Journal of Dynamical and Control Systems (2019) 25: 457–490
https://doi.org/10.1007/s10883-019-09435-8

Local (Sub)-Finslerian Geometry for the Maximum Norms in Dimension 2

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Received: 29 May 2017 / Revised: 7 February 2019 / Published online: 11 March 2019

Abstract
We consider specific sub-Finslerian structures in the neighborhood of 0 in $\mathbb{R}^2$, defined by fixing a family of vector fields (F_1, F_2) and considering the norm defined on the non-constant rank distribution $\Delta = \text{vect}\{F_1, F_2\}$ by

$$|G| = \inf_u \{\max\{|u_1|, |u_2|\} \mid G = u_1 F_1 + u_2 F_2\}.$$

If F_1 and F_2 are not proportional at p, then we obtain a Finslerian structure; if not, the structure is sub-Finslerian on a distribution with non-constant rank. We are interested in the study of the local geometry of these Finslerian and sub-Finslerian structures: generic properties, normal form, short geodesics, cut locus, switching locus, and small spheres.

Keywords Sub-Finslerian geometry · Maximum norm · Geodesics · Small spheres · Cut locus

Mathematics Subject Classification (2010) 53B40 · 53C22 · 49J15 · 49K15

1 Introduction

From the 1980s, the interest for the sub-Riemannian geometry increases with a lot of contributions in several domains as PDEs, analysis, probability, geometry, and control. One of the questions was to understand the local geometry of sub-Riemannian metrics, as the singularities of small spheres, local cut locus, local conjugate locus, and so on motivated in particular by new results on the heat kernel in the sub-Riemannian context (see [9, 10, 22, 23]). The contact and the Martinet cases were deeply studied (see [1, 2, 11, 12, 19]). The quasi-contact case in dimension 4 was also studied (see [15]). These results allowed to give

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new results on the asymptotics of the heat kernel at cut and conjugate loci in the 3D contact and 4D quasi-contact cases [5, 7].

In this article, we start the same work for Finslerian and sub-Finslerian metrics associated with a maximum norm: consider a manifold M, a vector bundle $\pi : E \to M$ with fibers of same dimension as M endowed with a maximum norm, and a morphism of vector bundles $f : E \to TM$ such that the map from $\Gamma(E) \to Vec(M)$ defined by $\sigma \mapsto f \circ \sigma$ is injective. An admissible curve is a curve γ in M such that exists a lift σ in E with $\dot{\gamma}(t) = f(\sigma(t))$ a.e. The length of such a curve is the infimum of the $\int_0^T |\sigma(t)|dt$ for all possible such σ and the distance between two points q_0 and q_1 is the infimum of the lengths of the curves joining q_0 and q_1. Remark that the map f itself is not assumed to be injective everywhere: at points where f is injective the structure is Finslerian when at points where it is not, it is sub-Finslerian.

Here, we concentrate our attention on the local study of such structures in dimension 2, that is when M and the fibers of E have dimension 2.

Equivalently, with a control point of view and since we are interested in local properties, we consider control systems in a neighborhood of 0 in $\mathbb{R}^2$ of the type

$$\dot{q} = u_1 F_1(q) + u_2 F_2(q) \tag{1}$$

where F_1 and F_2 are smooth vector fields and u_1 and u_2 are control functions satisfying

$$|u_1| \leq 1 \text{ and } |u_2| \leq 1. \tag{2}$$

Up to reparameterization, minimizing the distance in the geometric context is equivalent to minimizing the time of transfer in the control context.

We are interested in the study of the time optimal synthesis of such systems. Of course, the general situation cannot be completely described since singular cases may have very special behavior. For example, in the case $F_1 = \partial_x$ and $F_2 = \partial_y$, any admissible trajectory with $u_1 \equiv 1$ and $\int_0^1 u_2(t)dt = 0$ joins optimally $(0, 0)$ to $(1, 0)$. Hence, in the following, we will consider only "generic" situations as defined in Section 2.1.

Few works exist concerning sub-Finsler geometry since it is a new subject. Let us mention the works [17, 18] for dimension 3, considering norms which are assumed to be smooth outside the zero section. In [14], the sphere of a left invariant sub-Finsler structure associated with a maximum norm in the Heisenberg group is described. In the preprint [6], the authors describe the extremals (and discuss in particular their number of switches before the loss of optimality) for the Heisenberg, Grushin, and Martinet distributions. In the preprint [4], we describe, in the 3D generic contact case, the small spheres and the local cut locus.

The paper is organized as follows.

In Section 2, we recall Thom's transversality theorem and some of its corollaries, define what we mean by generic, give generic properties of the couples of vector fields on two-dimensional manifolds and give a normal form for the generic couples.

In Section 3, we give first general results about the optimal synthesis; recalling classical results as Chow-Rashevski, Filippov, and Pontryagin theorems; analyzing the possibilities for extremals to switch or to be singular depending on their initial conditions; giving details on the weights of coordinates in the normal form and on the associated nilpotent approximation.

In Section 4, we present the local synthesis in all the generic cases presented in the normal form of Section 2.

2 Normal Form

In this section, the goal is to give a list of properties of generic couples (F_1, F_2) and to construct a normal form for the couple (G_1, G_2) defined by $G_1 = F_1 + F_2$ and $G_2 = F_1 - F_2$. As we will see, $\pm G_1$ and $\pm G_2$ are the velocities of a large class of the minimizers of the optimal control system defined by Eqs. 1 and 2.

In all this article, we will consider the following sets. We define

$$\begin{aligned}\Delta_A &= \{q \in M \mid F_1(q) \text{ and } F_2(q) \text{ are colinear}\},\\ \Delta_1 &= \{q \in M \mid F_1(q) \text{ and } [F_1, F_2](q) \text{ are colinear}\},\\ \Delta_2 &= \{q \in M \mid F_2(q) \text{ and } [F_1, F_2](q) \text{ are colinear}\}.\end{aligned}$$

In order to give the list of properties, we use the Thom's transversality theorem and some of its corollaries.

2.1 Generic Properties of Couples of Smooth Vector Fields on 2D Manifolds

2.1.1 Thom's Transversality Theorem

Denote $J^k(M, N)$ the set of k-jets of maps from M to N.

Theorem 1 (Thom Transversality Theorem, [21], **Page 82)** *Let M, N be smooth manifolds and $k \geq 1$ an integer. If $S_1, \cdots, S_r$ are smooth submanifolds of $J^k(M, N)$ then the set*

$$\{f \in C^\infty(M, N) : J^k f \pitchfork S_i \text{ for } i = 1, 2, \cdots, r\},$$

is residual in the C^∞-Whitney topology.

Corollary 1 *Assume that $codim\ S_i > \dim M$ for $i = 1, \cdots, r$ and $k \geq 1$. Then, the set*

$$\{f \in C^\infty(M, N) : J^k f(M) \cap S_i = \emptyset \text{ for } i = 1, \cdots, r\},$$

is residual in the C^∞-Whitney topology.

Corollary 2 *For every f in the residual set defined in Theorem 1, the inverse images $\tilde{S}_i := (J^k f)^{-1}(S_i)$ are a smooth submanifold of M and $codim\ S_i = codim\ \tilde{S}_i$ for $i = 1, \cdots, r$.*

Remark 1 Let φ be a diffeomorphism of M and ϕ be a diffeomorphism of N. The map

$$\sigma_{\varphi,\phi} : \begin{cases} C^\infty(M, N) & \longrightarrow & C^\infty(M, N) \\ f & \longmapsto & \varphi \circ f \circ \phi \end{cases}$$

induces a diffeomorphism $\sigma^*_{\varphi,\phi}$ of $J^k(M, N)$ sending submanifolds of $J^k(M, N)$ on submanifolds of $J^k(M, N)$. Moreover, f is in the residual set defined in theorem 1, if and only if $\sigma_{\varphi,\phi}(f)$ is in the residual set

$$\{g \in C^\infty(M, N) : J^k g \pitchfork \sigma^*_{\varphi,\phi}(S_i) \text{ for } i = 1, \cdots, r\}.$$

This remark is important to facilitate the presentation of the proofs of the generic properties given in the next section.

Definition 1 In the following, we will say that a property of maps is generic if it is true on a residual set for the C^∞-Whitney topology.

2.1.2 First Generic Properties

Here, we give a list of generic properties for couples of vector fields on 2D manifolds. In order to use Thom transversality theorem, we work locally in coordinates: we fix a point and consider the Taylor series of a couple of vector fields at this point. Locally, one can consider such a couple as the data of a map

$$g : \begin{cases} U \subset \mathbb{R}^2 & \to & \mathbb{R}^2 \times \mathbb{R}^2 \\ (x, y) & \mapsto & ((g_1(x, y), g_2(x, y)), (g_3(x, y), g_4(x, y))) \end{cases}$$

and the k-jet at $q = (0, 0) \in U$ of g as the data of the map

$$J^k g : \begin{cases} \mathbb{R}^2 & \to & \mathbb{R}_k[x, y]^4 \\ (x, y) & \mapsto & (P_1(x, y), \ldots, P_4(x, y)) \end{cases}$$

where P_i $(1 \leq i \leq 4)$ is the Taylor series of order k of g_i at q.

In order to describe submanifolds of $\mathbb{R}_k[x, y]^4$ in coordinates, we write:

$$P_\ell(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} p_{\ell,i,j} x^i y^j, \quad \forall \ell = 1, .., 4.$$

In the following, (g_1, g_2) are the coordinates of G_1 and (g_3, g_4) the coordinates of G_2 in a local coordinate system.

Generic property 1 (GP1): for generic couples of vector fields (F_1, F_2) on M, the set of points where $G_1 = G_2 = 0$ is empty.

Indeed in coordinates, such points correspond to jets with $p_{1,0,0} = p_{2,0,0} = p_{3,0,0} = p_{4,0,0} = 0$ which form a submanifold of $\mathbb{R}_k[x, y]^4$ of codimension 4. Hence, thanks to corollary 1, the property is proven.

Denote J^k_N the set of $k - jets$ such that $P_1 \equiv 1$ and $P_2 \equiv 0$. Once assumed that we choose a coordinate system such that $G_1 = (1, 0)$, then $J^k g$ is in J^k_N.

Assume that a set S of $J^k(\mathbb{R}^2, \mathbb{R}^4)$ is defined as the zero level of a finite number of functions h_i, $i = 1 \ldots k$, whose differentials form a free family when restricted to $T J^k_N$. Then locally, the differentials of the functions h_i form a free family and hence, close to $J^k_N \cap S$, the set S is locally a submanifold. In this context, the codimension of S in $J^k(\mathbb{R}^2, \mathbb{R}^4)$ is equal to the codimension of $S' = S \cap J^k_N$ in J^k_N.

Thanks to remark 1, up to a permutation between $\pm F_1$ and $\pm F_2$ and a good choice of coordinates, we will assume in all the following that $G_1 \equiv (1, 0)$ locally. This implies that is $g_1 \equiv 1$ and $g_2 \equiv 0$ and that the jet of (G_1, G_2) is in J^k_N. As a consequence, if a set S is defined by a finite number of functions h_i, $i = 1 \ldots k$, whose differentials form a free family when restricted to $T J^k_N$, then to apply Thom's theorem and its corollaries we are reduced to apply them to the map

$$g : \begin{cases} U \subset \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (g_3(x, y), g_4(x, y)) \end{cases}$$

and the set $S' = S \cap J^k_N$ seen as a submanifold of $J^k(\mathbb{R}^2, \mathbb{R}^2)$.

Generic property 2 (GP2): for generic couples of vector fields (F_1, F_2) on M, the set of points where $G_2 = 0$ is a discrete set. The same holds for the set where $F_1 = 0$ or the set where $F_2 = 0$.

Indeed, such points correspond to jets with $p_{3,0,0} = p_{4,0,0} = 0$ which is a submanifold of $\mathbb{R}_k[x, y]^2$ of codimension 2. Hence, thanks to corollary 2, the set where $G_2 = 0$ is generically a submanifold of M of codimension 2 that is a discrete set. For $F_2 = 0$, the equations are $p_{3,0,0} = 1$ and $p_{4,0,0} = 0$ and for $F_1 = 0$ the equations are $p_{3,0,0} = -1$ and $p_{4,0,0} = 0$.

Generic property 3 (GP3): for generic couples of vector fields (F_1, F_2) on M, the set Δ_A of points where G_2 is parallel to G_1 is an embedded submanifold of codimension 1.

Indeed, Δ_A is exactly the set of points where $g_4 = 0$, corresponding to jets with $p_{4,0,0} = 0$. This last set is an embedded submanifold of $\mathbb{R}_k[x, y]^2$ of codimension 1. Thanks to (GP1) and to corollary 2, we can conclude that generically Δ_A is an embedded submanifold of codimension 1.

Generic property 4 (GP4): for generic couples of vector fields (F_1, F_2) on M, the set Δ_1 of points where F_1 is parallel to $[F_1, F_2]$ is an embedded submanifold of codimension 1. The same holds for Δ_2 where F_2 is parallel to $[F_1, F_2]$.

In order to prove (GP4), compute $[F_1, F_2]$ and describe Δ_1 in coordinates. $[F_1, F_2] = -\frac{1}{2}[G_1, G_2]$ hence it has coordinates $-\frac{1}{2}p_{3,1,0}$ and $-\frac{1}{2}p_{4,1,0}$ and F_1 has coordinates $\frac{1}{2}(1 + p_{3,0,0})$ and $\frac{1}{2}p_{4,0,0}$. Hence, Δ_1 corresponds to jets satisfying

$$\begin{vmatrix} -\frac{1}{2}p_{3,1,0} & \frac{1}{2}(1 + p_{3,0,0}) \\ -\frac{1}{2}p_{4,1,0} & \frac{1}{2}p_{4,0,0} \end{vmatrix} = 0.$$

The differential of this determinant is not degenerate hence the set of $\mathbb{R}_k[x, y]^2$ satisfying this equality is an embedded submanifold of codimension 1. Hence, generically, Δ_1 is the preimage of an immersed submanifold of codimension 1 which, thanks to corollary 2, permits to conclude that Δ_1 is an immersed submanifold of codimension 1.

Generic property 5 (GP5): for generic couples of vector fields (F_1, F_2) on M, the sets $(\Delta_A \cap \Delta_1)$, $(\Delta_A \cap \Delta_2)$ and $(\Delta_1 \cap \Delta_2)$ are discrete.

Since $G_1 = (1, 0)$, the set $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$ is the set of points where (F_1, F_2) is free and $[F_1, F_2] = 0$ that is

$$\begin{aligned} p_{4,0,0} &\neq 0, \\ p_{3,1,0} &= 0 \\ p_{4,1,0} &= 0. \end{aligned}$$

This set is an immersed submanifold of codimension 2 of $\mathbb{R}_k[x, y]^2$ hence, thanks to corollary 2, the set $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$ is generically a discrete set.

The set $(\Delta_A \cap \Delta_2) \setminus \Delta_1$ is a set of points where $F_2 = 0$. By *(GP2)* it is a discrete set. The same holds for $(\Delta_A \cap \Delta_1) \setminus \Delta_2$ which is a set of points where $F_1 = 0$.

The set $\Delta_A \cap \Delta_1 \cap \Delta_2$ is the union of the subset where $F_1 \neq 0$ and $F_1 /\!\!/ F_2 /\!\!/ [F_1, F_2]$ with a subset where $F_1 = 0$. The second is discrete. The first set is also defined by $G_1 /\!\!/ G_2 /\!\!/ [G_1, G_2]$ that is $p_{4,0,0} = 0$ and $p_{4,1,0} = 0$. Hence, thanks to corollary 2, the set where $F_1 \neq 0$ and $F_1 /\!\!/ F_2 /\!\!/ [F_1, F_2]$ is a submanifold of codimension 2 that is a discrete set.

Generic property 6 (GP6): for generic couples of vector fields (F_1, F_2) on M, the set of points where $G_1 /\!\!/ G_2 /\!\!/ [G_1, G_2] /\!\!/ [G1, [G_1, G_2]]$ is empty.

The set where $G_1 \mathbin{/\!/} G_2 \mathbin{/\!/} [G_1, G_2] \mathbin{/\!/} [G1, [G_1, G_2]]$ is such that $p_{4,0,0} = p_{4,1,0} = p_{4,2,0} = 0$. Hence, thanks to corollary 2, it is a submanifold of codimension 3 that is an empty set.

Generic property 7 (GP7): for generic couples of vector fields (F_1, F_2) on M, at the points q where $G_1(q) \mathbin{/\!/} G_2(q) \mathbin{/\!/} [G_1, G_2](q)$ one gets $G_1(q) \in T_q\Delta_A$.

The property $G_1(q) \mathbin{/\!/} G_2(q) \mathbin{/\!/} [G_1, G_2](q)$ implies that $p_{4,0,0} = p_{4,1,0} = 0$. If $p_{4,0,1} \neq 0$ then Δ_A can be written $p_{4,0,1}y = o(x)$ that is Δ_A is tangent to the x axis and $G_1 \in T_q\Delta_A$. Hence, the set of points where $G_1(q) \mathbin{/\!/} G_2(q) \mathbin{/\!/} [G_1, G_2](q)$ and $G_1(q) \notin T_q\Delta_A$ corresponds to jets with $p_{4,0,0} = p_{4,1,0} = p_{4,0,1} = 0$ which is a submanifold of codimension 3. Hence, generically, at the points q where $G_1(q) \mathbin{/\!/} G_2(q) \mathbin{/\!/} [G_1, G_2](q)$, one has $G_1(q) \in T_q\Delta_A$.

One can even detail more the generic properties: using Thom transversality theorem and its corollaries, we can prove that generically

Generic property 8 (GP8): along $\Delta_1 \setminus (\Delta_2 \cup \Delta_A)$, the points where G_1 or G_2 is tangent to Δ_1 are isolated. The same holds true for $\Delta_2 \setminus (\Delta_1 \cup \Delta_A)$.

Generic property 9 (GP9): at points of $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$, neither G_1 nor G_2 are tangent to Δ_1 or Δ_2.

Generic property 10 (GP10): along $\Delta_A \setminus (\Delta_1 \cup \Delta_2)$, the set of points where $G_2 = 0$ or $G_2 = \pm G_1$ is discrete.

2.2 Normal Form

Thanks to the generic properties established in the previous section, we prove

Theorem 2 (Normal form) *For a generic couple of vector fields (F_1, F_2) on a 2d manifold M, at each point q of M, up to an exchange between $\pm F_1$ and $\pm F_2$, we get that $G_1(q) \neq 0$ and that it exists a unique coordinate system (x, y) centered at q such that one of the following normal form holds*

(NF_1) $G_1(x, y) = \partial_x$,

$$G_2(x, y) = \partial_y + x(a_{10} + a_{20}x + a_{11}y + o(x, y))\partial_x + x(b_{10} + b_{20}x + b_{11}y + o(x, y))\partial_y,$$

and $q \notin \Delta_A$.

(NF_2) $G_1(x, y) = \partial_x$,

$$G_2(x, y) = (a_0 + a_{10}x + a_{01}y + o(x, y))\partial_x + x(1 + x(b_{20} + O(x, y)))\partial_y,$$

with $0 \leq a_0 \leq 1$, *and* $q \in \Delta_A \setminus \Delta_1$.

(NF_3) $G_1(x, y) = \partial_x$,

$$G_2(x, y) = (a_0 + o(1))\partial_x + (b_{01}y + \tfrac{1}{2}x^2 + b_{11}xy + b_{02}y^2 + o(x^2, y^2))\partial_y,$$

with $b_{01} > 0$ *and* $0 < a_0 < 1$, $q \in \Delta_A \cap \Delta_1 \cap \Delta_2$ *and* $G_1(q) \in T_q\Delta_A$.

For $(NF1)$ and $(NF2)$ one of the following subcases holds

(NF_{1a}) *(NF_1) holds with $a_{10} - b_{10} \neq 0$ and $a_{10} + b_{10} \neq 0$. It corresponds to $q \notin \Delta_A \cup \Delta_1 \cup \Delta_2$.*

(NF_{1b}) *(NF_1) holds with $a_{10} - b_{10} = 0$ and $a_{10} + b_{10} \neq 0$. It corresponds to $q \in \Delta_1 \setminus (\Delta_A \cup \Delta_2)$.*

(NF_{1c}) *(NF_1) holds with $a_{10} - b_{10} \neq 0$ and $a_{10} + b_{10} = 0$. It corresponds to $q \in \Delta_2 \setminus (\Delta_A \cup \Delta_1)$.*

(NF_{1d}) (NF_1) *holds with* $a_{10} = b_{10} = 0$. *It corresponds to* $q \in (\Delta_1 \cap \Delta_2) \setminus \Delta_A$.
(NF_{2a}) (NF_2) *holds with* $0 \le a_0 < 1$. *It corresponds to* $q \in \Delta_A \setminus (\Delta_1 \cup \Delta_2)$.
(NF_{2b}) (NF_2) *holds with* $a_0 = 1$. *It corresponds to* $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$ *that is to* $q \in \Delta_A \setminus \Delta_1$ *such that* $F_2(q) = 0$.

Such coordinate system is called the normal coordinate system associated with F_1 *and* F_2.

Proof In order to prove this normal form, we construct in each situation a coordinate chart by mean of the flow of some linearly independent vector fields associated with the sub-Finslerian structure. More precisely, once identified such a couple of vector fields (X, Y), we define the coordinate system by defining the map $(x, y) \mapsto e^{xX}e^{yY}q$.

We assume that all the generic properties given before are satisfied. Thanks to *(GP1)*, and thanks to the fact that we are working locally, we can assume that G_1 is not zero.

Thanks to *(GP3)*, we know that Δ_A is a submanifold of dimension 1. Let us start by considering a point q outside Δ_A. We define the map φ which to (x, y) in a neighborhood U of $(0, 0)$ in $\mathbb{R}^2$ associates the point reached by starting at q and following G_2 during time y and then G_1 during time x that is

$$\varphi : \begin{cases} U \to M \\ (x, y) \mapsto e^{xG_1}e^{yG_2}q \end{cases}$$

Since $\partial_x\varphi(0, 0) = G_1(q)$ and $\partial_y\varphi(0, 0) = G_2(q)$, φ is a local diffeomorphism hence defines a local coordinate system. One proves easily that at each point of coordinates (x, y) the vector $G_1(x, y) = (1, 0)$. Moreover, along the y axis, since $\varphi(0, y) = e^{yG_2}q$ then $G_2(0, y) = (0, 1)$. This implies the normal form (NF_1). With the normal form (NF_1), one gets that

$$\begin{aligned} [F_1, F_2](0) &= -\frac{1}{2}[G_1, G_2](0) = -\frac{1}{2}(a_{10}, b_{10}), \\ F_1(0) &= \frac{1}{2}(G_1(0) + G_2(0)) = (\frac{1}{2}, \frac{1}{2}), \\ F_2(0) &= \frac{1}{2}(G_1(0) - G_2(0)) = (\frac{1}{2}, -\frac{1}{2}) \end{aligned}$$

which implies that

$$[F_1, F_2](0) = -\frac{a_{10} + b_{10}}{2}F_1(0) - \frac{a_{10} - b_{10}}{2}F_2(0).$$

The subcases follow immediately.

Assume now that $q \in \Delta_A \setminus \Delta_1$. Hence, $G_1(q)$ and $G_2(q)$ are parallel and since we assume that $G_1(q)$ is not 0, we can assume up to a change of role that $G_2(q) = \alpha G_1(q)$ with $\alpha \in [0, 1]$. Since $q \notin \Delta_1$, $G_1(q)$ and $[G_1, G_2](q)$ are not parallel. This implies that G_1 is not tangent to Δ_A. As a consequence, one can choose a local parameterization $\gamma(t)$ of Δ_A such that $\gamma(0) = q$ and $\dot\gamma(t)$ has second coordinate 1 in the basis $(G_1(\gamma(t)), [G_1, G_2](\gamma(t)))$. We can know define the map φ which to (x, y) in a neighborhood U of $(0, 0)$ in $\mathbb{R}^2$ associates the point reached by starting at $\gamma(y)$ and following G_1 during time x that is

$$\varphi : \begin{cases} U \to M \\ (x, y) \mapsto e^{xG_1}\gamma(y) \end{cases}$$

In this coordinate system, Δ_A is the y axis, $G_1(x, y) = (1, 0)$ and the second coordinate of G_2 is null at $x = 0$ hence it is the product of the function $(x \mapsto x)$ with a smooth function g. Moreover, thanks to the property of γ, $g(0, y) = 1$ which implies that $g(x, y) =$

$1 + xh(x, y)$ with h a smooth function. This is exactly (NF_2). If $0 \le a_0 < 1$ then $F_1(q)$ and $F_2(q)$ are not null and since they are parallel but not parallel to $[F_1, F_2](q)$ then $q \in \Delta_A \setminus (\Delta_1 \cup \Delta_2)$. If $a_0 = 1$ then $F_2(q) = 0$ and $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$.

The case where $q \in (\Delta_A \cap \Delta_1) \setminus \Delta_2$ can de treated by exchanging the roles of G_1 and G_2 since in this case $G_2(q) \neq 0$.

Assume finally that $q \in \Delta_A \cap \Delta_1 \cap \Delta_2$. Thanks to *(GP6)* and *(GP7)* at such a point G_1 and $[G_1, [G_1, G_2]]$ are not parallel. Hence, we can define the map φ which to (x, y) in a neighborhood U of $(0, 0)$ in $\mathbb{R}^2$ associates the point reached by starting at q and following $[G_1, [G_1, G_2]]$ during time y and then G_1 during time x that is

$$\varphi : \begin{cases} U & \to M \\ (x, y) & \mapsto e^{xG_1} e^{y[G_1,[G_1,G_2]]} q \end{cases}$$

The fact that G_2 and $[G_1, G_2]$ are parallel to G_1 implies $b_0 = 0$ and $b_{10} = 0$. The fact that, along the y axis, $[G_1, [G_1, G_2]] = (0, 1)$ implies in particular that $b_{20} = \frac{1}{2}$. $\square$

3 General Facts About the Computation of the Optimal Synthesis

3.1 Local Controllability and Existence of Minimizers

In the three cases of the normal form (NF_1), (NF_2) and (NF_3) one checks that

$$\text{span}(F_1, F_2, [F_1, F_2], [F_1, [F_1, F_2]], [F_2, [F_1, F_2]]) = \mathbb{R}^2.$$

Hence, as a consequence of Chow-Rashevski theorem (see [3, 16, 25]), generically such a control system is locally controllable that is locally, for any two points, always exists an admissible curve joining the two points.

Moreover, since at each point, the set of admissible velocities is convex and compact, thanks to Filippov theorem (see [3, 20]), locally for any two points, always exists at least a minimizer.

3.2 Pontryagin Maximum Principle

The Pontryagn Maximum Principle (PMP for short, see [3, 24]) gives necessary conditions for a curve to be a minimizer of a control problem. For our problem, it takes the following form.

Theorem 3 (**PMP**) *Define the Hamiltonian*

$$H(q, \lambda, u, \lambda_0) = u_1 \lambda.F_1(q) + u_2 \lambda.F_2(q) + \lambda_0$$

where $q \in \mathbb{R}^2$, $\lambda \in T^*\mathbb{R}^2$, $u \in \mathbb{R}^2$ *and* $\lambda_0 \in \mathbb{R}$. *For any minimizer* $(q(t), u(t))$, *there exist a never vanishing Lipschitz covector* $\lambda : t \mapsto \lambda(t) \in T^*_{q(t)}\mathbb{R}^2$ *and a constant* $\lambda_0 \le 0$ *such that*

- $\dot{q}(t) = \frac{\partial H}{\partial \lambda}(q(t), \lambda(t), u(t), \lambda_0)$,
- $\dot{\lambda}(t) = -\frac{\partial H}{\partial q}(q(t), \lambda(t), u(t), \lambda_0)$,
- $0 = H(q(t), \lambda(t), u(t), \lambda_0) = \max_v \{H(q, \lambda, v, \lambda_0) \mid |v_i| \le 1 \text{ for } i = 1, 2\}$.

A couple (q, λ) *satisfying the previous conditions is called an extremal. If* $\lambda_0 = 0$, *it is called abnormal, if not, normal. A curve* q *may be associated with both abnormal and normal extremals.*

Proposition 1 *For a generic SF metric on a 2D manifold defined with a maximum norm, there is no non trivial abnormal extremal. Hence, we can fix $\lambda_0 = -1$. This is our choice in the following.*

Proof It is a classical fact that an abnormal extremal should correspond to a covector $\lambda \neq 0$ orthogonal to F_1, F_2, and $[F_1, F_2]$. This implies that along the trajectory the three vectors are parallel. But generically this happens only on a discrete set, which forbids to get a non trivial curve. □

3.3 Switchings

In this section, we follow the ideas of [13]. Recall that Δ_A is the set of point q where $F_1(q)$ and $F_2(q)$ are collinear, Δ_1 is the set of point q where $F_1(q)$ and $[F_1, F_2](q)$ are collinear, and Δ_2 is the set of point q where $F_2(q)$ and $[F_1, F_2](q)$ are collinear.

Definition 2 For an extremal triplet $(q(.), \lambda(.), u(.))$, define the *switching functions*

$$\phi_i(t) =< \lambda(t), F_i(q(t)) >, i = 1, 2,$$

and the function $\phi_3(t) =< \lambda(t), [F_1, F_2](q(t)) >$.

Thanks to $\lambda_0 = -1$, the ϕ_i functions satisfy

$$u_1(t)\phi_1(t) + u_2(t)\phi_2(t) = 1, \qquad \text{for a.e. } t.$$

A direct consequence of the maximality condition is

Proposition 2 *If $\phi_i(t) > 0$ (resp. $\phi_i(t) < 0$) then $u_i(t) = 1$ (resp. $u_i(t) = -1$).*
If $\phi_i(t) = 0$ and $\dot\phi_i(t) > 0$ (resp. $\dot\phi_i(t) < 0$) then ϕ_i changes sign at time t and the control u_i switches from -1 to $+1$(resp. from $+1$ to -1).

Definition 3 We call *bang* an extremal trajectory corresponding to constant controls with value 1 or -1 and *bang-bang* an extremal which is a finite concatenation of bangs. We call *u_i-singular* an extremal corresponding to a null switching function ϕ_i. A time t is said to be a *switching time* if u is not bang in any neighborhood of t.

Definition 4 Outside Δ_A, define the functions f_1 and f_2 by

$$[F_1, F_2](q) = f_2(q)F_1(q) - f_1(q)F_2(q).$$

It is clear that

$$\Delta_1 \setminus \Delta_A = f_1^{-1}(0), \quad \Delta_2 \setminus \Delta_A = f_2^{-1}(0).$$

Proposition 3 (Switching rules) *Outside $\Delta_A \cup \Delta_1 \cup \Delta_2$ the possible switches of the controls are*

- *if $f_1 > 0$ then u_1 can only switch from -1 to +1 when ϕ_1 goes to 0,*
- *if $f_1 < 0$ then u_1 can only switch from +1 to -1 when ϕ_1 goes to 0,*
- *if $f_2 > 0$ then u_2 can only switch from -1 to +1 when ϕ_2 goes to 0,*
- *if $f_2 < 0$ then u_2 can only switch from +1 to -1 when ϕ_2 goes to 0.*

Proof The fact that $\dot{\phi}_1(t) = -u_2.\lambda.[F_1, F_2]$ and $\dot{\phi}_2(t) = u_1.\lambda.[F_1, F_2]$ implies that, outside $\Delta_A \cup \Delta_1 \cup \Delta_2$,

$$\dot{\phi}_1(t) = u_2(t)\,(f_1(q(t))\phi_2(t) - f_2(q(t))\phi_1(t)) = -u_2(t)\phi_3(t), \tag{3}$$

$$\dot{\phi}_2(t) = u_1(t)\,(f_2(q(t))\phi_1(t) - f_1(q(t))\phi_2(t)) = u_1(t)\phi_3(t). \tag{4}$$

Now, if $\phi_1(t) = 0$ then $|\phi_2(t)| = 1$ which implies $u_2(t)\phi_2(t) = 1$ and hence $\dot{\phi}_1(t)$ and $f_1(q(t))$ have same sign and the sign of $f_1(q(t))$ determines the switch.

The same holds true for f_2, ϕ_2 and u_2. □

As a consequence, on each connected component of the complement of $\Delta_A \cup \Delta_1 \cup \Delta_2$, each control u_i can take only values − 1 and + 1 and can switch only once from − 1 to + 1 if $f_i > 0$ or from + 1 to - 1 if $f_i < 0$.

Proposition 4 *At any point q outside Δ_A it exists a $\tau > 0$ such that for any extremal issued from q and of length less than τ, only one of the two controls may switch.*

Proof If $\phi_1(t) = 0$ then $|\phi_2(t)| = 1$. Hence, if $\phi_1(t) = 0$ and $\phi_2(t') = 0$ then ϕ_1 passes from value 0 to ± 1 in time $t' - t$ which implies that $|\dot{\phi}_1|$ takes values larger than $\frac{1}{|t'-t|}$. But, since $\dot{\phi}_1(t) = -u_2(f_2(q(t))\phi_1(q(t)) - f_1(q(t))\phi_2(q(t)))$, we have $|\dot{\phi}_1(t)| \leq |f_1(q(t))| + |f_2(q(t))|$. As a consequence, if locally $|f_1 + f_2| < M$ then $|t' - t|$ cannot be smaller than $1/M$. □

A consequence of the previous proposition is

Proposition 5 *At any point q outside Δ_A, consider the normal coordinate system centered at q. Any local extremal stays in one of the following domains: $\mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{R}_+ \times \mathbb{R}_-$, $\mathbb{R}_- \times \mathbb{R}_+$ or $\mathbb{R}_- \times \mathbb{R}_-$.*

Proof Thanks to previous proposition, only one control may switch in short time. Assume that $u_1 \equiv 1$. Then at each time $u_1 F_1 + u_2 F_2 = F_1 + u_2 F_2$ hence the dynamics takes the form $\alpha G_1 + (1 - \alpha)G_2$ with $\alpha \in [0, 1]$. This dynamics leaves invariant the set $\mathbb{R}_+ \times \mathbb{R}_+$, hence the extremal does not leave this set. By the same argument one proves that if $u_1 \equiv -1$ then the extremal stays in $\mathbb{R}_- \times \mathbb{R}_-$, if $u_2 \equiv 1$ then the extremal stays in $\mathbb{R}_+ \times \mathbb{R}_-$ and that if $u_2 \equiv -1$ then the extremal stays in $\mathbb{R}_- \times \mathbb{R}_+$. □

3.4 Initial Conditions and Their Parameterization

On proves easily that in the (NF_1) case, $\max(|\lambda_x(0)|, |\lambda_y(0)|) = 1$. Hence, the set of initial conditions λ is compact and extremals switching in short time or singular extremals should have a ϕ_i null or close to zero. Moreover, only one control can switch in short time (see Proposition 4).

In the (NF_2) and (NF_3) cases $|\lambda_x(0)| = 1$ and there is no condition on λ_y. Hence, the set of initial condition is not compact. This allows to consider initial conditions with $|\lambda_y| >> 1$ and hence will appear optimal extremals along which the two controls switch. It is not in contradiction with the Proposition 4 since in this case the base point belongs to Δ_A.

In the (NF_{2a}) and (NF_3) cases, $\phi_1(0) = \pm\frac{1+a_0}{2}$ and $\phi_2(0) = \pm\frac{1-a_0}{2}$. Hence, if one considers a compact set of initial conditions, the corresponding extremals do not switch in short time. And are not singular. As a consequence, to consider the extremal switching at least once, one should consider initial conditions with $|\lambda_y(0)| >> 1$.

Let us give an idea of how to estimate the $|\lambda_y(0)|$ corresponding to a u_1-switch at small time t and the consequence in terms of choice of change of coordinates.

In the (NF_2) case, $\phi_1(0) = \frac{1+a_0}{2} \geq \frac{1}{2}$. Hence, if along an extremal the control u_1 switches for t small hence one gets, since $x(t) = O(t)$ and $y(t) = O(t^2)$,

$$0 = \lambda(t).F_1(x(t), y(t)) = \frac{1+a_0}{2} + \lambda_y(0)\frac{x(t)}{2} + O(t)$$

and it implies that if an extremal sees its control u_1 switching at τ then $\lambda_y(0)$ should be like $\frac{1}{\tau}$. Hence, in order to make estimations of the corresponding extremals, it is natural to choose as small parameter $r_0 = \frac{1}{\lambda_y(0)}$, to make the change of coordinate $r = \frac{1}{\lambda_y}$, the change of time $s = \frac{t}{r}$ and the change of coordinate $p_x = r\lambda_x$. This is what we do in the Sections 4.2 and 4.3.

In the (NF_3) case, $\phi_1(0) = \frac{1+a_0}{2} \geq \frac{1}{2}$. Hence, if along an extremal the control u_1 switches for t small hence one gets, since $x(t) = O(t)$ and $y(t) = O(t^3)$,

$$0 = \lambda(t).F_1(x(t), y(t)) = \frac{1+a_0}{2} + \lambda_y(0)\frac{x^2(t)}{4} + O(t)$$

and it implies that if an extremal sees its control u_1 switching at τ then $\lambda_y(0)$ should be like $\frac{1}{\tau^2}$. Hence, in order to make estimations of the corresponding extremals, it is natural to choose as small parameter r_0 such that $\lambda_y(0) = \pm\frac{1}{r_0^2}$, to make the change of coordinate $r = \frac{\pm 1}{\sqrt{|\lambda_y|}}$ and the change of time $s = \frac{t}{r}$. This is what we do in the Section 4.4.

3.5 Weights, Orders and Nilpotent Approximation

Privileged coordinates and nilpotent approximations are well-known notions in SR Geometry. Their definitions being too long and classical we refer to [8]. The coordinates we constructed in the normal form are privileged coordinates.

In the (NF_1) case, x and y have weight 1 and ∂_x and ∂_y have weight -1 as operators of derivation. In the (NF_2) case x has weight 1 and y has weight 2, ∂_x has weight -1 and ∂_y have weight -2. In the $(NF3)$ case, x has weight 1 and y has weight 3, ∂_x has weight -1 and ∂_y have weight -3.

In privileged coordinates, one way to understand the weights of the variables naturally is to estimate how they vary with time in small time along an admissible curve. As seen before, in the (NF_1) case x and y are $O(t)$ (and may be not $o(t)$), in the (NF_2) case $x = O(t)$ and $y = O(t^2)$ and in the (NF_3) case $x = O(t)$ and $y = O(t^3)$.

In the following, $o_k(x, y)$ will denote a function whose valuation at 0 has order larger than k respectively to the weights of x and y. For example x^7 has always weight 7 and y^3 has weight 3 in the (NF_1) case but 9 in the (NF_3) case.

With this notion of weights, we define the nilpotent approximation of our normal forms in the three cases

$$\begin{aligned}
(NF_1) \quad & G_1(x, y) = \partial_x, \\
& G_2(x, y) = \partial_y, \\
(NF_2) \quad & G_1(x, y) = \partial_x, \\
& G_2(x, y) = a_0\partial_x + x\partial_y, \\
(NF_3) \quad & G_1(x, y) = \partial_x, \\
& G_2(x, y) = a_0\partial_x + \frac{1}{2}x^2\partial_y,
\end{aligned}$$

which corresponds to an approximation up to order -1. In the following, when we will compute developments with respect to the parameter r_0, that is for $|\lambda_y(0)| >> 1$, we will need the approximation up to order 0 for (NF_{2a}) and (NF_3), and the approximation up to order 1 for (NF_{2b})

$$
\begin{aligned}
(NF_{2a}) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (a_0 + a_{10}x)\partial_x + x(1 + b_{20}x)\partial_y, \\
(NF_{2b}) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (1 + a_{10}x + a_{01}y + a_{20}x^2)\partial_x + x(1 + b_{20}x + b_{30}x^2)\partial_y, \\
(NF_3) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (a_0 + a_{10}x)\partial_x + \left(\frac{x^2}{2} + b_{01}y + b_{30}x^3\right)\partial_y,
\end{aligned}
$$

In the (NF_1) case, we will need the approximation up to order 2 in order to compute the cut locus, when present

$$
\begin{aligned}
(NF_1) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2)\partial_x + \\
&\quad + (1 + x(b_{10} + b_{20}x + b_{11}y + b_{30}x^2 + b_{21}xy + b_{12}y^2))\partial_y,
\end{aligned}
$$

3.6 Symbols of Extremals

As we will see in the following, the local extremals will be finite concatenations of bang arcs and u_i-singular arcs. In order to facilitate the presentation, a bang arc following $\pm G_i$ will be symbolized by $[[\pm G_i]]$, a u_1-singular arc with control $u_2 \equiv 1$ will be symbolized by $[[S_1^+]]$, a u_1-singular arc with control $u_2 \equiv -1$ will be symbolized by $[[S_1^-]]$, and we will combine these symbols in such a way that $[[-G_1, G_2, S_2^+]]$ symbolizes the concatenation of a bang arc following $-G_1$ with a bang arc following G_2 and a u_2-singular arc with control $u_1 \equiv 1$.

3.7 Symmetries

One can change the roles of the vectors F_1 and F_2 and look at the effect on the functions f_i or on the invariants appearing in the normal form. For this last part, one should be careful that changing the role of F_1 and F_2 implies changing G_1 and G_2 and hence changing the coordinates x and y.

First look at the effect on the functions f_i on an example: $\bar{F}_1 = -F_1$ and $\bar{F}_2 = F_2$. If we define the control system with $(\bar{F}_1, \bar{F}_2)$, it defines the same SF structure. We compute easily that

$$[\bar{F}_1, \bar{F}_2] = [-F_1, F_2] = -[F_1, F_2] = -(f_2F_1 - f_1F_2) = f_2\bar{F}_1 - (-f_1)\bar{F}_2$$

hence $\bar{f}_1 = -f_1$ and $\bar{f}_2 = f_2$. With this choice $\bar{G}_1 = -G_2$ and $\bar{G}_2 = -G_1$. Of course, with such a change on the vectors G_1 and G_2 the change on the invariants is not so trivial to compute.

In the following, we consider changes that send G_1 to $\pm G_1$ and G_2 to $\pm G_2$. These changes are interesting from a calculus point of view. Effectively, once computed the jet of a bang-bang extremals with symbol $[[G_1, G_2]]$ and of its switching times, we are able to get the expressions for the bang-bang extremals with symbols $[[\pm G_1, \pm G_2]]$ without new computations. For example, if one gets the expression of an extremal with symbol $[[G_1, G_2]]$ as function of the initial conditions, one gets the expression of an extremal with symbol $[[-G_1, G_2]]$ by respecting the effect on the coordinates and the invariants a_0, a_{10}, etc. of the corresponding change of role of F_1 and F_2.

3.7.1 $\bar{G}_1 = -G_1$ and $\bar{G}_2 = G_2$

Consider the change $\bar{F}_1 = -F_2$ and $\bar{F}_2 = -F_1$. Then $\bar{G}_1 = -G_1$ and $\bar{G}_2 = G_2$, $[\bar{G}_1, \bar{G}_2] = -[G_1, G_2]$ and $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = [G_1, [G_1, G_2]]$. With this choice,

$$[\bar{F}_1, \bar{F}_2] = [-F_2, -F_1] = -[F_1, F_2] = -(f_2 F_1 - f_1 F_2) = (-f_1)\bar{F}_1 - (-f_2)\bar{F}_2$$

hence $\bar{f}_1 = -f_2$ and $\bar{f}_2 = -f_1$.

We can know consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF_1) In this case, $\bar{x} = -x$ and $\bar{y} = y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = \partial_y$ and

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (a_{10}\bar{x} - a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (1 - b_{10}\bar{x} + b_{20}\bar{x}^2 - b_{11}\bar{x}\bar{y} + o_2(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

(NF_2) In this case, $\bar{x} = -x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (-a_0 + a_{10}\bar{x} - a_{01}\bar{y} - a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (\bar{x} - b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

(NF_3) In this case, $\bar{x} = -x$ and $\bar{y} = y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = \partial_y$ and

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (-a_0 + a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (\bar{x}^2/2 + b_{01}\bar{y} - b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

3.7.2 $\bar{G}_1 = G_1$ and $\bar{G}_2 = -G_2$

Consider the change $\bar{F}_1 = F_2$ and $\bar{F}_2 = F_1$. Then $\bar{G}_1 = G_1$, $\bar{G}_2 = -G_2$, $[\bar{G}_1, \bar{G}_2] = -[G_1, G_2]$ and $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = -[G_1, [G_1, G_2]]$. With this choice,

$$[\bar{F}_1, \bar{F}_2] = [F_2, F_1] = -[F_1, F_2] = -(f_2 F_1 - f_1 F_2) = (f_1)\bar{F}_1 - (f_2)\bar{F}_2$$

hence $\bar{f}_1 = f_2$ and $\bar{f}_2 = f_1$.

We can know consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF_1) In this case, $\bar{x} = x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = \partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (-a_{10}\bar{x} - a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (1 + b_{10}\bar{x} + b_{20}\bar{x}^2 - b_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

(NF_2) In this case, $\bar{x} = x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = \partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (-a_0 - a_{10}\bar{x} + a_{01}\bar{y} - a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (\bar{x} + b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

(NF_3) In this case, $\bar{x} = x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = \partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (-a_0 - a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (\bar{x}^2/2 - b_{01}\bar{y} + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

3.7.3 $\bar{G}_1 = -G_1$ and $\bar{G}_2 = -G_2$

Consider the change $\bar{F}_1 = -F_1$ and $\bar{F}_2 = -F_2$. Then $\bar{G}_1 = -G_1$, $\bar{G}_2 = -G_2$, $[\bar{G}_1, \bar{G}_2] = [G_1, G_2]$ and $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = -[G_1, [G_1, G_2]]$. With this choice,

$$[\bar{F}_1, \bar{F}_2] = [-F_1, -F_2] = [F_1, F_2] = (f_2F_1 - f_1F_2) = (-f_2)\bar{F}_1 - (-f_1)\bar{F}_2$$

hence $\bar{f}_1 = -f_1$ and $\bar{f}_2 = -f_2$.

We can know consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF_1) In this case, $\bar{x} = -x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = -\partial_y$. Moreover

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (-a_{10}\bar{x} + a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (1 - b_{10}\bar{x} + b_{20}\bar{x}^2 + b_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

(NF_2) In this case, $\bar{x} = -x$ and $\bar{y} = y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = \partial_y$. Moreover

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (a_0 - a_{10}\bar{x} + a_{01}\bar{y} + a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (\bar{x} - b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

(NF_3) In this case, $\bar{x} = -x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = -\partial_y$. Moreover

$$\begin{aligned}\bar{G}_1 &= \partial_{\bar{x}},\\ \bar{G}_2 &= (a_0 - a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} +\\ &\quad (\bar{x}^2/2 - b_{01}\bar{y} - b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.\end{aligned}$$

4 The Generic Local Optimal Synthesis

We present for generic couples (F_1, F_2) the local synthesis issued from a point q. The coordinates (x, y), centered at q, are those which have been constructed in the corresponding normal form in Section 2.

4.1 (NF_1) Case

At points q where (NF_1) holds, one can compute that

$$\begin{aligned}
f_1(x,y) &= \frac{1}{2}(a_{10}-b_{10}) \\
&+(2(a_{20}-b_{20})-b_{10}(a_{10}-b_{10}))\frac{x}{2}+(a_{11}-b_{11})\frac{y}{2} \\
&+(3(a_{30}-b_{30})-b_{10}(a_{20}-b_{20})-(2b_{20}-b_{10}^2)(a_{10}-b_{10}))\frac{x^2}{2} \\
&+(2(a_{21}-b_{21})-b_{11}(a_{10}-b_{10})-b_{10}(a_{11}-b_{11}))\frac{xy}{2} \\
&+(a_{12}-b_{12})\frac{y^2}{2}+o_2(x,y), \\
f_2(x,y) &= -\frac{1}{2}(a_{10}+b_{10}) \\
&-(2(a_{20}+b_{20})-b_{10}(a_{10}+b_{10}))\frac{x}{2}-(a_{11}+b_{11})\frac{y}{2} \\
&-(3(a_{30}+b_{30})-b_{10}(a_{20}+b_{20})-(2b_{20}-b_{10}^2)(a_{10}+b_{10}))\frac{x^2}{2} \\
&-(2(a_{21}+b_{21})-b_{11}(a_{10}+b_{10})-b_{10}(a_{11}+b_{11}))\frac{xy}{2} \\
&-(a_{12}+b_{12})\frac{y^2}{2}+o_2(x,y).
\end{aligned}$$

Hence, thanks to Proposition 3, if $a_{10}-b_{10}>0$ (resp. <0) then u_1 is bang-bang and the only possible switch is $-1\to+1$ (resp $+1\to-1$) and if $a_{10}+b_{10}<0$ (resp. >0) then u_2 is bang-bang and the only possible switch is $-1\to+1$ (resp $+1\to-1$).

Remark 2 (*Generic invariants*) Remark that generically, in the (NF_1) case, one of the following situation occurs

- $|a_{10}|\neq|b_{10}|$ (NF_{1a}),
- $a_{10}=b_{10}\neq0$ and $a_{20}-b_{20}\neq0$ and $a_{11}-b_{11}\neq0$,
- $a_{10}=b_{10}\neq0$ and $a_{20}-b_{20}=0$ and $a_{30}-b_{30}\neq0$ and $a_{11}-b_{11}\neq0$,
- $a_{10}=b_{10}\neq0$ and $a_{20}-b_{20}\neq0$ and $a_{11}-b_{11}=0$ and $a_{12}-b_{12}\neq0$,
- $a_{10}=-b_{10}\neq0$ and $a_{20}+b_{20}\neq0$ and $a_{11}+b_{11}\neq0$,
- $a_{10}=-b_{10}\neq0$ and $a_{20}+b_{20}=0$ and $a_{30}+b_{30}\neq0$ and $a_{11}+b_{11}\neq0$,
- $a_{10}=-b_{10}\neq0$ and $a_{20}+b_{20}\neq0$ and $a_{11}+b_{11}=0$ and $a_{12}+b_{12}\neq0$.
- $a_{10}=b_{10}=0$ and $a_{20}+b_{20}\neq0$ and $a_{11}+b_{11}\neq0$.

4.1.1 Singular Extremals

We consider now the properties of singular extremals and their support.

Proposition 6 *Under the generic assumption that Δ_A, Δ_1 and Δ_2 are submanifolds transversal by pair then*

1. *The support of a u_i-singular is included in Δ_i.*

2. *A u_1-singular extremal can follow Δ_1 being optimal only if, at each point $q(t)$ of the singular, $G_1(q(t))$ and $G_2(q(t))$ are pointing on the same side of Δ_1 (or one is tangent to Δ_1) where $f_1 > 0$.*
3. *A u_2-singular extremal can follow Δ_2 being optimal only if, at each point $q(t)$ of the singular, $G_1(q(t))$ and $-G_2(q(t))$ are pointing on the same side of Δ_2 (or one is tangent to Δ_2) where $f_2 > 0$.*
4. *Consider a u_i-singular $q(.)$ satisfying 2 or 3. If it does not intersect Δ_A and if at each time $G_1(q(t))$ and $G_2(q(t))$ are not tangent to Δ_i then $q(.)$ is a local minimizer that is at each time t exists ϵ such that $q(.)$ realizes the SF-distance between $q(t_1)$ and $q(t_2)$ for any t_1 and t_2 in $]t-\epsilon, t+\epsilon[$.*

Proof 1. Outside $\Delta_A \cup \Delta_i$, ϕ_i has isolated zero hence any u_i-singular should live in $\Delta_A \cup \Delta_i$. Moreover, since generically the set of points of Δ_A where the dynamics is tangent to Δ_A is isolated, a u_i-singular crosses Δ_A only at isolated times, which are consequently also in Δ_i.
2. Same proof as for point 3.
3. If a u_2-singular $q(.)$ has $u_1 = 1$ then its speed is $F_1(q(t)) + u_2(t)F_2(q(t))$ which is tangent to Δ_2. But $u_2 \in [-1, 1]$ hence either $|u_2(t)| = 1$ and G_1 or G_2 are tangent to Δ_2 or $|u_2(t)| < 1$ and $G_2(q(t)) = F_1(q(t)) - F_2(q(t))$ and $G_1(q(t)) = F_1(q(t)) + F_2(q(t))$ point on opposite sides.

   Now, assume that Δ_2 is such that G_1 and $-G_2$ point in the same side where $f_2 < 0$ at q and that the u_2-singular is optimal. Consider the normal coordinate system centered at q and the domain $\mathbb{R}_+ \times \mathbb{R}_+$. One can show, with the previous analysis, that the only possible extremals issued form q and entering the domain are the singular arc $[[S_2^+]]$ following Δ_2 and the bang-bang extremals starting with symbol $[[G_1, G_2]]$ or $[[G_2, G_1]]$.

   Let us prove that these last ones do not switch again before crossing Δ_2. If an extremal starts with $[[G_2, G_1]]$, switching for the first time at $t = \epsilon$ and hence at $y = \epsilon$ then along the second bang $x = t - \epsilon$, $y = \epsilon$, $\lambda \equiv (1, 1)$ and one computes easily that for $t > \epsilon$

$$\phi_2(t) = -\frac{1}{2}((a_{20} + b_{20})(t-\epsilon)^2 + (a_{11} + b_{11})(t-\epsilon)\epsilon + o_2(\epsilon, (t-\epsilon))).$$

   If $(a_{20}+b_{20})(a_{11}+b_{11}) < 0$ then the second time of switch satisfies $t-\epsilon = -\frac{a_{11}+b_{11}}{a_{20}+b_{20}}\epsilon + o(\epsilon)$ and hence the second switching locus has the form $(-\frac{a_{11}+b_{11}}{a_{20}+b_{20}}\epsilon, \epsilon)$. But Δ_2 satisfies that $x = -\frac{1}{2}\frac{a_{11}+b_{11}}{a_{20}+b_{20}}y + o(y)$ and hence the second bang crosses Δ_2 before ending. In the case $a_{20} + b_{20} = 0$ hence $(a_{11} - b_{11})(a_{30} + b_{30}) < 0$ and one shows that the second switching locus has the form $(\sqrt{-\frac{a_{11}+b_{11}}{a_{30}+b_{30}}\epsilon}, \epsilon)$ and Δ_2 satisfies that $x = \sqrt{-\frac{a_{11}+b_{11}}{3(a_{30}+b_{30})}y} + o(y)$ hence again the second bang crosses Δ_2 before ending. The same kind of computations show the same result when $a_{11} + b_{11} = 0$ and $(a_{20} + b_{20})(a_{12} + b_{12}) < 0$. The same holds for extremal starting by $[[G_1, G_2]]$.

   Finally, the different extremals with symbol $[[G_1, G_2]]$ do not intersect each other after their first switch hence they cannot lose optimality by crossing each other. Idem for those with symbol $[[G_2, G_1]]$. Hence, they can lose optimality by crossing the singular extremal or extremals with the other symbol.

   The last argument implies that optimal extremals are coming back to Δ_2 with symbol $[[G_1, G_2]]$ or $[[G_2, G_1]]$. But this is not possible since in this case, since $[[S_2^+]]$ is optimal, an optimal extremal with symbol $[[G_1, G_2, S_2^+]]$ would exist which is not the case since the switching is coming strictly after the crossing with Δ_2 as seen before.

Hence, in this case, the u_2-singular is not optimal.

4. Assume that the u_2-singular satisfies 3, that it does not intersect Δ_A and that $G_1(q)$ and $G_2(q)$ are pointing on opposite sides of Δ_2. Let construct normal coordinates centered at q, then the only local extremals entering the domains $\{xy > 0\}$ are the one starting by a u_2-singular and switching or not locally only once to $u_2 = \pm 1$. For example, if we consider $\mathbb{R}_+ \times \mathbb{R}_+$, since $G_1(q)$ points in the side of the domain $\{q' \mid f_2(q') > 0\}$ then an extremal starting by G_1 enters the domain $\{q' \mid f_2(q') > 0\}$ and hence u_2 cannot switch and the extremal stays on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$ and do not enter it. As a consequence the u_2-singular is locally optimal.

□

Remark 3 For what concerns the point 4, assume that q is a point where G_1 or G_2 is tangent to Δ_1 and $\Delta_1 \cap \{xy < 0\}$ is such that at each point G_1 and G_2 are transverse to Δ_1 and point in the domain $\{f_1 > 0\}$. Then, starting from q, a u_1-singular can run on $\Delta_1 \cap \{xy < 0\}$ and is locally optimal. The same arguments as those exposed at point 4 work.

Definition 5 If a connected part of Δ_1 (resp. Δ_2) is such that at each point G_1 and G_2 (resp. G_1 and $-G_2$) point on the same side where $f_1 > 0$ (resp. $f_2 > 0$), it is called a turnpike. If it does not at each point, it is called an anti-turnpike (see [13]).

Remark 4 Along a u_i-singular extremal the control u_i is completely determined by the fact that the dynamics should be tangent to Δ_i.

4.1.2 Optimal Synthesis in the Domain $\mathbb{R}_+ \times \mathbb{R}_+$

Consider a point q and the normal coordinate system (x, y) centered at q. The dynamics entering $\mathbb{R}_+^* \times \mathbb{R}_+^*$ is with $u_1 \equiv 1$ since u_2 switches (Propositions 4 and 5). Three different cases can be identified.

1st. case. $\Delta_2 \cap (\mathbb{R}_+ \times \mathbb{R}_+ \setminus \{0\})$ is empty locally. Thanks to proposition 6, no u_2-singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1c}) and (NF_{1d}) where $a_{10} + b_{10} = 0$ and

- $(a_{20} + b_{20})(a_{11} + b_{11}) > 0$,
- or $a_{20} + b_{20} = 0$ and $(a_{30} + b_{30})(a_{11} + b_{11}) > 0$,
- or $a_{11} + b_{11} = 0$ and $(a_{20} + b_{20})(a_{12} + b_{12}) > 0$.

Only one u_2-switch can occur along the extremal. One has $f_2 > 0$ in the domain if

- $a_{10} + b_{10} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$,

and in this case the possible extremals of the domain have symbol $[[G_1]]$ or $[[G_2]]$ or $[[G_2, G_1]]$. One has $f_2 < 0$ in the domain if

- $a_{10} + b_{10} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$.

and in this case, the possible extremals of the domain have symbol $[[G_1]]$ or $[[G_2]]$ or $[[G_1, G_2]]$.

In this case 1, the picture of the synthesis is given in Fig. 1.

2nd. case. $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_+^*)$ is not empty locally and is a turnpike. We are in the context of point 4 of proposition 6. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} < 0$ and $a_{11} + b_{11} > 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$ and $a_{30} + b_{30} < 0$,
- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} < 0$ and $a_{12} + b_{12} > 0$.

Then $f_2 > 0$ locally along $\{x > 0, y = 0\}$ and $f_2 < 0$ along $\{x = 0, y > 0\}$. Hence, no bang-bang extremal with symbol $[[G_1, G_2]]$ or $[[G_2, G_1]]$ exists and any extremal entering the domain starts with a u_2-singular arc. If it switches to G_1 then it enters the domain $(\mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \{f_2 > 0\}$ which is invariant by G_1 hence it does not switch anymore. If it switches to G_2 it enters the domain $(\mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \{f_2 < 0\}$ which is invariant by G_2 hence it does not switch anymore.

As a consequence, the only possible symbols for extremals are $[[G_1]]$, $[[G_2]]$, $[[S_2^+, G_1]]$ and $[[S_2^+, G_2]]$.

In this case 2, the picture of the synthesis is given in Fig. 2.

3rd. case. $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_+^*)$ is not empty locally and is a anti-turnpike. Then, thanks to proposition 6, no singular can enter the domain. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} > 0$ and $a_{11} + b_{11} < 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$ and $a_{30} + b_{30} > 0$,
- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} > 0$ and $a_{12} + b_{12} < 0$.

Then, as seen in Proposition 6, no u_2-singular is extremal. Hence, the possible beginning of symbols entering the domain are $[[G_1, G_2]]$ and $[[G_2, G_1]]$. In order to complete the synthesis in this case, we have to compute the cut time and cut locus. In fact, the two kinds of extremals intersect before their second switching time. Let us prove it.

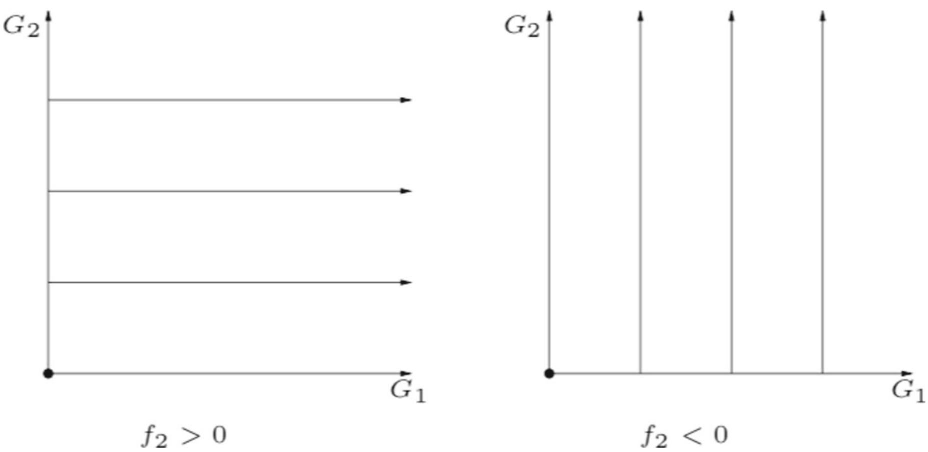

Fig. 1 The syntheses when $f_2 \neq 0$ in $(\mathbb{R}_+ \times \mathbb{R}_+) \setminus \{0\}$

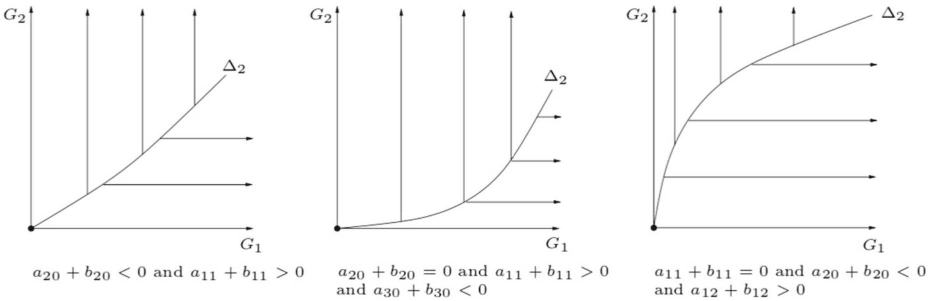

Fig. 2 The syntheses when $a_{10}+b_{10}=0$ and Δ_2 is a turnpike

Fix an $\epsilon_2>0$ and consider at time $t>\epsilon_2$ the extremal with symbol $[[G_2,G_1]]$ switching at time ϵ_2. One computes easily that $x(t)=t-\epsilon_2$ and $y(t)=\epsilon_2$. For an $\epsilon_1>0$ and the extremal with symbol $[[G_1,G_2]]$ switching at time ϵ_1, one gets by integrating the equations

$$\begin{aligned}
x(t) &= \epsilon_1 + a_{10}\epsilon_1(t-\epsilon_1) + a_{20}\epsilon_1^2(t-\epsilon_1) + \frac{1}{2}(a_{10}^2+a_{11})\epsilon_1(t-\epsilon_1)^2\\
&\quad + a_{30}\epsilon_1^3(t-\epsilon_1) + \frac{1}{2}(3a_{10}a_{20}+a_{21}+a_{11}b_{10})\epsilon_1^2(t-\epsilon_1)^2\\
&\quad + \frac{1}{3}(\frac{1}{2}a_{10}^3 + \frac{3}{2}a_{10}a_{11} + a_{12})\epsilon_1(t-\epsilon_1)^3\\
y(t) &= (t-\epsilon_1) + b_{10}\epsilon_1(t-\epsilon_1) + b_{20}\epsilon_1^2(t-\epsilon_1) + \frac{1}{2}(a_{10}b_{10}+b_{11})\epsilon_1(t-\epsilon_1)^2\\
&\quad + b_{30}\epsilon_1^3(t-\epsilon_1) + \frac{1}{2}(a_{20}b_{10}+b_{10}b_{11}+2a_{10}b_{20}+b_{21})\epsilon_1^2(t-\epsilon_1)^2\\
&\quad + \frac{1}{3}(\frac{1}{2}(a_{10}^2+a_{11})b_{10}+a_{10}b_{11}+b_{12})\epsilon_1(t-\epsilon_1)^3
\end{aligned}$$

Assume first that $a_{20}+b_{20}>0$ and $a_{11}+b_{11}<0$. Along the first front (depending on ϵ_2) $x+y=t$ when along the second $x+y=t+\epsilon_1(t-\epsilon_1)((a_{20}+b_{20})\epsilon_1+\frac{1}{2}(a_{11}+b_{11})$, hence, they are transverse at

$$\epsilon_1 = \frac{t}{1-\frac{2(a_{20}+b_{20})}{a_{11}+b_{11}}}$$

and they intersect at a point such that $y=-2\frac{a_{20}-b_{20}}{a_{11}-b_{11}}x+o(x)$. As seen previously, the switching locus for extremals with symbol $[[G_2,G_1]]$ satisfies $y=-\frac{a_{20}-b_{20}}{a_{11}-b_{11}}x+o(x)$ hence it stops to be optimal before switching. The same holds true for the extremals with symbol $[[G_1,G_2]]$. Finally, the cut locus satisfies

$$y_{cut} = -2\frac{a_{20}-b_{20}}{a_{11}-b_{11}}x_{cut}+o(x_{cut})$$

and is tangent to Δ_2.

The same computations can be done when G_1 or G_2 is tangent to Δ_2. Then one computes that the extremals lose optimality by crossing the cut before the second switch and that

- if $a_{20}+b_{20}=0$ then

$$y_{cut} = -3\frac{a_{30}+b_{30}}{a_{11}+b_{11}}x_{cut}^2+o(x_{cut}^2),$$

- if $a_{11} + b_{11} = 0$ then

$$x_{cut} = -\frac{1}{2}\frac{a_{12} + b_{12}}{a_{20} + b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases, the cut is tangent to Δ_2 and the contact is of order 2 when $(a_{20} + b_{20})(a_{11} + b_{11}) = 0$.

In this case 3, the picture of the synthesis is given in Fig. 3.

Remark 5 Using the symmetries presented in Section 3.7, one can obtain from the optimal synthesis in the domain $\mathbb{R}_+ \times \mathbb{R}_+$ the optimal synthesis in the three other domains.

4.1.3 Optimal Synthesis in the Domain $\mathbb{R}_- \times \mathbb{R}_-$

The dynamics entering $\mathbb{R}_-^* \times \mathbb{R}_-^*$ is with $u_1 \equiv -1$ since u_2 switches (Propositions 4 and 5). Three different cases can be identified.

1st. case. $\Delta_2 \cap (\mathbb{R}_- \times \mathbb{R}_- \setminus \{0\})$ is empty locally. No u_2-singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1c}) and (NF_{1d}) where $a_{10} + b_{10} = 0$ and

- $(a_{20} + b_{20})(a_{11} + b_{11}) > 0$,
- or $a_{20} + b_{20} = 0$ and $(a_{30} + b_{30})(a_{11} + b_{11}) < 0$,
- or $a_{11} + b_{11} = 0$ and $(a_{20} + b_{20})(a_{12} + b_{12}) < 0$.

Only one u_2-switch can occur along the extremal. One has $f_2 > 0$ in the domain if

- $a_{10} + b_{10} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$,

and in this case the possible extremals of the domain have symbol $[[-G_1]]$ or $[[-G_2]]$ or $[[-G_1, -G_2]]$. One has $f_2 < 0$ in the domain if

- $a_{10} + b_{10} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$.

and in this case, the possible extremals of the domain have symbol $[[-G_1]]$ or $[[-G_2]]$ or $[[-G_2, -G_1]]$.

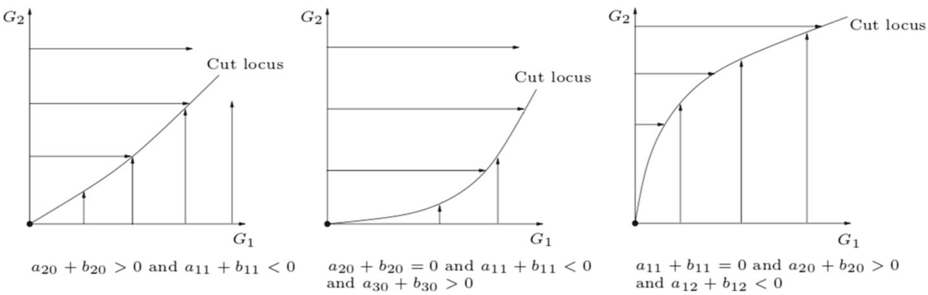

Fig. 3 The syntheses when $a_{10} + b_{10} = 0$ and Δ_2 is not a turnpike

2nd. case. $\Delta_2 \cap (\mathbb{R}^*_- \times \mathbb{R}^*_-)$ is not empty locally and is a turnpike. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} < 0$ and $a_{11} + b_{11} > 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$ and $a_{30} + b_{30} > 0$,
- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} < 0$ and $a_{12} + b_{12} < 0$.

In this case, the possible symbols for extremals are $[[-G_1]]$, $[[-G_2]]$, $[[S_2^-, -G_1]]$ and $[[S_2^-, -G_2]]$.

3rd. case. $\Delta_2 \cap (\mathbb{R}^*_- \times \mathbb{R}^*_-)$ is not empty locally and is a anti-turnpike. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} > 0$ and $a_{11} + b_{11} < 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$ and $a_{30} + b_{30} < 0$,
- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} > 0$ and $a_{12} + b_{12} > 0$.

The only optimal symbols are $[[-G_1]]$, $[[-G_2]]$, $[[-G_1, -G_2]]$, and $[[-G_2, -G_1]]$. Moreover

- if $a_{20} + b_{20} > 0$ and $a_{11} + b_{11} < 0$, the cut locus satisfies
$$y_{cut} = -2\frac{a_{20} + b_{20}}{a_{11} + b_{11}} x_{cut} + o(x_{cut}),$$
- if $a_{20} + b_{20} = 0$ then
$$y_{cut} = -3\frac{a_{30} + b_{30}}{a_{11} + b_{11}} x_{cut}^2 + o(x_{cut}^2),$$
- if $a_{11} + b_{11} = 0$ then
$$x_{cut} = -\frac{1}{2}\frac{a_{12} + b_{12}}{a_{20} + b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases, the cut is tangent to Δ_2 and the contact is of order 2 when $(a_{20} + b_{20})(a_{11} + b_{11}) = 0$.

4.1.4 Optimal Synthesis in the Domain $\mathbb{R}_+ \times \mathbb{R}_-$

The dynamics entering $\mathbb{R}^*_+ \times \mathbb{R}^*_-$ is with $u_2 \equiv 1$ since u_1 switches (Propositions 4 and 5). Three different cases can be identified.

1st. case. $\Delta_1 \cap (\mathbb{R}_+ \times \mathbb{R}_- \setminus \{0\})$ is empty locally. No u_1-singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1b}) and (NF_{1d}) where $a_{10} - b_{10} = 0$ and

- $(a_{20} - b_{20})(a_{11} - b_{11}) < 0$,
- or $a_{20} - b_{20} = 0$ and $(a_{30} - b_{30})(a_{11} - b_{11}) < 0$,
- or $a_{11} - b_{11} = 0$ and $(a_{20} - b_{20})(a_{12} - b_{12}) > 0$.

Only one u_1-switch can occur along the extremal. One has $f_1 > 0$ in the domain if

- $a_{10} - b_{10} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} + b_{11} < 0$,

and in this case, the possible extremals of the domain have symbol $[[G_1]]$ or $[[-G_2]]$ or $[[-G_2, G_1]]$. One has $f_1 < 0$ in the domain if

- $a_{10} - b_{10} < 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} < 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} > 0$.

and in this case, the possible extremals of the domain have symbol $[[G_1]]$ or $[[-G_2]]$ or $[[G_1, -G_2]]$.

2nd. case. $\Delta_1 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$ is not empty locally and is a turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} > 0$ and $a_{11} - b_{11} > 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} > 0$ and $a_{30} - b_{30} > 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} > 0$ and $a_{12} - b_{12} < 0$.

In this case, the possible symbols for extremals are $[[G_1]]$, $[[-G_2]]$, $[[S_1^+, G_1]]$, and $[[S_1^+, -G_2]]$.

3rd. case. $\Delta_1 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$ is not empty locally and is a anti-turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} < 0$ and $a_{11} - b_{11} < 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} < 0$ and $a_{30} - b_{30} < 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} < 0$ and $a_{12} - b_{12} > 0$.

Then, the only optimal symbols are $[[G_1]]$, $[[-G_2]]$, $[[G_1, -G_2]]$, and $[[-G_2, G_1]]$. Moreover,

- if $a_{20} - b_{20} < 0$ and $a_{11} + b_{11} < 0$, the cut locus satisfies

$$y_{cut} = -2\frac{a_{20} - b_{20}}{a_{11} - b_{11}} x_{cut} + o(x_{cut}),$$

- if $a_{20} - b_{20} = 0$ then

$$y_{cut} = -3\frac{a_{30} - b_{30}}{a_{11} - b_{11}} x_{cut}^2 + o(x_{cut}^2),$$

- if $a_{11} - b_{11} = 0$ then

$$x_{cut} = -\frac{1}{2}\frac{a_{12} - b_{12}}{a_{20} - b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases, the cut is tangent to Δ_1 and the contact is of order 2 when $(a_{20} - b_{20})(a_{11} - b_{11}) = 0$.

4.1.5 Optimal Synthesis in the Domain $\mathbb{R}_- \times \mathbb{R}_+$

The dynamics entering $\mathbb{R}_-^* \times \mathbb{R}_+^*$ is with $u_2 \equiv -1$ since u_1 switches (Propositions 4 and 5). Three different cases can be identified.

1st. case. $\Delta_1 \cap (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0\})$ is empty locally. No u_1-singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1b}) and (NF_{1d}) where $a_{10} - b_{10} = 0$ and

- $(a_{20} - b_{20})(a_{11} - b_{11}) < 0$,
- or $a_{20} - b_{20} = 0$ and $(a_{30} - b_{30})(a_{11} - b_{11}) > 0$,
- or $a_{11} - b_{11} = 0$ and $(a_{20} - b_{20})(a_{12} - b_{12}) < 0$.

Only one u_1-switch can occur along the extremal. One has $f_1 > 0$ in the domain if

- $a_{10} - b_{10} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} < 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} + b_{11} > 0$,

and in this case, the possible extremals of the domain have symbol $[[-G_1]]$ or $[[G_2]]$ or $[[-G_1, G_2]]$. One has $f_1 < 0$ in the domain if

- $a_{10} - b_{10} < 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} < 0$,

and in this case, the possible extremals of the domain have symbol $[[-G_1]]$ or $[[G_2]]$ or $[[G_2, -G_1]]$.

2nd. case. $\Delta_1 \cap (\mathbb{R}^*_- \times \mathbb{R}^*_+)$ is not empty locally and is a turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} > 0$ and $a_{11} - b_{11} > 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} > 0$ and $a_{30} - b_{30} < 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} > 0$ and $a_{12} - b_{12} > 0$.

In this case, the possible symbols for extremals are $[[-G_1]]$, $[[G_2]]$, $[[S_1^-, -G_1]]$, and $[[S_1^-, G_2]]$.

3rd. case. $\Delta_1 \cap (\mathbb{R}^*_- \times \mathbb{R}^*_+)$ is not empty locally and is a anti-turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} < 0$ and $a_{11} - b_{11} < 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} < 0$ and $a_{30} - b_{30} > 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} < 0$ and $a_{12} - b_{12} < 0$.

Then the only optimal symbols are $[[-G_1]]$, $[[G_2]]$, $[[-G_1, G_2]]$, and $[[G_2, -G_1]]$. Moreover

- if $a_{20} - b_{20} < 0$ and $a_{11} + b_{11} < 0$, the cut locus satisfies
$$y_{cut} = -2\frac{a_{20} - b_{20}}{a_{11} - b_{11}} x_{cut} + o(x_{cut}),$$
- if $a_{20} - b_{20} = 0$ then
$$y_{cut} = -3\frac{a_{30} - b_{30}}{a_{11} - b_{11}} x_{cut}^2 + o(x_{cut}^2),$$
- if $a_{11} - b_{11} = 0$ then
$$x_{cut} = -\frac{1}{2}\frac{a_{12} - b_{12}}{a_{20} - b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases, the cut is tangent to Δ_1 and the contact is of order 2 when $(a_{20} - b_{20})(a_{11} - b_{11}) = 0$.

4.2 (NF_{2a}) Case

Recall that the normal form (NF_{2a}) gives

$$G_1(x, y) = \partial_x, \quad G_2(x, y) = (a_0 + a_{10}x + o_1(x, y))\partial_x + (x + b_{20}x^2 + o(x, y))\partial_y,$$

with $0 \leq a_0 < 1$.

A point where the normal form is given by (NF_{2a}) is neither in Δ_1 nor in Δ_2. Hence, no singular extremal will appear in the study of the local synthesis.

One can compute easily that, for any extremal starting at 0, $\phi_1(0) = \frac{1}{2}\lambda_x(0)(1+a_0)$ and $\phi_2(0) = \frac{1}{2}\lambda_x(0)(1-a_0)$. With $H = 0$ it gives $|\lambda_x(0)| = 1$. Hence, since $\dot\phi_1 = -u_2\phi_3$ and $\dot\phi_2 = u_1\phi_3$, if we want to study extremals that switch in short time, we need to consider ϕ_3 large that is $|\lambda_y|$ large.

Moreover, since along an extremal issued from 0 $|\dot x(t)| \le 1$ for t small, one gets easily that $|x(t)| \le t$ and $|y(t)| \le t^2$ for t small enough. Hence $\phi_1(t) = \frac{1+a_0}{2}\lambda_x(0) + x(t)\lambda_y(0) + o(t, x(t)\lambda_y(0))$ and $\phi_2(t) = \frac{1-a_0}{2}\lambda_x(0)+x(t)\lambda_y(t)+o(t, x(t)\lambda_y(t))$. This implies that if one wants to consider an extremal switching at time τ small, he should consider initial conditions $\lambda_y(0) \sim \frac{1}{\tau}$. Inversing the point of view, if we consider an initial condition $\lambda_y(0) = \frac{1}{r_0}$ with r_0 small, the switching time should be of order 1 in r_0. This motivates the following change of coordinates on the fibers of the cotangent: $r = \frac{1}{\lambda_y}$, $p = r\lambda_x$ and the change of time $s = t/r$.

4.2.1 Equations of the Dynamics

With the new variables (x, y, p, r) and the new time s, the Hamiltonian equations become

$$\begin{aligned} x' &= r\frac{\partial H}{\partial \lambda_x}(x, y, p, -1), \quad & p' &= -r\frac{\partial H}{\partial x}(x, y, p, -1) + rp\frac{\partial H}{\partial y}(x, y, p, -1), \\ y' &= r\frac{\partial H}{\partial \lambda_y}(x, y, p, -1), \quad & r' &= r^2\frac{\partial H}{\partial y}(x, y, p, -1). \end{aligned}$$

Now, looking for the solutions as taylor series in r_0, that is under the form

$$\begin{aligned} x(r_0, s) &= x_1(s)r_0 + x_2(s)r_0^2 + o(r_0^2), \quad & p(r_0, s) &= p_1(s)r_0 + p_2(s)r_0^2 + o(r_0^2), \\ y(r_0, s) &= y_2(s)r_0^2 + y_3(s)r_0^3 + o(r_0^3), \quad & r(r_0, s) &= r_0 + r_2(s)r_0^2 + o(r_0^2), \end{aligned}$$

one finds the equations

$$\begin{aligned} x_1' &= \tfrac{u_1+u_2}{2} + \tfrac{u_1-u_2}{2}a_0, \quad & x_2' &= \tfrac{u_1-u_2}{2}a_{10}x_1, \\ y_2' &= \tfrac{u_1-u_2}{4}x_1, & y_3' &= \tfrac{u_1-u_2}{2}(b_{20}x_1^2 + x_2), \\ p_1' &= -\tfrac{u_1-u_2}{2}x_1, & p_2' &= -\tfrac{u_1-u_2}{2}(a_{10}p_1 + 2b_{20}x_1), \\ r_2' &= 0, \end{aligned}$$

with the initial conditions $x_1(0) = x_2(0) = y_2(0) = y_3(0) = p_2(0) = r_2(0) = 0$ and $p_1(0) = \pm 1$.

4.2.2 Computation of the Jets

Using these equations, we are able to compute the jets with respect to r_0 of four types of extremals: depending on the sign of $p_1(0) = \pm 1$ and of r_0. For each of these types, we can compute the functions $x_1, x_2, y_2, y_3, p_1, p_2$ of the variable s for the first bang. We can then compute the jets of ϕ_1 and ϕ_2 for the first bang and look for the first switching time under the form $s_1 = s_{10} + s_{11}r_0$ and then repeat the procedure for the second bang and so on. Finally, if we denote $\delta_p = \mathrm{sign}(p(0))$ and $\delta_r = \mathrm{sign}(r_0)$ then the controls during the first bang are $u_1 = u_2 = \delta_p$. The first time of switch is

$$s_1 = \delta_r(1 - \delta_r a_0) - \delta_p(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20})r_0 + o(r_0)$$

and corresponds to $\phi_2(s_1) = 0$ if $\delta_r = 1$ or $\phi_1(s_1) = 0$ if $\delta_r = -1$. The second bang corresponds to $u_1 = \delta_p\delta_r$ and $u_2 = -\delta_p\delta_r$ and the second switch is at

$$s_2 = \delta_r(3 - \delta_r a_0) - \delta_p((1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20})r_0 + o(r_0)$$

where $\phi_1(s_2) = 0$ if $\delta_r = 1$ and $\phi_2(s_2) = 0$ if $\delta_r = -1$. At this time

$$x(s_2) = \delta_p(\delta_r + a_0)r_0 - \delta_r(\delta_r + a_0)(-\delta_r a_{10} + b_{20} + \delta_r a_0 b_{20})r_0^2 + o(r_0^2),$$
$$y(s_2) = 2\delta_r r_0^2 - \delta_p \frac{4}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})r_0^3 + o(r_0^3).$$

The third bang corresponds to $u_1 = u_2 = -1$ if $\delta_p = 1$ and to $u_1 = u_2 = 1$ if $\delta_p = -1$. The third switching time satisfies $s_3 = \delta_r(5 - \delta_r a_0) + O(r_0)$ and the corresponding time t_3 is larger than the cut time as we will see later.

Let us analyze a little the situation in terms of cut locus for these extremals: if we consider the extremals with $\delta_p = \delta_r = 1$, they all start following G_1, without loosing optimality. Then they switch to G_2 at $t = r_0(1 - a_0) + o(r_0)$. During this second bang, they do not intersect one each other since they are all following G_2 with a different initial condition on $\{x > 0, y = 0\}$. Then they switch to $-G_1$ but at a different y hence again they cannot intersect. The loss of optimality cannot come from an intersection with extremals with $\delta_r = -1$ since these last one live in $\{y \leq 0\}$. As we will see in the following, the loss of optimality will come from the intersection with an extremal with $-\delta_p = \delta_r = 1$ during the third bang. Of course, the same occurs for extremals with $\delta_r = -1$.

Fix a small parameter $\rho > 0$. Since the dynamics during the third bang of all the extremals is given by $\pm G_1 = \pm\partial_x$, y is constant during these third bangs. Hence, for the extremals with $\delta_r = 1$, we can look for the r_0, as a jet in ρ, such that $y = 2\rho^2$ during the third bang, and for the extremals with $\delta_r = -1$, we can look for the r_0, as a jet in ρ, such that $y = -2\rho^2$ during the third bang. The result is

$$r_0 = \delta_r \rho + \delta_r \delta_p \frac{1}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2)$$

which allows to compute

$$t_2 = (3 - \delta_r a_0)\rho - \delta_r \delta_p \frac{3a_{10} - a_0^2 a_{10} + \delta_r 6b_{20} - 3a_0 b_{20} + a_0^3 b_{20}}{3}\rho^2 + o(\rho^2).$$

Hence, we can compute $x(t) = x(t_2) + (t - t_2)$ for this r_0 that is

$$x(t) = -\delta_p t + \delta_p 4\rho - \frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2).$$

We are now in situation to complete the computation of the jet of the cut locus: an extremal intersects an extremal of same length at the time $t_{cut} = 4\rho + o(\rho^2)$ which is less than $t_3 = (5 - \delta_r a_0)\rho$ hence t_{cut} is the cut time. When $\delta_r = 1$ the cut point satisfies

$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = 2\rho^2,$$

and when $\delta_r = -1$ the cut point satisfies

$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = -2\rho^2.$$

Finally, if one wants to describe the sphere at time t small, one have that the first switching time is

$$t_1 = \delta_r(1 - \delta_r a_0)r_0 - \delta_p(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20})r_0^2 + o(r_0^2)$$

and hence, at t small, the r_0 corresponding to a first switching point is

$$r_1 = \frac{t}{\delta_r(1 - \delta_r a_0)} + \delta_r \delta_p \frac{\delta_r a_{10} + b_{20}(1 - \delta_r a_0)}{(1 - a_0)^2} t^2 + o(t^2).$$

The second switching time is

$$t_2 = \delta_r(3 - \delta_r a_0)r_0 - \delta_p((1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20})r_0^2 + o(r_0^2)$$

which implies that, at t small, the r_0 corresponding to a second switching point is

$$r_2 = \frac{t}{\delta_r(3 - \delta_r a_0)} + \delta_p \frac{(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20}}{\delta_r(3 - \delta_r a_0)^3} t^2 + o(t^2).$$

And the cut time is

$$t_{cut} = 4\delta_r(r_0 - \delta_r \delta_p \frac{1}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})r_0^2) + o(r_0^2)$$

which implies that at t small the r_0 corresponding to a cut point is

$$r_{cut} = \frac{\delta_r}{4}(t + \frac{\delta_p}{12}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})t^2) + o(t^2).$$

4.3 (NF_{2b}) Case

Recall that the normal form (NF_{2b}) gives $G_1(x, y) = \partial_x$, and

$$G_2(x, y) = (1 + a_{10}x + a_{01}y + a_{20}x^2 + o_2(x, y))\partial_x + (x + b_{20}x^2 + b_{30}x^3 + o_3(x, y))\partial_y.$$

In this case, the extremals with initial condition $|\lambda_y(0)| >> 1$ are the limit when a_0 goes to 1 of the extremal presented in the case (NF_{2a}). If $\lambda_y(0) >> 1$ then the symbol starts with $[[G_2, -G_1]]$ or with $[[-G_2, G_1]]$ and if $-\lambda_y(0) >> 1$ then the symbol starts with $[[G_1, -G_2]]$ or with $[[-G_1, G_2]]$ (Fig. 4).

But $F_2(0) = 0$ then for all extremals $\phi_2(0) = 0$. Hence, an extremal may also, depending on the invariants, have symbol starting by $[[G_2, G_1]]$, $[[G_1, G_2]]$, $[[S_2^+, G_1]]$ or $[[S_2^+, G_2]]$ if $\lambda_x(0) = 1$, or starting by $[[-G_2, -G_1]]$, $[[-G_1, -G_2]]$, $[[S_2^-, -G_1]]$ or $[[S_2^-, -G_2]]$ if $\lambda_x(0) = -1$.

If $\lambda_x(0) = 1$ then at least for small time $u_1(t) = 1$ and $x(t) = t + o(t)$ and $y(t) = o(t)$. Then, computing ϕ_2 one finds $\phi_2(t) = -\lambda_x(t)\frac{a_{10}}{2} - \lambda_y(t)\frac{x(t)}{2} + o(t) = -(\frac{a_{10}+\lambda_y(0)}{2})t + o(t)$. Hence if $\lambda_y(0) > -a_{10}$ then, since $\phi_2(0) < 0$ for small time, the extremal starts by a bang following G_2. If $\lambda_y(0) < -a_{10}$ then $\phi_2(0) > 0$ for small time and the extremal starts by a bang following G_1.

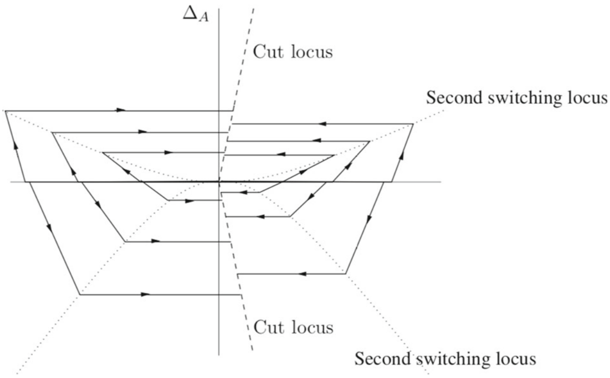

Fig. 4 The optimal synthesis in the (NF_{2a}) case

If $\lambda_x(0) = -1$ then at least for small time $u_1(t) = -1$ and $x(t) = -t + o(t)$ and $y(t) = o(t)$. Then $\phi_2(t) = (\frac{a_{10}-\lambda_y(0)}{2})t + o(t)$. Hence if $\lambda_y(0) > a_{10}$ then, since $\phi_2(0) < 0$ for small time, the extremal starts by a bang following $-G_1$. If $\lambda_y(0) < a_{10}$ then $\phi_2(0) > 0$ for small time and the extremal starts by a bang following $-G_2$.

In coordinates, one can compute that

$$\det(F_2, [F_1, F_2])(x, y) = \frac{1}{4}((a_{10}b_{20} - a_{20})x^2 + a_{01}y) + o_2(x, y)$$

where x has weight 1 and y has weight 2. Since generically at such points (which are isolated points) $a_{01} \neq 0$ then an equation for Δ_2 is given by

$$y = \frac{a_{20} - a_{10}b_{20}}{a_{01}}x^2 + o(x^2).$$

Remark that generically $\frac{a_{20}-a_{10}b_{20}}{a_{01}}$ is neither 0 nor $\frac{1}{2}$. Moreover

$$f_2(x, y) = \frac{\det(F_2, [F_1, F_2])(x, y)}{\det(F_2, F_1)(x, y)} = \frac{((a_{10}b_{20} - a_{20})x^2 + a_{01}y) + o_2(x, y)}{2x}.$$

Recall that an equation of the support of the integral curve of G_1 passing by 0 is $y = 0$ and that an equation for the support of the integral curve of G_2 passing by 0 is $y = \frac{x^2}{2} + o(x^2)$.

If $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < 0$ or if $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > \frac{1}{2}$ then Δ_2 does not enter the domain $\mathcal{D} = \{x > 0,\ 0 < y < \frac{x^2}{2}\}$ and along it G_1 and G_2 point on the same side of Δ_2 hence Δ_2 is not a turnpike. In these cases

- if $a_{10}b_{20} - a_{20} > 0$ then $f_2 > 0$ in $\mathcal{D}$ and the new extremals, that are not described as limit of the case NF_{2a}, have symbol $[[G_2, G_1]]$.
- if $a_{10}b_{20} - a_{20} < 0$ then $f_2 < 0$ in $\mathcal{D}$ and the new extremals, that are not described as limit of the case NF_{2a}, have symbol $[[G_1, G_2]]$.

If $0 < \frac{a_{20}-a_{10}b_{20}}{a_{01}} < \frac{1}{2}$ then Δ_2 enters $\mathcal{D}$ and along it G_1 and G_2 point on opposite sides of Δ_2. In this case:

- if $a_{10}b_{20} - a_{20} > 0$ then, along $\Delta_2 \cap \mathcal{D}$, G_1 points in direction of $f_2 > 0$ and Δ_2 is a turnpike. Then, the only extremals entering the domain $\mathcal{D}$ start with a singular arc and have symbols $[[S_2^+]]$, $[[S_2^+, G_1]]$ or $[[S_2^+, G_2]]$.
- if $a_{10}b_{20} - a_{20} < 0$ then, along $\Delta_2 \cap \mathcal{D}$, G_1 points in direction of $f_2 < 0$ and Δ_2 is not a turnpike. In this case, the symbols start with $[[G_1, G_2]]$ and $[[G_2, G_1]]$. One can compute, with the same techniques that in Section 4.2.2, the switching times and the second switching locus for extremals that enter the domain $\mathcal{D}$, that is for extremal with initial condition $\lambda_y(0) = -a_{10} + \delta\epsilon$ with $\epsilon > 0$ small and $\delta = \pm 1$. If $\delta < 0$ then the symbol is $[[G_1, G_2, G_1]]$ and the switching times are $t_1 = \frac{\epsilon}{a_{20}-a_{10}b_{20}}$ and $t_2 = t_1 + \frac{2\epsilon}{a_{01}-2a_{20}+2a_{10}b_{20}}$, the second switching locus being

$$x(\epsilon) = \frac{a_{01}\epsilon}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})},$$
$$y(\epsilon) = \frac{2(a_{01} - a_{20} + a_{10}b_{20})\epsilon^2}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})^2}.$$

If $\delta > 0$ then the symbol is $[[G_2, G_1, G_2]]$ and the switching times are $t_1 = \frac{2\epsilon}{a_{01}-2a_{20}+2a_{10}b_{20}}$ and $t_2 = t_1 + \frac{\epsilon}{a_{20}-a_{10}b_{20}}$, the second switching locus being

$$x(\epsilon) = \frac{a_{01}\epsilon}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})},$$
$$y(\epsilon) = \frac{2\epsilon^2}{(a_{01} - 2a_{20} + 2a_{10}b_{20})^2}.$$

In fact, these extremals lose optimality before the second switching. Effectively, the two fronts intersect before creating cut locus. In order to compute this cut locus, one can compute the jets of the two corresponding families of curves: the first one following G_1 during time ϵ_{11} then following G_2 during time ϵ_{12}, with $\epsilon_{11} + \epsilon_{12} = t$; the second one following G_2 during time ϵ_{21} then following G_1 during time ϵ_{22}, with $\epsilon_{21} + \epsilon_{22} = t$. Writing $\epsilon_{11} = t - \epsilon_{12}$ and $\epsilon_{22} = t - \epsilon_{21}$ and then $\epsilon_{12} = s_1 t + s_2 t^2 + o(t^2)$ and $\epsilon_{21} = t_1 t + t_2 t^2 + o(t^2)$, one can compute the jets with respect to t of both families and compute the cut locus. On finds

$$t_1 = \frac{a_{01} - 6(\alpha - a_{10}\beta) - 2(a_{01} - 3(\alpha - a_{10}\beta))s_1 + (a_{01} - 2(\alpha - a_{10}\beta))s_1^2}{2(\alpha - a_{10}\beta)(-2 + s_1)}$$

and

$$s_1 = \frac{1 - 2\gamma + 8\gamma^2 - 2\sqrt{\gamma}}{1 + 4\gamma^2}$$

where $\alpha = \frac{a_{01}+2a_{20}+a_{10}^2}{6}$, $\beta = \frac{2b_{20}+a_{10}}{6}$, $\gamma = \frac{\alpha - a_{10}\beta}{a_{01}-2(\alpha-a_{10}\beta)}$. Under the hypotheses of this case, one proves easily that $\frac{1}{4} \leq \gamma \leq 1$ which allows to prove that the expression in the formula of s_1 varies between 0 and 1. Finally, one gets the formula for the cut locus

$$x(t) = t + \frac{1}{2}(2a_{10}s_1 - a_{10}s_1^2)t^2 + o(t^2), \quad y(t) = \frac{1}{2}(2s_1 - s_1^2)t^2 + o(t^2).$$

Hence, the only optimal symbols entering the domain $\mathcal{D}$ are $[[G_1, G_2]]$ and $[[G_2, G_1]]$.

Pictures for the (NF_{2b}) case are in Figs. 5 and 6.

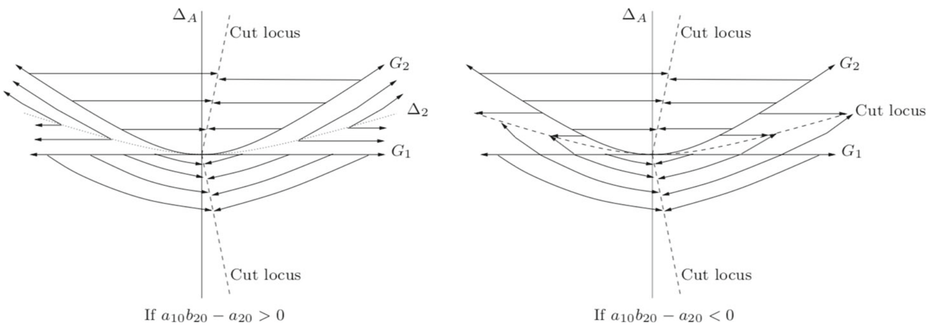

Fig. 5 (NF_{2b}) case: when $0 < \frac{a_{20}-a_{10}b_{20}}{a_{01}} < \frac{1}{2}$

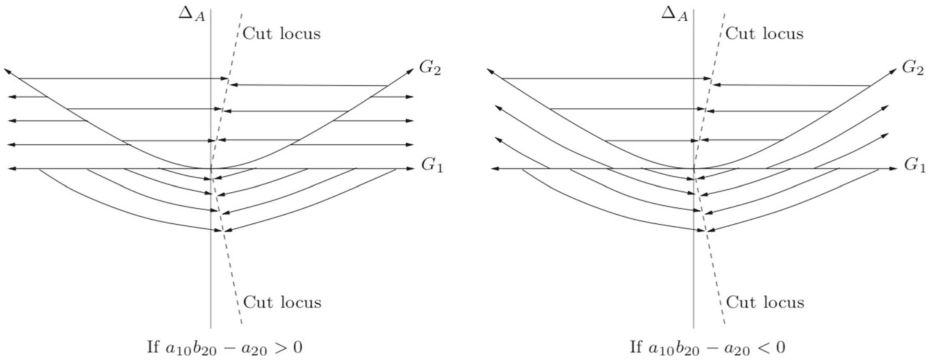

Fig. 6 (NF_{2b}) case: when $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < 0$ or $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > \frac{1}{2}$

4.4 (NF_3) Case

Recall that in the (NF_3) case, x has weight 1 and y has weight 3. Hence, we can write

$$\begin{aligned} G_1(x, y) &= \partial_x \\ G_2(x, y) &= (a_0 + a_{10}x + o(x, y))\partial_x + \left(\frac{x^2}{2} + b_{01}y + b_{30}x^3 + o_3(x, y)\right)\partial_y \end{aligned}$$

with $b_{0,1} \neq 0$ and $0 < a_0 < 1$, where $o_k(x, y)$ has the meaning given in Section 3.5. As in the (NF_{2b}) case, for any extremal starting at 0,

$$\phi_1(0) = \frac{1}{2}\lambda_x(0)(1 + a_0) \quad \text{and} \quad \phi_2(0) = \frac{1}{2}\lambda_x(0)(1 - a_0).$$

And for the same reasons, if we want to study extremals that switch in short time, we need to consider $|\lambda_y|$ large.

The set of initial condition is $\{(\lambda_x(0), \lambda_y(0)) \mid \lambda_x(0) = \pm 1\}$. We parameterize the upper part of this set by setting $\lambda_y(0) = \frac{1}{r_0^2}$ and the lower part by $\lambda_y(0) = -\frac{1}{r_0^2}$.

As explained in Section 3.4, in order to compute extremals with $\lambda_y(0) >> 1$ we make the change of coordinates $r = \frac{1}{\sqrt{\lambda_y}}$, $X = \frac{x}{r}$, $Y = \frac{y}{r^3}$ and the change of time $s = \frac{t}{r}$.

Now, looking for the solutions as taylor series in r_0, that is under the form

$$\begin{aligned} X(r_0, s) &= X_0(s) + r_0 X_1(s) + o(r_0), & \lambda_x(r_0, s) &= \lambda_{x0}(s) + r_0\lambda_{x1}(s) + o(r_0), \\ Y(r_0, s) &= Y_0(s) + r_0 Y_1(s) + o(r_0), & r(r_0, s) &= r_0 + r_0^2\, r_2(s) + o_2(r_0) \end{aligned}$$

one finds the equations

$$
\begin{aligned}
X_0'(s) &= \frac{1}{2}(u_1+u_2) + \frac{a_0}{2}(u_1-u_2),\\
X_1'(s) &= \frac{(u_1-u_2)}{4}(2a_{10}-b_{01})X_0(s),\\
Y_0'(s) &= \frac{1}{4}(u_1-u_2)X_0^2(s),\\
Y_1'(s) &= \frac{(u_1-u_2)}{4}(2b_{30}X_0^3(s)+2X_0(s)X_1(s)-b_{01}Y_0(s)),\\
\lambda_{x0}'(s) &= -\frac{1}{2}(u_1-u_2)X_0(s),\\
\lambda_{x1}'(s) &= -\frac{(u_1-u_2)}{2}(a_{10}\lambda_{x0}(s)+3b_{30}X_0^2(s)+X_1(s)),\\
r_2'(s) &= \frac{b_{01}}{4}(u_1-u_2),
\end{aligned}
$$

For an initial condition $\lambda_x(0) = 1$, one find $\phi_1(0) > 0$ and $\phi_2(0) > 0$, hence $u_1(0) = u_2(0) = 1$. One can integrate the equations and look for the first switching time as a Taylor series $s^1 = s_0^1 + r_0 s_1^1 + o(r_0)$ and compute $\phi_2(r_0, s_0^1 + r_0 s_1^1 + o(r_0))$ in order to compute

$$s_0^1 = \sqrt{2}\sqrt{1-a_0} \quad \text{and} \quad s_1^1 = -a_{10} - 2b_{30}(1-a_0).$$

At the switching time

$$
\begin{aligned}
X(s^1) &= \sqrt{2}\sqrt{1-a0} - (a_{10}+2b_{30})(1-a_0)r_0, & \lambda_x(s^1) &= 1,\\
Y(s^1) &= 0, & r(s^1) &= r_0.
\end{aligned}
$$

After this first switch $\phi_1(0) > 0$ and $\phi_2(0) < 0$, hence $u_1(0) = 1$ and $u_2(0) = -1$. We can compute and look for the next switching time and one finds that ϕ_1 goes to 0 at $s^2 = s_0^2 + r_0 s_1^2 + o(r_0)$ with

$$
\begin{aligned}
s_0^2 &= s_0^1 + \sqrt{2}\frac{\sqrt{1+a_0}-\sqrt{1-a_0}}{a_0},\\
s_1^2 &= s_1^1 + \frac{b_{01}((1-a_0)^{\frac{3}{2}} - \sqrt{1+a_0}(1-2a_0)) - 12b_{30}a_0^2\sqrt{1+a_0}}{3a_0^2\sqrt{1+a_0}}.
\end{aligned}
$$

At the second switching time

$$
\begin{aligned}
X(s^2) &= \sqrt{2}\sqrt{1+a_0}\\
&\quad + \frac{3a_{10}a_0\sqrt{1+a_0} + b_{01}((1-a_0)^{\frac{3}{2}} - (1+a_0)^{\frac{3}{2}}) - 6b_{30}a_0(1+a_0)^{\frac{3}{2}}}{3a_0\sqrt{1+a_0}}\, r_0,\\
Y(s^2) &= \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0}\\
&\quad - \frac{2b_{01}(1-a_0+a_0^2-(1-a_0)^{\frac{3}{2}}\sqrt{1+a_0}) + 12a_0^2 b_{30}}{3a_0^2}\, r_0,\\
\lambda_x(s^2) &= -1,\\
r(s^2) &= r_0 + \frac{(\sqrt{1+a_0}-\sqrt{1-a_0})b_{01}}{\sqrt{2}a_0}\, r_0^2.
\end{aligned}
$$

After this second switch, $\phi_1(0) < 0$ and $\phi_2(0) < 0$, hence $u_1(0) = u_2(0) = -1$. One can compute the third switch as being $s^3 = s_0^3 + r_0 s_1^3 + o(r_0)$ with

$$s_0^3 = s_0^2 + 2\sqrt{2}\sqrt{1+a0}, \quad s_1^3 = s_1^2 - \frac{2((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})b_{01}}{3a_0\sqrt{1+a_0}}.$$

At this time $X(s^3) = -\sqrt{2}\sqrt{1+a_0} + O(r_0)$ and we will see that this third switching time comes after the cut time.

The same computations can be done for the extremals starting with $\lambda_x(0) = -1$. We use the notation $\bar{z}$ for variables z corresponding to these extremals. During the first bang the controls are $\bar{u}_1 = \bar{u}_2 = -1$, during the second $\bar{u}_1 = 1$ and $\bar{u}_2 = -1$ and during the third one $\bar{u}_1 = \bar{u}_2 = 1$. The switching times are $\bar{s}^1$ and $\bar{s}^2$ satisfying

$$\begin{aligned} \bar{s}_0^1 &= \sqrt{2}\sqrt{1+a_0}, & \bar{s}_1^1 &= -a_{10} + 2b_{30}(1+a_0), \\ \bar{s}_0^2 &= \bar{s}_0^1 + \sqrt{2}\tfrac{\sqrt{1+a_0}-\sqrt{1-a_0}}{a_0}, & \bar{s}_1^2 &= \bar{s}_1^1 + \tfrac{b_{01}((1+a_0)^{\frac{3}{2}} - \sqrt{1-a_0}(1+2a_0)) + 12b_{30}a_0^2\sqrt{1-a_0}}{3a_0^2\sqrt{1-a_0}}. \end{aligned}$$

And at the second switching time

$$\begin{aligned} \bar{X}(\bar{s}^2) &= -\sqrt{2}\sqrt{1-a_0} \\ &+ \frac{-3a_{10}a_0\sqrt{1-a_0} + b_{01}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}}) - 6b_{30}a_0(1-a_0)^{\frac{3}{2}}}{3a_0\sqrt{1-a_0}}\bar{r}_0, \\ \bar{Y}(\bar{s}^2) &= \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0} \\ &- \frac{2b_{01}(1+a_0+a_0^2 - \sqrt{1-a_0}(1+a_0)^{\frac{3}{2}}) - 12a_0^2 b_{30}}{3a_0^2}\bar{r}_0, \\ \bar{\lambda}_x(\bar{s}^2) &= -1, \\ \bar{r}(\bar{s}^2) &= \bar{r}_0 + \frac{(\sqrt{1+a_0} - \sqrt{1-a_0})b_{01}}{\sqrt{2}a_0}\bar{r}_0^2. \end{aligned}$$

One can compute that at the third switching time $\bar{X}(\bar{s}^3) = \sqrt{2}\sqrt{1-a_0} + O(r_0)$.

We are now ready to compute the cut locus. As one can estimate easily, an extremal starting with $\lambda_x(0) > 0$ intersects an extremal starting with $\lambda_x(0) < 0$, both during their third bang. Moreover, since $Y(s^2) = \bar{Y}(\bar{s}^2) + o(r_0)$ one have that $\bar{r}_0 = r_0 + o(r_0)$.

Fix ρ and look for the extremals intersecting at $y = \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3$. We write $r_0 = \rho + R_{cut}\rho^2 + o(\rho^2)$ and look for R_{cut} such that $r_0 Y(s^2) = \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3 + o(\rho^4)$. We find

$$R_{cut} = \frac{\sqrt{2}((-2a_0^2 + (2+a_0)(-1+\sqrt{1-a_0^2}))b_{01} + 6a_0^2 b_{30})}{3a_0((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}.$$

For $\bar{r}_0 = \rho + \bar{R}_{cut}\rho^2 + o(\rho^2)$ one finds

$$\bar{R}_{cut} = \frac{\sqrt{2}((-2a_0^2 + (2-a_0)(-1+\sqrt{1-a_0^2}))b_{01} - 6a_0^2 b_{30})}{3a_0((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}.$$

With these values, we can compute the second switching times $t^2 = rs^2 = t_1^2\rho + t_2^2\rho^2 + o(\rho^3)$ and $\bar{t}_2 = \bar{r}\bar{s}^2 = \bar{t}_1^2\rho + \bar{t}_2^2\rho^2 + o(\rho^3)$ with

$$
\begin{aligned}
t_1^2 &= \sqrt{2}\left(\sqrt{1-a_0} + \frac{\sqrt{1+a_0}-\sqrt{1-a_0}}{a_0}\right)\\
t_2^2 &= -a_{10} + \frac{2(-4+a_0-2a_0^2+a_0^3+(-1+a_0)\sqrt{1-a_0^2})}{3+a_0^2}b_{30}\\
&\quad+\frac{(-5+2a_0-6a_0^2+a_0^3)\sqrt{1+a_0}-(-5-3a_0+a_0^2+3a_0^3)\sqrt{1-a_0}}{3a_0^2\sqrt{1+a_0}(2+\sqrt{1-a_0^2})}b_{01}\\
\bar{t}_1^2 &= \sqrt{2}\left(\sqrt{1+a_0} + \frac{\sqrt{1+a_0}-\sqrt{1-a_0}}{a_0}\right)\\
\bar{t}_2^2 &= -a_{10} + \frac{2(4+a_0+2a_0^2+a_0^3)+(1+a_0)\sqrt{1-a_0^2})}{3+a_0^2}b_{30}\\
&\quad-\frac{(-5+3a_0+a_0^2-3a_0^3)\sqrt{1+a_0}+(5+2a_0+6a_0^2+a_0^3)\sqrt{1-a_0}}{3a_0^2\sqrt{1-a_0}(2+\sqrt{1-a_0^2})}b_{01}
\end{aligned}
$$

and the x coordinates of the point of second switching under the form $x = x_1\rho + x_2\rho^2 + o(\rho^3)$ and $\bar{x} = \bar{x}_1\rho + \bar{x}_2\rho^2 + o(\rho^3)$ with

$$
\begin{aligned}
x_1 &= \frac{2\sqrt{2}(1+3a_0^2-(1-a_0^2)^{\frac{3}{2}})}{a_0((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})},\\
x_2 &= -\frac{5+a_0+5a_0^2-(5+a_0)\sqrt{1-a_0^2}}{3a_0^2}b_{01} - 4b_{30},\\
\bar{x}_1 &= -\frac{2\sqrt{2}(1+3a_0^2-(1-a_0^2)^{\frac{3}{2}})}{a_0((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})},\\
\bar{x}_2 &= \frac{5-a_0+5a_0^2+(-5+a_0)\sqrt{1-a_0^2}}{3a_0^2}b_{01} - 4b_{30}.
\end{aligned}
$$

One find easily that the cut locus is at $x_c = \frac{x_1+\bar{x}_1}{2}\rho + \frac{x_2+\bar{x}_2}{2}\rho^2 + o(\rho^2)$ that is

$$
\begin{aligned}
x_{cut}^{+} &= -\left(\frac{a_0}{3(1+\sqrt{1-a_0^2})}b_{01} + 4b_{30}\right)\rho^2 + o(\rho^2),\\
y_{cut}^{+} &= \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}}-(1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3.
\end{aligned}
$$

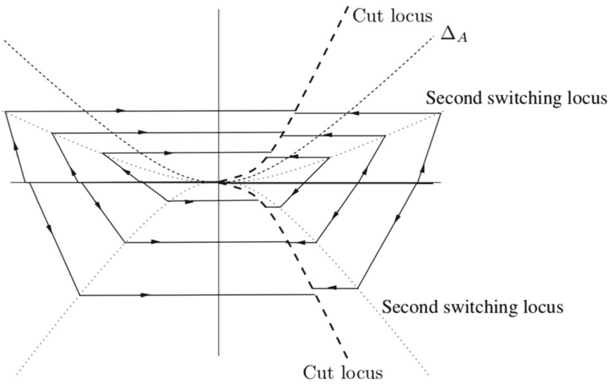

Fig. 7 The synthesis in the (NF_3) case

When $-\lambda_y(0) >> 1$, then we set $r = \frac{1}{\sqrt{-\lambda_y}}$. Equations are changed but the final result is very similar

$$x^-_{cut} = -\left(\frac{a_0}{3(1+\sqrt{1-a_0^2})}b_{01} + 4b_{30}\right)\rho^2 + o(\rho^2),$$

$$y^-_{cut} = -\frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3.$$

Finally, the cut locus appears to be a cusp whose tangent at the singular point is the tangent to Δ_A, see Fig. 7.

Funding information This research has been supported by ANR-15-CE40-0018.

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