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General and Optimal Decay Result for a Viscoelastic Problem with Nonlinear Boundary Feedback

Mohammad M. Al-Gharabli[1] · **Adel M. Al-Mahdi**[2] · **Salim A. Messaoudi**[2]

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Abstract
In this paper, we consider a viscoleastic equation with a nonlinear feedback localized on a part of the boundary and a relaxation function satisfying $g'(t) \leq -\xi(t)G(g(t))$. We establish an explicit and general decay rate results, using the multiplier method and some properties of the convex functions. Our results are obtained without imposing any restrictive growth assumption on the damping term. This work generalizes and improves earlier results in the literature, in particular those of Messaoudi (Topological Methods in Nonlinear Analysis 51(2):413–427, 2018), Messaoudi and Mustafa (Nonlinear Analysis: Theory Methods & Applications 72(9–10):3602–3611, 2010), Mustafa (Mathematical Methods in the Applied Sciences 41(1): 192–204, 2018) and Wu (Zeitschrift für angewandte Mathematik und Physik 63(1):65–106, 2012).

Keywords Viscoelasticity · Optimal decay · Relaxation functions · Convexity

Mathematics Subject Classification (2010) 35B35 · 35L55 · 75D05 · 74D10 · 93D20

1 Introduction

In this paper, we consider the following viscoelastic problem:

$$\begin{cases} u_{tt}(t) - \Delta u(t) + \int_0^t g(t-s)\Delta u(s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial \nu}(t) - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + h(u_t(t)) = 0, & \text{on } \Gamma_1 \times \mathbb{R}^+ \\ u(t) = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+ \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+ \end{cases} \tag{1.1}$$

✉ Mohammad M. Al-Gharabli
mahfouz@kfupm.edu.sa

Extended author information available on the last page of the article.

where u denotes the transverse displacement of waves, Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that Γ_0 and Γ_1 are closed and disjoint, with meas. $(\Gamma_0) > 0$, ν is the unit outer normal to $\partial\Omega$, and g and h are specific functions.

During the last half century, a great attention has been devoted to the study of viscoelastic problems and many existence and long-time behavior results have been established. We start with the pioneer work of Dafermos [5, 6], where he considered a one-dimensional viscoelastic problem of the form

$$\rho u_{tt} = c u_{xx} - \int_{-\infty}^{t} g(t-s) u_{xx}(s) ds,$$

and established various existence results and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as t goes to infinity. However, no rate of decay has been specified. Hrusa [7] considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - c u_{xx} + \int_{0}^{t} m(t-s)(\psi(u_x(s)))_x ds = f(x,t),$$

and proved several global existence results for large data. He also proved an exponential decay result for strong solutions when $m(s) = e^{-s}$ and ψ satisfies certain conditions. In [8] Dassios and Zafiropoulos considered a viscoelatic problem in $\mathbb{R}^3$ and proved a polynomial deacy result for exponentially decaying kernels. In their book, Fabrizio and Morro [9] established a uniform stability of some problems in linear viscoelasticity. In all the above mentioned works, the rates of decay in relaxation functions were either of exponential or polynomial type. In 2008, Messaoudi [10, 11] generalized the decay rates allowing an extended class of relaxation functions and gave general decay rates from which the exponential and the polynomial decay rates are only special cases. However, the optimality in the polynomial decay case was not obtained. Precisely, he considered relaxation functions that satisfy

$$g'(t) \leq -\xi(t) g(t), \ t \geq 0, \tag{1.2}$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function and showed that the rate of the decay of the energy is the same rate of decay of g, which is not necessarily of exponential or polynomial decay type. After that a series of papers using Eq. 1.2 has appeared see, for instance, [12–18] and [19].

Inspired by the experience with frictional damping initiated in the work of Lasiecka and Tataru [20], another step forward was done by considering relaxation functions satisfying

$$g'(t) \leq -\chi(g(t)). \tag{1.3}$$

This condition, where χ is a positive function, $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and strictly convex near the origin, with some additional constraints imposed on χ, was used by several authors with different approaches. We refer to previous studies [21–27] and [28], where general decay results in terms of χ were obtained. Here, it should be mentioned that, in [26], it was the first time where Lasiecka and Wang established not only general but also optimal results in which the decay rates are characterized by an ODE of the same type as the one generated by the inequality (1.3) satisfied by g. Mustafa and Messaoudi [29] established an explicit and general decay rate for relaxation function satisfying

$$g'(t) \leq -H(g(t)), \tag{1.4}$$

where $H \in C^1(\mathbb{R})$, with $H(0) = 0$ and H is linear or strictly increasing and strictly convex function C^2 near the origin. In [30], Cavalcanti et al. considered the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.5}$$

with a relaxation function satisfying Eq. 1.4 and the additional requirement:

$$\lim \inf_{x \to 0^+} x^2 H'' - xH' + H(x) \geq 0,$$

and that $y^{1-\alpha_0} \in L^1(1, \infty)$, for some $\alpha_0 \in [0, 1)$, where $y(t)$ is the solution of the problem

$$y'(t) + H(y(t)) = 0, \;\; y(0) = g(0) > 0.$$

They characterized the decay of the energy by the solution of a corresponding ODE as in [20]. Recently, Messaoudi and Al-Khulaifi [31] treated (1.5) with a relaxation function satisfying

$$g'(t) \leq -\xi(t) g^p(t), \;\; \forall t \geq 0, \;\; 1 \leq p < \frac{3}{2}. \tag{1.6}$$

They obtained a more general stability result for which the results of [10, 11] are only special cases. Moreover, the optimal decay rate for the polynomial case is achieved without any extra work and conditions as in [25] and [20]. For stabilization by mean of boundary feedback, Cavalcanti et al. [32] studied (1.1) and proved a global existence result for weak and strong solutions. Moreover, they gave some uniform decay rate results under some restrictive assumptions on both the kernel g and the damping function h. These restrictions had been relaxed by Cavalcanti et al. [33] and further they established a uniform stability depending on the behavior of h near the origin and on the behavior of g at infinity. In the absence of the viscoelastic term ($g = 0$), problem (1.1) has been investigated by many authors and several stability results were established. We refer the reader to the work of Lasiecka and Tataru [20], Alabau-Boussouira [34], Cavalcanti et al. [35], Guesmia [36, 37], Cavalcanti [38] and the references therein.

2 Preliminaries

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms and denote by V the following space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}.$$

Throughout this paper, c is used to denote a generic positive constant.

We consider the following hypotheses:

$(A1)$ $\quad g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = \ell > 0, \tag{2.1}$$

and there exists a C^1 function $G : (0, \infty) \to (0, \infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r_1]$, $r_1 \leq g(0)$, with $G(0) = G'(0) = 0$, such that

$$g'(t) \leq -\xi(t) G(g(t)), \qquad \forall t \geq 0, \tag{2.2}$$

where $\xi(t)$ is a positive nonincreasing differentiable function.

$(A2)$ $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing function $h_0 \in C^1(\mathbb{R}^+)$, with $h_0(0) = 0$, and positive constants c_1, c_2, ε such that

$$\begin{aligned} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|) \qquad &\text{for all} \qquad |s| \le \varepsilon, \\ c_1|s| \le |h(s)| \le c_2|s| \qquad &\text{for all} \qquad |s| \ge \varepsilon. \end{aligned} \tag{2.3}$$

In addition, we assume that the function H, defined by $H(s) = \sqrt{s}h_0(\sqrt{s})$, is a strictly convex C^2 function on $(0, r_2]$, for some $r_2 > 0$, when h_0 is nonlinear.

Remark 2.1 It is worth noting that condition (2.3) was considered first in [20].

Remark 2.2 Hypothesis $(A2)$ implies that $sh(s) > 0$, for all $s \neq 0$.

Remark 2.3 If G is a strictly increasing and strictly convex C^2 function on $(0, r_1]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is strictly increasing and strictly convex C^2 function on $(0, \infty)$. For instance, if $G(r_1) = a$, $G'(r_1) = b$, $G''(r_1) = c$, we can define $\overline{G}$, for $t > r_1$, by

$$\overline{G}(t) = \frac{c}{2}t^2 + (b - cr_1)t + \left(a + \frac{c}{2}{r_1}^2 - br_1\right). \tag{2.4}$$

The same remark can be established for $\overline{H}$.

For completeness we state, without proof, the existence result of [32].

Proposition 2.4 *Let $(u_0, u_1) \in V \times L^2(\Omega)$ be given. Assume that $(A1)$ and $(A2)$ are satisfied, then the problem (1.1) has a unique global (weak) solution*

$$u \in C(\mathbb{R}^+; V) \cap C^1(\mathbb{R}^+; L^2(\Omega)).$$

Moreover, if

$$(u_0, u_1) \in (H^2(\Omega) \cap V) \times V,$$

and satisfies the compatibility condition

$$\frac{\partial u_0}{\partial \nu} + h(u_1) = 0 \ \text{on} \ \Gamma_1,$$

then the solution

$$u \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap V) \cap W^{1,\infty}(\mathbb{R}^+; V) \cap W^{2,\infty}(\mathbb{R}^+; L^2(\Omega)).$$

We introduce the “modified” energy associated to problem (1.1):

$$E(t) = \frac{1}{2}||u_t(t)||_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) || \nabla u(t)||_2^2 + \frac{1}{2}(g \circ \nabla u)(t), \tag{2.5}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s)||\nabla u(t) - \nabla u(s)||_2^2 ds.$$

Direct differentiation, using Eq. 1.1, leads to

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_1} u_t(t)h(u_t(t))d\Gamma \le 0. \tag{2.6}$$

3 Technical Lemmas

In this section, we establish several lemmas needed for the proof of our main result.

Lemma 3.1 *Under the assumptions* $(A1)$ *and* $(A2)$*, the functional*

$$\psi_1(t) := \int_\Omega u(t)u_t(t)dx$$

satisfies, along the solution of Eq. 1.1*, the estimate*

$$\psi_1'(t) \le -\frac{\ell}{2}||\nabla u(t)||_2^2 + ||u_t(t)||_2^2 + \frac{cC_\alpha}{2\ell}(k \circ \nabla u)(t) + c\int_{\Gamma_1} h^2(u_t(t))d\Gamma, \quad \forall t \in \mathbb{R}^+, \tag{3.1}$$

where, for any $0 < \alpha < 1$,

$$C_\alpha = \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)}ds \quad and \quad k(t) = \alpha g(t) - g'(t). \tag{3.2}$$

Proof Direct computations, using Eq. 1.1, yield

$$\begin{aligned}\psi'(t) &= \int_\Omega u_t^2 dx + \int_\Omega u\Delta u dx - \int_\Omega u\int_0^t g(t-s)\Delta u(s)dsdx \\ &= \int_\Omega u_t^2 dx - \left(1 - \int_0^t g(s)ds\right)\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_1} uh(u_t)d\Gamma \\ &\quad + \int_\Omega \nabla u.\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))dsdx.\end{aligned} \tag{3.3}$$

Using Young's and Cauchy Schwarz' inequalities, we obtain

$$\begin{aligned}&\int_\Omega \nabla u.\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))dsdx \\ &\le \delta\int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\right)^2 dx \\ &\le \delta\int_\Omega |\nabla u|^2 dx + \\ &\frac{1}{4\delta}\int_\Omega\left(\int_0^t \frac{g(t-s)}{\sqrt{\alpha g(t-s) - g'(t-s)}}\sqrt{\alpha g(t-s) - g'(t-s)}|\nabla u(s) - \nabla u(t)|ds\right)^2 dx \\ &\le \delta\int_\Omega |\nabla u|^2 dx + \\ &\frac{1}{4\delta}\left(\int_0^t \frac{g^2(s)}{\alpha g(s) - g'(s)}ds\right)\int_\Omega\int_0^t \left[\alpha g(t-s) - g'(t-s)\right]|\nabla u(s) - \nabla u(t)|^2 dsdx \\ &\le \delta\int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta}C_\alpha(k \circ \nabla u)(t).\end{aligned} \tag{3.4}$$

Also, use of Young's and Poincaré's inequalities and the trace theorem gives

$$\begin{aligned}-\int_{\Gamma_1} uh(u_t)d\Gamma &\le \delta\int_{\Gamma_1} u^2 d\Gamma + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma \\ &\le c_p\delta\int_\Omega |\nabla u|^2 d\Gamma + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma.\end{aligned} \tag{3.5}$$

From Eqs. 3.4 and 3.5, we have

$$\begin{aligned}\int_\Omega \nabla u.\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))dsdx - \int_{\Gamma_1} uh(u_t)d\Gamma\\ \leq (1+c_p)\delta\int_\Omega |\nabla u|^2dx + \frac{1}{4\delta}C_\alpha(k\circ\nabla u)(t) + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma\\ \leq c\delta\int_\Omega |\nabla u|^2dx + \frac{1}{4\delta}C_\alpha(k\circ\nabla u)(t) + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma \end{aligned} \tag{3.6}$$

Combining Eqs. 3.3 and 3.6 and choosing $\delta = \frac{\ell}{2c}$ leads to Eq. 3.1. □

Lemma 3.2 *Under the assumptions* $(A1)$ *and* $(A2)$, *the functional*

$$\psi_2(t) := -\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, along the solution of Eq. 1.1, *the estimate*

$$\begin{aligned}\psi_2'(t) \leq \delta||\ \nabla u(t)||_2^2 - \left(\int_0^t g(s)ds-\delta\right)||u_t(t)||_2^2 + \left((\tfrac{3c}{\delta}+1)C_\alpha + \tfrac{c}{\delta}\right)(ko\nabla u)(t)\\ +c\int_{\Gamma_1} h^2(u_t(t))d\Gamma, \qquad \forall t\in\mathbb{R}^+ \ \ and \ \ \forall\delta>0. \end{aligned} \tag{3.7}$$

Proof By exploiting Eq. 1.1 and performing integration by parts, we arrive at

$$\begin{aligned}\psi_2'(t) &= \int_\Omega \nabla u.\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\ &\quad - \int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right).\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\ &\quad + \int_{\Gamma_1}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)h(u_t)d\Gamma\\ &\quad - \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega {u_t}^2dx\\ &= \left(1-\int_0^t g(s)ds\right)\int_\Omega \nabla u.\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\ &\quad + \int_\Omega\left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2dx\\ &\quad + \int_{\Gamma_1}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)h(u_t)d\Gamma\\ &\quad - \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u_t^2dx. \end{aligned}$$

Using Young's inequality and performing similar calculations as in Eq. 3.4, we obtain

$$\left(1-\int_0^t g(s)ds\right)\int_\Omega \nabla u.\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))dsdx \leq \delta\int_\Omega |\nabla u|^2dx + \frac{cC_\alpha}{\delta}(k\circ\nabla u)(t)$$

and

$$\int_{\Gamma_1}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)h(u_t)d\Gamma \leq \frac{cC_\alpha}{\delta}(k\circ\nabla u)(t) + \delta\int_{\Gamma_1} h^2(u_t)d\Gamma.$$

Also,

$$
\begin{aligned}
&-\int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx = \int_\Omega u_t \int_0^t k(t-s)(u(t)-u(s))dsdx\\
&-\int_\Omega u_t \int_0^t \alpha g(t-s)(u(t)-u(s))dsdx \le \delta \int_\Omega u_t^2 dx\\
&+\frac{1}{2\delta}\int_\Omega \left(\int_0^t \sqrt{k(t-s)}\sqrt{k(t-s)}|u(t)-u(s)|ds\right)^2 dx\\
&+\frac{\alpha^2}{2\delta}\int_\Omega \left(\int_0^t g(t-s)|u(t)-u(s)|ds\right)^2 dx \le \delta \int_\Omega u_t^2 dx\\
&+\frac{\left(\int_0^t k(s)ds\right)}{2\delta}(k\circ u)(t) + \frac{\alpha^2 C_\alpha}{2\delta}(k\circ \nabla u)(t) \le \delta \int_\Omega u_t^2 dx\\
&+\frac{c}{\delta}(k\circ \nabla u)(t) + \frac{cC_\alpha}{\delta}(k\circ\nabla u)(t).
\end{aligned}
$$

Combining all the above estimates, Eq. 3.7 is established. □

Lemma 3.3 *Under the assumptions* $(A1)$ *and* $(A2)$*, the functional*

$$
\psi_3(t) = \int_\Omega \int_0^t r(t-s)|\nabla u(s)|^2 dsdx, \tag{3.8}
$$

satisfies, along the solution of Eq. 1.1*, the estimate*

$$
\psi_3'(t) \le -\frac{1}{2}(g\circ\nabla u)(t) + 3(1-\ell)\int_\Omega |\nabla u(t)|^2 dx. \tag{3.9}
$$

where $r(t) = \int_t^{+\infty} g(s)ds$.

Proof By Young's inequality and the fact that $r'(t) = -g(t)$, we see that

$$
\begin{aligned}
\psi_3'(t) &= r(0)\int_\Omega |\nabla u(t)|^2 dx - \int_\Omega \int_0^t g(t-s)|\nabla u(s)|^2 dx\\
&= -\int_\Omega \int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2 dsdx\\
&\quad -2\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))dsdx + r(t)\int_\Omega |\nabla u(t)|^2 dx.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&-2\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))dsdx\\
&\quad \le 2(1-\ell)\int_\Omega |\nabla u(t)|^2 dx + \frac{\int_0^t g(s)ds}{2(1-\ell)}\int_\Omega \int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2 dsdx.
\end{aligned}
$$

Using the facts that $r(t) \le r(0) = 1-\ell$ and $\int_0^t g(s)ds \le 1-\ell$, Eq. 3.9 is established. □

Lemma 3.4 *There exist positive constants* d *and* t_1 *such that*

$$
g'(t) \le -dg(t), \quad \forall t \in [0, t_1]. \tag{3.10}
$$

Proof By $(A1)$, we easily deduce that $\lim_{t\to+\infty} g(t) = 0$. Hence, there is $t_1 \geq 0$ large enough such that

$$g(t_1) = r$$

and

$$g(t) \leq r, \qquad \forall t \geq t_1.$$

As g and ξ are positive nonincreasing continuous and H is a positive continuous function, then, for all $t \in [0, t_1]$,

$$\begin{cases} 0 < g(t_1) \leq g(t) \leq g(0) \\ 0 < \xi(t_1) \leq \xi(t) \leq \xi(0), \end{cases}$$

which implies that there are two positive constants a and b such that

$$a \leq \xi(t) H(g(t)) \leq b.$$

Consequently, for all $t \in [0, t_1]$,

$$g'(t) \leq -\xi(t)H(g(t)) \leq -\frac{a}{g(0)} g(0) \leq -\frac{a}{g(0)} g(t). \tag{3.11}$$

□

Remark 3.5 Using the fact that $\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$ and recalling the Lebesgue dominated convergence theorem, we can easily deduce that

$$\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds \to 0 \text{ as } \alpha \to 0. \tag{3.12}$$

Lemma 3.6 *Assume that* $(A1)$ *and* $(A2)$ *hold. Then there exist constants* $N, N_1, N_2, m, m_0, c > 0$ *such that the functional*

$$L(t) = NE(t) + N_1\psi_1(t) + N_2\psi_2(t) + m_0 E(t)$$

satisfies, for all $t \geq t_1$,

$$L'(t) \leq -mE(t) + c\int_{t_1}^t g(t-s)\int_\Omega |\nabla u(t) - \nabla u(s)|^2 dxds + c\int_{\Gamma_1} h^2(u_t(t))d\Gamma \tag{3.13}$$

Proof By using Eqs. 2.6, 3.1 and 3.7, recalling that $g' = (\alpha g - k)$ and taking $\delta = \frac{\ell}{4N_2}$, we easily see that

$$\begin{aligned} L'(t) \leq &-\left(\frac{\ell}{2}N_1 - \frac{\ell}{4}\right) \|\nabla u\|_2^2 - \left(N_2 g_1 - \frac{\ell}{4} - N_1\right) \|u_t\|_2^2 + \frac{\alpha}{2} N(g \circ \nabla u)(t) \\ &-\left(\frac{1}{2}N - \frac{4c}{\ell}N_2^2 - C_\alpha\left(\frac{c}{2\ell}N_1 + \frac{12c}{\ell}N_2^2 + N_2\right)\right)(k \circ \nabla u)(t)] \\ &+c(N_1 + N_2)\int_{\Gamma_1} h^2(u_t)d\Gamma + m_0 E'(t). \end{aligned} \tag{3.14}$$

At this point, we choose N_1 large enough so that

$$\frac{\ell}{2}N_1 - \frac{\ell}{4} > 4(1-\ell)$$

and then N_2 large enough so that

$$N_2 g_1 - \frac{\ell}{4} - N_1 - 1 > 0.$$

Now, using Remark 3.5, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8\left(\frac{cN_1}{2\ell} + \frac{12cN_2^2}{\ell} + N_2\right)}. \tag{3.15}$$

Now, we choose N large enough and α so that

$$\frac{1}{4}N - \frac{4c}{N_2^2} > 0 \text{ and } \alpha = \frac{1}{2N} < \alpha_0,$$

which gives

$$\frac{1}{2}N - \frac{4c}{\ell}N_2^2 - C_\alpha\left(\frac{c}{2\ell}N_1 + \frac{12c}{\ell}N_2^2 + N_2\right) > 0.$$

Therefore, we arrive at

$$L'(t) \le -4(1-\ell)|| \nabla u||_2^2 - ||u_t||_2^2 + \frac{1}{4}(g \circ \nabla u)(t) + c\int_{\Gamma_1} h^2(u_t)d\Gamma + m_0 E'(t). \tag{3.16}$$

Using Eqs. 2.6 and 3.10 we conclude that, for any $t \ge t_1$,

$$\int_0^{t_1} g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le \frac{-1}{d}\int_0^{t_1} g'(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds$$
$$\le -cE'(t) \tag{3.17}$$

Combining Eqs. 3.16 and 3.17 and selecting a suitable choice of m_0, Eq. 3.13 is established. On the other hand (see [39]), we can choose N even larger (if needed) so that

$$L \sim E. \tag{3.18}$$

□

4 Stability

In this section, we state and prove the main result of our work. For this purpose, we have the following lemmas and remarks.

Lemma 4.1 *Under the assumptions* $(A1)$ *and* $(A2)$, *the solution of* Eq. 1.1 *satisfies the estimates*

$$\int_{\Gamma_1} h^2(u_t)d\Gamma \le c\int_{\Gamma_1} u_t h(u_t)d\Gamma, \qquad \text{if } h_0 \text{ is linear} \tag{4.1}$$

$$\int_{\Gamma_1} h^2(u_t)d\Gamma \le cH^{-1}(J(t)) - cE'(t), \qquad \text{if } h_0 \text{ is nonlinear} \tag{4.2}$$

where

$$J(t) := \frac{1}{|\Gamma_{12}|}\int_{\Gamma_{12}} u_t(t)h(u_t(t))d\Gamma \le -cE'(t) \tag{4.3}$$

and

$$\Gamma_{12} = \{x \in \Gamma_1 : |u_t(t)| \le \varepsilon_1\}.$$

Proof **Case 1**: h_0 is linear. Then, using $(A2)$ we have

$$c_1'|u_t| \le |h(u_t)| \le c_2'|u_t|,$$

and hence

$$h^2(u_t) \le c_2' u_t h(u_t), \tag{4.4}$$

So, Eq. 4.1 is established.

Case 2: h_0 is nonlinear on $[0, \varepsilon]$.

We establish this case, borrowing some ideas from [20]. So, we first assume that $\max\{r_2, h_0(r_2)\} < \varepsilon$; otherwise we take r_2 smaller. Let $\varepsilon_1 = \min\{r_2, h_0(r_2)\}$. Using $(A2)$, we have, for $\varepsilon_1 \leq |s| \leq \varepsilon$,

$$|h(s)| \leq \frac{h_0^{-1}(|s|)}{|s|}|s| \leq \frac{h_0^{-1}(|\varepsilon|)}{|\varepsilon_1|}|s|$$

and

$$|h(s)| \geq \frac{h_0(|s|)}{|s|}|s| \geq \frac{h_0(|\varepsilon_1|)}{|\varepsilon|}|s|$$

So, we deduce that

$$\begin{cases} h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|) & \text{for all } \ |s| < \varepsilon_1 \\ c_1'|s| \leq |h(s)| \leq c_2'|s| & \text{for all } \ |s| \geq \varepsilon_1 \end{cases} \tag{4.5}$$

Then Eq. 4.5, yields, for all $|s| \leq \varepsilon_1$,

$$H(h^2(s)) = |h(s)|h_0(|h(s)|) \leq sh(s)$$

which gives

$$h^2(s) \leq H^{-1}(sh(s)) \quad \text{for all } \ |s| \leq \varepsilon_1. \tag{4.6}$$

Now, we define the following partition which was first introduced by Komornik [40]:

$$\Gamma_{11} = \{x \in \Gamma_1 : |u_t(t)| > \varepsilon_1\}, \qquad \Gamma_{12} = \{x \in \Gamma_1 : |u_t(t)| \leq \varepsilon_1\}$$

Using Eq. 4.5, we get on Γ_{12}

$$u_t h(u_t(t)) \leq \varepsilon_1 h_0^{-1}(\varepsilon_1) \leq h_0(r_2)r_2 = H(r_2^2). \tag{4.7}$$

Then, Jensen's inequality gives (note that H^{-1} is concave)

$$H^{-1}\left(J(t)\right) \geq c\int_{\Gamma_{12}} H^{-1}(u_t(t)h(u_t(t)))d\Gamma. \tag{4.8}$$

Thus, combining Eqs. 4.6 and 4.8, we arrive at

$$\begin{aligned} \int_{\Gamma_1} h^2(u_t(t))d\Gamma &= \int_{\Gamma_{12}} h^2(u_t(t))d\Gamma + \int_{\Gamma_{11}} h^2(u_t(t))d\Gamma \\ &\leq \int_{\Gamma_{12}} H^{-1}\left(u_t h(u_t(t))\right)d\Gamma + \int_{\Gamma_{11}} h^2(u_t(t))d\Gamma \\ &\leq cH^{-1}(J(t)) - cE'(t) \end{aligned} \tag{4.9}$$

□

Lemma 4.2 *Assume that* $(A1)$ *and* $(A2)$ *hold and* h_0 ***is linear****. Then, the energy functional satisfies the following estimate*

$$\int_0^{+\infty} E(s)ds < \infty \tag{4.10}$$

Proof Let $F(t) = L(t) + \psi_3(t)$, then using Eqs. 3.9 and 3.16, we obtain

$$F'(t) \leq -(1-\ell)\int_\Omega |\nabla u|dx - \int_\Omega u_t^2 dx - \frac{1}{4}(go\nabla u)(t) + c\int_{\Gamma_1} h^2(u_t)d\Gamma \tag{4.11}$$

Using Eqs. 2.6, 4.1 and 4.11, we obtain

$$\begin{aligned} F'(t) &\le -bE(t) + c\int_\Omega u_t h(u_t)dx \\ &\le -bE(t) - cE'(t), \end{aligned}$$

where b is some positive constant. Therefore,

$$b\int_{t_1}^t E(s)ds \le F_1(t_1) - F_1(t) \le F_1(t_1) < \infty, \tag{4.12}$$

where $F_1(t) = F(t) + cE(t) \sim E$. □

Let's define

$$I(t) := -\int_{t_1}^t g'(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le -cE'(t), \tag{4.13}$$

Lemma 4.3 *Under the assumptions* $(A1)$ *and* $(A2)$*, we have the following estimates*

$$\int_{t_1}^t g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le \frac{1}{q}\overline{G}^{-1}\left(\frac{qI(t)}{\xi(t)}\right), \quad \text{if } h_0 \text{ is linear} \tag{4.14}$$

$$\int_{t_1}^t g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le \frac{(t-t_1)}{q}\overline{G}^{-1}\left(\frac{qI(t)}{(t-t_1)\xi(t)}\right), \quad \text{if } h_0 \text{ is nonlinear} \tag{4.15}$$

where $q \in (0,1)$ *and* $\overline{G}$ *is an extension of* G *such that* $\overline{G}$ *is strictly increasing and strictly convex* C^2 *function on* $(0, \infty)$*; see Remark 2.3.*

Proof First we establish (4.14). For this, we define the following quantity

$$\lambda(t) := q\int_{t_1}^t \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds,$$

where, by Eq. 4.10, q is chosen so small that, for all $t \ge t_1$,

$$\lambda(t) < 1. \tag{4.16}$$

Since G is strictly convex on $(0, r_1]$ and $G(0) = 0$, then

$$G(\theta z) \le \theta G(z), \;\; 0 \le \theta \le 1 \text{ and } z \in (0, r_1]. \tag{4.17}$$

The use of Eqs. 2.2, 4.16, and 4.17 and Jensen's inequality leads to

$$\begin{aligned} I(t) &= \frac{1}{q\lambda(t)}\int_{t_1}^t \lambda(t)(-g'(s))\int_\Omega q|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\ge \frac{1}{q\lambda(t)}\int_{t_1}^t \lambda(t)\xi(s)G(g(s))\int_\Omega q|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\ge \frac{\xi(t)}{q\lambda(t)}\int_{t_1}^t G(\lambda(t)g(s))\int_\Omega q|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\ge \frac{\xi(t)}{q}G\left(q\int_{t_1}^t g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds\right) \\ &= \frac{\xi(t)}{q}\overline{G}\left(q\int_{t_1}^t g(s)\int_\Omega |\nabla u(t) - \nabla u(s)|^2 dxds\right) \end{aligned} \tag{4.18}$$

This gives (4.14).

For the proof of (4.15), we define the following

$$\lambda_1(t) := \frac{q}{(t-t_1)} \int_{t_1}^{t} \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds,$$

then using (2.5) and (2.6), we easily see that

$$\lambda_1(t) \leq \frac{8qE(0)}{\ell},$$

then choosing $q \in (0,1)$ small enough so that, for all $t \geq t_1$,

$$\lambda_1(t) < 1. \tag{4.19}$$

The use of Eqs. 2.2, 4.17 and 4.19 and Jensen's inequality leads to

$$\begin{aligned} I(t) &= \frac{1}{q\lambda_1(t)} \int_{t_1}^{t} \lambda_1(t)(-g'(s)) \int_{\Omega} q|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{1}{q\lambda_1(t)} \int_{t_1}^{t} \lambda_1(t)\xi(s)G(g(s)) \int_{\Omega} q|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{\xi(t)}{q\lambda_1(t)} \int_{t_1}^{t} G(\lambda_1(t)g(s)) \int_{\Omega} q|\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{(t-t_1)\xi(t)}{q} G\left(\frac{q}{(t-t_1)} \int_{t_1}^{t} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds\right) \\ &= \frac{(t-t_1)\xi(t)}{q} \overline{G}\left(\frac{q}{(t-t_1)} \int_{t_1}^{t} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds\right). \end{aligned} \tag{4.20}$$

This implies that

$$\int_{t_1}^{t} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{(t-t_1)}{q} \overline{G}^{-1}\left(\frac{qI(t)}{(t-t_1)\xi(t)}\right)$$

□

Theorem 4.4 *Let $(u_0, u_1) \in V \times L^2(\Omega)$ be given. Assume that $(A1)$ and $(A2)$ are satisfied and h_0 **is linear**. Then there exist strictly positive constants c_1, c_2, k_1 and k_2 such that the solution of* Eq. 1.1 *satisfies, for all $t \geq t_1$,*

$$E(t) \leq c_1 e^{-c_2 \int_{t_1}^{t} \xi(s)ds}, \quad \textit{if } G \textit{ is linear} \tag{4.21}$$

$$E(t) \leq k_2 G_1^{-1}\left(k_1 \int_{t_1}^{t} \xi(s)ds\right), \quad \textit{if } G \textit{ is nonlinear,} \tag{4.22}$$

where $G_1(t) = \int_t^{r_1} \frac{1}{sG'(s)} ds$.

Proof **Case 1**: G is linear

Multiplying (3.13) by $\xi(t)$ and using Eqs. 2.2, 2.6, 4.1, 4.3 and 4.13, we get

$$
\begin{aligned}
&\xi(t)L'(t) \leq \\
&-m\xi(t)E(t) + c\xi(t)\int_{t_1}^{t} g(t-s)\int_{\Omega}|\nabla u(t)-\nabla u(s)|^2 dxds + c\xi(t)\int_{\Gamma_1} h^2(u_t(t))d\Gamma \\
&\leq -m\xi(t)E(t) + c\int_{t_1}^{t}\xi(s)g(s)\int_{\Omega}|\nabla u(t)-\nabla u(s)|^2 dxds + c\xi(t)\int_{\Gamma_1} h^2(u_t(t))d\Gamma \\
&\leq -m\xi(t)E(t) - c\int_{t_1}^{t} g'(s)\int_{\Omega}|\nabla u(t)-\nabla u(s)|^2 dxds + c\xi(t)\int_{\Gamma_1} u_t h(u_t(t))d\Gamma \\
&\leq -m\xi(t)E(t) - 2cE'(t)
\end{aligned}
$$

which gives, as $\xi(t)$ is non-increasing,

$$(\xi L + 2cE)' \leq -m\xi(t)E(t), \forall t \geq t_1. \tag{4.23}$$

Hence, using the fact that $\xi L + 2cE \sim E$, we easily obtain

$$E(t) \leq c' e^{-\bar{c}\int_{t_1}^{t}\xi(s)ds}. \tag{4.24}$$

Case 2: G is non-linear.

Using Eqs. 3.13, 4.1 and 4.14, we obtain

$$L'(t) \leq -mE(t) + c\left(\overline{G}\right)^{-1}\left(\frac{qI(t)}{\xi(t)}\right) - cE'(t), \tag{4.25}$$

Let $\mathcal{F}_1(t) = L(t) + cE(t) \sim E$, then Eq. 4.25 becomes

$$\mathcal{F}_1'(t) \leq -mE(t) + c\left(\overline{G}\right)^{-1}\left(\frac{qI(t)}{\xi(t)}\right), \tag{4.26}$$

we find that the functional $\mathcal{F}_2$, defined by

$$\mathcal{F}_2(t) := \overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$.

$$\alpha_1\mathcal{F}_2(t) \leq E(t) \leq \alpha_2\mathcal{F}_2(t) \tag{4.27}$$

and

$$
\begin{aligned}
\mathcal{F}_2'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}\overline{G}''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1(t) + \overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1{}'(t) \\
&\leq -mE(t)\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + c\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{G}^{-1}\left(\frac{qI(t)}{\xi(t)}\right).
\end{aligned} \tag{4.28}
$$

Let $\overline{G}^*$ be the convex conjugate of $\overline{G}$ in the sense of Young (see [41]), then

$$\overline{G}^*(s) = s(\overline{G}')^{-1}(s) - \overline{G}\left[(\overline{G}')^{-1}(s)\right], \quad \text{if } s \in (0, \overline{G}'(r_1)] \tag{4.29}$$

and $\overline{G}^*$ satisfies the following generalized Young inequality

$$AB \leq \overline{G}^*(A) + \overline{G}(B), \quad \text{if } A \in (0, \overline{G}'(r_1)],\ B \in (0, r_1]. \tag{4.30}$$

So, with $A = \overline{G}'\left(\varepsilon_0 \frac{E'(t)}{E(0)}\right)$ and $B = \overline{G}^{-1}\left(\frac{qI(t)}{\xi(t)}\right)$ and using Eqs. 2.6 and 4.28–4.30, we arrive at

$$\begin{aligned}\mathcal{F}_2'(t) &\le -mE(t)\overline{G}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\overline{G}^*\left(\overline{G}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{qI(t)}{\xi(t)}\right)\\ &\le -mE(t)\overline{G}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\overline{G}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\left(\frac{qI(t)}{\xi(t)}\right). \qquad (4.31)\end{aligned}$$

So, multiplying Eq. 4.31 by $\xi(t)$ and using the fact that $\varepsilon_0 \frac{E(t)}{E(0)} < r_1$, $\overline{G}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$, gives

$$\begin{aligned}\xi(t)\mathcal{F}_2'(t) &\le -m\xi(t)E(t)G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\xi(t)\varepsilon_0 \frac{E(t)}{E(0)}G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cqI(t)\\ &\le -m\xi(t)E(t)G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\xi(t)\varepsilon_0 \frac{E(t)}{E(0)}G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t)\end{aligned}$$

Consequently, with a suitable choice of ε_0, we obtain, for all $t \ge t_1$,

$$\mathcal{F}_3'(t) \le -k\xi(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -k\xi(t)G_2\left(\frac{E(t)}{E(0)}\right), \qquad (4.32)$$

where $\mathcal{F}_3 = \xi\mathcal{F}_2 + cE \sim E$ and $G_2(t) = tG'(\varepsilon_0 t)$. Since $G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 t G''(\varepsilon_0 t)$, then, using the strict convexity of G on $(0, r_1]$, we find that $G_2'(t), G_2(t) > 0$ on $(0, 1]$. Thus, with

$$R(t) = \varepsilon \frac{\alpha_1 \mathcal{F}_3(t)}{E(0)}, \quad 0 < \varepsilon < 1,$$

taking in account (4.27) and (4.32), we have

$$R(t) \sim E(t) \qquad (4.33)$$

and, for some $k_1 > 0$.

$$R'(t) \le -k_1\xi(t)G_2(R(t)), \quad \forall t \ge t_1.$$

Then, the integration over (t_1, t) yields

$$\int_{t_1}^{t} \frac{-R'(s)}{G_2(R(s))} ds \ge k_1 \int_{t_1}^{t} \xi(s) ds.$$

Hence, by an approprite change of variable, we get

$$\int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_1)} \frac{1}{\tau G'(\tau)} d\tau \ge k_1 \int_{t_1}^{t} \xi(s) ds$$

Thus, we have

$$R(t) \le \frac{1}{\varepsilon_0} G_1^{-1}\left(k_1 \int_{t_1}^{t} \xi(s) ds\right), \qquad (4.34)$$

where $G_1(t) = \int_t^{r_1} \frac{1}{sG'(s)} ds$. Here, we have used the fact that G_1 is strictly decreasing on $(0, r_1]$. Therefore Eq. 4.22 is established by virtue of Eq. 4.33. □

Remark 4.5 The decay rate of $E(t)$ given by Eq. 2.2 is optimal because it is consistent with the decay rate of $g(t)$ given by Eq. 4.22. In fact,

$$g(t) \le G_0^{-1}\left(\int_{g^{-1}(r_1)}^{t} \xi(s) ds\right), \quad \forall t \ge g^{-1}(r_1),$$

where $G_0(t) = \int_t^r \frac{1}{G(s)}$.
Using the properties of G, G_0 and G_1, we can see that

$$G_1(t) = \int_t^{r_1} \frac{1}{sG'(s)} ds \leq \int_t^{r_1} \frac{1}{G(s)} ds = G_0(t).$$

This implies

$$G_1^{-1}(t) \leq G_0^{-1}(t).$$

This shows that Eq. 4.22 provides the best decay rates expected under the very general assumption (2.2).

Theorem 4.6 *Let $(u_0, u_1) \in V \times L^2(\Omega)$ be given. Assume that $(A1)$ and $(A2)$ are satisfied and h_0 **is nonlinear**. Then there exist strictly positive constants c_3, c_4, k_2, k_3 and ε_2 such that the solution of* Eq. 1.1 *satisfies, for all $t \geq t_1$,*

$$E(t) \leq H_1^{-1}\left(c_3 \int_{t_1}^t \xi(s) ds + c_4\right), \text{ if } G \text{ is linear,} \tag{4.35}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds$.

$$E(t) \leq k_3(t - t_1) W_2{}^{-1}\left(\frac{k_2}{(t - t_1)\int_{t_1}^t \xi(s) ds}\right), \text{ if } G \text{ is non-linear,} \tag{4.36}$$

where $W_2(t) = tW'(\varepsilon_2 t)$ and $W = \left(\overline{G}^{-1} + \overline{H}^{-1}\right)^{-1}$.

Proof **Case 1**: G is linear

Multiplying Eq. 3.13 by $\xi(t)$ and using Eq. 4.2, we get

$$\begin{aligned}
\xi(t)L'(t) &\leq -m\xi(t)E(t) + c\xi(t)\int_{t_1}^t g(t-s)\int_\Omega |\nabla u(t) - \nabla u(s)|^2 dx ds \\
&\quad + c\xi(t)\int_{\Gamma_1} h^2(u_t(t)) d\Gamma \\
&\leq -m\xi(t)E(t) - cE'(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t(t)) d\Gamma \\
&\leq -m\xi(t)E(t) - cE'(t) + c\xi(t)H^{-1}(J(t)) - c\xi(t)E'(t) \\
&\leq -m\xi(t)E(t) - cE'(t) + c\xi(t)H^{-1}(J(t)) - c\xi(0)E'(t) \\
&\leq -m\xi(t)E(t) - cE'(t) + c\xi(t)H^{-1}(J(t))
\end{aligned}$$

which gives, as $\xi(t)$ is non-increasing,

$$(\xi L + cE)' \leq -m\xi(t)E(t) + c\xi(t)H^{-1}(J(t)), \forall t \geq t_1. \tag{4.37}$$

Therefore, Eq. 4.37 becomes

$$\mathcal{L}'(t) \leq -m\xi(t)E(t) + c\xi(t)H^{-1}(J(t)), \forall t \geq t_1, \tag{4.38}$$

where $\mathcal{L} := \xi L + 2cE$, which is clearly equivalent to E. Now, for $\varepsilon_1 < r_2$ and $c_0 > 0$, using Eq. 4.38 and the fact that $E' \leq 0$, $H' > 0$, $H'' > 0$ on $(0, r_2]$, we find that the functional $\mathcal{L}_1$, defined by

$$\mathcal{L}_1(t) := H'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right)\mathcal{L}(t) + c_0 E(t)$$

satisfies, for some $\alpha_3, \alpha_4 > 0$.

$$\alpha_3 \mathcal{L}_1(t) \leq E(t) \leq \alpha_4 \mathcal{L}_1(t) \tag{4.39}$$

and

$$\begin{aligned}\mathcal{L}_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{L}(t) + H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{L}'(t) + c_0 E'(t) \\ &\leq -mE(t) H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\xi(t) H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}(J(t)) + c_0 E'(t) \end{aligned} \tag{4.40}$$

Let H^* be the convex conjugate of H in the sense of Young (see [41]), then, as in Eqs. 4.29 and 4.30, with $A = H'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right)$ and $B = H^{-1}(J(t))$, using Eqs. 2.6 and 4.7, we arrive at

$$\begin{aligned}\mathcal{L}_1'(t) &\leq -mE(t) H'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + c\xi(t) H^*\left(H'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right)\right) + c\xi(t) J(t) + c_0 E'(t) \\ &\leq -mE(t) H'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + c\varepsilon_1 \xi(t) \frac{E(t)}{E(0)} H'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) - cE'(t) + c_0 E'(t)\end{aligned}$$

Consequently, with a suitable choice of ε_1 and c_0, we obtain, for all $t \geq t_1$,

$$\mathcal{L}_1'(t) \leq -c\xi(t) \frac{E'(t)}{E(0)} H'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) = -c\xi(t) H_2\left(\varepsilon_1 \frac{E(t)}{E(0)}\right), \tag{4.41}$$

where $H_2(t) = tH'(\varepsilon_1 t)$. Since $H_2'(t) = H'(\varepsilon_1 t) + \varepsilon_1 t H''(\varepsilon_1 t)$, then, using the strict convexity of H on $(0, r_2]$, we find that $H_2'(t), H_2(t) > 0$ on $(0, 1]$. Thus, with

$$R_1(t) = \varepsilon \frac{\alpha_3 \mathcal{L}_1(t)}{E(0)}, \quad 0 < \varepsilon < 1,$$

taking in account (4.39) and (4.41), we have

$$R_1(t) \sim E(t) \tag{4.42}$$

and, for some $c_3 > 0$.

$$R_1'(t) \leq -c_3 \xi(t) H_2(R_1(t)), \quad \forall t \geq t_1.$$

Then, a simple integration gives, for some $c_4 > 0$,

$$R_1(t) \leq H_1^{-1}(c_3 \int_{t_1}^{t} \xi(s) ds + c_4), \quad \forall t \geq t_1, \tag{4.43}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds$.

Case 2. G is non-linear.

Using Eqs. 3.13, 4.2 and 4.15, we obtain

$$L'(t) \leq -mE(t) + c(t - t_1) \left(\overline{G}\right)^{-1} \left(\frac{qI_1(t)}{(t - t_1)\xi(t)}\right) + cH^{-1}(J(t)) - cE'(t). \tag{4.44}$$

Since $\lim_{t \to +\infty} \frac{1}{t - t_1} = 0$, there exists $t_2 > t_1$ such that $\frac{1}{t - t_1} < 1$ whenever $t > t_2$. Combining this with the strictly increasing and strictly convex properties of $\overline{H}$, setting $\theta = \frac{1}{t - t_1} < 1$ and using Eq. 4.17, we obtain

$$\overline{H}^{-1}(J(t)) \leq (t - t_1) \overline{H}^{-1}\left(\frac{J(t)}{(t - t_1)}\right), \forall t \geq t_2$$

and, then, Eq. 4.44 becomes

$$L'(t) \leq -mE(t) + c(t-t_1)\left(\overline{G}\right)^{-1}\left(\frac{qI_1(t)}{(t-t_1)\xi(t)}\right) + c(t-t_1)\overline{H}^{-1}\left(\frac{J(t)}{(t-t_1)}\right)$$
$$-cE'(t), \qquad \forall t \geq t_2. \tag{4.45}$$

Let $L_1(t) = L(t) + cE(t) \sim E$, then Eq. 4.45 takes the form

$$L_1'(t) \leq -mE(t)+c(t-t_1)\left(\overline{G}\right)^{-1}\left(\frac{qI_1(t)}{(t-t_1)\xi(t)}\right)+c(t-t_1)\overline{H}^{-1}\left(\frac{J(t)}{(t-t_1)})\right), \tag{4.46}$$

Let $r_0 = \min\{r_1, r_2\}$, $\chi(t) = \max\{\frac{qI_1(t)}{(t-t_1)\xi(t)}, \frac{J(t)}{(t-t_1)}\}$ and $W = \left(\left(\overline{G}\right)^{-1} + \overline{H}^{-1}\right)^{-1}$.
So, Eq. 4.46 reduces to

$$L_1'(t) \leq -mE(t) + c(t-t_1)W^{-1}(\chi(t)), \forall t \geq t_2 \tag{4.47}$$

Now, for $\varepsilon_2 < r_0$ and using Eq. 4.44 and the fact that $E' \leq 0$, $W' > 0$, $W'' > 0$ on $(0, r_0]$, we find that the functional L_2, defined by

$$L_2(t) := W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)L_1(t), \quad \forall t \geq t_2,$$

satisfies, for some $\alpha_5, \alpha_6 > 0$.

$$\alpha_5 L_2(t) \leq E(t) \leq \alpha_6 L_2(t) \tag{4.48}$$

and, for all $t \geq t_2$,

$$\begin{aligned} L_2'(t) &= \left(\frac{-\varepsilon_2}{(t-t_1)^2}\frac{E(t)}{E(0)} + \frac{\varepsilon_2}{(t-t_1)}\frac{E'(t)}{E(0)}\right)W''\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)L_1(t) \\ &\quad + W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)L_1'(t) \leq -mE(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) \\ &\quad + c(t-t_1)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)W^{-1}(\chi(t)). \end{aligned} \tag{4.49}$$

Let W^* be the convex conjugate of W in the sense of Young (see [41]), then, as in Eqs. 4.29 and 4.30, and with $A = W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)$ and $B = W^{-1}(\chi(t))$, using (2.6), we arrive at

$$\begin{aligned} L_2'(t) &\leq -mE(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c(t-t_1)W^*\left(W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\right) \\ &\quad + c(t-t_1)\chi(t) \\ &\leq -mE(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c(t-t_1)\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) \\ &\quad + c(t-t_1)\chi(t). \end{aligned} \tag{4.50}$$

Using Eqs. 4.3 and 4.13, we observe that

$$\begin{aligned} (t-t_1)\xi(t)\chi(t) &\leq qI(t) + \xi(t)J(t) \\ &\leq qI(t) + \xi(0)J(t) \\ &\leq -cE'(t) - cE'(t) \\ &\leq -cE'(t) \end{aligned}$$

So, multiplying Eq. 4.50 by $\xi(t)$ and using the fact that, $\varepsilon_2 \frac{E(t)}{E(0)} < r_0$, give

$$\xi(t)L_2'(t) \le -m\xi(t)E(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c\varepsilon_2\xi(t)\cdot\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) -cE'(t), \forall t \ge t_2.$$

Using the non-increasing property of ξ, we obtain, for all $t \ge t_2$,

$$(\xi L_2 + cE)'(t) \le -m\xi(t)E(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c\varepsilon_2\xi(t)\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)$$

Therefore, by setting $L_3 := \xi L_2 + cE \sim E$, we get

$$L_3'(t) \le -m\xi(t)E(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right) + c\varepsilon_2\xi(t)\cdot\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)$$

This gives, for a suitable choice of ε_2,

$$L_3'(t) \le -k\xi(t)\left(\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right), \qquad \forall t \ge t_2$$

or

$$k\left(\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\xi(t) \le -L_3'(t), \qquad \forall t \ge t_2 \tag{4.51}$$

An integration of Eq. 4.51 yields

$$\int_{t_2}^{t} k\left(\frac{E(s)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{s-t_1}\cdot\frac{E(s)}{E(0)}\right)\xi(s)ds \le -\int_{t_2}^{t} L_3'(s)ds \le L_3(t_2). \tag{4.52}$$

Using the facts that $W', W'' > 0$ and the non-increasing property of E, we deduce that the map $t \mapsto E(t)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)$ is non-increasing and consequently, we have

$$k\left(\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\xi(s)ds \le \int_{t_2}^{t} k\left(\frac{E(s)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{s-t_1}\cdot\frac{E(s)}{E(0)}\right)\xi(s)ds \le L_3(t_2), \qquad \forall t \ge t_2 \tag{4.53}$$

Multiplying each side of Eq. 4.53 by $\frac{1}{t-t_1}$, we have

$$k\left(\frac{1}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)W'\left(\frac{\varepsilon_2}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\xi(s)ds \le \frac{k_2}{t-t_1}, \qquad \forall t \ge t_2 \tag{4.54}$$

Next, we set $W_2(s) = sW'(\varepsilon_2 s)$ which is strictly increasing, then we obtain,

$$kW_2\left(\frac{1}{t-t_1}\cdot\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\xi(s)ds \le \frac{k_2}{t-t_1}, \qquad \forall t \ge t_2 \tag{4.55}$$

Finally, for two positive constants k_2 and k_3, we obtain

$$E(t) \le k_3(t-t_1){W_2}^{-1}\left(\frac{k_2}{(t-t_1)\int_{t_2}^{t}\xi(s)ds}\right). \tag{4.56}$$

This finishes the proof. □

Example 4.7 The following examples illustrate our results:

1. h_0 and G are linear

   Let $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$ is small enough so that Eq. 2.1 is satisfied, then $g'(t) = -\xi(t)G(g(t))$ where $G(t) = t$ and $\xi(t) = b$. For the frictional nonlinearity, assume that $h_0(t) = ct$ and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. Therefore,, we can use Eq. 4.21 to deduce

$$E(t) \leq c_1 e^{-c_2 t} \tag{4.57}$$

   which is the exponential decay.

2. h_0 is linear and G is non-linear

   Let $g(t) = ae^{-t^q}$, where $0 < q < 1$ and $a > 0$ is small enough so that g satisfies (2.1), then $g'(t) = -\xi(t)G(g(t))$ where $\xi(t) = 1$ and $G(t) = \frac{q^t}{(ln(a/t))^{\frac{1}{q}-1}}$. For, the boundary feedback, let $h_0(t) = ct$, and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. Since

$$G'(t) = \frac{(1-q) + qln(a/t)}{(ln(a/t))^{1/q}}$$

   and

$$G''(t) = \frac{(1-q)\,(ln(a/t) + 1/q)}{(ln(a/t))^{\frac{1}{q}+1}}.$$

   then the function G satisfies the condition (A1) on $(0, r_1]$ for any $0 < r_1 < a$.

$$\begin{aligned}
G_1(t) &= \int_t^{r_1} \frac{1}{sG'(s)} ds = \int_t^{r_1} \frac{\left[\ln \frac{a}{s}\right]^{\frac{1}{q}}}{s\left[1 - q + q\ln \frac{a}{s}\right]} ds \\
&= \int_{\ln \frac{a}{r_1}}^{\ln \frac{a}{t}} \frac{u^{\frac{1}{q}}}{1 - q + qu} du \\
&= \frac{1}{q} \int_{\ln \frac{a}{r_1}}^{\ln \frac{a}{t}} u^{\frac{1}{q}-1} \left[\frac{u}{\frac{1-q}{q} + u}\right] du \\
&\leq \frac{1}{q} \int_{\ln \frac{a}{r_1}}^{\ln \frac{a}{t}} u^{\frac{1}{q}-1} du \leq \left(\ln \frac{a}{t}\right)^{\frac{1}{q}}.
\end{aligned}$$

   Then, Eq. 4.22 gives

$$E(t) \leq ke^{-kt^q} \tag{4.58}$$

3. h_0 is non-linear and G is linear

   Let $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$ is small enough so that Eq. 2.1 is satisfied, then $g'(t) = -\xi(t)G(g(t))$ where $G(t) = t$ and $\xi(t) = b$. Also, assume that $h_0(t) = ct^q$, where $q > 1$ and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct^{\frac{q+1}{2}}$. Then,

$$H_1{}^{-1}(t) = (ct + 1)^{\frac{-2}{q-1}}.$$

   Therefore, applying Eq. 4.35, we obtain

$$E(t) \leq (c_1 t + c_2)^{\frac{-2}{q-1}} \tag{4.59}$$

4. h_0 is non-linear and G is non-linear

   Let $g(t) = \frac{a}{(1+t)^2}$, where a is chosen so that hypothesis (2.1) remains valid. Then

$$g'(t) = -bG(g(t)), \qquad \text{with} \qquad G(s) = s^{\frac{3}{2}},$$

where b is a fixed constant. For the boundary feedback, let $h_0(t) = ct^5$ and $H(t) = ct^3$. Then,

$$W(s) = (G^{-1} + H^{-1})^{-1} = \left(\frac{-1 + \sqrt{1+4s}}{2} \right)^3$$

and

$$\begin{aligned} W_2(s) &= \frac{3s}{\sqrt{1+4s}} \left(\frac{-1 + \sqrt{1+4s}}{2} \right)^2 \\ &= \frac{3s}{2\sqrt{1+4s}} + \frac{3s^2}{\sqrt{1+4s}} - \frac{3s}{2} \\ &\leq \frac{3s}{2} + \frac{3s^2}{2\sqrt{s}} - \frac{3s}{2} = cs^{\frac{3}{2}} \end{aligned}$$

Therefore, applying Eq. 4.36, we obtain

$$E(t) \leq \frac{c}{(t - t_1)^{\frac{1}{3}}}$$

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References

1. Messaoudi SA. Al-khulaifi General and optimal decay for a viscoelastic equation with boundary feedback. Topological Methods in Nonlinear Analysis. 2018;51(2):413–427.
2. Messaoudi SA, Mustafa MI. On convexity for energy decay rates of a viscoelastic equation with boundary feedback. Nonlinear Analysis: Theory Methods & Applications. 2010;72(9-10):3602–3611.
3. Mustafa MI. Optimal decay rates for the viscoelastic wave equation. Mathematical Methods in the Applied Sciences. 2018;41(1):192–204.
4. Wu S-T. General decay and blow-up of solutions for a viscoelastic equation with nonlinear boundary damping-source interactions. Zeitschrift für angewandte Mathematik und Physik. 2012;63(1):65–106.
5. Dafermos CM. Asymptotic stability in viscoelasticity. Archive for rational mechanics and analysis. 1970;37(4):297–308.
6. Dafermos CM. An, abstract volterra equation with applications to linear viscoelasticity. Journal of Differential Equations. 1970;7(3):554–569.
7. Hrusa WJ. Global existence and asymptotic stability for a semilinear hyperbolic volterra equation with large initial data. SIAM journal on mathematical analysis. 1985;16(1):110–134.
8. Dassios G, Zafiropoulos F. Equipartition of energy in linearized 3-d viscoelasticity. Q Appl Math. 1990;48(4):715–730.
9. Fabrizio M, Morro A. Mathematical problems in linear viscoelasticity, vol. 12. Siam. 1992.
10. Messaoudi SA. General decay of the solution energy in a viscoelastic equation with a nonlinear source. Nonlinear Analysis: Theory Methods & Applications. 2008;69(8):2589–2598.
11. Messaoudi SA. General decay of solutions of a viscoelastic equation. J Math Anal Appl. 2008;341(2):1457–1467.
12. Han X, Wang M. General decay of energy for a viscoelastic equation with nonlinear damping. Mathematical Methods in the Applied Sciences. 2009;32(3):346–358.
13. Liu W. General decay of solutions to a viscoelastic wave equation with nonlinear localized damping. In: Annales academiScientiarum fennicæ. Mathematica, vol 34, pp 291–302. 2009.
14. Liu W. General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms. Journal of Mathematical Physics. 2009;50(11):113506.

15. Messaoudi SA, Mustafa MI. On the control of solutions of viscoelastic equations with boundary feedback. Nonlinear Analysis: Real World Applications. 2009;10(5):3132–3140.
16. Mustafa MI. Uniform decay rates for viscoelastic dissipative systems. J Dyn Control Syst. 2016;22(1):101–116.
17. Mustafa MI. Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations. Nonlinear Analysis: Real World Applications. 2012;13(1):452–463.
18. Park JY, Park SH. General decay for quasilinear viscoelastic equations with nonlinear weak damping. Journal of Mathematical Physics. 2009;50(8):083505.
19. Wu S-T. General decay for a wave equation of kirchhoff type with a boundary control of memory type. Boundary Value Problems. 2011;2011(1):55.
20. Lasiecka I, Tataru D, et al. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. Differential Integral Equations. 1993;6(3):507–533.
21. Alabau-Boussouira F, Cannarsa P. A general method for proving sharp energy decay rates for memory-dissipative evolution equations. Comptes Rendus Mathématique,. 2009;347(15-16):867–872.
22. Cavalcanti MM, Domingos Cavalcanti VN, Lasiecka I, Falcao Nascimento FA. Intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects. Discrete & Continuous Dynamical Systems-Series B. 2014;19(7):1987–2011.
23. Cavalcanti MM, Cavalcanti AD, Lasiecka I, Wang X. Existence and sharp decay rate estimates for a von karman system with long memory. Nonlinear Analysis: Real World Applications. 2015;22:289–306.
24. Guesmia A. Asymptotic stability of abstract dissipative systems with infinite memory. J Math Anal Appl. 2011;382(2):748–760.
25. Lasiecka I, Messaoudi SA, Mustafa MI. Note on intrinsic decay rates for abstract wave equations with memory. Journal of Mathematical Physics. 2013;54(3):031504.
26. Lasiecka I, Wang X. Intrinsic decay rate estimates for semilinear abstract second order equations with memory. In: New prospects in direct, inverse and control problems for evolution equations, pp 271–303. Springer. 2014.
27. Mustafa MI. On the control of the wave equation by memory-type boundary condition. Discrete & Continuous Dynamical Systems-A. 2015;35(3):1179–1192.
28. Xiao T-J, Liang J. Coupled second order semilinear evolution equations indirectly damped via memory effects. Journal of Differential Equations. 2013;254(5):2128–2157.
29. Mustafa MI, Messaoudi SA. General stability result for viscoelastic wave equations. Journal of Mathematical Physics. 2012;53(5):053702.
30. Cavalcanti MM, Cavalcanti VND, Lasiecka I, Webler CM. Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. Advances in Nonlinear Analysis. 2017;6(2):121–145.
31. Messaoudi SA, Al-Khulaifi W. General and optimal decay for a quasilinear viscoelastic equation. Appl Math Lett. 2017;66:16–22.
32. Cavalcanti M, Cavalcanti VD, Prates Filho J, Soriano J, et al. Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. Differential and integral equations. 2001;14(1):85–116.
33. Cavalcanti M, Cavalcanti VD, Martinez P. General decay rate estimates for viscoelastic dissipative systems. Nonlinear Analysis: Theory Methods & Applications. 2008;68(1):177–193.
34. Alabau-Boussouira F. Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. Appl Math Optim. 2005;51(1):61–105.
35. Cavalcanti MM, Cavalcanti VND, Lasiecka I. Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping–source interaction. Journal of Differential Equations. 2007;236(2):407–459.
36. Guesmia A. Well-posedness and optimal decay rates for the viscoelastic kirchhoff equation. Boletim da Sociedade Paranaense de Matemática,. 2017;35(3):203–224.
37. Guesmia A. A new approach of stabilization of nondissipative distributed systems. SIAM journal on control and optimization. 2003;42(1):24–52.
38. Cavalcanti M, Guesmia A, et al. General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type. Differential and Integral equations. 2005;18(5):583–600.
39. Berrimi S, Messaoudi SA. Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping., Electronic Journal of Differential Equations (EJDE)[electronic only], vol. 2004, pp Paper–No. 2004.
40. Komornik V. Exact controllability and stabilization: the multiplier method, vol. 36. Masson. 1994.
41. Arnol'd VI. Mathematical methods of classical mechanics, vol. 60. Springer Science & Business Media. 2013.

Affiliations

Mohammad M. Al-Gharabli[1] · Adel M. Al-Mahdi[2] · Salim A. Messaoudi[2]

Adel M. Al-Mahdi
almahdi@kfupm.edu.sa

Salim A. Messaoudi
messaoud@kfupm.edu.sa

[1] Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

[2] The Preparatory Year Program, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia