Journal of Dynamical and Control Systems (2020) 26:311–331
https://doi.org/10.1007/s10883-019-09452-7

Existence and Stability of Traveling Waves for Infinite-Dimensional Delayed Lattice Differential Equations

Ge Tian[1] · Lili Liu[1] · Zhi-Cheng Wang[1]

Received: 22 October 2018 / Revised: 2 June 2019 / Published online: 3 July 2019

Abstract
In this paper, we study the existence and stability of traveling waves of infinite-dimensional lattice differential equations with time delay, where the equation may be not quasi-monotone. Firstly, by applying Schauder's fixed point theorem, we get the existence of traveling waves with the speed $c > c_*$ (here c_* is the minimal wave speed). Using a limiting argument, the existence of traveling waves with wave speed $c = c_*$ is also established. Secondly, for sufficiently small initial perturbations, the asymptotic stability of the traveling waves $\mathbf{\Phi} := \{\Phi(n + ct)\}_{n\in\mathbb{Z}}$ with the wave speed $c > c_*$ is proved. Here we emphasize that the traveling waves $\mathbf{\Phi} := \{\Phi(n + ct)\}_{n\in\mathbb{Z}}$ may be non-monotone.

Keywords Lattice differential equations · Traveling wave solutions · Stability · Weighted energy · Time delay

Mathematics Subject Classification (2010) 34A33 · 35C07 · 35B35 · 92D25

1 Introduction

This paper is concerned with the traveling waves of infinite-dimensional lattice differential equations with time delay

$$\begin{cases} \frac{d}{dt} w_n(t) = \rho(J \star w - w)_n(t) - \delta w_n(t) + (R \otimes f(w))_n(t - \tau), & t > 0, \\ w_n(s) = w_n^0(s), \quad s \in [-\tau, 0], \quad n \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $(J \star w)_n(t) := \Sigma_{i\in\mathbb{Z}\setminus\{0\}} J(i) w_{n-i}(t)$ and $(R \otimes f(w))_n(t) := \Sigma_{i\in\mathbb{Z}} R(i) f(w_{n-i}(t))$. Here $w_n(t)$ represents the matured population density in the n-th patch environment at

✉ Zhi-Cheng Wang
wangzhch@lzu.edu.cn

[1] School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, Gansu, People's Republic of China

the time t, $\rho > 0$ represents the diffusion coefficient of matured population, and τ is the maturation delay. The kernels $J(\cdot)$ and $R(\cdot)$ satisfy

(K1) $J(\cdot) : \mathbb{Z} \to \mathbb{R}^+$ and $R(\cdot) : \mathbb{Z} \to \mathbb{R}^+$ are even.
(K2) $\Sigma_{i\in\mathbb{Z}\setminus\{0\}} J(i) = 1,\ \Sigma_{i\in\mathbb{Z}} R(i) = 1$.
(K3) There is $\hat{\lambda}$ such that $\Sigma_{i\in\mathbb{Z}\setminus\{0\}} e^{\lambda i} J(i)$ and $\Sigma_{i\in\mathbb{Z}} e^{\lambda i} R(i)$ are convergent for every $\lambda \in [0, \hat{\lambda})$, and at least one of $\lim_{\lambda\uparrow\hat{\lambda}} \Sigma_{i\in\mathbb{Z}\setminus\{0\}} e^{\lambda i} J(i) = +\infty$ and $\lim_{\lambda\uparrow\hat{\lambda}} \Sigma_{i\in\mathbb{Z}} e^{\lambda i} R(i) = +\infty$ hold, where $\hat{\lambda}$ may be $+\infty$.

The term $\rho \Sigma_{i\in\mathbb{Z}\setminus\{0\}} J(k-i) w_i(t)$ indicates the individuals jump from all other points to point k, and the population mobile from point k to all other points is denoted as $-\rho w_k(t)$. The function $f(\cdot)$ denotes the birth function, and the death rate is denoted by δ.

As we know, the traveling waves can reveal certain dynamical behavior of the scientific inquiry. Thus, it is significant to investigate traveling wave solutions of Eq. 1.1. A traveling wave solution of Eq. 1.1 is a solution in the form of $\mathbf{w}(t) = \{w_n(t)\}_{n\in\mathbb{Z}} = \{\Phi(n+ct)\}_{n\in\mathbb{Z}}$, and it satisfies

$$\begin{cases} c\Phi'(\xi) = \rho(J \star \Phi - \Phi)(\xi) - \delta\Phi(\xi) + (R \otimes f(\Phi))(\xi - c\tau), \\ \lim_{\xi\to-\infty} \Phi(\xi) = 0, \quad \liminf_{\xi\to+\infty} \Phi(\xi) > 0, \end{cases} \tag{1.2}$$

where $(J \star \Phi)(\xi) := \Sigma_{i\in\mathbb{Z}\setminus\{0\}} J(i)\Phi(\xi - i)$ and $(R \otimes f(\Phi))(\xi - c\tau) := \Sigma_{i\in\mathbb{Z}} R(i) f(\Phi(\xi - i - c\tau))$.

The traveling wave solutions of lattice differential equations with or without delay have been widely studied, see [1–8, 10, 12–14, 22–26]. Chen and Guo [1, 2] proved the existence and uniqueness of traveling fronts in the following lattice equations with monostable nonlinearity

$$\frac{d}{dt} w_n(t) = (\Delta g(w))_n(t) + f(w_n)(t), \quad n \in \mathbb{Z},$$

where $(\Delta g(w))_n(t) := g(w_{n+1})(t) - 2g(w_n)(t) + g(w_{n-1})(t)$, see also in [3]. Weng et al. [22] proposed and studied a model which describes the growth of a single specie with age structure living in a patchy environment

$$\frac{d}{dt} u_n(t) = \frac{\rho}{2}(\Delta w)_n(t) - \delta w_n(t) + (R \otimes f(w))_n(t - \tau), \tag{1.3}$$

where $(R \otimes f(w))_n(t-\tau) = \Sigma_{i\in\mathbb{Z}} R(i) f(w_{n-i}(t-\tau))$. When the birth function $f(\cdot)$ in Eq. 1.3 is monostable and monotone, they showed that traveling fronts with speed $c > c_*$ exist and the minimal wave speed c_* is also the spreading speed of Eq. 1.3. Ma and Zou [12] also studied the traveling wave solutions of Eq. 1.3 with quasi-monotone bistable nonlinearity. When the birth function $f(\cdot)$ is monostable and non-monotone, Fang et al. [6] established the existence of traveling wave solutions and the spreading speed of Eq. 1.3. It is clear that in Eq. 1.3, spatial diffusion occurs only in the local effects of adjacent patches. To model the effects of the arbitrary movement of the population, Ma et al. [14] proposed the more general lattice differential Eq. 1.1. When the function $f(\cdot)$ is monostable and monotone, they established the spreading speed and existence of traveling fronts of Eq. 1.1. However, when the function $f(\cdot)$ is monostable and non-monotone, the spreading speed and existence of traveling waves of Eq. 1.1 are unknown so far. This is our first objective in this paper.

We first study the spreading speed by comparison arguments and a fluctuation method, and then establish the existence of traveling waves by Schauder's fixed point theorem and a limiting process. Although the method used in this paper is standard and similar to that of Fang et al. [6], the construction of our super- and subsolutions to prove the existence of traveling waves is different from that in [6]. In fact, Fang et al. [6] constructed super- and subsolutions by using traveling waves of two auxiliary systems, while in this paper, we construct super- and subsolutions by using the eigenfunction of the linearized equation. Hence, we can get the exact exponential asymptotic behavior of traveling waves at minus infinity (see Eq. 1.4), which will be very useful to establish the uniqueness and stability of traveling waves. The following assumptions are needed to establish the spreading speed and existence of traveling waves.

(H1) 0 and K are two equilibrium points, namely, $f(0) = f(K) - \delta K = 0$. Furthermore, assume that $f'(0) > \delta$, $f'(K) < \delta$ and $f(w) \neq \delta w$ for $w \geq 0$ with $w \neq 0, K$.
(H2) $f(\cdot) : [0, \infty) \to \mathbb{R}^+$ is of $\mathbb{C}^2$, and $f'(0)w \geq f(w) > 0$ for any $w > 0$.

The assumption (H1) shows that Eq. 1.1 is a monostable system, while the birth function $f(\cdot)$ in Eq. 1.1 may be non-monotone. Define $f^*(w) := \max_{v\in[0,w]} f(v)$ for $w \geq 0$. According to the assumptions (H1) and (H2), the equation $f^*(w) = \delta w$ has the smallest positive root $K^* \geq K > 0$. Define $f_*(w)$ by $f_*(w) = \min_{v\in[w,K^*]} f(v)$ for $w \in [0, K^*]$ and $f_*(w) = f(w)$ for $w > K^*$. Then the equation $f_*(w) = \delta w$ has the smallest positive solution $K_* \in [0, K^*]$. Based on the above assumptions, we have the following conclusions, which will be proved in Section 2.

Theorem 1.1 (**Spreading speed**) *Suppose that* (K1)-(K3) *and* (H1)-*(H2) hold. Let* $\mathbf{w}(t) := \{w_n(t)\}_{n\in\mathbb{Z}}$ *be the unique global solution of* Eq. 1.1 *with the initial value* $\mathbf{w^0} := \{w_n^0(s)\}_{n\in\mathbb{Z}}$*, where* $u_n^0(s) \in [0, K^*]$ *for* $s \in [-\tau, 0]$*. Then we get:*

(**1**) *When* $w_n^0(s) = 0, \forall\, s \in [-\tau, 0]$, $|n| \geq k > 0$, *there holds* $\lim\limits_{t\to\infty, |n|\geq ct} w_n(t) = 0, \ \forall c > c_*$;
(**2**) *When* $w_n^0(\cdot) \not\equiv 0$ *on* $[-\tau, 0]$ *for some* $n \in \mathbb{Z}$*, there holds*

$$K^* \geq \limsup_{t\to\infty, |n|\leq ct} w_n(t) \geq \liminf_{t\to\infty, |n|\leq ct} w_n(t) \geq K_*, \ \forall\, c \in (0, c_*).$$

Furthermore, $\lim_{t\to\infty, |n|\leq ct} w_n(t) = K$ *once the following assumption holds,*

(F) $\frac{f(u)}{u} < \frac{f(v)}{v}$ *for* $u, v \in [K_*, K^*]$ *satisfying* $u > v$*. In particular, there must be* $u = v$ *provided that* u, v *satisfy* $K^* \geq u \geq K \geq v \geq K_*$, $\delta v \geq f(u)$*, and* $\delta u \leq f(v)$*.*

Theorem 1.2 (**Existence of traveling waves**) *Suppose that* (K1)-(K3) *and* (H1)-*(H2) hold. Then* Eq. 1.1 *has a traveling wave solution* $\mathbf{\Phi} := \{w_n(t)\}_{n\in\mathbb{Z}} = \{\Phi(n + ct)\}_{n\in\mathbb{Z}}$ *with* $c \geq c_*$ *satisfying*

$$\lim_{\xi\to-\infty} e^{-\lambda_1\xi}\Phi(\xi) = 1 \quad \text{and} \quad K^* \geq \limsup_{\xi\to+\infty} \Phi(\xi) \geq \liminf_{\xi\to+\infty} \Phi(\xi) \geq K_* > 0, \qquad (1.4)$$

where $\xi = n + ct$*. Furthermore, if the assumption* (F) *holds, then* $\lim_{\xi\to+\infty} \Phi(\xi) = K$*. In addition, if* $c \in (0, c_*)$*,* Eq. 1.1 *has no traveling wave satisfying* $0 \leq \Phi(\xi) \leq K^*$ *for all* $\xi \in \mathbb{R}$ *and* $\liminf_{\xi\to-\infty} \Phi(\xi) < K_*$*.*

Here we would like to note that functions $f(w) = pwe^{-aw}$ and $f(w) = \frac{pw}{a+aw^q}$, where $p > 0$, $q > 0$, and $a > 0$, are typical examples satisfying the assumptions (H1)-(H2) and (F).

Besides the spreading speed and existence of traveling waves, the stability is also a central question in the study of traveling waves. For lattice differential Eqs. 1.1 and 1.3, if the spatial non-local effects were not considered, there have been many results about the stability of traveling waves [1, 2, 5, 13, 21, 24] whether the function f is monotone or non-monotone. For Eqs. 1.1 and 1.3 with the spatial non-local effects, the global stability of traveling waves was studied by Zhang [26] only for the case when the function $f(\cdot)$ is monostable and monotone. However, when the function $f(\cdot)$ is *non-monotone* and monostable, there are few results on the stability of traveling waves of Eqs. 1.1. Therefore, our second objective in this paper is to solve the issue.

In fact, when f is not monotone, the method in Zhang [26] is invalid, where they used the comparison principle together with the semi-discrete Fourier transform. In addition, though there have been some results studying the stability of traveling waves of non-monotone delayed equations without spatial non-local effects by weighted energy method (see [5, 15, 17]), they usually used a piecewise weighted function (that is, $\omega(\xi) := \min\{e^{-2\lambda(\xi-\xi_0)}, 1\}$). However, for Eqs. 1.1 and 1.3 with the spatial non-local effects, we can only prove the stability of the traveling waves with sufficiently large speeds due to the influences of the non-local terms if we choose such a piecewise weighted function. Indeed, a sufficiently large speed c is needed to ensure that some term in the l^2-estimates is positive. Therefore, in this paper, we choose the non-piecewise weighted function $\{\omega_n(t)\} := \{e^{-2\lambda(n+ct)}\}_{n\in\mathbb{Z}}$ with $\lambda \in (\lambda_1, \lambda_2)$ to establish the expected energy estimates, which can be done for any $c > c^*$. By applying the anti-weighted energy method and the nonlinear Halanay's inequality [11], we could obtain that for any given $c > c_*$, the solution $\mathbf{w}(t)$ of Eq. 1.1 converges to the corresponding traveling waves $\mathbf{\Phi}(n + ct)$ in the given space. Here we emphasize that some similar works have been done for the non-local dispersal equations in continuous media, see [9, 16, 18, 23].

Now we state our results on the stability of traveling waves, which will be proved in Section 3. The notations appeared in the following theorem can also be found in Section 3. The following hypothesis is needed:

(H3) $f'(0) \geq |f'(w)|$ for $w \in (0, +\infty)$.

Theorem 1.3 (**Stability**) *Suppose that* (K1)-(K3), (H1)-(H3), *and* (F) *hold. Let* $\{\Phi(n + ct)\}_{n\in\mathbb{Z}} = \mathbf{\Phi}(\xi)(c > c_*)$ *be the traveling waves satisfying* $\Phi(-\infty) = 0$, $\Phi(+\infty) = K$. *Suppose that* $\mathbf{W}^0(s) = \mathbf{w}^0(s) - \mathbf{\Phi}(n + cs) \in C([-\tau, 0]; l^\infty)$, $\sqrt{\omega(s)}\mathbf{W}^0(s) \in C([-\tau, 0]; l^2) \cap L^2([-\tau, 0]; l^2)$. *Then there exist constants* $\delta_0 > 0$, $C > 1$, *and* $\alpha > 0$ *such that when*

$$\sup_{s\in[-\tau,0]} \left(\left\| \mathbf{W}^0(s) \right\|_{l^\infty}^2 + \left\| \mathbf{W}^0(s) \right\|_{l^2_\omega}^2 + \int_{-\tau}^0 \left\| \mathbf{W}^0(s) \right\|_{l^2_\omega}^2 ds \right) \leq \delta_0,$$

the solution $\mathbf{w}(t) = \{w_n(t)\}_{n\in\mathbb{Z}}$ *of equation* (1.1) *globally exists and satisfies*

$$\|\mathbf{W}(t)\|_{l^\infty} \leq Ce^{-\alpha t}, \quad 0 \leq t < \infty,$$

where $\mathbf{W}(t) := \{w_n(t) - \Phi(n + ct)\}_{n\in\mathbb{Z}}$.

Finally, for the sake of convenience, in the remainder of this paper, we always use the following notations:

$$\Sigma_{i\in\mathbb{Z}\setminus\{0\}} J(i) w_{n-i}(t) = (J \star w)_n(t), \quad \Sigma_{i\in\mathbb{Z}} R(i) w_{n-i}(t) = (R \otimes w)_n(t),$$
$$\Sigma_{i\in\mathbb{Z}\setminus\{0\}} J(i) V(\xi - i) = (J \star V)(\xi), \quad \Sigma_{i\in\mathbb{Z}} R(i) V(\xi - i) = (R \otimes V)(\xi),$$
$$\Sigma_{i\in\mathbb{Z}\setminus\{0\}} J(i) e^{\lambda(\xi-i)} = (J \star \exp(\lambda))(\xi), \quad \Sigma_{i\in\mathbb{Z}} R(i) e^{\lambda(\xi-i)} = (R \otimes \exp(\lambda))(\xi).$$

2 Spreading Speed and Existence of the Traveling Waves

In this section, we are dedicated to solving the spreading speed and existence of the traveling waves of Eq. 1.1. The characteristic equation of the linearized equation of Eq. 1.2 at zero equilibrium is as follows

$$\mathcal{E}(\lambda, c) = -\rho\Sigma_{i\in\mathbb{Z}\setminus\{0\}}e^{-\lambda i}J(i) + \rho + \delta + c\lambda - f'(0)e^{-\lambda c\tau}\Sigma_{i\in\mathbb{Z}}e^{-\lambda i}R(i) = 0.$$

Lemma 2.1 *Suppose that* (K1)-(K3) *hold and* $f'(0) > \delta$. *Then there are positive constants* c_* *and* λ_* *such that*

$$\mathcal{E}(\lambda_*, c_*) = 0, \quad \frac{\partial}{\partial\lambda}\mathcal{E}(\lambda, c_*)\Big|_{\lambda=\lambda_*} = 0.$$

In addition, when $c > c_*$, *the equation* $\mathcal{E}(\lambda, c) = 0$ *admits two distinct roots which satisfy* $0 < \lambda_1(c) < \lambda_* < \lambda_2(c) < \hat{\lambda}$, $\mathcal{E}(\lambda, c) > 0$ *for* $\lambda_1(c) < \lambda < \lambda_2(c)$, $\mathcal{E}(\lambda, c) < 0$ *for* $\lambda \in (0, \hat{\lambda})\setminus(\lambda_1(c), \lambda_2(c))$.

Proof Since the proof is similar to Ma et al. [14, Lemma 2.2], here we omit it. □

Define $\mathcal{C} := C([-\tau, 0]; \mathbb{R})$ with the maximum norm $\|\cdot\|$, $\mathcal{D} := \{\mathbf{u^0}(s) = \{u_n^0(s)\}_{n\in\mathbb{Z}} : u_n^0(s) \in \mathcal{C}\}$ with the supremum norm. For $\mathbf{u^0}(\cdot)$, $\mathbf{v^0}(\cdot) \in \mathcal{D}$, $\mathbf{u^0}(\cdot) \leq \mathbf{v^0}(\cdot)$ means that $u_n^0(s) \leq v_n^0(s)$, $\forall\, s \in [-\tau, 0]$, $n \in \mathbb{Z}$. Let $\mathcal{X} = \{\phi \in C(\mathbb{R}; \mathbb{R}) | \sup_{x\in\mathbb{R}} |\phi(x)| < \infty\}$ with the supremum norm. For any $\alpha > 0$, define $\mathcal{C}_\alpha := \{v(s) \in \mathcal{C} : v(s) \in [0, \alpha]$ for $s \in [-\tau, 0]\}$, $\mathcal{D}_\alpha := \{\mathbf{u^0}(s) \in \mathcal{D} : 0 \leq u_n^0(s) \leq \alpha, \forall\, n \in \mathbb{Z}, \forall\, s \in [-\tau, 0]\}$, $\mathcal{X}_\alpha := \{\phi \in \mathcal{X} : 0 \leq \phi(x) \leq \alpha, \forall\, x \in \mathbb{R}\}$.

From the definition of $f^*(w)$ and $f_*(w)$, there exists a $\eta \in (0, K)$ such that $f^*(w) = f_*(w) = f(w)$ for $w \in [0, \eta]$. Clearly, $f^*(w)$ and $f_*(w)$ are non-decreasing and Lipschitz continuous in $[0, K^*]$, and satisfy $0 < f_*(w) \leq f(w) \leq f^*(w) \leq f'(0)w$ for $w \in (0, K^*]$. Note that $f^*(\cdot)$ (or $f_*(\cdot)$) satisfies the assumption (H1) with $f(\cdot) = f^*(\cdot)$ (or $f_*(\cdot)$) and $K = K^*$(or K_*), respectively, and $f^*(\cdot)$(or $f_*(\cdot)$) has the same linearization as that of $f(\cdot)$ at 0. In particular, the following two auxiliary quasi-monotone systems could be obtained

$$\frac{d}{dt}w_n(t) = \rho(J \star w - w)_n(t) - \delta w_n(t) + (R \otimes f^*(w))_n(t - \tau), \tag{2.1}$$

$$\frac{d}{dt}w_n(t) = \rho(J \star w - w)_n(t) - \delta w_n(t) + (R \otimes f_*(w))_n(t - \tau). \tag{2.2}$$

Proposition 2.2 *Suppose that* (K1)-(K3) *and* (H1)*-(H2) hold. For any* $\mathbf{w^0} \in \mathcal{D}_{K^*}$, Eqs. 1.1, 2.1*, and* 2.2 *have unique solution* $\mathbf{w}(t, \mathbf{w^0}) = \{w_n(t)\}_{n\in\mathbb{Z}}$, $\bar{\mathbf{w}}(t, \mathbf{w^0}) = \{\bar{w}_n(t)\}_{n\in\mathbb{Z}}$, $\underline{\mathbf{w}}(t, \mathbf{w^0}) = \{\underline{w}_n(t)\}_{n\in\mathbb{Z}}$ *with* $w_n(t), \bar{w}_n(t), \underline{w}_n(t) \in C^1([0, +\infty), [0, K^*])$*, respectively. Furthermore, for any* $\mathbf{w_1^0}, \mathbf{w_2^0} \in \mathcal{D}_{K^*}$ *with* $\mathbf{w_1^0} \leq \mathbf{w_2^0}$*, there hold* $\bar{w}_n(t, \mathbf{w_1^0}) \leq \bar{w}_n(t, \mathbf{w_2^0})$, $\underline{w}_n(t, \mathbf{w_1^0}) \leq \underline{w}_n(t, \mathbf{w_2^0})$*, respectively. In addition, for any* $\bar{\mathbf{w}}^\mathbf{0}, \mathbf{w^0}, \underline{\mathbf{w}}^\mathbf{0} \in \mathcal{D}_{K^*}$*, if* $\underline{\mathbf{w}}^\mathbf{0} \leq \mathbf{w^0} \leq \bar{\mathbf{w}}^\mathbf{0}$*, then* $0 \leq \underline{w}_n(t, \underline{\mathbf{w}}^\mathbf{0}) \leq w_n(t, \mathbf{w^0}) \leq \bar{w}_n(t, \bar{\mathbf{w}}^\mathbf{0}) \leq K^*, \forall\, n \in \mathbb{Z}, t \geq 0$.

Here we omit the proof, since it is similar to Ma et al. [14, Lemma 2.1]. The following conclusion indicates that the spreading speed of the Eq. 1.1 is c_*.

Proof of Theorem 1.1 Since $f^*(w)$ and $f_*(w)$ are non-decreasing in $[0, K^*]$ and satisfy $f'(0)w \geq f^*(w) \geq f_*(w) \geq 0$ for $w \geq 0$ and $f(w) = f^*(w) = f_*(w)$ for $0 \leq w \leq \eta$, it follows from Ma et al. [15, Theorem 1.1] that Eqs. 2.1 and 2.2 admit the same spreading speed c_*. From Proposition 2.2, for any $\mathbf{w^0}, \bar{\mathbf{w}}^\mathbf{0}, \underline{\mathbf{w}}^\mathbf{0} \in \mathcal{D}_{K^*}$ with $\underline{\mathbf{w}}^\mathbf{0} \leq \mathbf{w^0} \leq \bar{\mathbf{w}}^\mathbf{0}$, Eqs. 1.1, 2.1, and 2.2 have solutions $\mathbf{w}(t, \mathbf{w^0})$, $\bar{\mathbf{w}}(t, \bar{\mathbf{w}}^\mathbf{0})$, $\underline{\mathbf{w}}(t, \underline{\mathbf{w}}^\mathbf{0})$ with $\underline{w}_n(t, \underline{\mathbf{w}}^\mathbf{0}) \leq w_n(t, \mathbf{w^0}) \leq \bar{w}_n(t, \bar{\mathbf{w}}^\mathbf{0})$ respectively. In particular, there holds $0 \leq \underline{w}_n(t, \underline{\mathbf{w}}^\mathbf{0}) \leq w_n(t, \mathbf{w^0}) \leq \bar{w}_n(t, \bar{\mathbf{w}}^\mathbf{0}) \leq K^*, \forall\ t \in [-\tau, \infty), n \in \mathbb{Z}$. Therefore, the spreading speed of Eq. 1.1 is c_*, which implies (**1**) and the first part of (**2**).

The upward convergence, namely, the second part of (ii), can be proved by the same arguments as those in Thieme [20, §3.9] and Fang et al. [6]. This completes the proof. □

Proof of Theorem 1.2 We will give the proof by three steps.

Step 1: Fix $c > c_*$. Take $\gamma > \frac{\rho}{c} + \frac{\delta}{c}$. Define

$$\Gamma(\Phi)(\xi) := \left(\gamma - \frac{\rho}{c} - \frac{\delta}{c}\right)\Phi(\xi) + \frac{1}{c}(R \otimes f(\Phi))(\xi - c\tau) + \frac{\rho}{c}(J \star \Phi)(\xi), \quad (2.3)$$

then Eq. 1.2 can be expressed as

$$\Phi'(\xi) + \gamma\Phi(\xi) = \Gamma(\Phi)(\xi), \quad \forall \xi \in \mathbb{R}. \quad (2.4)$$

We can define Γ^* and Γ_* by substituting f with f^* and f_* in Eq. 2.3, respectively. From the definition of f^* and f_*, we can conclude that

$$\Gamma(K) = \gamma K, \quad \Gamma^*(K^*) = \gamma K^*, \quad \Gamma_*(K_*) = \gamma K_*, \quad \Gamma(0) = \Gamma^*(0) = \Gamma_*(0) = 0$$

and Γ^* and Γ_* are non-decreasing, that is, for any $\Phi, \Psi \in C(\mathbb{R}, [0, K^*])$ with $\Phi(\xi) \geq \Psi(\xi), \forall\ \xi \in \mathbb{R}$, there are $\Gamma^*[\Phi](\xi) \geq \Gamma^*[\Psi](\xi)$ and $\Gamma_*[\Phi](\xi) \geq \Gamma_*[\Psi](\xi)$ for all $\xi \in \mathbb{R}$. It follows from the definition of $\mathcal{X}$ that $\Phi(\xi) = e^{-\gamma\xi}\int_{-\infty}^{\xi} e^{\gamma y}\Gamma(\Phi)(y)dy$ satisfies Eq. 2.4 for $\Phi \in \mathcal{X}$. Therefore, we can define an operator $F : \mathcal{X} \to \mathcal{X}$ by

$$F(\Phi)(\xi) = \int_{-\infty}^{\xi} e^{-\gamma(\xi - y)}\Gamma(\Phi)(y)dy.$$

In a same way, by virtue of Γ^* and Γ_*, we can similarly define $F^* : \mathcal{X} \to \mathcal{X}$ and $F_* : \mathcal{X} \to \mathcal{X}$. Obviously, $F^*(K^*) = K^*, F_*(K_*) = K_*, F(K) = K, F^*(w) \geq F(w) \geq F_*(w)$ for $0 < w < K^*$. F^* and F_* are also non-decreasing, that is, for $\Phi, \Psi \in C(\mathbb{R}, [0, K^*])$ with $\Phi(\xi) \geq \Psi(\xi), \forall\ \xi \in \mathbb{R}$, we have $F^*[\Phi](\xi) \geq F^*[\Psi](\xi)$ and $F_*[\Phi](\xi) \geq F_*[\Psi](\xi)$ for $\xi \in \mathbb{R}$.

Define

$$V^+(\xi) = \min_{\xi \in \mathbb{R}}\{e^{\lambda_1\xi}, K^*\} \quad \text{and} \quad V^-(\xi) = e^{\lambda_1\xi}\max_{\xi \in \mathbb{R}}\{1 - \zeta e^{\varepsilon\xi}, 0\}.$$

where $0 < \varepsilon < \frac{\lambda_1}{2}, \lambda_1 + \varepsilon < \lambda_2$, and $\zeta > 1$ are parameters. From the definition of $V^-(\xi)$, by calculating, $\max_{\xi \in \mathbb{R}} V^-(\xi) = V^-(\frac{1}{\varepsilon}\ln\frac{\lambda_1}{\zeta(\lambda_1+\varepsilon)}) = \varepsilon(\frac{\lambda_1}{\zeta})^{(\lambda_1/\varepsilon)}(\frac{1}{\lambda_1+\varepsilon})^{(\lambda_1/\varepsilon+1)}$. Thus, there exists a constant $\eta > 0$ such that for $\zeta > 1$ large enough, $0 \leq V^-(\xi) < \eta$ for any $\xi \in \mathbb{R}$. Let

$$N^*(\Phi)(\xi) := \frac{d\Phi}{d\xi} + \gamma\Phi(\xi) - \Gamma^*(\Phi)(\xi), \quad N_*(\Phi)(\xi) := \frac{d\Phi}{d\xi} + \gamma\Phi(\xi) - \Gamma_*(\Phi)(\xi).$$

When $\xi \geq \frac{1}{\varepsilon}\ln\frac{1}{\zeta}$, we have $V^-(\xi) = 0$ and note the fact that $f(w)$ is nonnegative; hence,

$$\begin{aligned} N_*(V^-)(\xi) &= (V^-)' + \frac{\rho + \delta}{c}V^- - \frac{\rho}{c}(J \star V^-)(\xi) - \frac{1}{c}(R \otimes f(V^-))(\xi - c\tau) \\ &\leq -\frac{\rho}{c}(J \star V^-)(\xi) - \frac{1}{c}(R \otimes f(V^-))(\xi - c\tau) \leq 0. \end{aligned}$$

When $\xi < \frac{1}{\varepsilon}\ln\frac{1}{\zeta}$, we have $V^-(\xi) = e^{\lambda_1\xi}(1-\zeta e^{\varepsilon\xi})$ and $V^-(\xi-i) \geq e^{\lambda_1(\xi-i)}(1-\zeta e^{\varepsilon(\xi-i)})$. By the Taylor expansion, we get $f'(0)w - Mw^2 \leq f(w)$, $\forall\ w \in [0,\eta)$, where $M = \max_{w\in[0,\eta]}|f''(w)|$. Then

$$\begin{aligned}
N_*(V^-)(\xi) &= (V^-)' + \frac{\rho+\delta}{c}V^- - \frac{\rho}{c}(J\star V^-)(\xi) - \frac{1}{c}(R\otimes f(V^-))(\xi - c\tau)\\
&\leq \frac{1}{c}\Big[c\lambda_1 e^{\lambda_1\xi} - c(\lambda_1+\varepsilon)\zeta e^{(\lambda_1+\varepsilon)\xi} + (\rho+\delta)e^{\lambda_1\xi} - (\rho+\delta)\zeta e^{(\lambda_1+\varepsilon)\xi}\\
&\quad -\rho(J\star\exp(\lambda_1))(\xi) + \rho\zeta(J\star\exp(\lambda_1+\varepsilon))(\xi)\\
&\quad -f'(0)(R\otimes V^-)(\xi-c\tau) + M(R\otimes(V^-)^2)(\xi-c\tau)\Big]\\
&= \frac{1}{c}\left(e^{\lambda_1\xi}\mathcal{E}(\lambda_1,c) - \zeta\mathcal{E}(\lambda_1+\varepsilon,c)e^{(\lambda_1+\varepsilon)\xi} + M(R\otimes(V^-)^2)(\xi-c\tau)\right)\\
&= \frac{1}{c}\left(-\zeta\mathcal{E}(\lambda_1+\varepsilon,c)e^{(\lambda_1+\varepsilon)\xi} + M(R\otimes(V^-)^2)(\xi-c\tau)\right).
\end{aligned}$$

From the definition of $V^-(\xi)$, for $\zeta > 1$ large enough, when $\zeta e^{\varepsilon(\xi-i-c\tau)} < 1$, it yields

$$e^{\varepsilon(\xi-i-c\tau)} < \zeta^{-1}, \quad e^{(\lambda_1-\varepsilon)(\xi-i-c\tau)} = (e^{\varepsilon(\xi-i-c\tau)})^{\frac{\lambda_1-\varepsilon}{\varepsilon}} \leq \zeta^{-\frac{\lambda_1-\varepsilon}{\varepsilon}} < 1.$$

Consequently, we have

$$\begin{aligned}
(R\otimes(V^-)^2)(\xi-c\tau) &= \Sigma_{i\in\mathbb{Z}}R(i)e^{2\lambda_1(\xi-i-c\tau)}\left(\max\{0,1-\zeta e^{\varepsilon(\xi-i-c\tau)}\}\right)^2\\
&\leq \Sigma_{i\in\mathbb{Z}}R(i)e^{(\lambda_1+\varepsilon)(\xi-i-c\tau)}e^{(\lambda_1-\varepsilon)(\xi-i-c\tau)}\left(\max\{0,1-\zeta e^{\varepsilon(\xi-i-c\tau)}\}\right)^2\\
&\leq (R\otimes\exp(\lambda_1+\varepsilon))(\xi-c\tau) = e^{(\lambda_1+\varepsilon)\xi}\Sigma_{i\in\mathbb{Z}}R(i)e^{-(\lambda_1+\varepsilon)(i+c\tau)},
\end{aligned}$$

and hence,

$$\begin{aligned}
N_*(V^-)(\xi) &= \frac{1}{c}\left(-\zeta e^{(\lambda_1+\varepsilon)\xi}\mathcal{E}(\lambda_1+\varepsilon,c) + M(R\otimes(V^-)^2)(\xi-c\tau)\right)\\
&\leq \frac{1}{c}\left(-\zeta\mathcal{E}(\lambda_1+\varepsilon,c) + M\Sigma_{i\in\mathbb{Z}}R(i)e^{-(\lambda_1+\varepsilon)i}\right)e^{(\lambda_1+\varepsilon)\xi}.
\end{aligned}$$

Finally, when ζ is sufficiently large, we always have that $N_*(V^-)(\xi) < 0$ for any $\xi \in \mathbb{R}$. Since F_* are non-decreasing, similar to [22, Lemma 3.3], we can obtain that $F_*(V^-)(\xi) \geq V^-(\xi)$ for any $\xi \in \mathbb{R}$. Similarly, we can show $F^*(V^+)(\xi) \leq V^+(\xi)$ for any $\xi \in \mathbb{R}$.

For $\lambda \in (0, \lambda_1(c))$, define a Banach space $(X_\lambda, ||\cdot||_{X_\lambda})$,

$$X_\lambda = \left\{\Phi \in C(\mathbb{R},\mathbb{R})\Big|\sup_{\xi\in\mathbb{R}} e^{-\lambda\xi}|\Phi(\xi)| < +\infty\right\}, \quad \|\Phi(\xi)\|_{X_\lambda} = \sup_{\xi\in\mathbb{R}} e^{-\lambda\xi}|\Phi(\xi)|.$$

Clearly, V^+ and V^- are elements of X_λ. Let

$$Y := \{\Phi \in X_\lambda : V^- \leq \Phi \leq V^+\} \subset X_\lambda.$$

Obviously, Y is convex and closed. Since

$$V^+ \geq F^*(V^+) \geq F^*(w) \geq F(w) \geq F_*(w) \geq F_*(V^-) \geq V^-, \quad w \in Y,$$

we have $Y \supset F(Y)$.

Similar to Fang et al. [6, Theorem 4.1] and Ma et al. [14, Theorem 3.1], we get that F is compact on Y. Thus, F has a fixed point Φ in Y by using the Schauder's fixed point theorem. Obviously, $\lim_{\xi\to-\infty}\Phi(\xi)e^{-\lambda_1\xi}=1$ and Φ is non-trivial. Therefore, $\mathbf{\Phi}=\{\Phi(n+ct)\}_{n\in\mathbb{Z}}$ is a traveling wave solution satisfying $\Phi(-\infty)=0$. Because of $\{w_n(t)\}_{n\in\mathbb{Z}}=\{\Phi(n+ct)\}_{n\in\mathbb{Z}}$ is the solution of the Eq. 1.1, from Theorem 1.1 (**1**), we have

$$K^*\geq \limsup_{t\to\infty,|n|\leq\bar{c}t}\Phi(n+ct)\geq \liminf_{t\to\infty,|n|\leq\bar{c}t}\Phi(n+ct)\geq K_*,\quad 0<\bar{c}<c_*.$$

Especially, we have $K^*\geq\limsup_{t\to\infty}\Phi(ct)\geq\liminf_{t\to\infty}\Phi(ct)\geq K_*$. Let $\xi=ct$, then we have $K^*\geq\limsup_{\xi\to\infty}\Phi(\xi)\geq\liminf_{\xi\to\infty}\Phi(\xi)\geq K_*$. If the assumption (F) holds, we further have $\lim_{\xi\to+\infty}\Phi(\xi)=K$.

Step 2: Here we demonstrate the existence of the critical traveling waves ($c=c_*$). Taking a sequence $\{c_j\}_{j\in\mathbb{N}}$ which satisfies $c_*+1>c_j>c_{j+1}>c_*$ and $\lim\limits_{j\to+\infty}c_j=c_*$. From **Step 1**, we know that Eq. 1.1 admits a traveling wave $\mathbf{\Phi_j}:=\{\Phi_j(n+c_jt)\}_{n\in\mathbb{Z}}$ which satisfies $\Phi_j(-\infty)=0$ and $K^*\geq\limsup_{\xi\to+\infty}\Phi_j(\xi)\geq\liminf_{\xi\to+\infty}\Phi_j(\xi)\geq K_*$. Then for some $\alpha\in(0,K_*)$, by a shift we can assume that $\Phi_j(0)=\alpha<K_*$ and $\Phi_j(\xi)\leq\alpha$, $\forall\,\xi<0$, $j\in\mathbb{N}$. From Eq. 1.2, we obtain that for any ξ,

$$c_j\frac{d}{d\xi}\Phi_j(\xi)=\rho(J\star\Phi_j-\Phi_j)(\xi)-\delta\Phi_j(\xi)+(R\otimes f(\Phi_j))(\xi-c_j\tau).$$

It follows from $\Phi_j(\xi)\in[0,K^*]$ that there exists a constant $C_1>0$ such that $|\Phi_j'(\xi)|\leq C_1$, $\forall\,\xi\in\mathbb{R}$, $j\in\mathbb{N}$. Differentiating the above equation with respect to ξ, we can find another constant $C_2>0$ such that $|\Phi_j''(\xi)|\leq C_2$, $\forall\,j\in\mathbb{N},\xi\in\mathbb{R}$. Consequently, up to a subsequence, we have that $\Phi_j(\xi)$ converges to $\Phi_*(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $j\to\infty$. Note that

$$\Phi_j(\xi)=\int_{-\infty}^0 e^{\gamma y}\Gamma[\Phi_j](\xi+y)dy,\quad \forall\,j\in\mathbb{N},\ \xi\in\mathbb{R}. \tag{2.5}$$

Let $j\to+\infty$ in Eq. 2.5, it holds that $F(\Phi_*)(\xi)=\Phi_*(\xi)$ ($c=c_*$, $\xi\in\mathbb{R}$) by applying the dominated convergence theorem. In addition, we have $\Phi_*(0)=\alpha$, $\Phi_*(\xi)\leq\alpha$ for any $\xi<0$. Similar to **Step 1**, we also have $K^*\geq\limsup_{\xi\to+\infty}\Phi_*(\xi)\geq\liminf_{\xi\to+\infty}\Phi_*(\xi)\geq K_*$.

Next, we prove that $\Phi_*(-\infty)=0$. Suppose $\limsup_{\xi\to-\infty}\Phi_*(\xi)=\beta>0$, then there must be $\beta\leq\alpha$. Choose $\xi_j\to-\infty$ satisfying $\lim_{j\to+\infty}\Phi_*(\xi_j)=\beta$. Let $\Phi_{*,j}(\xi)=\Phi_*(\xi+\xi_j)$ and $\Phi_\natural(\xi)=\lim_{j\to+\infty}\Phi_{*,j}(\xi)$ up to a subsequence, then it yields $\Phi_\natural(0)=\beta\leq\alpha$ and $\Phi_\natural(\xi)\leq\beta$. Since $\Phi_\natural(\xi)$ satisfies

$$c_*\frac{d}{d\xi}\Phi_\natural(\xi)=\rho(J\star(\Phi_\natural)-\Phi_\natural)(\xi)-\delta\Phi_\natural(\xi)+(R\otimes f(\Phi_\natural))(\xi-i-c_*\tau)),$$

it follows from Theorem 1.1 that $\liminf_{t\to\infty,|n|\leq\tilde{c}t}\Phi_\natural(n+c_*t)\geq K_*, 0<\tilde{c}<c_*$, which means $K_*\leq\liminf_{t\to\infty}\Phi_\natural(c_*t)$, namely, $\liminf_{\xi\to\infty}\Phi_\natural(\xi)\geq K_*>\beta$. This is contradictory to $\Phi_\natural(\xi)\leq\beta$ above. Therefore, $\Phi_*(-\infty)=0$ is proved. Using the analogous arguments as above, if the assumption (F) holds, we can show that $\Phi_*(+\infty)=K$.

Step 3: For the non-existence of the traveling wave solution, since the proof is similar to that of Fang et al. [6, Theorem 3.4], we omit it for simplicity.

□

3 Stability of Traveling Waves

We have already proved that Eq. 1.1 admits traveling wave $\{\Phi(n+ct)\}_{n\in\mathbb{Z}}$ with $c \geq c_*$ in Section 2. Based on the fact and the assumptions of (H1)-(H3) and (F), in this section, we mainly study the stability of the noncritical traveling waves $\{\Phi(n+ct)\}_{n\in\mathbb{Z}}$ $(c > c_*)$ satisfying $\Phi(-\infty) = 0$ and $\Phi(+\infty) = K$. First of all, we need take some transforms to Eq. 1.1.

Define $W_n(t) = w_n(t) - \Phi(n+ct)$ for $t \geq 0$, and $W_n^0(s) = w_n^0(s) - \Phi(n+cs)$ for $s \in [-\tau, 0]$, where $n \in \mathbb{Z}$. Then system (1.1) reduces to

$$\begin{cases} \frac{dW_n}{dt}(t) - \rho(J\star W - W)_n(t) + \delta W_n(t) - (R\otimes(f'(\Phi)W))_n(t-\tau) \\ \qquad = (R\otimes Q(W))_n(t-\tau)), \quad t>0, \quad n\in\mathbb{Z}, \\ W_n(s) = W_n^0(s), \quad s\in[-\tau,0], \quad n\in\mathbb{Z}, \end{cases} \tag{3.1}$$

where

$$\begin{aligned}(Q(W))_n(t-\tau) := \; & f(\Phi(n+ct-c\tau)+W_n(t-\tau)) - f(\Phi(n+ct-c\tau)) \\ & -f'(\Phi(n+ct-c\tau))W_n(t-\tau).\end{aligned}$$

By Taylor's formula, it holds

$$|(Q(W))_n| \leq \Lambda|W_n|^2, \quad \forall\, n\in\mathbb{Z}, \tag{3.2}$$

where $\Lambda > 0$ depends on the bound of the second derivative of f and the value of $\|\mathbf{W}\|_{l^\infty}$.

Before presenting the results about the stability, we introduce some notations. In the following, a generic constant is denoted as $C > 0$ and a specific constant is denoted as $C_k > 0$ $(k = 1, 2, \cdots)$. Denote $\mathfrak{B}$ as a Banach space with a norm $\|\cdot\|_{\mathfrak{B}}$ and $T > 0$ as a number. Furthermore, we define:

$$C^0([0,T];\mathfrak{B}) := \{\phi : [0,T]\to\mathfrak{B} \text{ is continuous}\}.$$

$$L^1([0,T];\mathfrak{B}) := \left\{\phi \text{ maps } [0,T] \text{ to } \mathfrak{B}, \int_0^T \|\phi(t)\|_{\mathfrak{B}}dt < \infty\right\}.$$

$$l^\infty = \{\mathbf{u} = \{u_n\}_{n\in\mathbb{Z}} : u_n\in\mathbb{R}, \|\mathbf{u}\|_{l^\infty} < \infty\}, \quad \|\mathbf{u}\|_{l^\infty} = \sup_{n\in\mathbb{Z}}|u_n|.$$

$$l^1 = \{\mathbf{u} = \{u_n\}_{n\in\mathbb{Z}} : u_n\in\mathbb{R}, \|\mathbf{u}\|_{l^1} < \infty\}, \quad \|\mathbf{u}\|_{l^1} = \sum_{n\in\mathbb{Z}}|u_n|.$$

$$l^2 = \{\mathbf{u} = \{u_n\}_{n\in\mathbb{Z}} : u_n\in\mathbb{R}, \|\mathbf{u}\|_{l^2} < \infty\}, \quad \|\mathbf{u}\|_{l^2} = \left(\sum_{n\in\mathbb{Z}} u_n^2\right)^{\frac{1}{2}}.$$

$$l^2_\omega = \{\mathbf{u} = \{u_n\}_{n\in\mathbb{Z}} : u_n\in\mathbb{R}, \|\mathbf{u}\|_{l^2_\omega} < \infty\}, \quad \|\mathbf{u}\|_{l^2_\omega} = \left(\sum_{n\in\mathbb{Z}} \omega_n u_n^2\right)^{\frac{1}{2}}, \quad \omega = \{\omega_n\}_{n\in\mathbb{Z}}.$$

Define the weight function as

$$\omega(t) := \{\omega_n(t)\}_{n\in\mathbb{Z}} := \left\{e^{-2\lambda(n+ct)}\right\}_{n\in\mathbb{Z}}, \quad \lambda\in(\lambda_1,\lambda_2).$$

For $0 < T \leq \infty$, define

$$C_{unif}[-\tau, T] := \{\mathbf{w}(t) := \{w_n(t)\}_{n\in\mathbb{Z}} \in C([-\tau, T]; l^\infty), \text{ and } \lim_{n\to+\infty} w_n(t) \text{ exists uniformly in } t \in [-\tau, T]\},$$

$$X(-\tau, T) := \Big\{\mathbf{W}|\mathbf{W}(t) = \{W_n(t)\}_{n\in\mathbb{Z}} \in C_{unif}[-\tau, T], \text{ and } \mathbf{W}(t) \in C([-\tau, T]; l^2_\omega) \cap L^2([-\tau, T]; l^2_\omega)\Big\},$$

with the norm

$$\|\mathbf{W}\|^2_{X(-\tau,T)} := \sup_{t\in[-\tau,T]} \left(\|\mathbf{W}(t)\|^2_{l^\infty} + \|\mathbf{W}(t)\|^2_{l^2_\omega} + \int_{-\tau}^t \|\mathbf{W}(t)\|^2_{l^2_\omega} dt\right),$$

where $\|\mathbf{W}(t)\|_{l^2_\omega} := \left(\sum_{n\in\mathbb{Z}} \omega_n(t) W_n^2(t)\right)^{\frac{1}{2}}$.

Meanwhile, give the definition of discrete Fourier transform (refer to [19]) as follows: For $\mathbf{v} = \{v_j\}_{j\in\mathbb{Z}} \in l^2$, the Fourier transform of $\mathbf{v}$ is given by

$$\mathcal{F}[\mathbf{u}](\eta) = \hat{\mathbf{u}}(\eta) = \frac{1}{\sqrt{2\pi}} \Sigma_{j\in\mathbb{Z}} e^{-\mathbf{i}\eta j} u_j, \quad \eta \in [-\pi, \pi].$$

The inverse Fourier transform of $\hat{\mathbf{u}}$ is denoted as

$$\mathcal{F}^{-1}[\hat{\mathbf{u}}] = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{\mathbf{i}\eta j} \hat{\mathbf{u}}(\eta) d\eta, \quad j \in \mathbb{Z}, \quad \mathbf{i}^2 = -1.$$

3.1 Local Existence and Uniqueness

In this subsection, our main goal is to give the proof of the local existence of the solution $\mathbf{W}(t)$ of system (3.1).

Theorem 3.1 *Suppose that* (K1)-(K3)*,* (H1)-(H3)*, and* (F) *hold. Let* $\{\Phi(n + ct)\}_{n\in\mathbb{Z}} = \mathbf{\Phi}(\xi)$*,* $(c > c_*)$ *be the traveling waves which satisfy* $\Phi(-\infty) = 0$*,* $\Phi(+\infty) = K$*. For any* $\delta_1 > 0$*, suppose* $\mathbf{W}^0(s) := \left\{W_n^0(s)\right\}_{n\in\mathbb{Z}} \in X(-\tau, 0)$ *satisfying* $\left\|\mathbf{W}^0(s)\right\|_{X(-\tau,0)} \leq \delta_1$*, then there exist a sufficiently small* $t_0 = t_0(\delta_1)$ *such that the solution* $\mathbf{W}(t)$ *of the perturbed equation* (3.1) *unique exists for* $-\tau \leq t \leq t_0$*, and satisfies* $\mathbf{W}(t) \in X(-\tau, t_0)$ *and* $\|\mathbf{W}(t)\|_{X(-\tau,t_0)} < C_1 \left\|\mathbf{W}^0(s)\right\|_{X(-\tau,0)}$ *for some constant* $C_1 > 1$*, where* C_1 *is independent of* δ_1 *and* t_0*.*

Proof Fix $\mathbf{W}^0(s) \in X(-\tau, 0)$. For $t_0 > 0$, let

$$Y(-\tau, t_0) = \{\mathbf{W}(t) \in X(-\tau, t_0) \mid \mathbf{W}(s) = \mathbf{W}^0(s), s \in [-\tau, 0]\}. \tag{3.3}$$

For $\mathbf{W}(t) \in Y(-\tau, t_0)$, define $\hat{\mathbf{W}}(t) = \mathcal{T}(\mathbf{W})(t)$ by

$$\begin{cases} \dfrac{d}{dt}\hat{\mathbf{W}}(t) + (\rho + \delta)\hat{\mathbf{W}}(t) = \mathbf{g}(\mathbf{W})(t), & t > 0, \\ \hat{\mathbf{W}}(s) = \mathbf{W}^0(s), & s \in [-\tau, 0], \end{cases} \tag{3.4}$$

where $\hat{\mathbf{W}}(t) = \{\hat{W}_n(t)\}_{n\in\mathbb{Z}}$, $\mathbf{g}(\mathbf{W})(t) := \{g_n(\mathbf{W})(t)\}_{n\in\mathbb{Z}}$, and

$$g_n(\mathbf{W})(t) = \rho(J \star W)_n(t) + (R \otimes [f(\Phi + W) - f(\Phi)])_n(t - \tau).$$

Clearly, $\hat{\mathbf{W}}(t)$ is well defined. And the Eq. 3.4 is equivalent to

$$\hat{\mathbf{W}}(t) = \mathbf{W}^0(0)e^{-(\rho+\delta)t} + e^{-(\rho+\delta)t}\int_0^t e^{(\rho+\delta)s}\mathbf{g}(\mathbf{W})(s)ds, \quad t \in [0, t_0]. \tag{3.5}$$

Step 1. We prove that the mapping $\mathcal{T}$ satisfies $\mathcal{T}(Y(-\tau, t_0)) \subset Y(-\tau, t_0)$.

(i) Firstly, we show $\hat{\mathbf{W}}(t) \in C_{unif}[-\tau, t_0]$. It follows from $\mathbf{W}(t) \in C_{unif}[-\tau, t_0]$ that there exists $W_\infty(t) \in C[-\tau, t_0]$ satisfying $\lim_{n\to\infty} W_n(t) := W_\infty(t)$ uniformly for $t \in [-\tau, t_0]$. Then by virtue of the assumptions on $J(\cdot)$, $R(\cdot)$, and $f(\cdot)$, we have that

$$\lim_{n\to\infty} g_n(\mathbf{W})(t) = \rho W_\infty(t) + f(K + W_\infty(t-\tau)) - f(K)$$

uniformly for $t \in [0, t_0]$. By Eq. 3.5, we get

$$\lim_{n\to\infty} \hat{W}_n(t) = e^{-(\rho+\delta)t} W^0_\infty(0) + e^{-(\rho+\delta)t}\int_0^t e^{(\rho+\delta)s}\left[\rho W_\infty(s) + f(K + W_\infty(s-\tau)) - f(K)\right]ds \tag{3.6}$$

uniformly for $t \in [0, t_0]$. From Eq. 3.5, we can also obtain

$$\left\|\hat{\mathbf{W}}(t)\right\|_{l^\infty} \le \left\|\mathbf{W}^0(0)\right\|_{l^\infty} + C't_0 \sup_{t\in[-\tau,t_0]} \|\mathbf{W}(t)\|_{l^\infty}, \quad t \in [0, t_0], \tag{3.7}$$

where $C' := \rho + f'(0) > 0$. For any $0 \le t_1 \le t_2 \le t_0$, we have

$$\begin{aligned}
&\left\|\hat{\mathbf{W}}(t_1) - \hat{\mathbf{W}}(t_2)\right\|_{l^\infty} \\
&\le \left\|\mathbf{W}^0(0)e^{-(\rho+\delta)t_1}(1 - e^{-(\rho+\delta)(t_2-t_1)})\right\|_{l^\infty} + \left\|\int_{t_1}^{t_2} e^{-(\rho+\delta)(t_2-s)}\mathbf{g}(\mathbf{W})(s)ds\right\|_{l^\infty} \\
&\quad + \left\|\int_0^{t_1} e^{-(\rho+\delta)(t_1-s)}(1 - e^{-(\rho+\delta)(t_2-t_1)})\mathbf{g}(\mathbf{W})(s)ds\right\|_{l^\infty} \\
&\le \left|1 - e^{-(\rho+\delta)(t_2-t_1)}\right|\left(\left\|\mathbf{W}^0(0)\right\|_{l^\infty} + C't_0 \sup_{s\in[-\tau,t_0]} \|\mathbf{W}(s)\|_{l^\infty}\right) \\
&\quad + C'|t_1 - t_2| \sup_{s\in[-\tau,t_0]} \|\mathbf{W}(s)\|_{l^\infty},
\end{aligned}$$

which combining Eqs. 3.6 and 3.7 and the fact that $\hat{\mathbf{W}}(s) = \mathbf{W}^0(s)$ $(-\tau \le s \le 0)$ imply that $\hat{\mathbf{W}}(t) \in C_{unif}[-\tau, t_0]$.

(ii) Secondly, we show the energy estimates for $\hat{\mathbf{W}}(t) \in C\left([-\tau, t_0]; l^2_\omega\right) \cap L^2\left([-\tau, t_0]; l^2_\omega\right)$. By taking the regular energy estimates $\sum_{n\in\mathbb{Z}} \int_0^t \omega_n(s)\hat{W}_n(s) \times$ (3.4)ds, we get

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \int_0^t \omega_n(s)\frac{d\hat{W}_n(s)}{ds}\hat{W}_n(s)ds + \int_0^t (\rho+\delta)\sum_{n\in\mathbb{Z}}\omega_n(s)\hat{W}_n(s)\hat{W}_n(s)ds \\
&= \sum_{n\in\mathbb{Z}} \int_0^t (J\star W)_n(s)\omega_n(s)\hat{W}_n(s)ds \\
&\quad + \sum_{n\in\mathbb{Z}} \int_0^t \omega_n(s)\hat{W}_n(s)(R\otimes(f(\Phi+W)-f(\Phi)))_n(s-\tau)ds \\
&:= P_1(t)+P_2(t).
\end{aligned} \tag{3.8}$$

For any $t \in [0, t_0]$, a direct computation gives

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \int_0^t \frac{d\hat{W}_n(s)}{ds}\omega_n(s)\hat{W}_n(s)ds \\
&= \frac{1}{2}\left\|\sqrt{\omega(t)}\hat{\mathbf{W}}(t)\right\|^2_{l^2} - \frac{1}{2}\left\|\sqrt{\omega(0)}\mathbf{W}^0(0)\right\|^2_{l^2} + \lambda c\int_0^t \left\|\sqrt{\omega(s)}\hat{\mathbf{W}}(s)\right\|^2_{l^2} ds.
\end{aligned} \tag{3.9}$$

Applying Young's inequality $2ab \le \eta a^2 + \frac{1}{\eta}b^2$ for any $\eta > 0$, we have

$$\begin{aligned}
P_2(t) &\le \sum_{n\in\mathbb{Z}} \int_0^t f'(0)\omega_n(s)(R\otimes|W|)_n(s-\tau)|\hat{W}_n(s)|ds \\
&\le \frac{\varepsilon}{2}\int_0^t \left\|\hat{\mathbf{W}}(s)\right\|^2_{l^2_\omega} ds + \frac{C_0(f'(0))^2}{2\varepsilon}\left(\int_{-\tau}^0 \|\mathbf{W}(s)\|^2_{l^2_\omega} ds + \int_0^t \|\mathbf{W}(s)\|^2_{l^2_\omega} ds\right),
\end{aligned} \tag{3.10}$$

for $t \in [0, t_0]$, where $C_0 = \sum_{i\in\mathbb{Z}} R(i)\frac{\omega_n(s)}{\omega_{n-i}(s-\tau)} = \sum_{i\in\mathbb{Z}} R(i)e^{-2\lambda(i+c\tau)}$ and $\varepsilon > 0$ is a constant which will be determined later. Similarly, we have

$$P_1(t) \le \frac{C_0'\rho}{2}\int_0^t \|\mathbf{W}(s)\|^2_{l^2_\omega} ds + \frac{\rho}{2}\int_0^t \left\|\hat{\mathbf{W}}(s)\right\|^2_{l^2_\omega} ds, \tag{3.11}$$

for $t \in [0, t_0]$, where $C_0' = \sum_{i\in\mathbb{Z}\setminus\{0\}} J(i)\frac{\omega_n(s)}{\omega_{n-i}(s)} = \sum_{i\in\mathbb{Z}\setminus\{0\}} J(i)e^{-2\lambda i}$. Substituting Eqs. 3.9, 3.10, and 3.11 into 3.8, we obtain

$$\begin{aligned}
&\left\|\hat{\mathbf{W}}(t)\right\|^2_{l^2_\omega} + 2\mathcal{A}\int_0^t \left\|\hat{\mathbf{W}}(s)\right\|^2_{l^2_\omega} ds \\
&\le \left\|\mathbf{W}^0(0)\right\|^2_{l^2_\omega} + C_0'\rho\int_0^t \|\mathbf{W}(s)\|^2_{l^2_\omega} ds \\
&\quad + \frac{C_0(f'(0))^2}{\varepsilon}\left(\int_{-\tau}^0 \|\mathbf{W}(s)\|^2_{l^2_\omega} ds + \int_0^t \|\mathbf{W}(s)\|^2_{l^2_\omega} ds\right),
\end{aligned}$$

where $\mathcal{A} = \lambda c + \delta + \frac{\rho}{2} - \frac{\varepsilon}{2}$. Choose $\varepsilon = \rho$, then $\mathcal{A} := \lambda c + \delta > 0$. Consequently, there exists $C > 0$, which depends on λ, c, δ, C_0, C_0', ρ, and $f'(0)$, such that

$$\left\|\hat{\mathbf{W}}(t)\right\|_{l^2_\omega}^2 + \int_0^t \left\|\hat{\mathbf{W}}(s)\right\|_{l^2_\omega}^2 ds \le C\left(\left\|\mathbf{W}^0(0)\right\|_{l^2_\omega}^2 + \int_{-\tau}^0 \left\|\mathbf{W}^0(s)\right\|_{l^2_\omega}^2 ds + \int_0^t \|\mathbf{W}(s)\|_{l^2_\omega}^2 ds\right) \tag{3.12}$$

for $t \in [0, t_0]$, which implies that $\hat{\mathbf{W}}(t) \in l^2_\omega$, and $\hat{\mathbf{W}}(t) \in L^2([-\tau, t_0]; l^2_\omega)$. In addition, for any $0 \le t_1 \le t_2 \le t_0$, it holds

$$\begin{aligned}
&\left\|\sqrt{\omega(t_1)}\hat{\mathbf{W}}(t_1) - \sqrt{\omega(t_2)}\hat{\mathbf{W}}(t_2)\right\|_{l^2}^2\\
\le& \sum_{n\in\mathbb{Z}}\Bigg(W_n^0(0)e^{-\lambda(n+ct_1)}e^{-(\rho+\delta)t_1} + \int_0^{t_1} e^{-\lambda(n+ct_1)}e^{-(\rho+\delta)(t_1-s)}g_n(\mathbf{W})(s)ds\\
&-W_n^0(0)e^{-\lambda(n+ct_2)}e^{-(\rho+\delta)t_2} - \int_0^{t_2} e^{-\lambda(n+ct_2)}e^{-(\rho+\delta)(t_2-s)}g_n(\mathbf{W})(s)ds\Bigg)^2\\
\le& \sum_{n\in\mathbb{Z}} 3\Bigg[\left(W_n^0(0)e^{-(\rho+\delta)t_1}e^{-\lambda(n+ct_1)}\left(1 - e^{-(\rho+\delta)(t_2-t_1)}e^{-\lambda c(t_2-t_1)}\right)\right)^2\\
&+\left(\int_0^{t_1} e^{-(\rho+\delta)(t_1-s)}e^{-\lambda(n+ct_1)}\left(1-e^{-(\rho+\delta)(t_2-t_1)}e^{-\lambda c(t_2-t_1)}\right)g_n(\mathbf{W})(s)ds\right)^2\\
&+\left(\int_{t_1}^{t_2} e^{-(\rho+\delta)(t_2-s)}e^{-\lambda(n+ct_2)}g_n(\mathbf{W})(s)ds\right)^2\Bigg]\\
=& J_1(t_1,t_2) + J_2(t_1,t_2) + J_3(t_1,t_2).
\end{aligned}$$

The estimates of $J_1(t_1, t_2)$–$J_3(t_1, t_2)$ are given below. Firstly,

$$J_1(t_1, t_2) = 3\left\|\mathbf{W}^0(0)\right\|_{l^2_\omega}^2 \left(e^{-(\rho+\delta+\lambda c)t_1}\left(1 - e^{-(\rho+\delta)(t_2-t_1)}e^{-\lambda c(t_2-t_1)}\right)\right)^2,$$

thus, $J_1(t_1, t_2) \to 0$ as $|t_1 - t_2| \to 0$. Secondly,

$$\begin{aligned}
J_2(t_1,t_2) \le\ & 6\left(1 - e^{-(\rho+\delta)(t_2-t_1)}e^{-\lambda c(t_2-t_1)}\right)^2\Bigg[\rho^2\sum_{n\in\mathbb{Z}}\left(\int_0^{t_1} e^{-\lambda(n+ct_1)}(J\star W)_n(s)ds\right)^2\\
&+\left(f'(0)\right)^2\sum_{n\in\mathbb{Z}}\left(\int_0^{t_1} e^{-\lambda(n+ct_1)}(R\otimes W)_n(s-\tau)ds\right)^2\Bigg].
\end{aligned}$$

Denote $\dot{C} = \sum_{i\in\mathbb{Z}\setminus\{0\}} J(i)e^{-2\lambda i}$, $\ddot{C} = \sum_{i\in\mathbb{Z}} R(i)e^{-2\lambda i}$. It follows from the assumption (K3) that $\dot{C}$ and $\ddot{C}$ are bounded. Since $\sum_{i\in\mathbb{Z}\setminus\{0\}} J(i) = \sum_{i\in\mathbb{Z}} R(i) = 1$, and $\mathbf{W}(t) \in X(-\tau, t_0)$, it yields

$$\begin{aligned}
J_2(t_1,t_2) \le\ & 6\left(1 - e^{-(\rho+\delta)(t_2-t_1)}e^{-\lambda c(t_2-t_1)}\right)^2\left(\rho^2\dot{C}\int_0^{t_1} \|\mathbf{W}(s)\|_{l^2_\omega}^2 ds\right.\\
&\left.+\left(f'(0)\right)^2\ddot{C}\int_0^{t_1}\|\mathbf{W}(s-\tau)\|_{l^2_\omega}^2 ds\right) \longrightarrow 0, \quad \text{as} \quad |t_1 - t_2| \to 0.
\end{aligned}$$

Finally, calculated as above, we have

$$J_3(t_1,t_2) \leq 6\rho^2\dot{C}\,\|\mathbf{W}(s)\|_{X(-\tau,t_0)}\,|t_2-t_1| + 6(f'(0))^2\ddot{C}\,\|\mathbf{W}(s-\tau)\|_{X(-\tau,t_0)}\,|t_2-t_1| \longrightarrow 0, \quad \text{as} \quad |t_1-t_2|\to 0.$$

Thus, we get $\hat{\mathbf{W}}(t) \in C([-\tau,t_0];l^2_\omega)$. Based on the proof of (i) and (ii), it holds that $\hat{\mathbf{W}} = \mathcal{T}(\mathbf{V})$ maps from $Y(-\tau,t_0)$ to $Y(-\tau,t_0)$.

In addition, it follows from Eqs. 3.7 and 3.12 that there exists a constant $\hat{C} > 0$, which depends on $\lambda, c, \delta, C_0, C_0', \rho$, and $f'(0)$, such that

$$\|\hat{\mathbf{W}}(t)\|^2_{X(-\tau,t_0)} \leq \hat{C} \sup_{s\in[-\tau,0]} \left(\left\|\mathbf{W}^0(s)\right\|^2_{l^\infty} + \left\|\mathbf{W}^0(s)\right\|^2_{l^2_\omega} + \int_{-\tau}^0 \left\|\mathbf{W}^0(s)\right\|^2_{l^2_\omega} ds \right) + \hat{C}t_0\,\|\mathbf{W}(t)\|^2_{X(-\tau,t_0)}. \quad (3.13)$$

Step 2. We prove that $\mathcal{T}$ is a contraction mapping on $Y(-\tau,t_0)$. For any $\mathbf{W^1}(t), \mathbf{W^2}(t) \in Y(-\tau,t_0)$, define $\hat{\mathbf{W}}^\mathbf{1} = \mathcal{T}\mathbf{W^1}, \hat{\mathbf{W}}^\mathbf{2} = \mathcal{T}\mathbf{W^2}$. By a series of calculations similar to Step 1, we have $\|\hat{\mathbf{W}}^\mathbf{1} - \hat{\mathbf{W}}^\mathbf{2}\|^2_{X(-\tau,t_0)} \leq C_4t_0\|\mathbf{W_1}-\mathbf{W_2}\|^2_{X(-\tau,t_0)}$, where $C_4 > 0$ is a constant depending on $\lambda, c, \delta, C_0, C_0', \rho$, and $f'(0)$. Take $0 < t_0 < \min\left\{\frac{1}{C_4}, \frac{1}{2\hat{C}}\right\}$, then

$$\|\hat{\mathbf{W}}^\mathbf{1} - \hat{\mathbf{W}}^\mathbf{2}\|^2_{X(-\tau,t_0)} = \|\mathcal{T}\mathbf{W^1} - \mathcal{T}\mathbf{W^2}\|^2_{X(-\tau,t_0)} \leq \iota\|\mathbf{W_1}-\mathbf{W_2}\|^2_{X(-\tau,t_0)},$$

where $\iota < 1$. Thus, $\mathcal{T}$ is a contraction mapping on given space. Hence, the local existence of the solution in $Y(-\tau,t_0)$ (see Eq. 3.3 for the definition of $Y(-\tau,t_0)$) can be proved by using the Banach fixed point theorem. Furthermore, by the similar calculation as above (see Eq. 3.13), we get $\|\mathbf{W}\|_{X(-\tau,t_0)} < C_1\left\|\mathbf{W}^0\right\|_{X(-\tau,0)}$ for some constant $C_1 > 1$, which depends on $\lambda, c, \delta, C_0, C_0', \rho$, and $f'(0)$. Clearly, the constant $C_1 > 1$ is independent of δ_1 and t_0. This completes the proof.

□

3.2 Key Estimate

In Section 3.1, we have proved the local existence of solutions of Eq. 3.1. In this subsection, we give a key estimate for local solutions of Eq. 3.1 when the solutions are sufficiently small.

Theorem 3.2 *Suppose that* (K1)-(K3)*,* (H1)-(H3)*, and* (F) *hold. Let* $\mathbf{W}(t) \in X(-\tau,T)$ *be a local solution of system* (3.1) *on* $[0,T]$ *for a given constant* $T > 0$*. Then there exist constants* $\alpha > 0$, $\tilde{C} > 1$*, and* $\varrho \in (0,1)$*, which are independent of* T *and* $\mathbf{W}(t) \in X(-\tau,T)$*, such that, when* $\|\mathbf{W}\|_{X(-\tau,T)} \leq \varrho$*, there holds*

$$\|\mathbf{W}(t)\|^2_{l^\infty} + \|\mathbf{W}(t)\|^2_{l^2_\omega} + \int_0^t e^{-2\alpha(t-s)}\,\|\mathbf{W}(s)\|^2_{l^2_\omega}\,ds$$
$$\leq \tilde{C}e^{-2\alpha t}\sup_{s\in[-\tau,0]}\left(\left\|\mathbf{W}^0(s)\right\|^2_{l^\infty} + \left\|\mathbf{W}^0(s)\right\|^2_{l^2_\omega} + \int_{-\tau}^0 \left\|\mathbf{W}^0(s)\right\|^2_{l^2_\omega} ds\right) \qquad \forall t \in [0,T].$$

To prove this theorem, we first show four lemmas in the following.

Lemma 3.3 *Suppose* $\|\mathbf{W}(\cdot)\|_{X(-\tau,T)} \le \varrho_1$ *for some* $\varrho_1 \in (0,1)$ *small enough. Then there exist constants* $C_5 > 0$, $\varepsilon \in (0, \frac{\delta}{2})$ *and an integer* $n_0 \gg 1$, *which are independent of* T, *satisfying*

$$\|\mathbf{W}(t)\|_{l^\infty[n_0-[cT]-1,+\infty)} \le C_5 e^{-\varepsilon t} \sup_{s\in[-\tau,0]} \|\mathbf{W}^0(s)\|_{l^\infty}, \quad \forall t \in [0,T].$$

Proof We have $\lim_{n\to+\infty} W_n(t)$ exists uniformly with respect to $t \in [-\tau, T]$ due to the fact that $\mathbf{W}(t) := \{W_n(t)\}_{n\in\mathbb{Z}} \in X(-\tau, T)$. Let $\lim_{n\to+\infty} W_n(t) := W_\infty(t)$ for $-\tau \le t \le T$ and $\lim_{n\to+\infty} W_n^0(s) := W_\infty^0(s)$ for any $s \in [-\tau, 0]$. Taking the limits to Eq. 3.1, we can obtain

$$\begin{cases} \frac{d}{dt} W_\infty(t) + \delta W_\infty(t) - f'(K) W_\infty(t-\tau) = Q(W_\infty(t-\tau)), \quad 0 < t \le T, \\ W_\infty(s) = W_\infty^0(s), \quad s \in [-\tau, 0]. \end{cases}$$

It is clear that $\| W_\infty(\cdot)\|_{L^\infty[-\tau,0]} \le \|\mathbf{W}(t)\|_{X(-\tau,0)}$. Using the nonlinear Halanay's inequality (see [11]), we have that there exist $\varrho_1 \in (0,1)$ small enough, $0 < \varepsilon < \frac{\delta}{2}$, and $C > 0$ such that

$$|W_\infty(t)| \le C\|W_\infty^0(\cdot)\|_{L^\infty[-\tau,0]} e^{-2\varepsilon t} \le C \sup_{s\in[-\tau,0]} \left\|\mathbf{W}^0(s)\right\|_{l^\infty} e^{-2\varepsilon t}, \quad t > 0, \tag{3.14}$$

provided $\|\mathbf{W}(t)\|_{X(-\tau,T)} < \varrho_1$. In particular, the constants ϱ_1, ε, and $C > 0$ are independent of $\mathbf{W}(t)$. Multiplying both sides of Eq. 3.1 by $e^{\delta t}$ and integrating the two sides of the equation on $[0, t]$ yield

$$\begin{aligned} W_n(t) &= e^{-\delta t}\left(W_n^0(0) + \rho \int_0^t e^{\delta s}(J \star W - W)_n(s)ds \right. \\ &\quad \left. + \int_0^t e^{\delta s}(R \otimes f'(\Phi)W)_n(s-\tau)ds + \int_0^t e^{\delta s}(R \otimes Q(W))_n(s-\tau)ds\right). \end{aligned}$$

Furthermore, multiplying both sides of the above equation by $e^{\varepsilon t}$, and taking the limit of the above equation as $n \to +\infty$, we can obtain

$$\begin{aligned} &\lim_{n\to+\infty} e^{\varepsilon t} W_n(t) \\ &\le e^{-(\delta-\varepsilon)t}\left(W_\infty^0(0) + f'(K)\int_0^t e^{\delta s} W_\infty(s-\tau)ds + \Lambda \int_0^t e^{\delta s}|W_\infty(s-\tau)|^2 ds\right) \\ &\le Ce^{-(\delta-\varepsilon)t} \sup_{s\in[-\tau,0]} \left\|\mathbf{W}^0(s)\right\|_{l^\infty} \left(1 + f'(K)\int_0^t e^{\delta s} e^{-2\varepsilon(s-\tau)} ds + \Lambda \int_0^t e^{\delta s} e^{-2\varepsilon(s-\tau)} ds\right) \\ &\le Ce^{-\varepsilon t} \sup_{s\in[-\tau,0]} \left\|\mathbf{W}^0(s)\right\|_{l^\infty}, \ \text{uniformly in } \ t \ge 0, \end{aligned}$$

where we have used the inequality (3.2) with $\Lambda := \max_{u\in[0,K^*+1]} |f''(u)|$. In particular, the last constant $C > 0$ is independent of t and the choosing of the constants $\varrho_1 \in (0,1)$ and $\varepsilon \in (0, \frac{\delta}{2})$. Thus, for each $\epsilon > 0$, there is $n_0 = n_0(\epsilon) \gg 1$, which is independent of t, satisfying

$$\left|e^{\varepsilon t} W_n(t) - e^{\varepsilon t} W_\infty(t)\right| < \epsilon \quad \forall\, n \ge n_0 - 1 - [cT],$$

which together with Eq. 3.14 yields

$$e^{\varepsilon t}|W_n(t)| \le C \sup_{s\in[-\tau,0]} \left\|\mathbf{W}^0(s)\right\|_{l^\infty} + \epsilon, \quad \forall\, n \ge n_0 - 1 - [cT].$$

Letting $\epsilon = \sup_{s\in[-\tau,0]} \left\|\mathbf{W}^0(s)\right\|_{l^\infty}$, we have

$$\sup_{n\in[n_0-[cT]-1,+\infty)} |W_n(t)| \le C_5 e^{-\varepsilon t} \sup_{s\in[-\tau,0]} \left\|\mathbf{W}^0(s)\right\|_{l^\infty}, \quad \forall\, t \ge 0.$$

Clearly, $C_5 > 0$ and $\varepsilon \in (0, \frac{\delta}{2})$ only depend on ϱ_1. The proof is completed. □

Clearly, Lemma 3.3 gives an estimate of $W_n(t)$ for $n \ge n_0 - 1 - [cT]$. In the following, we derive a similar estimate of $W_n(t)$ for $n \le n_0-1-[cT]$. For $n_0 \in \mathbb{Z}$ given in Lemma 3.3, define

$$\overline{W}_n(t) = \sqrt{\omega_n(t)}W_{n+n_0}(t) = e^{-\lambda(n+ct)}W_{n+n_0}(t),$$

and let $\overline{\mathbf{W}}(t) := \{\overline{W}_n(t)\}_{n\in\mathbb{Z}}$. Substituting $W_{n+n_0}(t) = \frac{1}{\sqrt{\omega_n(t)}}\overline{W}_n(t)$ into Eq. 3.1, we derive the following equation:

$$\begin{cases} \overline{W}_n'(t) - \rho((J\cdot\exp(-\lambda))\star W)_n(t) + (\rho+\delta+c\lambda)\overline{W}_n(t) \\ -e^{-\lambda c\tau}((R\cdot\exp(-\lambda))\otimes f'(\Phi)\overline{W})_n(t-\tau) = \sqrt{\omega_n(t)}(R\otimes Q(W))_{n+n_0}(t-\tau), \quad t>0, \\ \overline{W}_n(s) = \overline{W}_n^0(s), \quad s\in[-\tau,0], \end{cases} \tag{3.15}$$

where $((J\cdot\exp(-\lambda))\star W)_n(t) = \sum_{i\in\mathbb{Z}\setminus\{0\}} J(i)e^{-\lambda i}W_{n-i}(t)$ and $((R\cdot\exp(-\lambda))\otimes W)_n(t) = \sum_{i\in\mathbb{Z}} R(i)e^{-\lambda i}W_{n-i}(t)$.

Lemma 3.4 *Suppose* (K1)-(K3), (H1)-(H3), *and* (F) *hold. Then*

$$\frac{1}{2}\frac{d\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2}{dt} + \mu\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2 + C_6\left(\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2 - \left\|\overline{\mathbf{W}}(t-\tau)\right\|_{l^2}^2\right) \le I_1(t), \quad 0\le t\le T, \tag{3.16}$$

where

$$\mu := -\rho\sum_{i\in\mathbb{Z}\setminus\{0\}} e^{-\lambda i}J(i) + c\lambda + \rho + \delta - f'(0)e^{-\lambda c\tau}\sum_{i\in\mathbb{Z}} e^{-\lambda i}R(i) > 0,$$

$$C_6 := \frac{1}{2}f'(0)e^{-\lambda c\tau}\sum_{i\in\mathbb{Z}} e^{-\lambda i}R(i), \quad I_1(t) := \sum_{n\in\mathbb{Z}}\sqrt{\omega_n(t)}(R\otimes Q(W))_{n+n_0}(t-\tau)\overline{W}_n(t).$$

Proof Taking the regular energy estimates $\sum_{n\in\mathbb{Z}}\overline{W}_n(t)\times(3.15)$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2}{dt} + (\rho+\delta+c\lambda)\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2 - \rho\sum_{n\in\mathbb{Z}}((J\cdot\exp(-\lambda))\star\overline{W})_n(t)\overline{W}_n(t) \\ &-\sum_{n\in\mathbb{Z}} e^{-\lambda c\tau}((R\cdot\exp(-\lambda))\otimes f'(\Phi)\overline{W})_n(t-\tau)\overline{W}_n(t) \\ =& \sum_{n\in\mathbb{Z}}\sqrt{\omega_n(t)}(R\otimes Q(W))_{n+n_0}(t-\tau)\overline{W}_n(t) := I_1(t). \end{aligned} \tag{3.17}$$

Define $I_2(t) := \rho\sum_{n\in\mathbb{Z}}((J\cdot\exp(-\lambda))\otimes\overline{W})_n(t)\overline{W}_n(t)$ and

$$I_3(t) := \sum_{n\in\mathbb{Z}} e^{-\lambda c\tau}((R\cdot\exp(-\lambda))\otimes f'(\Phi)\overline{W})_n(t-\tau)\overline{W}_n(t).$$

Via the Hölder inequality and Fourier transform, we have

$$
\begin{aligned}
|I_2(t)| &\le \rho \left\|\overline{\mathbf{W}}(t)\right\|_{l^2} \left\|\left\{((J\cdot\exp(-\lambda))\star\overline{W})_n(t)\right\}_{n\in\mathbb{Z}}\right\|_{l^2}\\
&= \rho \left\|\overline{\mathbf{W}}(t)\right\|_{l^2} \left\|\mathcal{F}\left[\left\{((J\cdot\exp(-\lambda))\star\overline{W})_n(t)\right\}_{n\in\mathbb{Z}}\right]\right\|_{L^2[-\pi,\pi]}\\
&= \rho \left\|\overline{\mathbf{W}}(t)\right\|_{l^2} \left\|\sqrt{2\pi}\mathcal{F}\left[\left\{J(i)e^{-\lambda i}\right\}_{i\in\mathbb{Z}\backslash\{0\}}\right]\cdot\mathcal{F}\left[\overline{\mathbf{W}}(t)\right]\right\|_{L^2[-\pi,\pi]}\\
&\le \rho \left\|\overline{\mathbf{W}}(t)\right\|_{l^2} \left(\int_{-\pi}^{\pi}\left|\sum_{i\in\mathbb{Z}\backslash\{0\}} J(i)e^{-\lambda i}\right|^2\cdot\left|\mathcal{F}\left[\overline{\mathbf{W}}(t)\right]\right|^2 d\xi\right)^{\frac{1}{2}}\\
&= \rho\left(\sum_{i\in\mathbb{Z}\backslash\{0\}} e^{-\lambda i}J(i)\right)\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2.
\end{aligned}
\tag{3.18}
$$

By the hypothesis (H3) and the calculations similar to Eq. 3.18, we can obtain

$$
|I_3(t)| \le f'(0)\left(\textstyle\sum_{i\in\mathbb{Z}} R(i)e^{-\lambda i-\lambda c\tau}\right)\left(\frac{1}{2}\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2+\frac{1}{2}\left\|\overline{\mathbf{W}}(t-\tau)\right\|_{l^2}^2\right). \tag{3.19}
$$

Substituting Eqs. 3.18 and 3.19 into Eq. 3.17, we get

$$
\begin{aligned}
I_1(t) \ge\ & \frac{1}{2}\frac{d\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2}{dt}+\frac{1}{2}f'(0)\left(\sum_{i\in\mathbb{Z}} R(i)e^{-\lambda i-\lambda c\tau}\right)\left(\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2-\left\|\overline{\mathbf{W}}(t-\tau)\right\|_{l^2}^2\right)\\
&+\left(-\rho\sum_{i\in\mathbb{Z}\backslash\{0\}} e^{-\lambda i}J(i)+c\lambda+\rho+\delta-f'(0)\left(\sum_{i\in\mathbb{Z}} e^{-\lambda(i+c\tau)}R(i)\right)\right)\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2.
\end{aligned}
$$

The proof is completed. □

Lemma 3.5 *Suppose* (K1)-(K3), (H1)-(H3), *and* (F) *hold. Then there exist constants* $\varrho_2 \in (0,1)$, $\sigma \in \left(0, \frac{\mu}{2}\right)$, *and* $C_7 > 0$ *such that*

$$
\begin{aligned}
&\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2 + e^{-2\sigma t}\int_0^t e^{2\sigma s}\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2 ds\\
&\le C_7 e^{-2\sigma t}\left(\left\|\overline{\mathbf{W}}^0(0)\right\|_{l^2}^2+\int_{-\tau}^0 e^{2\sigma s}\left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 ds\right), \quad t\in[0,T],
\end{aligned}
$$

provided $\|\mathbf{W}(t)\|_{X(-\tau,T)} \le \varrho_2$, *where* μ *is defined in Lemma 3.4. In particular, constants* ϱ_2, σ, *and* C_7 *are independent of* T *and* $\mathbf{W}(t)$.

Proof Multiplying inequality (3.16) by $e^{2\sigma t}$ and integrating it from 0 to t, that is

$$
\begin{aligned}
&e^{2\sigma t}\left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2+2(\mu-\sigma)\int_0^t e^{2\sigma s}\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2 ds\\
&+2C_6\int_0^t e^{2\sigma s}\left(\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2-\left\|\overline{\mathbf{W}}(s-\tau)\right\|_{l^2}^2\right)ds \le \|\overline{\mathbf{W}}^0(0)\|_{l^2}^2+2\int_0^t e^{2\sigma s}I_1(s)ds.
\end{aligned}
\tag{3.20}
$$

By changing variables, we get

$$
\begin{aligned}
& 2C_6\int_0^t e^{2\sigma s}\left\|\overline{\mathbf{W}}(s-\tau)\right\|_{l^2}^2 ds \\
\leq\ & 2C_6\int_{-\tau}^0 e^{2\sigma(s+\tau)}\left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 ds + 2C_6\int_0^t e^{2\sigma(s+\tau)}\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2 ds.
\end{aligned}
$$

Substituting the above inequality into Eq. 3.20, we can obtain

$$
\begin{aligned}
& \left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2 + 2\tilde{\mathcal{A}}\int_0^t e^{-2\sigma(t-s)}\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2 ds \\
\leq\ & C'e^{-2\sigma t}\left(\left\|\overline{\mathbf{W}}^0(0)\right\|_{l^2}^2 + \int_{-\tau}^0 e^{2\sigma s}\left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 ds\right) + 2\int_0^t e^{-2\sigma(t-s)} I_1(s)ds,
\end{aligned} \tag{3.21}
$$

where $C' = 1 + 2C_6 e^{2\sigma\tau}$ and $\tilde{\mathcal{A}} = (\mu - \sigma) + C_6(1 - e^{2\sigma\tau})$.

Now we estimate $I_1(t)$. It follows from $\mathbf{W}(t) := \{W_n(t)\}_{n\in\mathbb{Z}} \in X(-\tau, T)$ that $\mathbf{W}(t) \in C([-\tau, T]; l^2)$ and $\sup_{t\in[-\tau,T]}\left|W_{n+n_0}(t)\right| \leq \|\mathbf{W}(t)\|_{X(-\tau,T)} \leq \varrho_2 < 1$. Obviously,

$$
\begin{aligned}
& \overline{W}_n(t) = \sqrt{\omega_n(t)}W_{n+n_0}(t) = e^{-\lambda n-\lambda ct}W_{n+n_0}(t), \\
& \overline{W}_{n-i}(t-\tau) = \sqrt{\omega_{n-i}(t-\tau)}W_{n-i+n_0}(t-\tau) = e^{-\lambda(n-i)-\lambda c(t-\tau)}W_{n-i+n_0}(t-\tau).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
& 2\int_0^t e^{-2\sigma(t-s)} I_1(s)ds \\
\leq\ & 2\Lambda\int_0^t e^{-2\sigma(t-s)}\sum_{n\in\mathbb{Z}}\sqrt{\omega_n(s)}(R\otimes W^2)_{n+n_0}(s-\tau)\overline{W}_n(s)ds \\
=\ & 2\Lambda\int_0^t e^{-2\sigma(t-s)}\sum_{n\in\mathbb{Z}}\left|\overline{W}_n(s)\right|\left(\sum_{i\in\mathbb{Z}}R(i)e^{-\lambda(i+c\tau)}\left|\overline{W}_{n-i}(s-\tau)\right|\left|W_{n+n_0-i}(s-\tau)\right|\right)ds \\
\leq\ & \Lambda\left(\sum_{i\in\mathbb{Z}}e^{-\lambda(i+c\tau)}R(i)\right)\|\mathbf{W}(t)\|_{X(-\tau,T)}e^{-2\sigma t}\int_0^t e^{2\sigma s}\left(\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2 + \left\|\overline{\mathbf{W}}(s-\tau)\right\|_{l^2}^2\right)ds \\
\leq\ & C''\|\mathbf{W}(t)\|_{X(-\tau,T)}\left(e^{-2\sigma t}\int_0^t e^{2\sigma s}\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2 ds + e^{-2\sigma t}\int_{-\tau}^0 e^{2\sigma s}\left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 ds\right),
\end{aligned} \tag{3.22}
$$

where $C'' = \Lambda\sum_{i\in\mathbb{Z}}e^{-\lambda(i+c\tau)}R(i)$ and $\Lambda := \max_{u\in[0,K^*+1]}|f''(u)|$. Substitute Eq. 3.22 into Eq. 3.21, that is

$$
\begin{aligned}
& \left\|\overline{\mathbf{W}}(t)\right\|_{l^2}^2 + 2\left(\tilde{\mathcal{A}} - C''\|\mathbf{W}(t)\|_{X(-\tau,T)}\right)e^{-2\sigma t}\int_0^t e^{2\sigma s}\left\|\overline{\mathbf{W}}(s)\right\|_{l^2}^2 ds \\
\leq\ & C'''e^{-2\sigma t}\left(\left\|\overline{\mathbf{W}}^0(0)\right\|_{l^2}^2 + \int_{-\tau}^0 e^{2\sigma s}\left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 ds\right),
\end{aligned}
$$

where $C''' = C' + C''$. Here, we can choose a sufficiently small $0 < \sigma < \frac{\mu}{2}$ such that $\tilde{\mathcal{A}} = (\mu - \sigma) + C_6(1 - e^{2\sigma\tau}) \geq \frac{\mu}{2}$, where $\mu > 0$ is defined in Lemma 3.4. Take $\varrho_2 \in (0, 1)$ satisfying $C''\varrho_2 \leq \frac{\mu}{4}$. Therefore, there exists a constant $C_7 > 0$ such that

$$\|\overline{\mathbf{W}}(t)\|_{l^2}^2 + \int_0^t e^{-2\sigma(t-s)} \|\overline{\mathbf{W}}(s)\|_{l^2}^2 \, ds \leq C_7 e^{-2\sigma t} \left(\left\|\overline{\mathbf{W}}^0(0)\right\|_{l^2}^2 + \int_{-\tau}^0 e^{2\sigma s} \left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 ds \right). \tag{3.23}$$

Clearly, ϱ_2, σ, and C_7 are independent of T and $\mathbf{W}(t)$. The proof is completed. □

Lemma 3.6 *Suppose* (K1)-(K3)*,* (H1)-(H3)*, and* (F) *hold. Let ϱ_2, σ, and C_7 be defined in Lemma 3.5. Then there exists $C_8 > 0$ such that*

$$\|\mathbf{W}(t)\|_{l^\infty(-\infty, n_0 - [cT] - 1]} \leq C_8 \kappa e^{-\sigma t}, \quad t > 0, \tag{3.24}$$

provided $\|\mathbf{W}(t)\|_{X(-\tau,T)} \leq \varrho_2$, where $n_0 \gg 1$ is defined in Lemma 3.3 and

$$\kappa^2 = \left\|\mathbf{W}^0(0)\right\|_{l^2_\omega}^2 + \int_{-\tau}^0 e^{2\sigma s} \left\|\mathbf{W}^0(s)\right\|_{l^2_\omega}^2 ds.$$

Proof It follows from Eq. 3.23 that

$$\|\overline{\mathbf{W}}(t)\|_{l^2}^2 + \int_0^t e^{-2\sigma(t-s)} \|\overline{\mathbf{W}}(s)\|_{l^2}^2 \, ds \leq C_7 e^{-2\sigma t} \left(\left\|\overline{\mathbf{W}}^0(0)\right\|_{l^2}^2 + \int_{-\tau}^0 e^{2\sigma s} \left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 ds \right).$$

By Sobolev's embedding inequality $l^2 \hookrightarrow l^\infty$, it yields

$$\left\|\overline{\mathbf{W}}(t)\right\|_{l^\infty} \leq \left\|\overline{\mathbf{W}}(t)\right\|_{l^2} \leq \sqrt{C_7} e^{-\sigma t} \left(\left\|\overline{\mathbf{W}}^0(0)\right\|_{l^2}^2 + \int_{-\tau}^0 \left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 e^{2\sigma s} ds \right)^{\frac{1}{2}}.$$

Since $\overline{W}_n(t) = \sqrt{\omega_n(t)} W_{n+n_0}(t) = e^{-\lambda(n+ct)} W_{n+n_0}(t) \geq e^{-\lambda(n+[cT]+1)} W_{n+n_0}(t)$, and $e^{-\lambda(n+[cT]+1)} \geq 1$ for any $n + [cT] + 1 \in (-\infty, 0]$, we can obtain

$$\sup_{n+[cT]+1 \in (-\infty,0]} |W_{n+n_0}(t)| \leq \sqrt{C_7} e^{-\sigma t} \left(\left\|\overline{\mathbf{W}}^0(0)\right\|_{l^2}^2 + \int_{-\tau}^0 \left\|\overline{\mathbf{W}}^0(s)\right\|_{l^2}^2 e^{2\sigma s} ds \right)^{\frac{1}{2}}.$$

Consequently, we have

$$\sup_{n \in (-\infty, n_0 - [cT] - 1]} |W_n(t)| \leq C_8 \kappa e^{-\sigma t}, \quad t \in [0, T]$$

where $C_8 > 0$ is a constant. Thus, Eq. 3.24 is proved. □

Proof of Theorem 3.2 Combining Lemmas 3.3, 3.5, and 3.6, there are $0 < \varrho \leq \min\{\varrho_1, \varrho_2\}$, $\alpha = \min\{\varepsilon, \sigma\}$, and $\tilde{C} > 1$ such that

$$\begin{aligned} &\|\mathbf{W}(t)\|_{l^\infty}^2 + \|\mathbf{W}(t)\|_{l^2_\omega}^2 + \int_0^t e^{-2\alpha(t-s)} \|\mathbf{W}(s)\|_{l^2_\omega}^2 \, ds \\ &\leq \tilde{C} e^{-2\alpha t} \sup_{s \in [-\tau, 0]} \left(\left\|\mathbf{W}^0(s)\right\|_{l^\infty}^2 + \left\|\mathbf{W}^0(s)\right\|_{l^2_\omega}^2 + \int_{-\tau}^0 \left\|\mathbf{W}^0(s)\right\|_{l^2_\omega}^2 ds \right), \end{aligned}$$

provided $\|\mathbf{W}(\cdot)\|_{X(-\tau,T)} < \varrho$, $t \in [0, T]$. This completes the proof. □

3.3 Asymptotic Stability

Proof of Theorem 1.3 According to the local existence (Theorem 3.1) and the key estimate (Theorem 3.2), we prove the theorem via the continuity extension method [15]. Let α, $\tilde{C}$, and ϱ be defined in Theorem 3.2, which are independent of T and $\mathbf{W}(t)$. Let C_1 be defined in Theorem 3.1. Set

$$\delta_0 = \min\left\{\frac{\varrho}{C_1}, \frac{\varrho}{\sqrt{\tilde{C}}C_1}\right\}, \quad \delta_1 = \max\{\delta_0, \varrho\}, \tag{3.25}$$

$$\|\mathbf{W}(s)\|_{X(-\tau,0)} \le \delta_0 < \delta_1. \tag{3.26}$$

By Theorem 3.1, there exists $t_0 = t_0(\delta_1) > 0$ so that $\mathbf{W}(t) \in X(-\tau, t_0)$ and

$$\|\mathbf{W}(t)\|_{X(-\tau,t_0)} \le C_1 \left\|\mathbf{W^0}(s)\right\|_{X(-\tau,0)} \le C_1\delta_0 \le \varrho.$$

It follows from Theorem 3.2 that

$$\|\mathbf{W}(t)\|_{X(0,t_0)} \le \sqrt{\tilde{C}}e^{-2\alpha t}\|\mathbf{W^0}(s)\|_{X(-\tau,0)} \le \sqrt{\tilde{C}}\delta_0 \le \frac{\varrho}{C_1}. \tag{3.27}$$

Now consider Eq. 3.1 on the initial time interval $[t_0-\tau, t_0]$. Combining Eqs. 3.25, 3.26, and 3.27, we have

$$\begin{aligned}\|\mathbf{W}(t)\|_{X(t_0-\tau,t_0)} &\le \max\left\{\|\mathbf{W^0}(s)\|_{X(-\tau,0)}, \|\mathbf{W}(t)\|_{X(0,t_0)}\right\}\\ &\le \max\left\{\|\mathbf{W^0}(s)\|_{X(-\tau,0)}, \frac{\varrho}{C_1}\right\} \le \delta_1.\end{aligned}$$

Applying Theorem 3.1 once more, we obtain that $\mathbf{W}(t) \in X(-\tau, 2t_0)$ and $\|\mathbf{W}(t)\|_{X(t_0-\tau,2t_0)} \le C_1\|\mathbf{W}(s)\|_{X(t_0-\tau,t_0)}$. In addition, $\|\mathbf{W}(t)\|_{X(t_0-\tau,t_0)} \le \max\left\{\delta_0, \frac{\varrho}{C_1}\right\} \le \frac{\varrho}{C_1}$, which indicates $\|\mathbf{W}(t)\|_{X(t_0-\tau,2t_0)} \le \varrho$. Thus,

$$\begin{aligned}\|\mathbf{W}(t)\|_{X(-\tau,2t_0)} &\le \max\left\{\|\mathbf{W}(t)\|_{X(t_0-\tau,2t_0)}, \|\mathbf{W}(t)\|_{X(0,t_0-\tau)}, \|\mathbf{W^0}(s)\|_{X(-\tau,0)}\right\}\\ &\le \max\left\{\delta_0, \frac{\varrho}{C_1}, \varrho\right\} \le \varrho.\end{aligned}$$

Then, by Theorem 3.2, for $t \in [0, 2t_0]$, there is

$$\|\mathbf{W}(t)\|_{X(0,2t_0)} \le \tilde{C}e^{-2\alpha t}\|\mathbf{W}(s)\|_{X(-\tau,0)} \le \tilde{C}\delta_0 \le \frac{\varrho}{C_1}.$$

Repeating this process step by step, we can obtain the solution $\mathbf{W}(t)$ exists globally in $X(-\tau, \infty)$ and satisfies

$$\|\mathbf{W}(t)\|_{l^\infty} \le Ce^{-\alpha t}, \quad 0 \le t < \infty.$$

The proof is completed. □

Acknowledgments The authors are grateful to the anonymous referee for her/his very valuable comments and suggestions helping to the improvement of the manuscript.

Funding Information This work was supported by NNSF of China (11371179).

References

1. Chen X, Guo J-S. Existence and asymptotic stability of travelling waves of discrete quasilinear monostable equations. J Differ Equ. 2002;184:549–69.
2. Chen X, Guo J-S. Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. Math Ann. 2003;326:123–46.
3. Chen X, Fu S-C, Guo J-S. Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices. SIAM J Math Anal. 2006;38:233–58.
4. Cheng C-P, Li W-T, Wang Z-C. Spreading speeds and traveling waves in a delayed population model with stage structure on a two-dimensional spatial lattice. IMA J Appl Math. 2008;73:592–618.
5. Cheng C-P, Li W-T, Wang Z-C. Asymptotic stability of traveling wavefronts in a delayed population model with stage structure on a two dimensional spatial lattice. Discrete Contin Dyn Syst Ser B. 2010;13:559–75.
6. Fang J, Wei J, Zhao X. Spreading speeds and traveling waves for non-monotone time-delayed lattice equation. Proc Roy Soc Edinburgh Sect A. 2010;466:1919–34.
7. Fang J, Wei J, Zhao X. Uniqueness of traveling waves for nonlocal lattice equations. Proc Am Math Soc. 2011;139:1361–73.
8. Guo S, Zimmer J. Stability of traveling wavefronts in discrete reaction-diffusion equations with nonlocal delay effects. Nonlinearity. 2015;28:463–92.
9. Huang R, Mei M, Zhang K-J, Zhang Q-F. Asymptotic stability of non-monotone traveling waves for time-delayed nonlocal dispersion equations. Discrete Contin Dyn Syst. 2016;36:1331–53.
10. Lin G, Li W-T, Pan S. Traveling wavefronts in delayed lattice dynamical systems with global interaction. J Difference Equ Appl. 2010;16:1429–46.
11. Lin C-K, Lin C-T, Lin Y-P, Mei M. Exponential stability of nonmonotone traveling waves for Nicholson's blowflies equation. SIAM J Math Anal. 2014;46:1053–84.
12. Ma S, Zou X. Propagation and its failure in a lattice delayed differential equation with global interaction. J Differ Equ. 2005;212:129–90.
13. Ma S, Zou X. Existence, uniqueness and stability of travelling waves in a discrete reaction-diffusion monostable equation with delay. J Differ Equ. 2005;217:54–87.
14. Ma S, Weng P, Zou X. Asymptotic speed of propagation and traveling wavefronts in a non-local delayed lattice differential equation. Nonlinear Anal. 2006;65:1858–90.
15. Mei M, So J-W-H, Li M-Y, Shen S-S-P. Asymptotic stability of traveling waves for the Nicholson's blowflies equation with diffusion. Proc R Soc Edinb Sect A. 2004;134:579–94.
16. Mei M, So J-W-H. Stability of strong traveling waves for a non-local time-delayed reaction-diffusion equation. Proc Roy Soc Edinburgh Sect A. 2008;138:551–68.
17. Mei M, Lin C-K, So J-W-H. Traveling wavefronts for time-delayed reaction-diffusion equation: (I) local nonlinearity. J Differ Equ. 2009;247:495–510.
18. Mei M, Ou C, Zhao X-Q. Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations. SIAM J Math Anal. 2010;42:2762–90.
19. Titchmarsh E-C. Introduction to the theory of fourier integrals. Oxford: Oxford University Press; 1948.
20. Thieme H-R. Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread. J Math Biol. 1979;8:173–87.
21. Tian G, Zhang G-B, Yang Z-X. Stability of non-monotone critical traveling waves for spatially discrete reaction-diffusion equations with time delay. Turkish J Math. 2017;41:655–80.
22. Weng P-X, Huang H-X, Wu J. Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction. IMA J Appl Math. 2003;68:409–39.
23. Wang Z-C, Li W-T, Wu J. Entire solutions in delayed lattice differential equations with monostable nonlinearity. SIAM J Math Anal. 2009;40:2392–420.
24. Yu Z-X, Mei M. Uniqueness and stability of traveling waves for cellular neural networks with multiple delays. J Differ Eq. 2016;260:241–67.
25. Yang Z-X, Zhang G-B. Stability of non-monotone traveling waves for a discrete diffusion equation with monostable convolution type nonlinearity. Sci China Math. 2018;61:1789–806.
26. Zhang G-B. Global stability of traveling wave fronts for non-local delayed lattice differential equations. Nonlinear Anal Real World Appl. 2012;13:1790–801.