]$ and $\phi \in C([-\tau, 0], X)$. Function g will be suitably defined in the subsequent section.

Some of the real-world problems can be adequately modeled by functional differential equations or delay differential equations. Often, it has been observed that the delays are either distributed delays or fixed constants. Recently, many authors have shown their interest in the time delay of both kinds, finite and infinite [1, 6, 21, 23]. In [3], Babiarz et al. mentioned that there are many unsolved problems on controllability concepts for different types of dynamical systems with delay terms, which serve as a motivation to this manuscript. So far, the study of the approximate controllability of non-autonomous second-order differential equation with finite delay was an untreated topic, which has been dealt with in this manuscript. Here, we consider the above (1.1) control problem described by a second-order differential equation in a Banach space.

2 Preliminaries

In this segment, we recall some basic concepts, notations and properties that would be needed to establish our controllability results. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Hilbert spaces and $\mathcal{L}(Y, X)$ be the Banach space of bounded linear operators from Y into X endowed with its natural topology. In particular, we prefer the notation $\mathcal{L}(X)$ when $Y = X$.

Let us take the abstract non-autonomous second-order problem with initial conditions

$$\begin{aligned} v''(t) &= A(t)v(t) + g(t), \quad 0 \le t, s \le T, \\ v(s) &= v_0, \quad v'(s) = y_0, \end{aligned} \tag{2.1}$$

where $A(t) : \Omega(A(t)) \subset X \to X, t \in I = [0, T]$ is a closed dense operator and $g : I \to X$ is a suitable function. Author refers the reader to [12] and the references therein. Often in the literature, the existence of solutions to the problem (2.1) is related to the existence of an evolution operator $\Psi(t, s)$ for the homogeneous equation

$$v''(t) = A(t)v(t), \quad 0 \le t, s \le T. \tag{2.2}$$

Let us take that the domain of $A(t)$ is a subspace Ω which is dense in X and independent of t and for each $v \in \Omega$ the function $t \to A(t)v$ is continuous. For the fundamental solution of second-order evolution problem (2.2), we refer to [10].

We will use the following concept of evolution operator for the development of our results.

Definition 2.1 A family Ψ of bounded linear operators $\Psi(t, s) : I \times I \to \mathcal{L}(X)$ is called an evolution operator for (2.2) if the following conditions are satisfied:

(D1) For all $v \in X$, the mapping $(t, s) \in [0, T] \times [0, T] \to \Psi(t, s)v \in X$ is of class C^1 and

- (i) For all $t \in [0, T]$, $\Psi(t, t) = 0$,
- (ii) For all $t, s \in [0, T]$, and for each $v \in X$,

$$\frac{\partial}{\partial t}\Psi(t, s)v|_{t=s} = v, \qquad \frac{\partial}{\partial s}\Psi(t, s)v|_{t=s} = -v.$$

(D2) For all $t, s \in [0, T]$, and if $v \in \Omega(A)$, then $\Psi(t, s)v \in \Omega(A)$, the mapping $(t, s) \in [0, T] \times [0, T] \to \Psi(t, s)v \in X$ is a class C^2 and

- (i) $\frac{\partial^2}{\partial t^2}\Psi(t, s)v = A(t)\Psi(t, s)v$,
- (ii) $\frac{\partial^2}{\partial s^2}\Psi(t, s)v = \Psi(t, s)A(s)v$,
- (iii) $\frac{\partial}{\partial s}\frac{\partial}{\partial t}\Psi(t, s)v|_{t=s} = 0$,

(D3) For all $t, s \in [0, T]$, and if $v \in \Omega(A)$, then $\frac{\partial}{\partial s}\Psi(t, s)v \in \Omega(A)$, then $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}\Psi(t, s)v$, $\frac{\partial}{\partial s}\Psi(t, s)v \in \Omega(A)$, then $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}\Psi(t, s)v$ and

- (i) $\frac{\partial}{\partial s}\Psi(t, s)v \in \Omega(A)$, then $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}\Psi(t, s)v = A(t)\frac{\partial}{\partial s}\Psi(t, s)v$,
- (ii) $\frac{\partial}{\partial s}\Psi(t, s)v \in \Omega(A)$, then $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}\Psi(t, s)v = \frac{\partial}{\partial t}\Psi(t, s)A(s)v$, and the mapping $(t, s) \in [0, T] \times [0, T] \to A(t)\frac{\partial}{\partial s}\Psi(t, s)v$ is continuous.

In this entire paper, we consider that there exists an evolution operator $\Psi(t, s)$ associated with the operator $A(t)$. For the sake of convenience, we take the operator

$|\Phi(t,s) = -\frac{\partial \Psi(t,s)}{\partial s}$. Furthermore, we set N and $\tilde{N}$ for the positive constants such that $\sup_{0\leq t,s\leq T}\|\Psi(t,s)\| \leq \tilde{N}$ and $\sup_{0\leq t,s\leq T}\|\Phi(t,s)\| \leq N$. Also, we take a positive constant N_1 such that

$$\|\Psi(t+l,s) - \Psi(t,s)\| \leq N_1|l|$$

for all $s, t, t+l \in [0,T]$. If $g : I \to X$ is an integrable function, then the mild solution $v : [0,T] \to X$ of the problem (2.1) is given by

$$v(t) = \Phi(t,s)v_0 + \Psi(t,s)y_0 + \int_s^t \Psi(t,\tau)g(\tau)d\tau.$$

In the literature, an abundance of techniques have been used to formulate the existence of the evolution operator $\Psi(t,s)$. In particular, the quite well-known situation is that $A(t)$ is the perturbation of operator A that generates a cosine family. Because of this, we will be briefly reviewing definition of the theory of cosine family and related terms.

Definition 2.2 A one parameter family $(\Phi(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

(i) $\Phi(s+t) + \Phi(s-t) = 2\Phi(s)(t)$ for all $s, t \in \mathbb{R}$,
(ii) $\Phi(0) = \mathbb{I}$ (identity operator),
(iii) $\Phi(t)v$ is continuous in t on $\mathbb{R}$ for each fixed point $v \in X$.

Let $A : \Omega(A) \subset X \to X$ be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(\Phi(t))_{t\in\mathbb{R}}$ on Banach space X. We denote $(\Psi(t))_{t\in\mathbb{R}}$ is the sine function associated with the strongly continuous cosine family, $(\Phi(t))_{t\in\mathbb{R}}$ which is defined by

$$\Psi(t)v = \int_0^t \Phi(s)v\,ds, \quad v \in X, \quad t \in \mathbb{R}.$$

For more details, we refer the reader [7, 25]. The domain $\Omega(A)$ of the operator A is the Banach space, which is defined by

$$\Omega(A) = \{v \in X : \Phi(t)v \text{ is twice continuously differentiable in } t\}$$

endowed with norm

$$\|v\|_A = \|v\| + \|Av\|, v \in \Omega(A).$$

Define $\tilde{\Omega} = \{v \in X : \Phi(t)v$ is once continuously differentiable in $t\}$, endowed with norm

$$\|v\|_{\tilde{\Omega}} = \|v\| + \sup_{0\leq t\leq 1}\|A\Psi(t)v\|, v \in \tilde{\Omega}$$

is a Banach space.

The results related with the existence of solutions for the second-order abstract Cauchy problem

$$\begin{aligned} v''(t) &= Av(t) + \kappa(t), \quad s \leq t \leq T \\ v(s) &= v_0, \qquad v'(s) = y_0, \end{aligned} \tag{2.3}$$

where $\kappa : [0,T] \to X$ is an integrable function, can be found in [24]. The existence of the solutions of semilinear second-order abstract Cauchy problem has been treated in [25]. We only mention here that the function $v(\cdot)$ is given by

$$v(t) = \Phi(t-s)v_0 + \Psi(t-s)y_0 + \int_s^t \Psi(t-\tau)\kappa(\tau)d\tau, \quad 0 \leq t \leq T \tag{2.4}$$

is called a mild solution to the problem (2.3) and when $v_0 \in \tilde{\Omega}$, $v(\cdot)$ is continuously differentiable and

$$v'(t) = A\Psi(t-s)v_0 + \Phi(t-s)y_0 + \int_s^t \Psi(t-\tau)\kappa(\tau)d\tau, \quad 0 \le t \le T.$$

In addition, if $v_0 \in \Omega(A)$, $y_0 \in \tilde{\Omega}$ and g is continuously differentiable function, then the function $v(\cdot)$ is a solution of the initial value problem (2.3).

Let us take that $A(t) = A + \tilde{A}(t)$ where $\tilde{A}(\cdot) : \mathbb{R} \to \mathcal{L}(\tilde{\Omega}, X)$ is a map such that the function $t \to \tilde{A}(t)v$ is a continuously differentiable in X for each $v \in \tilde{\Omega}$. For more details, see [22], for each $(v_0, y_0) \in \Omega(A) \times \tilde{\Omega}$ the non-autonomous Cauchy problem

$$v''(t) = (A + \tilde{A}(t))v(t), \quad t \in \mathbb{R} \tag{2.5}$$

$$v(0) = v_0, \qquad v'(0) = y_0 \tag{2.6}$$

has a unique solution $v(\cdot)$ such that the function $t \to v(t)$ is continuously differentiable in $\tilde{\Omega}$. Following a similar argument, one can conclude that Eq. 2.5 with the initial condition of Eq. 2.3 has a unique solution $v(\cdot, s)$ such that the function $t \to v(t, s)$ is continuously differentiable in $\tilde{\Omega}$. It follows from (2.4) that

$$v(t,s) = \Phi(t-s)v_0 + \Psi(t-s)y_0 + \int_s^t \Psi(t-\tau)\tilde{A}(\tau)x(\tau,s)d\tau.$$

In particular, for $v_0 = 0$ we have

$$v(t,s) = \Psi(t-s)y_0 + \int_s^t \Psi(t-\tau)\tilde{A}(\tau)v(\tau,s)d\tau.$$

Consequently,

$$\|v(t,s)\|_1 \le \|\Psi(t-s)\|_{\mathcal{L}(X,\tilde{\Omega})}\|y_0\| + \int_s^t \|\Psi(t-\tau)\|_{\mathcal{L}(X,\tilde{\Omega})}\|\tilde{A}(\tau)\|_{\mathcal{L}(X,\tilde{\Omega})}\|v(\tau,s)\|_1 d\tau.$$

Gronwall's inequality implies that

$$\|v(t,s)\|_1 \le \tilde{M}\|y_0\|, \qquad \forall\, s, t \in I,$$

where $\tilde{M} = \|\Psi(t-s)\| \exp[\|\Psi(t-\tau)\|\|\tilde{A}(\tau)\|(t-s)]$.

Let us define the operator $\Psi(t,s)y_0 = v(t,s)$. It follows from the previous estimate that $\Psi(t,s)$ is a bounded linear map on $\tilde{\Omega}$. Since $\tilde{\Omega}$ is dense in X, we can extend $\Psi(t,s)$ to X. We keep the notation $\Psi(t,s)$ for this extension. It is a very well-known fact that the cosine family $\Phi(t)$ cannot be compact unless the $\dim(X) < \infty$. By contrast, for the cosine family that arise in specific applications, the sine family $\Psi(t)$ is very often a compact operator for all $t \in \mathbb{R}$.

Let $C([0,T], X)$ be the space of continuous functions $v : [0,T] \to X$. Also, assume that $C([0,T], X)$ endowed with uniform convergence norm.

Definition 2.3 A function $v : [-\tau, T] \to X$ is said to be mild solution of the control problem (1.1) if $v(\cdot) \in C(I, X)$, $v(t) = \phi(t)$ for $t \in [-\tau, 0]$ and solution of the following integral equation

$$v(t) = \Phi(t,0)\phi(0) + \Psi(t,0)y_0 + \int_0^t \Psi(t,s)[A_1 v_s + \mathcal{C}u(s) + g(s, v_s, v(s))]ds. \tag{2.7}$$

Definition 2.4 The system (1.1) is called approximately controllable if $\overline{E(T)} = X$ where

$$E(T) = \{v(T; u) \ : \ u(\cdot) \in L^2([0, T]; U)\}$$

and $v(t; u)$ is a mild solution of (1.1).

It is convenient at this point to define the operators

$$\Gamma_0^T = \int_0^T \Psi(T, s)\mathcal{C}\mathcal{C}^* \Psi^*(T, s)ds$$
$$R(\alpha, \Gamma_0^T) = (\alpha\mathbb{I} + \Gamma_0^T)^{-1},$$

where $\mathbb{I}$ is an identity operator. It can be easily seen that the operator Γ_0^T is a linear operator.

To investigate the approximate controllability of the problem (1.1), we take the following hypothesis: **H** $\alpha R(\alpha, \Gamma_0^T) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

The hypothesis **H** holds if and only if the following second-order linear control system

$$\begin{aligned} v''(t) &= Av(t) + \mathcal{C}u(t), \quad t \in I \\ v(0) &= v_0, \quad v'(0) = y_0, \end{aligned} \tag{2.8}$$

is approximately controllable on I. For more details, see [15].

In order to establish the controllability result of the system (1.1), we consider the following assumptions.

Assumptions:

(A1): $\Psi(t), t > 0$ is compact.

(A2): The function $g : I \times X \times X \to X$ satisfies the following conditions:

(i) Let $g(t, \xi, \nu)$ be strongly measurable for $\xi, \nu \in X$.
(ii) Let $g(t, \cdot, \cdot)$ be continuous for each $t \in I$.
(iii) For each $q > 0$, there exists a function $\lambda_q \in L^1(I, R^+)$ such that

$$\sup_{\|\nu\|, \|\xi\| \leq q} \|g(t, \xi, \nu)\| \leq \lambda_q(t), \quad \text{for } a.e. \ t \in I,$$

and

$$\lim_{q\to\infty} inf \int_0^T \frac{\lambda_q(t)}{q} dt = \delta < \infty.$$

(A3): Let $g : I \times X \times X \to X$ be a continuous function. Also, there exists $L > 0$ such that $\|g(t, \xi, \nu)\| \leq L$ for all $(t, \xi, \nu) \in I \times X \times X$.

3 Approximate Controllability Result

In this segment, we prove the approximate controllability of second-order non-autonomous finite delay system with deviated argument. For this, we first prove the existence of solutions of the problem (1.1) using Schauder's fixed-point theorem. After that, the approximate controllability of the problem (1.1) is derived by the fact that the linear system (2.8) is approximately controllable.

It is convenience to introduce some notations which will be useful for further manipulation.

$$K_A = \|A_1\|, \quad M_{\mathcal{C}} = \|\mathcal{C}\|, \quad K = \|v_T\| + N\|\phi(0)\| + \tilde{N}\|y_0\|,$$
$$K^* = N\|\phi(0)\| + \tilde{N}\|y_0\| + \frac{1}{\alpha}\tilde{N}^2 M_{\mathcal{C}}^2 T K, \quad \hat{\Delta} = \left(1 + \frac{1}{\alpha}\tilde{N}^2 M_{\mathcal{C}}^2 K_A T\right)\tilde{N}$$

Let $\mathcal{Z} = \{v \in C_{L_0}([-\tau, T], X) \ : \ v(0) = \phi(0)\}$ be the space endowed with uniform norm convergence. In space $\mathcal{Z}$, we consider a set $\mathcal{W} = \{v \in \mathcal{Z} \ : \ \|v\| \leq r\}$, where r is a positive constant.

For any $v \in \mathcal{W}$ and $0 \leq t \leq t_0$,

$$\|v_t\|_C = \sup_{-\tau \leq \theta \leq 0} \|v_t(\theta)\|_X \leq \sup_{-\tau \leq \zeta \leq t_0} \|v(\zeta)\|_X \leq r$$

Theorem 3.1 *The system* (1.1) *has solution on I if the assumptions (A1)-(A2) are satisfied and for all* $\alpha > 0$

$$\hat{\Delta}(K_A T + \delta) < 1.$$

Proof We define the feedback control function

$$u(t) = \mathcal{C}^*\Psi^*(T, t)R(\alpha, \Gamma_0^T)$$
$$\left[v_T - \Phi(T, 0)\phi(0) - \Psi(T, 0)y_0 - \int_0^T \Psi(T, s)\left[A_1 v_s + g(s, v_s, v(s))\right] ds\right].$$

For $\alpha > 0$, define the operator $\mathcal{F}_\alpha : \mathcal{Z} \to \mathcal{Z}$, which is given by

$$\mathcal{F}_\alpha v(t) = \Phi(t, 0)\phi(0) + \Psi(t, 0)y_0 + \int_0^t \Psi(t, s)[A_1 v_s + \mathcal{C}u(s) + g(s, v_s, v(s))]ds.$$

Proof of this theorem is divided into three steps. □

Step 1. It will be shown that for every $\alpha > 0$ the operator $\mathcal{F}_\alpha : \mathcal{Z} \to \mathcal{Z}$ has a fixed point. For $\alpha > 0$, we claim that there exists $r > 0$ such that $\mathcal{F}_\alpha(\mathcal{W}) \subset \mathcal{W}$. Suppose that our claim is false, then there exists $\alpha > 0$ such that for all $r > 0$, there exist $\tilde{v} \in \mathcal{W}$ and $t_0 \in I$ such that $r < \|\mathcal{F}_\alpha \tilde{v}(t_0)\|$.

For such $\alpha > 0$, we see that

$$r < \|\mathcal{F}_\alpha \tilde{v}(t_0)\|$$
$$\leq N\|\phi(0)\| + \tilde{N}\|y_0\| + \tilde{N} r K_A T + \tilde{N} M_{\mathcal{C}} \int_0^t \|u(s)\|ds + \tilde{N}\int_0^t \|g(s, \tilde{v}_s, \tilde{v}(s))\|ds$$

Hence,

$$\begin{aligned} r &\leq N\|\phi(0)\| + \tilde{N}\|y_0\| + \tilde{N} r K_A T \\ &\quad + \tilde{N} M_{\mathcal{C}} T\left[\frac{1}{\alpha}\tilde{N} M_{\mathcal{C}}(K + \tilde{N}\int_0^T \lambda_r(s)ds)\right] + \tilde{N}\int_0^T \lambda_r(s)ds \\ &\leq \left(1 + \frac{1}{\alpha} T\tilde{N}^2 M_{\mathcal{C}}^2\right)\tilde{N}\left[r K_A T + \int_0^T \lambda_r(s)ds\right] + K^* \\ &\leq \hat{\Delta}\left(r K_A T + \int_0^T \lambda_r(s)ds\right) + K^* \end{aligned}$$

As $r \to \infty$, we have

$$1 \leq \hat{\Delta}(K_A T + \delta).$$

Which contradicts our condition

$$\hat{\Delta}(K_A T + \delta) < 1.$$

Hence, $\mathcal{F}_\alpha(\mathcal{W}) \subset \mathcal{W}$.

Step 2. It is shown that for each $\alpha > 0$, the operator $\mathcal{F}_\alpha$ maps bounded set $\mathcal{W}$ into a relatively compact subset of $\mathcal{W}$.

We take set $\Pi(t) = \{\mathcal{F}_\alpha v(t) : v \in \mathcal{W}\}$.

For $t \in (0, T]$ and $0 < \varepsilon < t \leq T$ define

$$(\mathcal{F}_\alpha^\varepsilon v)(t) = C(t,0)\phi(0) + \Psi(t,0)y_0 + \int_0^{t-\varepsilon} \Psi(t,s)[A_1 v_s + \mathcal{C}u(s) + g(s, v_s, v(s))]ds.$$

Since sine family $\Psi(t)$ is compact, the set $\Pi_\varepsilon(t) = \{\mathcal{F}_\alpha^\varepsilon v(t) : v \in \mathcal{W}\}$ is relatively compact in X for each $\varepsilon,\ 0 < \varepsilon < t$. Moreover for each $0 < \varepsilon < t$, we have

$$\begin{aligned}\|(\mathcal{F}_\alpha v)(t) - (\mathcal{F}_\alpha^\varepsilon v)(t)\| \leq\ & \tilde{N} \int_{t-\varepsilon}^t \|A_1 v_s\| ds + \tilde{N} M_{\mathcal{C}} \int_{t-\varepsilon}^t \|u(s)\| ds \\ & + \tilde{N} \int_{t-\varepsilon}^t \|g(s, v_s, v(s))\| ds.\end{aligned}$$

Hence, there exist relatively compact set arbitrarily close to $\Pi(t) = \{\mathcal{F}_\alpha v(t) : v \in \mathcal{W}\}$ as $\varepsilon \to 0$. Since it is compact at $t = 0$, hence, set $\Pi(t)$ is relatively compact in $X\ \forall\ t \in [0, T]$.

Now we prove that $\Pi(t) = \{\mathcal{F}_\alpha v(t) : v \in \mathcal{W}\}$ is equicontinuous on $[0, T]$ for $0 < t_1 < t_2 < T$,

$$\begin{aligned}\|(\mathcal{F}_\alpha v)(t_2) - (\mathcal{F}_\alpha v)(t_1)\| \leq\ & \|\Phi(t_2,0) - \Phi(t_1,0)\| \|\phi(0)\| + \|\Psi(t_2,0) - \Psi(t_1,0)\| y_0\| \\ & + \tilde{N} K_A \int_{t_1}^{t_2} \|v_s\| ds + K_A \int_0^{t_1} \|\Psi(t_2,s) - \Psi(t_1,s)\| \|v_s\| ds \\ & + \tilde{N} M_{\mathcal{C}} \int_{t_1}^{t_2} \|u(s)\| ds + M_{\mathcal{C}} \int_0^{t_1} \|\Psi(t_2,s) - \Psi(t_1,s)\| \|u(s)\| ds \\ & + \tilde{N} \int_{t_1}^{t_2} \lambda_r(s) ds + \int_0^{t_1} \|\Psi(t_2,s) - \Psi(t_1,s)\| \lambda_r(s) ds.\end{aligned}$$

$$\begin{aligned}&\|(\mathcal{F}_\alpha v)(t_2) - (\mathcal{F}_\alpha v)(t_1)\| \\ &\leq \|\Phi(t_2,0) - \Phi(t_1,0)\| \|\phi(0)\| + \|\Psi(t_2,0) - \Psi(t_1,0)\| y_0\| \\ &\quad + \tilde{N} K_A \int_{t_1}^{t_2} \|v_s\| ds + K_A \int_0^{t_1} \|\Psi(t_2,s) - \Psi(t_1,s)\| \|v_s\| ds \\ &\quad + \frac{\tilde{N}^2 M_{\mathcal{C}}^2}{\alpha} \int_{t_1}^{t_2} \left(\|v_T\| + N\|\phi(0)\| + \tilde{N}\|y_0\| + \tilde{N} K_A \int_0^T \|v_s\| ds + \tilde{N} \int_0^T \lambda_r(s) ds \right) d\eta \\ &\quad + \frac{\tilde{N} M_{\mathcal{C}}^2}{\alpha} \int_0^{t_1} \|\Psi(t_2,s) - \Psi(t_1,s)\| \Big(\|v_T\| + N\|\phi(0)\| + \tilde{N}\|y_0\| \\ &\quad + \tilde{N} K_A \int_0^T \|v_s\| ds + \tilde{N} \int_0^T \lambda_r(s) ds \Big) d\eta \\ &\quad + \tilde{N} \int_{t_1}^{t_2} \lambda_r(s) ds + \int_0^{t_1} \|\Psi(t_2,s) - \Psi(t_1,s)\| \lambda_r(s) ds.\end{aligned}$$

Here, it can be seen that $\|(\mathcal{F}_\alpha v)(t_2) - (\mathcal{F}_\alpha v)(t_1)\| \to 0$ as $(t_1 - t_2) \to 0$. Also, the compactness of evolution operator $\Psi(t, s)$ implies the continuity in the uniform operator topology.

Thus, the set $\Pi(t) = \{\mathcal{F}_\alpha v(t) : v \in \mathcal{W}\}$ is equicontinuous on $[0, T]$.

Step 3. It is shown that the operator $\mathcal{F}_\alpha(\cdot)$ is continuous on $\mathcal{W}$.

Let $(v_n)_{n\in N}$ be a sequence in $\mathcal{W}$ and $v \in \mathcal{W}$ such that $v_n \to v$. From the condition, we find that $v_t^n \to v_t$ as $n \to \infty\ \forall\ t \in I$.

From the inequality, we see that

$$\begin{aligned}\|g(s, v_s^n, v^n(s)) \ - \ g(s, v_s, v(s))\| &\\ \leq \|g(s, v_s^n, v^n(s)) - g(s, v_s^n, v(s))\| &\\ +\|g(s, v_s^n, v(s)) - g(s, v_s, v(s))\| &\end{aligned}$$

We infer that

$$g(s, v_s^n, v^n(s)) \to g(s, v_s, v(s)) \ \ as \ \ n \to \infty \ \ \forall \ \ s \in I$$

By the help of Lebesgue dominated convergence theorem and assumption (A2), it can be asserted that $\mathcal{F}_\alpha v_n \to \mathcal{F}_\alpha v$ in $\mathcal{W}$. Hence, $\mathcal{F}_\alpha(\cdot)$ is continuous on $\mathcal{W}$. Thus by Schauder's fixed-point theorem, $\mathcal{F}_\alpha$ has a fixed point and the problem (1.1) has a solution on I.

Theorem 3.2 *Assume that the linear system* (2.8) *is approximately controllable on* I. *If the assumptions (A1)–(A3) are satisfied then the system* (1.1) *is approximately controllable.*

Proof Let $v^\alpha(\cdot)$ be a fixed point of $\mathcal{F}_\alpha$ in $\mathcal{W}$. Any fixed point of $\mathcal{F}_\alpha$ is a mild solution of the problem (1.1) under the control

$$u^\alpha(t) = \mathcal{C}^*\Psi^*(T, t)R(\alpha, \Gamma_0^T)p(v^\alpha),$$

where

$$p(v^\alpha) = v_T - \Phi(T, 0)\phi(0) - \Psi(T, 0)y_0 - \int_0^T \Psi(T, s)\left[A_1 v_s^\alpha + g(s, v_s^\alpha, v^\alpha(s))\right] ds$$

and satisfies the inequality

$$v^\alpha(T) = v_T + \alpha R(\alpha, \Gamma_0^T)p(v^\alpha).$$

By the assumption (A3)

$$\int_0^T \|A_1 v_s^\alpha + g(s, v_s^\alpha, v^\alpha(s))\|^2 ds \ \leq \ T(K_A r + L)^2.$$

Hence, the sequence $\{A_1 v_s^\alpha + g(s, v_s^\alpha, v^\alpha(s))\}$ is bounded in $L_2(I, X)$ and there exists a subsequence denoted by $\{A_1 v_s^\alpha + g(s, v_s^\alpha, v^\alpha(s))\}$ that weakly converges to $A_1 v(s) + g(s)$ in $L_2(I, X)$. By using infinite dimensional version of the Ascoli-Arzela theorem, an operator $l(\cdot) \to \int_0^{\cdot} \Psi(\cdot, s)l(s)ds \ : \ L_2(I, X) \to C(I, X)$ is compact.

We obtain

$$\begin{aligned}\|p(v^\alpha) - w\| &= \Bigg\| \int_0^T \Psi(T,s)\left[(A_1 v_s^\alpha + g(s, v_s^\alpha, v^\alpha(s)))\right. \\ &\quad \left. - (A_1 v_s + g(s, v_s, v(s)))ds\right] \Bigg\| \\ &\leq \sup_{t\in I} \Bigg\| \int_0^t \Psi(T,s)\left[(A_1 v_s^\alpha + g(s, v_s^\alpha, v^\alpha(s)))\right. \\ &\quad \left. - (A_1 v_s + g(s, v_s, v(s)))\right] ds \Bigg\| \to 0\end{aligned}$$

as $\alpha \to 0^+$, where

$$w = v_T - \Phi(T,0)\phi(0) - \Psi(T,0)y_0 - \int_0^T \Psi(T,s)[A_1 v_s + g(s, v_s, v(s))]ds.$$

Then,

$$\begin{aligned}\|v^\alpha(T) - v_T\| &\leq \|\alpha R(\alpha, \Gamma_0^T)(w)\| + \|\alpha R(\alpha, \Gamma_0^T)\| \|p(v^\alpha) - w\| \\ &\leq \|\alpha R(\alpha, \Gamma_0^T)(w)\| + \|p(v^\alpha) - w\|.\end{aligned}$$

It follows from the hypothesis (**H**) and above estimate that $\|v^\alpha(T) - v_T\| \to 0$ as $\alpha \to 0^+$. Hence, the approximate controllability of the problem (1.1) is proved. □

4 Second-order System with Non-instantaneous Impulses

In this segment, we prove the approximate controllability of the impulsive system non-instantaneous impulses.

$$\begin{cases} v''(t) = A(t)v(t) + A_1 v_t + \mathcal{C}u(t) + g(t, v_t, v(t)), \ t \in (s_i, t_{i+1}], \ i = 0, 1, \cdots, m, \\ v(t) = \psi_i^1(t, v(t_i^-)), \ t \in (t_i, s_i], \ i = 1, 2, \cdots, m, \\ v'(t) = \psi_i^2(t, v(t_i^-)), \ t \in (t_i, s_i], \ i = 1, 2, \cdots, m, \\ v(t) = \phi(t), \quad v'(0) = y_0, \quad t \in [-\tau, 0], \end{cases} \tag{4.1}$$

where $A, A_1, \mathcal{C}$ and g are defined as in Eq. 1.1 and $v(t)$ is a state function with time interval $0 = s_0 = t_0 < t_1 < s_1 < t_2, \cdots, t_m < s_m < t_{m+1} = T < \infty$. Consider the state function $v \in C((t_i, t_{i+1}], \mathbb{R}^n), i = 0, 1, \cdots, m$ and there exist $v(t_i^-)$ and $v(t_i^+)$, $i = 1, 2, \cdots, m$ with $v(t_i^-) = v(t_i)$. The functions $\psi_i^1(t, v(t_i^-))$ and $\psi_i^2(t, v(t_i^-))$ represent non-instantaneous impulses which occur in the intervals $(t_i, s_i]$, $i = 1, 2, \cdots, m$. Further, let us consider $PC([0, T], X)$ be the Banach space of piecewise continuous functions $v : [0, T] \to X$ endowed with the norm $\|v\|_{PC} = \sup_{t\in I} \|v(t)\|$.

Definition 4.1 A function $v \in PC([0, T], X)$ is called a mild solution of the impulsive problem (4.1) if it satisfies the following: relations $v(t) = \phi(t)$, $v'(0) = y_0$, the non-instantaneous impulse conditions $v(t) = \psi_i^1(t, v(t_i^-))$, $t \in (t_i, s_i]$, $i = 1, 2, \cdots, m$, $v'(t) = \psi_i^2(t, v(t_i^-))$, $t \in (t_i, s_i]$, $i = 1, 2, \cdots, m$, and $v(t)$ is the solution of the following integral equations

$$\begin{aligned}v(t) &= \Phi(t,0)\, \psi_i^1(t, v(t_i^-)) + \Psi(t,0)\psi_i^2(t, v(t_i^-)) \\ &\quad + \int_0^t \Psi(t,s)[A_1 v_s + \mathcal{C}u(s) + g(s, v_s, v(s))]ds.\end{aligned}$$

(A4) There exist positive constants $C_{\psi_i^1}$ and $C_{\psi_i^2}$, $i = 1, 2, \cdots, m$ such that $C_{\psi_i^1} = \max_{t \in I_i} \|\psi_i^1(t, \cdot)\|$ and $C_{\psi_i^2} = \max_{t \in I_i} \|\psi_i^2(t, \cdot)\|$, where $I_i := [t_i, s_i]$.

(A5) $\psi_i^k \in C(I_i \times \mathbb{R}, \mathbb{R})$ and there are positive constants $L_{\psi_i^k}$, $i = 1, 2, \cdots, m$, $k = 1, 2$, such that $\|\psi_i^k(t, u) - \psi_i^k(t, v)\| \leq L_{\psi_i^k}\|u - v\|$, $\forall\, t \in I_i$ and $u, v \in \mathbb{R}$.

Let $\overline{\mathcal{Z}} = \{v \in C_{L_0}([-\tau, T], X) \;:\; v(0) = \phi(0)\}$ be the space endowed with uniform norm convergence. In space $\overline{\mathcal{Z}}$, we consider a set $\overline{\mathcal{W}} = \{v \in \overline{\mathcal{Z}} \;:\; \|v\| \leq \overline{r}\}$, where $\overline{r}$ is a positive constant.

For any $v \in \overline{\mathcal{W}}$ and $0 \leq t \leq t_0$,

$$\|v_t\|_C = \sup_{-\tau \leq \theta \leq 0} \|v_t(\theta)\|_X \leq \sup_{-\tau \leq \zeta \leq t_0} \|v(\zeta)\|_X \leq \overline{r}$$

Theorem 4.2 *The system* (4.1) *is approximately controllable on* I *if the assumptions (A1)–(A5) are satisfied and for all* $\alpha > 0$

$$\hat{\Delta}(K_A T + \delta) < 1.$$

Proof For the proof of this theorem, we use some notations for convenience as follows

$$\tilde{K} = \|v_T\| + NC_{\psi_i^1} + \tilde{N}C_{\psi_i^2}, \quad \hat{\Delta} = \left(1 + \frac{1}{\alpha}\tilde{N}^2 M_{\mathcal{C}}^2 K_A T\right)\tilde{N}$$
$$\overline{K} = NC_{\psi_i^1} + \tilde{N}C_{\psi_i^2} + \frac{1}{\alpha}\tilde{N}^2 M_{\mathcal{C}}^2 T\tilde{K},$$

Now, we define the feedback control function

$$u(t) = \mathcal{C}^*\Psi^*(T, t)R(\alpha, \Gamma_0^T)\Big[v_T - \Phi(T, 0)\psi_i^1(t, v(t_i^-)) - \Psi(T, 0)\psi_i^2(t, v(t_i^-)) - \int_0^T \Psi(T, s)\,[A_1 v_s + g(s, v_s, v(s))]\,ds\Big].$$

For $\alpha > 0$, define the operator $\overline{\mathcal{F}}_\alpha : \overline{\mathcal{Z}} \to \overline{\mathcal{Z}}$, which is given by

$$\overline{\mathcal{F}}_\alpha v(t) = \Phi(t, 0)\,\psi_i^1(t, v(t_i^-)) + \Psi(t, 0)\psi_i^2(t, v(t_i^-)) + \int_0^t \Psi(t, s)[A_1 v_s + \mathcal{C}u(s) + g(s, v_s, v(s))]ds.$$

It will be shown that for every $\alpha > 0$, the operator $\overline{\mathcal{F}}_\alpha : \overline{\mathcal{Z}} \to \overline{\mathcal{Z}}$ has a fixed point. For $\alpha > 0$, we claim that there exists $\overline{r} > 0$ such that $\mathcal{F}_\alpha(\overline{\mathcal{W}}) \subset \overline{\mathcal{W}}$. Suppose that our claim is false, then there exists $\alpha > 0$ such that for all $\overline{r} > 0$, there exist $\tilde{v} \in \overline{\mathcal{W}}$ and $t_0 \in I$ such that $\overline{r} < \|\overline{\mathcal{F}}_\alpha\tilde{v}(t_0)\|$.

For such $\alpha > 0$, we see that

$$\begin{aligned}\overline{r} &< \|\overline{\mathcal{F}}_\alpha\tilde{v}(t_0)\| \\ &\leq N\|\psi_i^1(t, v(t_i^-))\| + \tilde{N}\|\psi_i^2(t, v(t_i^-))\| + \tilde{N}\overline{r}K_A T \\ &\quad + \tilde{N}M_{\mathcal{C}}\int_{s_i}^{t_i}\|u(s)\|ds + \tilde{N}\int_{s_i}^{t_i}\|g(s, \tilde{v}_s, \tilde{v}(s))\|ds\end{aligned}$$

Hence,

$$\begin{aligned}\overline{r} &\leq NC_{\psi_i^1} + \tilde{N}C_{\psi_i^2} + \tilde{N}\overline{r}K_AT \\ &\quad + \tilde{N}M_{\mathcal{C}}T\left[\frac{1}{\alpha}\tilde{N}M_{\mathcal{C}}(\widetilde{K} + \tilde{N}\int_{s_i}^{T}\lambda_{\overline{r}}(s)ds)\right] + \tilde{N}\int_{s_i}^{T}\lambda_{\overline{r}}(s)ds \\ &\leq \left(1 + \frac{1}{\alpha}T\tilde{N}^2M_{\mathcal{C}}^2\right)\tilde{N}\int_0^T\lambda_{\overline{r}}(s)ds + \frac{1}{\alpha}\tilde{N}^3M_{\mathcal{C}}^2K_AT^2\overline{r} + \tilde{N}\overline{r}K_AT + \overline{K} \\ &\leq \left(1 + \frac{1}{\alpha}T\tilde{N}^2M_{\mathcal{C}}^2\right)\tilde{N}\left[\overline{r}K_AT + \int_0^T\lambda_{\overline{r}}(s)ds\right] + \overline{K} \\ &\leq \hat{\Delta}\left(\overline{r}K_AT + \int_0^T\lambda_{\overline{r}}(s)ds\right) + \overline{K}\end{aligned}$$

As $\overline{r} \to \infty$, we have

$$1 \leq \hat{\Delta}(K_AT + \delta).$$

Which contradicts our condition

$$\hat{\Delta}(K_AT + \delta) < 1.$$

Hence, $\overline{\mathcal{F}}_\alpha(\overline{\mathcal{W}}) \subset \overline{\mathcal{W}}$. Further, we can easily prove that $\overline{\mathcal{F}}_\alpha$ has a fixed point for all $\alpha > 0$ by employing the technique used in Theorem (3.1). □

5 Second-order Integro-differential Equation

In this segment, we consider a control system represented by an integro-differential equation in the Banach space X.

$$\begin{aligned}&v''(t) = A(t)v(t) + A_1v_t + \mathcal{C}u(t) + g(t, v_t, v(t)) + \int_0^t\omega(t-s)f(s, v(s))ds, \quad t \in I = [0, T],\\ &v(t) = \phi(t), \quad v'(0) = y_0, \quad t \in [-\tau, 0] \qquad (5.1)\end{aligned}$$

where $A, A_1, \mathcal{C}$ and g are defined as in Eq. 1.1 and f and ω are the suitable functions to be specified later. In order to prove the approximate controllability of the integro-differential Eq. 5.1, we need the following assumptions:

(A6) $\omega_T = \int_0^t\omega(s)ds$

(A7) (i): $\|f(t, \cdot)\| \leq M_g$

(i): $\|f(t, u_1(t)) - f(t, u_2(t))\| \leq L_g\|(u_1(t)) - u_2(t))\|$

where M_g and L_g are positive constants.

Definition 5.1 A function $v : [-\tau, T] \to X$ is said to be mild solution of the control problem (5.1) if $v(\cdot) \in C(I, X)$, $v(t) = \phi(t)$ for $t \in [-\tau, 0]$ and solution of the following integral equation

$$\begin{aligned}v(t) &= \Phi(t,0)\phi(0) + \Psi(t,0)y_0 + \int_0^t\Psi(t,s)\Big[A_1v_s + \mathcal{C}u(s) + g(s, v_s, v(s)) \\ &\quad + \int_0^t\omega(t-\zeta)f(\zeta, v(\zeta))d\zeta\Big]ds. \qquad (5.2)\end{aligned}$$

Let $\hat{\mathcal{Z}} = \{v \in C_{L_0}([-\tau, T], X) \ : \ v(0) = \phi(0)\}$ be the space endowed with uniform norm convergence. In space $\hat{\mathcal{Z}}$, we consider a set $\hat{\mathcal{W}} = \{v \in \hat{\mathcal{Z}} \ : \ \|v\| \leq \hat{r}\}$, where $\hat{r}$ is a positive constant.

For any $v \in \hat{\mathcal{W}}$ and $0 \leq t \leq t_0$,

$$\|v_t\|_C = \sup_{-\tau \leq \theta \leq 0} \|v_t(\theta)\|_X \leq \sup_{-\tau \leq \zeta \leq t_0} \|v(\zeta)\|_X \leq \hat{r}$$

Theorem 5.2 *The system (5.1) is approximately controllable on I if the assumptions (A1)–(A3) and (A6)–(A7) are satisfied and for all* $\alpha > 0$

$$\hat{\Delta}(K_A T + \delta) < 1.$$

Proof We define the feedback control function for (5.1)

$$\begin{aligned} u(t) &= \mathcal{C}^*\Psi^*(T,t)R(\alpha,\Gamma_0^T)\Bigg[v_T - \Phi(T,0)\phi(0) - \Psi(T,0)y_0 \\ &\quad -\int_0^T \Psi(T,s)\bigg[A_1 v_s + g(s, v_s, v(s)) + \int_0^t \omega(t-\zeta)f(\zeta, v(\zeta))d\zeta\bigg]ds\Bigg]. \end{aligned}$$

For $\alpha > 0$, define the operator $\hat{\mathcal{F}} : \hat{\mathcal{Z}} \to \hat{\mathcal{Z}}$, which is given by

$$\begin{aligned} \hat{\mathcal{F}}_\alpha v(t) &= \Phi(t,0)\phi(0) + \Psi(t,0)y_0 + \int_0^t \Psi(t,s)\bigg[A_1 v_s + \mathcal{C}u(s) + g(s, v_s, v(s)) \\ &\qquad + \int_0^t \omega(t-\zeta)f(\zeta, v(\zeta))d\zeta\bigg]ds. \end{aligned}$$

It will be shown that for every $\alpha > 0$ the operator $\hat{\mathcal{F}} : \hat{\mathcal{Z}} \to \hat{\mathcal{Z}}$ has a fixed point. For $\alpha > 0$, we claim that there exists $\hat{r} > 0$ such that $\hat{\mathcal{F}}_\alpha(\hat{\mathcal{W}}) \subset \hat{\mathcal{W}}$. Suppose that our claim is false, then there exists $\alpha > 0$ such that for all $\hat{r} > 0$, there exist $\tilde{v} \in \hat{\mathcal{W}}$ and $t_0 \in I$ such that $\hat{r} < \|\hat{\mathcal{F}}_\alpha \tilde{v}(t_0)\|$.

For such $\alpha > 0$, we see that

$$\begin{aligned} \hat{r} &< \|\hat{\mathcal{F}}_\alpha \tilde{v}(t_0)\| \\ &\leq N\|\phi(0)\| + \tilde{N}\|y_0\| + \tilde{N}\hat{r}K_A t_0 + \tilde{N}M_{\mathcal{C}}\int_0^t \|u(s)\|ds \\ &+\tilde{N}\int_0^t \bigg[\|g(s, \tilde{v}_s, \tilde{v}(s))\|ds + \int_0^t \|\omega(t-\zeta)f(\zeta, v(\zeta))d\zeta\bigg] \end{aligned}$$

Hence,

$$\begin{aligned} \hat{r} &\leq N\|\phi(0)\| + \tilde{N}\|y_0\| + \tilde{N}\hat{r}K_A T \\ &+ \tilde{N}M_{\mathcal{C}}T\bigg[\frac{1}{\alpha}\tilde{N}M_{\mathcal{C}}(K + \tilde{N}\int_0^T \lambda_{\hat{r}}(s)ds + \omega_T M_g)\bigg] + \tilde{N}\int_0^T \lambda_{\hat{r}}(s)ds + \omega_T M_g \\ &\leq \left(1 + \frac{1}{\alpha}T\tilde{N}^2M_{\mathcal{C}}^2\right)\tilde{N}\bigg[\omega_T M_g + \hat{r}K_A T + \int_0^T \lambda_{\hat{r}}(s)ds\bigg] + K^* \\ &\leq \left(1 + \frac{1}{\alpha}T\tilde{N}^2M_{\mathcal{C}}^2\right)\tilde{N}\bigg[\hat{r}K_A T + \int_0^T \lambda_{\hat{r}}(s)ds\bigg] + K^* + \tilde{K} \end{aligned}$$

As $\hat{r} \to \infty$, we have

$$1 \leq \hat{\Delta}(K_A T + \delta)$$

Which contradicts our condition

$$\hat{\Delta}(K_A T + \delta) < 1.$$

Hence, $\hat{\mathcal{F}}_\alpha(\hat{\mathcal{W}}) \subset \hat{\mathcal{W}}$. Further, we can easily prove that $\hat{\mathcal{F}}_\alpha$ has a fixed point for all $\alpha > 0$ by employing the technique used in Theorem (3.1). □

6 Example

In this segment, we introduce a few technical terms to give an example. From Eqs. (2.5)–(2.6), we consider $A(t) = A + \tilde{A}(t)$ where A is the infinitesimal generator of a cosine function $\Phi(t)$ with associated sine function $\Psi(t)$ and $\tilde{A}(t) : \Omega(\tilde{A}(t)) \to X$ is a closed linear operator.

Let us take the space $X = L^2(\mathbb{T}, \mathbb{C})$ where group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. Also, we will use the identification between functions on $\mathbb{T}$ and 2π periodic functions on $\mathbb{R}$. Furthermore, we denote by $L^2(\mathbb{T}, \mathbb{C})$ the space of 2π periodic 2- integrable functions from $\mathbb{R}$ to $\mathbb{C}$. Besides, $H^2(\mathbb{T}, \mathbb{C})$ denotes the Sobolev space of 2π periodic from $\mathbb{R}$ to $\mathbb{C}$ such that $v'' \in L^2(\mathbb{T}, \mathbb{C})$.

We consider the operator $Av(t) = v''(t)$ with domain $\Omega(A) = H^2(\mathbb{T}, \mathbb{C})$. Operator A is an infinitesimal generator of a strongly continuous cosine family $\Phi(t)$ on X. Furthermore, A has discrete spectrum and the spectrum of A consists of eigenvalues $-n^2$ for $n \in \mathbb{Z}$, with associated normalized eigenvectors

$\vartheta_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}, \quad n \in \mathbb{Z}$, the set $\{\vartheta_n : n \in \mathbb{Z}\}$ is an orthonormal basis of X. In particular,

$$Av = -\sum_{n=1}^{\infty} n^2 \langle v, \vartheta_n \rangle \vartheta_n$$

for $v \in \Omega(A)$. The cosine function $\Phi(t)$ is given by

$$\Phi(t)v = \sum_{n=1}^{\infty} \cos(nt) \langle v, \vartheta_n \rangle \vartheta_n, \quad t \in \mathbb{R}$$

with associated sine function

$$\Psi(t)v = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle v, \vartheta_n \rangle \vartheta_n, \quad t \in \mathbb{R}$$

It is clear that $\|\Phi(t)\| \leq 1$ for all $t \in \mathbb{R}$. Hence, $\Phi(t)$ is uniformly bounded on $\mathbb{R}$.

We consider the second-order partial differential equation with control

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathcal{V}(t, y) &= \frac{\partial^2}{\partial y^2}\mathcal{V}(t, y) + b(t)\frac{\partial}{\partial y}\mathcal{V}(t, y) + \eta(t + \theta, y) + \mu(t, y) \\ &\quad + g_2(y, \mathcal{V}(t + \theta, y)) + g_3(t, y, \mathcal{V}(t, y))\end{aligned} \tag{6.1}$$

for $t \in I = [0, T], \quad \theta \in [-\tau, 0], \quad 0 \leq y \leq \pi$, subject to the IBCs

$$\begin{aligned}\mathcal{V}(t, 0) &= \mathcal{V}(t, \pi) = 0, \quad t \in [0, T],\ 0 < T < \infty, \\ \mathcal{V}(t, y) &= \phi(t, y), \quad \frac{\partial}{\partial t}\mathcal{V}(0, y) = \mathcal{V}_1(y), \quad y \in [0, \pi], \quad t \in [-\tau, 0]\end{aligned}$$

where $b : \mathbb{R} \to \mathbb{R},\ \mu : I \times [0, \pi] \to [0, \pi]$ are continuous functions. We fix $a > 0$ and set $\beta = \sup_{0 \leq t \leq a} |b(t)|$

We take $\tilde{A}(t)v(y) = b(t)v'(y)$ defined on $H^1(\mathbb{T}, \mathbb{C})$. It is easy to see that $A(t) = A + \tilde{A}(t)$ is a closed linear operator. Initially we will show that $A + \tilde{A}(t)$ generates an evolution operator. It is well-known that the solution of the scalar initial value problem

$$x''(t) = -n^2 x(t) + z(t).$$

$$x(s) = 0, \quad x'(s) = x_1,$$

is given by

$$x(t) = \frac{x_1}{n}\sin n(t-s) + \frac{1}{n}\int_s^t \sin n(t-y)z(y)dy.$$

Therefore, the solution of the scalar initial value problem

$$\begin{aligned} x''(t) &= -n^2 x(t) + inb(t)x(t), \\ x(s) &= 0, \quad x'(s) = x_1, \end{aligned} \tag{6.2}$$

is given by

$$x(t) = \frac{x_1}{n}\sin n(t-s) + i\int_s^t \sin n(t-y)b(y)x(y)dy.$$

Applying Gronwall-Bellman lemma, we have

$$|x(t)| \le \frac{|x_1|}{n}e^{c(t-s)} \tag{6.3}$$

for $s \le t$ and c is a constant. We denote by $x_n(t, s)$ the solution of (6.2).

We define

$$\Psi(t, s)v = \sum_{n=1}^{\infty} x_n(t, s)\langle v, \vartheta_n\rangle\vartheta_n.$$

It follows from the estimate (6.3) that $\Psi(t, s) : X \to X$ is well defined and satisfies the conditions of definition (2.1).

Equation 6.1 with IBCs can be reformulated as the following abstract equation in $X = L^2(\mathbb{T}, \mathbb{C})$:

$$\begin{aligned} &v'' = A(t)v(t) + A_1 v_t + \mathcal{C}u(t) + g(t, v_t, v(t)), \quad t \in I = [0, T], \\ &v(t) = \phi(t), \quad v'(0) = y_0, \quad t \in [-\tau, 0] \end{aligned}$$

where $v(t) = \mathcal{V}(t, \cdot)$ that is $v(t)(y) = \mathcal{V}(t, y)$, $y \in [0, \pi]$.

The function $g : \mathbb{R}_+ \times X \times X \to X$, is given by

$$g(t, \psi, \xi)(y) = g_2(y, \xi) + g_3(t, y, \psi),$$

where $g_2 : [0, \pi] \times X \to H_0^1(\mathbb{T}, \mathbb{C})$ is given by

$$g_2(y, \xi) = \int_0^y K(y, x)\xi(x)dx,$$

and

$$g_3(t, y, \mathcal{V}(t, y)) = \int_0^y K(y, s)\mathcal{V}(s)(c_1|\mathcal{V}(t, s)| + c_2|\mathcal{V}(t, s)|))ds.$$

Let us assume that $c_1, c_2 \ge 0$, $(c_1, c_2) \ne (0, 0)$ and $K : [0, \pi] \times [0, \pi] \to \mathbb{R}$. Also, we have

$$\|g_3(t, y, \psi)\| \le f(y, t)(1 + \|\psi\|_{H^2(\mathbb{T},\mathbb{C})})$$

with $f(\cdot, t) \in X$ and f is continuous in its second argument and $u : I \to U$ be defined as

$$\mathcal{C}(u(t))(y) = \mu(t, y), \quad y \in [0, \pi],$$

where $\mu : I \times [0, \pi] \to [0, \pi]$ is continuous. $H : [-\tau, 0] \times [0, \pi] \to [0, \pi]$ be defined by as

$$A_1 \mathcal{V}(t + \theta, y) = \eta(t + \theta, y), \quad y \in [0, \pi], \quad \theta \in [-\tau, 0],$$

where $\eta : [-\tau, 0] \times [0, \pi] \to [0, \pi]$ is continuous.

It can be easily verified that the function g satisfies the assumptions (A2)–(A3). For more details, see [19]. Thus, Theorem 3.2 can be applied to the problem (6.1).

Acknowledgements We are grateful to the anonymous reviewer and editor. Their valuable suggestions and comments helped us to improve the quality of this manuscript.

Funding Information This study was financially supported by the Council of Scientific and Industrial Research (CSIR), Government of India, under research project no-25(0268)/17/EMR-II.

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