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ORIGINAL RESEARCH

Blow-up of Solutions to a *p*-Kirchhoff-Type Parabolic Equation with General Nonlinearity

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Abstract

In this paper, finite time blow-up property of solutions to a p-Kirchhoff-type parabolic equation with general nonlinearity is considered. Some sufficient conditions are given for the weak solutions to blow up in finite time. An upper bound for the blow-up time is also derived. The results partially generalize some recent ones reported by Han and Li (Comput Math Appl. 2018;75:3283–3297).

Keywords p-Kirchhoff · Blowup · Upper bound · General nonlinearity

Mathematics Subject Classification (2010) 35K20; 35K92

1 Introduction

In the past few years, partial differential equations with different kinds of nonlocal terms have drawn more and more attentions to mathematicians, physicists, and biologist due to their wide applications in both physics and biology. For example, the following hyperbolic equation with a nonlocal coefficient are as follows:

$$\varepsilon u_{tt}^{\varepsilon} + u_t^{\varepsilon} - M\left(\int_{\Omega} |\nabla u^{\varepsilon}|^p \mathrm{d}x\right) \Delta_p u^{\varepsilon} = f(x, t, u^{\varepsilon}) \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ is a potential model for the damped small transversal vibrations of an elastic string ($n = 1$) or membrane ($n = 2$) with uniform density ε (see [1]). By taking $\varepsilon = 0$ in Eq. 1.1 formally, one obtains the following p-Kirchhoff-type parabolic equation:

$$u_t - M\left(\int_{\Omega} |\nabla u|^p \mathrm{d}x\right) \Delta_p u = f(x, t, u). \tag{1.2}$$

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One of the main features of Eqs. (1.1) and (1.2) is that the coefficient of $\Delta_p u^\varepsilon$ or $\Delta_p u$ depends on the integration of the gradient of the unknown, which means that the equation is no longer a pointwise identity. Since similar models were first proposed by Kirchhoff [2], such equations are usually referred to as p-Kirchhoff equations or nonlocal equations. When $p = 2$, we briefly call them Kirchhoff equations. Other justifications of nonlocal models lies in the fact that, in reality, the measurements are not made pointwise but through some local average. Interested readers may refer to [3–5] and the references therein for more interesting features of nonlocal problems.

Although Kirchhoff-type problems have been proposed for a long time, it was only after the pioneer work of Lions [6] that the existence, uniqueness, and regularities of solutions to Kirchhoff-type equations were well studied, for both the evolution cases and their elliptic counterpart as follows:

$$-M\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\Delta_p u = f(x,u). \tag{1.3}$$

We first review some works on Kirchhoff or p-Kirchhoff elliptic problems. When $p = 2$, Ma et al. [7] showed the existence of positive solutions to Eq. 1.3 (coupled with homogeneous Dirichlet boundary condition) under some restrictions on $M(s)$ and $f(x,u)$, by the variational method and minimization arguments. In 2005, Alves et al. [8] proved the existence of positive solutions to the same problem when $M(s)$ is a non-increasing function that does not grow too fast in a suitable interval near zero and when $f(x,u)$ satisfies the so-called Ambrosetti-Rabinowitz condition, by the truncation argument and uniform a priori estimates due to Gidas and Spruck [9]. When $M(s)$ is increasing, the existence of positive solutions was obtained in, for example [10], by applying Yang index. When $f(x,u) = \lambda h(x)|u|^{q-2}u + g(x)|u|^{r-2}u$ with $1 < q < 2 < r < 2^*$, Chen et al. [11] obtained the existence and multiplicity of positive solutions to Eq. 1.3 for $r < 4$, $r = 4$, and $r > 4$, respectively, by using Nehari's manifold and fibering maps.

As for the general p-Kirchhoff-type elliptic problems, by the Krasnoselskii's genus, Corrêa et al. [12] showed the existence and multiplicity of solutions to Eq. 1.3 when the nonlinearity is nonnegative and satisfies subcritical growth condition. By using the Fountain theorem and dual fountain theorem, Liu [13] established the existence of infinitely many solutions. When $f(x,u)$ satisfies critical growth condition, Hamydy et al. [14] showed that problem Eq. 1.3 admits at least one nontrivial solution by the variational method.

With regard to the study of Kirchhoff type parabolic problems, there are also some works. When the diffusion coefficient $M(s)$ satisfies $0 < m \le M(s) \le M_0$ for all $s \ge 0$ and the nonlinearity is replaced by $f(x)$, global existence, uniqueness, and asymptotic behavior of weak or strong solutions to Eq. 1.2 have been investigated by Chipot et al. [15] and Zheng et al. [16] for the case $p = 2$. Later, Chipot et al. [17] extended such results to p-Kirchhoff problems. In 2016, Fu et al. [18] obtained the local existence of weak solutions to the corresponding $p(x,t)$-Kirchhoff problem, by using Galerkin approximation and some a priori estimates in the framework of variable exponent Sobolev spaces. On the other hand, when $f(x,t,u)$ grows super-linearly in u at infinity, the solutions to problem Eq. 1.2 may blow up in finite time. Recently, for the case $p = 2$, $M(s) = a + bs$, $a, b > 0$, and $f(x,t,u) = |u|^{q-1}u$ with $q \in (3, 2^* - 1)$, Han and Li [19] obtained the threshold results for the existence of global or finite time blowing-up solutions to problem Eq. 1.2, when the initial energy is subcritical, critical, and supercritical, by applying the modified potential well method and some variational ideas.

Motivated mainly by [19], in this short paper, we will confine ourselves to the finite time blow-up of solutions to the following p-Kirchhoff parabolic problem:

$$\begin{cases} u_t - M(\int_\Omega |\nabla u|^p \mathrm{d}x)\Delta_p u = f(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $M(s) = a + bs$ with positive parameters a, b, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $p \geq 2$, $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and the initial datum u_0 satisfies $0 \not\equiv u_0 \in W_0^{1,p}(\Omega)$. The nonlinearity f satisfies the following two assumptions.

(A1) $f \in C^1(\mathbb{R})$, and there exists a constant $q \geq 2p - 1$ such as the following:

$$s[sf'(s) - qf(s)] \geq 0, \qquad \forall\, s \in \mathbb{R};$$

(A2) There exist a positive integer l and constants $a_k > 0 (1 \leq k \leq l)$ such as the following:

$$|f(s)| \leq \sum_{k=1}^{l} a_k |s|^{p_k}, \qquad \forall\, s \in \mathbb{R},$$

where,

$$1 < p_l < p_{l-1} < \cdots < p_1 < \frac{np - n + p}{n - p}, \qquad n > p;$$
$$1 < p_l < p_{l-1} < \cdots < p_1 < +\infty, \qquad n \leq p.$$

By applying the first-order differential inequality method and concavity argument, we will give some sufficient conditions for the solutions to problem Eq. 1.4 to blow-up in finite time. Moreover, an upper bound for the blow-up time is also derived. The main feature of this study is that we consider a quite general nonlinearity f, a prototype of which is a combination of some power type nonlinearities. In particular, our research includes the case that $f(s) = |s|^{q-1}s$ with $q \in (2p - 1, p^*)$. Therefore, even for the case $p = 2$, our results are more general than those obtained in [19]. There are also many important works dealing with blow-up properties of solutions by using first-order differential inequality method or concavity arguments, among which, we only mention [20, 21].

The remainder of this paper is organized as follows. Some necessary notations, definitions, and lemmas will be presented in Section 2, and the main results will be stated and proved in Section 3.

2 Preliminaries

We begin this section with some notations and definitions. Denote by $\|u\|_r$, the $L^r(\Omega)$ norm of a Lebesgue function $u \in L^r(\Omega)$ for $r \geq 1$ and by $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$. We use $W_0^{1,p}(\Omega)$ to denote the well-known Sobolev space such that both u and $|\nabla u|$ are in $L^p(\Omega)$ equipped with the norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$, which, due to Poincaré's inequality, is equivalent to the standard one. For $u \in W_0^{1,p}(\Omega)$, define the potential energy functional and the Nehari's functional, respectively, by the following:

$$J(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \int_\Omega F(u)\mathrm{d}x, \tag{2.1}$$

$$I(u) = a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} - \int_\Omega uf(u)\mathrm{d}x, \tag{2.2}$$

where $F(s) = \int_0^s f(t)\mathrm{d}t$. It is easily seen from (A2) that both $J(u)$ and $I(u)$ are well-defined and continuous on $W_0^{1,p}(\Omega)$. In order to investigate the relationship between $J(u)$ and $I(u)$, we need the following lemma.

Lemma 2.1 *If f satisfies (A1), then it holds, for any $s \in \mathbb{R}$, as follows;*

$$sf(s) \geq (q+1)F(s). \tag{2.3}$$

Proof First, we assume that $s \geq 0$. Then, it follows from (A1) that $sf'(s) \geq qf(s)$ and

$$sf(s) - \int_0^s f(t)\mathrm{d}t = \int_0^s tf'(t)\mathrm{d}t \geq q\int_0^s f(t)\mathrm{d}t = qF(s),$$

which yields Eq. 2.3.

When $s < 0$, one obtains from (A1) that $sf'(s) \leq qf(s)$. Integration of this inequality over $[s, 0]$ results in the same inequality. The proof is complete. □

Remark 2.1 For any $u \in W_0^{1,p}(\Omega)$, due to $q \geq 2p-1$, it can be checked directly from Eqs. 2.1– 2.3 as follows:

$$\begin{aligned} I(u) &\leq a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} - (q+1)\int_\Omega F(u)\mathrm{d}x \\ &= -\frac{a(q+1-p)}{p}\|\nabla u\|_p^p - \frac{b(q+1-2p)}{2p}\|\nabla u\|_p^{2p} + (q+1)J(u) \\ &\leq -\frac{a(q+1-p)}{p}\|\nabla u\|_p^p - \frac{b(q+1-2p)}{2p}(2\|\nabla u\|_p^p - 1) + (q+1)J(u) \\ &= -\frac{a(q+1-p)+b(q+1-2p)}{p}\|\nabla u\|_p^p + (q+1)J(u) + \frac{b(q+1-2p)}{2p}. \end{aligned} \tag{2.4}$$

Let $S_* = \inf_{u\in W_0^{1,p}(\Omega)\backslash\{0\}} \dfrac{\|\nabla u\|_p}{\|u\|_2}$. Then $S_* > 0$ and

$$S_*\|u\|_2 \leq \|\nabla u\|_p, \qquad \forall\, u \in W_0^{1,p}(\Omega). \tag{2.5}$$

Denote $C_0 = \dfrac{[a(q+1-p)+b(q+1-2p)]S_*^p}{p(q+1)}$, $C_1 = \dfrac{b(q+1-2p)}{2p(q+1)}$ and $C_2 = C_0 + C_1$. Then $C_0, C_2 > 0$, and $C_1 \geq 0$. Recalling that $p \geq 2$, we have $s^2 \leq s^p + 1$ for any $s \geq 0$, which, together with Eq. 2.4, yields the following:

$$\begin{aligned} I(u) &\leq -(q+1)[C_0\|u\|_2^p - J(u) - C_1] \\ &\leq -(q+1)[C_0(\|u\|_2^2 - 1) - J(u) - C_1] \\ &= -(q+1)[C_0\|u\|_2^2 - J(u) - C_2]. \end{aligned} \tag{2.6}$$

In this paper, we consider weak solutions to Problem Eq. 1.4. Local existence of weak solutions to Problem Eq. 1.4 can be proved by combining the standard Galerkin's approximation with Aubin-Lions compactness theorem. We refer to [18] for the proof of local solutions to a $p(x,t)$-Kirchhoff equation, which contains Eq. 1.4 as a special case.

Definition 2.1 A function $u = u(x,t) \in L^\infty(0,T; W_0^{1,p}(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ is called a weak solution to problem Eq. 1.4, if $u(x,0) = u_0 \in W_0^{1,p}(\Omega)$ and $u(x,t)$ satisfies

$$(u_t, \phi) + \left((a + b\int_\Omega |\nabla u|^p \mathrm{d}x)|\nabla u|^{p-2}\nabla u, \nabla\phi\right) = (f(u), \phi), \quad a.e.\ t \in (0,T), \quad (2.7)$$

for any $\phi \in W_0^{1,p}(\Omega)$.

We say that $u(x,t)$ blows up at a finite time T provided as follows:

$$\lim_{t\to T^-} \|u(x,t)\|_2^2 = +\infty.$$

To derive the upper bound of the blow-up time of $u(x,t)$ when the initial energy satisfies $J(u_0) + C_2 \geq 0$, the following basic lemma is needed.

Lemma 2.2 *(See [22]) Suppose that a positive, twice-differentiable function $\psi(t)$ satisfies the inequality as follows:*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0,$$

where $\theta > 0$. If $\psi(0) > 0$, $\psi'(0) > 0$, then $\psi(t) \to \infty$ as $t \to t_* \leq t^* = \dfrac{\psi(0)}{\theta\psi'(0)}$.

Lemma 2.3 *Let $T > 0$ be the maximal existence time of the solution $u(x,t)$ to problem* Eq. 1.4 *and let $J(u)$ and $I(u)$ be given in* Eqs. 2.1 *and* 2.2*, respectively. Then the following statements hold:*

(i)
$$\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + J(u(x,t)) = J(u_0), \qquad a.e.\ t \in (0,T), \qquad (2.8)$$

(ii)
$$\frac{d}{dt}\left(\frac{1}{2}\|u(x,t)\|_2^2\right) = -I(u(x,t)) = (u, u_t), \qquad a.e.\ t \in (0,T). \qquad (2.9)$$

Proof Equation 2.8 can be accomplished by taking u_t as a test function in Eq. 2.7 for smooth solutions. By approximation, it is not difficult to check that Eq. 2.8 also holds for weak solutions. In particular, it implies that $J(u(x,t))$ is non-increasing in t. Equation 2.9 follows by choosing u as a test function in Eq. 2.7. The proof is complete. □

3 Main results

With the help of the lemmas given in Section 2, we can now state and prove our main results. For brevity, we will always use $u(t)$ to denote the weak solution $u(x,t)$ to problem Eq. 1.4 if no confusion arises.

Theorem 3.1 *Suppose that $f(s)$ satisfies (A1)–(A2) and that $u(t)$ is a weak solution to problem* Eq. 1.4. *Suppose that one of the following statements holds:*

(i) $J(u_0) < 0$;
(ii) $0 \leq J(u_0) < C_0\|u_0\|_2^2 - C_2$, *where C_0 and C_2 are the constants given in Remark 2.1.*

Then $T < +\infty$, which means that $u(t)$ blows up in finite time. Moreover, an upper bound for T has the following form:

In case (i), $T \leq \dfrac{\|u_0\|_2^2}{(1-q^2)J(u_0)}$.

In case (ii), $T \leq \dfrac{4q\|u_0\|_2^2}{(q-1)^2(q+1)[C_0\|u_0\|_2^2 - J(u_0) - C_2]}$.

Proof (i) The proof of this case is inspired by some ideas from [23]. Set

$$L(t) = \frac{1}{2}\|u(t)\|_2^2, \quad K(t) = -J(u(t)),$$

then $L(0) > 0$, $K(0) > 0$. By Eq. 2.8, one has the following:

$$K'(t) = -\frac{d}{dt}J(u(t)) = \|u_t(t)\|_2^2 \geq 0,$$

which implies that $K(t) \geq K(0) > 0$ for all $t \in [0, T)$. Recalling Eqs. 2.4, 2.9, and noticing that $q \geq 2p-1$, one obtains, for any $t \in (0, T)$, that

$$\begin{aligned} L'(t) &= (u, u_t) = -I(u(t)) \\ &\geq \frac{a(q+1-p)}{p}\|\nabla u(t)\|_p^p + \frac{b(q+1-2p)}{2p}\|\nabla u(t)\|_p^{2p} - (q+1)J(u(t)) \\ &\geq (q+1)K(t). \end{aligned} \tag{3.1}$$

Combining Eq. 3.1 with Cauchy-Schwarz inequality, we get the following:

$$L(t)K'(t) = \frac{1}{2}\|u(t)\|_2^2\|u_t(t)\|_2^2 \geq \frac{1}{2}(u, u_t)^2 = \frac{1}{2}(L'(t))^2 \geq \frac{q+1}{2}L'(t)K(t). \tag{3.2}$$

It follows from Eq. 3.2 as follows:

$$\left(K(t)L^{-\frac{q+1}{2}}(t)\right)' = L^{-\frac{q+3}{2}}(t)\left(K'(t)L(t) - \frac{q+1}{2}K(t)L'(t)\right) \geq 0.$$

Therefore,

$$\begin{aligned} 0 < \kappa &\triangleq K(0)L^{-\frac{q+1}{2}}(0) \leq K(t)L^{-\frac{q+1}{2}}(t) \leq \frac{1}{q+1}L'(t)L^{-\frac{q+1}{2}}(t) \\ &= \frac{2}{1-q^2}\left(L^{\frac{1-q}{2}}(t)\right)'. \end{aligned} \tag{3.3}$$

Integrating Eq. 3.3 over $[0, t]$ for any $t \in (0, T)$ and noticing that $q \geq 2p-1 (> 1)$, we obtain the following:

$$\kappa t \leq \frac{2}{1-q^2}\left(L^{\frac{1-q}{2}}(t) - L^{\frac{1-q}{2}}(0)\right),$$

or equivalently

$$0 \leq L^{\frac{1-q}{2}}(t) \leq L^{\frac{1-q}{2}}(0) - \frac{q^2-1}{2}\kappa t, \qquad t \in (0, T). \tag{3.4}$$

Obviously, Eq. 3.4 cannot hold for all $t > 0$. Therefore, $T < +\infty$. Moreover, it can be inferred from Eq. 3.4 as follows:

$$T \leq \frac{2}{(q^2-1)\kappa}L^{\frac{1-q}{2}}(0) = \frac{\|u_0\|_2^2}{(1-q^2)J(u_0)}.$$

(ii) To deal with the case $0 \le J(u_0) < C_0\|u_0\|_2^2 - C_2$, we apply the concavity arguments from [24, 25]. Similar treatments have also been applied to investigate the blow-up properties of solutions to some other evolution problems ([26–28]). First, from Eq. 2.6 and the assumption (ii), we have the following:

$$I(u_0) \le -(q+1)[C_0\|u_0\|_2^2 - J(u_0) - C_2] < 0.$$

We claim that $I(u(t)) < 0$ for all $t \in [0, T)$. If not, by continuity, there would exist a $t_0 \in (0, T)$ such that $I(u(t)) < 0$ for all $t \in [0, t_0)$ and $I(u(t_0)) = 0$. By Eq. 2.9, $\|u(t)\|_2^2$ is strictly increasing in t for $t \in [0, t_0)$, and therefore,

$$0 \le J(u_0) < C_0\|u_0\|_2^2 - C_2 < C_0\|u(t_0)\|_2^2 - C_2. \tag{3.5}$$

On the other hand, from the monotonicity of $J(u(t))$ and Eq. 2.6, we obtain the following:

$$\begin{aligned} J(u_0) \ \ge\ J(u(t_0)) &\ge \frac{1}{q+1} I(u(t_0)) + C_0\|u(t_0)\|_2^2 - C_2 \\ &= C_0\|u(t_0)\|_2^2 - C_2, \end{aligned}$$

a contradiction. Therefore, $I(u(t)) < 0$ for all $t \in [0, T)$ as claimed, and $\|u(t)\|_2^2$ is strictly increasing on $[0, T)$.

For any $T^* \in (0, T)$, $\beta > 0$ and $\sigma > 0$, define the following:

$$F(t) = \int_0^t \|u(\tau)\|_2^2 \mathrm{d}\tau + (T^* - t)\|u_0\|_2^2 + \beta(t+\sigma)^2, \qquad t \in [0, T^*]. \tag{3.6}$$

Then, $F(0) = T^*\|u_0\|_2^2 + \beta\sigma^2 > 0$. Taking the derivative with respect to t, one has the following:

$$\begin{aligned} F'(t) &= \|u(t)\|_2^2 - \|u_0\|_2^2 + 2\beta(t+\sigma) = \int_0^t \frac{d}{d\tau}\|u(\tau)\|_2^2 \mathrm{d}\tau + 2\beta(t+\sigma) \\ &= 2\int_0^t (u, u_\tau)\mathrm{d}\tau + 2\beta(t+\sigma), \end{aligned} \tag{3.7}$$

and $F'(0) = 2\beta\sigma > 0$. Taking the derivative again, recalling Eqs. 2.6 and 2.8 and noticing the monotonicity of $\|u(t)\|_2^2$, one obtains the following:

$$\begin{aligned} F''(t) &= 2(u, u_t) + 2\beta = -2I(u(t)) + 2\beta (> 0) \\ &\ge 2(q+1)[C_0\|u(t)\|_2^2 - J(u(t)) - C_2] + 2\beta \\ &\ge 2(q+1)\left[C_0\|u_0\|_2^2 - J(u_0) - C_2 + \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau\right] + 2\beta. \end{aligned} \tag{3.8}$$

Since $F''(t) > 0$ on $[0, T^*]$, $F'(t)$ is monotone increasing on $[0, T^*]$ and $F'(t) \ge F'(0) > 0$, which further implies that $F(t)$ is strictly increasing on $[0, T^*]$. For $t \in [0, T^*]$, set

$$\xi(t) = \left(\int_0^t \|u(\tau)\|_2^2 \mathrm{d}\tau + \beta(t+\sigma)^2\right)\left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta\right) - \left(\int_0^t (u, u_\tau)\mathrm{d}\tau + \beta(t+\sigma)\right)^2.$$

It is easily checked by using Cauchy-Schwarz inequality and Hölder's inequality that $\xi(t)$ is nonnegative on $[0, T^*]$. Therefore, in view of Eqs. 3.6– 3.8, we have the following:

$$\begin{aligned}
& F(t)F''(t) - \frac{q+1}{2}(F'(t))^2 \\
= \; & F(t)F''(t) - 2(q+1)\left(\int_0^t (u, u_\tau)\mathrm{d}\tau + \beta(t+\sigma)\right)^2 \\
= \; & F(t)F''(t) + 2(q+1)\left[\xi(t) - (F - (T^* - t)\|u_0\|_2^2)(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta)\right] \\
\geq \; & F(t)F''(t) - 2(q+1)F(t)(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta) \\
\geq \; & 2(q+1)F(t)\left[C_0\|u_0\|_2^2 - J(u_0) - C_2 - \frac{q\beta}{q+1}\right] \\
\geq \; & 0,
\end{aligned} \tag{3.9}$$

for any $t \in [0, T^*]$ and $\beta \in (\, 0, \frac{q+1}{q}(C_0\|u_0\|_2^2 - J(u_0) - C_2)\,]$. Recalling Lemma 2.2, we see the following:

$$T^* \leq \frac{2F(0)}{(q-1)F'(0)} = \frac{2(T^*\|u_0\|_2^2 + \beta\sigma^2)}{2(q-1)\beta\sigma} = \frac{\|u_0\|_2^2}{(q-1)\beta\sigma}T^* + \frac{\sigma}{q-1},$$

or

$$T^*\left(1 - \frac{\|u_0\|_2^2}{(q-1)\beta\sigma}\right) \leq \frac{\sigma}{q-1}, \tag{3.10}$$

for any $\beta \in (\, 0, \frac{q+1}{q}(C_0\|u_0\|_2^2 - J(u_0) - C_2)\,]$ and $\sigma > 0$.

To derive an upper bound for T^*, we first fix a $\beta_0 \in (\, 0, \frac{q+1}{q}(C_0\|u_0\|_2^2 - J(u_0) - C_2)\,]$. Then for any $\sigma \in (\frac{\|u_0\|_2^2}{(q-1)\beta_0}, +\infty)$, we have $0 < \frac{\|u_0\|_2^2}{(q-1)\beta_0\sigma} < 1$, which, together with Eq. 3.10, implies the following:

$$T^* \leq \frac{\sigma}{q-1}\left(1 - \frac{\|u_0\|_2^2}{(q-1)\beta_0\sigma}\right)^{-1} = \frac{\beta_0\sigma^2}{(q-1)\beta_0\sigma - \|u_0\|_2^2}. \tag{3.11}$$

Minimizing the right-hand side of Eq. 3.11 for $\sigma \in \left(\frac{\|u_0\|_2^2}{(q-1)\beta_0}, +\infty\right)$, we get the following:

$$T^* \leq \frac{4\|u_0\|_2^2}{(q-1)^2\beta_0}, \quad \beta_0 \in \left(0, \frac{q+1}{q}(C_0\|u_0\|_2^2 - J(u_0) - C_2)\right]. \tag{3.12}$$

Minimizing the right-hand side of Eq. 3.12 for $\beta_0 \in (\, 0, \frac{q+1}{q}(C_0\|u_0\|_2^2 - J(u_0) - C_2)\,]$, we finally obtain the following:

$$T^* \leq \frac{4q\|u_0\|_2^2}{(q-1)^2(q+1)[C_0\|u_0\|_2^2 - J(u_0) - C_2]}.$$

By the arbitrariness of $T^* < T$, it follows the following:

$$T \leq \frac{4q\|u_0\|_2^2}{(q-1)^2(q+1)[C_0\|u_0\|_2^2 - J(u_0) - C_2]}.$$

The proof is complete. □

Remark 3.1 By assumptions (A1)–(A2) and applying similar argument to that in [24], we can show that for any given constant $D > 0$, there exists an initial datum u_0 such that $J(u_0) > D$ and (ii) in Theorem 3.1 holds. This means that problem Eq. 1.4 admits blow-up solutions at an arbitrarily high initial energy level.

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