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A General Decay for a Weakly Nonlinearly Damped Porous System

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Abstract In this paper, we consider a one-dimensional porous system damped with a single weakly nonlinear feedback. Without imposing any restrictive growth assumption near the origin on the damping term, we establish an explicit and general decay rate, using a multiplier method and some properties of convex functions in case of the same speed of propagation in the two equations of the system. The result is new and opens more research areas into porous-elastic system.

Keywords Porous system · General decay · Nonlinear damping

Mathematics Subject Classification (2010) 35B35 · 35B40 · 93D20

1 Introduction

In this paper, we consider the following porous system:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & x \in (0,1),\ t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + \mathrm{b}\, \mathrm{u}_x + \xi\phi + \alpha(t)g(\phi_t) = 0, & x \in (0,1),\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1), \\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), & x \in (0,1), \\ u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, & t \geq 0, \end{cases} \tag{1.1}$$

where the functions u is the displacement of the solid elastic material and ϕ is the volume fraction. The term $\alpha(t)g(\phi_t)$ is the nonlinear damping term, which acts only on the second equation. Here, g and α are specific functions to be specified later. The mass density ρ and

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the coefficient J are always assumed positive. The parameters μ, b, δ, ξ are constitutive constants, which satisfy the following conditions

$$\mu > 0,\ \delta > 0,\ \xi > 0,\ \mu\xi > b^2.$$

The conditions are imposed to guarantee the positivity of the internal energy (see Cowin and Nunziato [3] for details). As coupling is concerned, b must be different from 0, but its sign does not matter in the analysis. Our aim is to establish an explicit and a general decay rate result for the energy of system (1.1) in case of the same speed of propagation in the two equations of the system, that is

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{1.2}$$

In the last decades, a great deal of attention has been devoted to determine the asymptotic behavior of solutions for porous-elastic systems, since the pioneer work of Goodman and Cowin [1], in which they introduced the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter idea was introduced by Nunziato and Cowin [2] when they developed a nonlinear theory of elastic materials with voids. The importance of materials with microstructure has been demonstrated by the huge number of papers published in different fields of human endeavors most importantly, in petroleum industry, material science, soil mechanics, foundation engineering, powder technology, biology, and others. We refer the reader to [3, 4] and the references therein for more details.

The basic evolution equations for one-dimentional theories of porous materials are given by

$$\rho u_{\mathrm{tt}} = T_x, \quad J\phi_{\mathrm{tt}} = H_x + G, \tag{1.3}$$

where T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force. The constitutive equations with nonlinear damping term are as follows:

$$T = \mu u_x + b\phi, \quad H = \delta\phi_x, \quad G = -\,\mathrm{bu}_x - \xi\phi - \alpha(t)g(\phi_t). \tag{1.4}$$

By substituting (1.4) into (1.3), we obtain the first two equations in Eq. 1.1.

Replacing the nonlinear damping term in Eq. 1.1 with porous dissipation gives

$$\begin{aligned} \rho u_{\mathrm{tt}} &= \mu u_{\mathrm{xx}} + b\phi_x, \\ J\phi_{\mathrm{tt}} &= \delta\phi_{\mathrm{xx}} - \mathrm{b}\,\mathrm{u}_x - \xi\phi - \alpha\phi_t. \end{aligned} \tag{1.5}$$

Quintanilla [5] considered (1.5) with some initial and boundary conditions, using Hurtwitz theorem to prove that the damping through porous-viscosity ($\alpha\phi_t$) is not strong enough to obtain an exponential decay but only a slow (nonexponential) decay. However, Apalara [6] considered the same system and proved that the system is exponentially stable provided the wave speeds of the two systems are equal. Similarly, when the porous dissipation in Eq. 1.5 is replaced with memory term of the form $\int_0^t g(t-s)\phi_{\mathrm{xx}}(x,s)\mathrm{ds}$, Apalara [7] established a general decay result depending on the kernel of the memory term and the wave speeds of the system. We refer reader to [8, 9] for similar results. Relatedly, when $\alpha = 0$ and viscoelasticity ($-\gamma u_{\mathrm{txx}}$) is added to the elastic equation, that is

$$\begin{aligned} &\rho u_{\mathrm{tt}} - \mu u_{\mathrm{xx}} - b\phi_x - \gamma u_{\mathrm{txx}} = 0,\ x \in (0,1),\ t > 0, \\ &J\phi_{\mathrm{tt}} - \delta\phi_{\mathrm{xx}} + \mathrm{b}\,\mathrm{u}_x + \xi\phi = 0, \quad\ x \in (0,1),\ t > 0 \end{aligned}$$

with some initial and boundary conditions, Magaña and Quintanilla [10] proved that the viscoelasticity damping is not strong enough to bring about exponential stability. However, they showed that the presence of both porous-viscosity and viscoelasticity stabilizes the system exponentially. For various other damping mechanisms used and more results on porous-elastic system, we refer reader to [11–17] and the references therein.

It is to be noted that when $\mu = b = \xi$, then Eq. 1.1 becomes

$$\begin{array}{ll} \rho u_{tt} - \mu(u_x + \phi)_x = 0, & x \in (0,1),\ t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + \mu(u_x + \phi) + \alpha(t)g(\phi_t) = 0, & x \in (0,1),\ t > 0, \end{array} \tag{1.6}$$

which is a well-known Timoshenko system with nonlinear feedback control. Alabau-Boussouira [18] considered (1.6) with $\alpha(t) \equiv 1$ and established a general semi-explicit formula for the decay rate of the energy at infinity subject to Eq. 1.2. Mustafa and Messaoudi [19] considered (1.6) with all the coefficients (ρ, μ, J, δ) equal to one and obtained an explicit and general decay result, depending on α and g. See [20–22] for similar result. According to this observation and results, two questions naturally arise:

1. Is it possible to consider a porous system with a weakly nonlinear dissipation only on the second equation and obtain the same result as in Timoshenko system?
2. Is it possible to obtain the stability result with same conditions on α, g, and the assumption of equal wave speed as in Timoshenko system?

The aim of the present work is to give satisfactory answers to these questions by considering (1.1) and establish an explicit and general decay result depending on α and g and providing that Eq. 1.2 holds. In other words, we consider (1.1) and establish an explicit and general decay result subject to Eq. 1.2 and depending on α and g. Our result provides an explicit energy decay formula that allows for a larger class of functions α and g, from which the energy decay rates are not necessarily of exponential or polynomial types.

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. To avoid such case and also to be able to use Poincaré's inequality, we perform the following transformation.

From the first equation in Eq. 1.1, we observe that $\int_0^1 u_{tt}\,dx = 0$.

So, if $v(t) := \int_0^1 u\,dx$ then $v(0) = \int_0^1 u_0\,dx$ and $v'(0) = \int_0^1 u_1\,dx$. Thus, we have the following initial value problem

$$\begin{cases} v''(t) = 0, & \forall t \geq 0 \\ v(0) = \int_0^1 u_0\,dx, & v'(0) = \int_0^1 u_1\,dx. \end{cases} \tag{1.7}$$

Solving (1.7), we obtain

$$v = \int_0^1 u(x,t)\,dx = t\int_0^1 u_1(x)\,dx + \int_0^1 u_0(x)\,dx.$$

Consequently, if we let

$$\overline{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)\,dx - \int_0^1 u_0(x)\,dx, \tag{1.8}$$

we obtain

$$\int_0^1 \overline{u}(x,t)\,dx = 0, \quad \forall t \geq 0.$$

Therefore, the use of Poincaré's inequality in the sequel is justified. In addition, a simple substitution shows that $(\overline{u}, \phi)$ satisfies system (1.1) with initial data for $\overline{u}$ given as

$$\overline{u}_0(x) = u_0(x) - \int_0^1 u_0(x)\mathrm{dx} \ \text{ and } \ \overline{u}_1(x) = u_1(x) - \int_0^1 u_1(x)\mathrm{dx}.$$

Henceforth, we work with $\overline{u}$ instead of u but write u for simplicity of notation.

The paper is organized as follows. In Section 2, we give some notations and material needed for our work. In addition, we state, without proof, the well-posedness of system (1.1). In Section 3, we state and proof some technical lemmas needed in the proof of our main result. Section 4 is devoted to the statements and proofs of our stability result. We use c throughout this paper to denote a generic positive constant.

2 Preliminaries

In this section, we present some materials needed in the proof of our result.

We assume α and g satisfy the following hypotheses:

(H1) $\alpha : R_+ \to R_+$ is a non-increasing differentiable function;

(H2) $g : R \to R$ is a non-decreasing C^0-function such that there exist positive constants c_1, c_2, ϵ, and a strictly increasing function $G \in C^1([0, +\infty))$, with $G(0) = 0$, and G is linear or strictly convex C^2-function on $(0, \epsilon]$ such that

$$\begin{cases} s^2 + g^2(s) \leq \ G^{-1}(sg(s)) \text{ for all } |s| \leq \epsilon, \\ c_1|s| \leq |g(s)| \leq c_2|s| \text{ for all } |s| \geq \epsilon. \end{cases}$$

Remark 2.1
- Hypothesis (H2) implies that $sg(s) > 0$, for all $s \neq 0$.
- According to our knowledge, hypothesis (H2) with $\epsilon = 1$ was first introduced by Lasiecka and Tataru [23]. They established a decay result, which depends on the solution of an explicit nonlinear ordinary differential equation. Furthermore, they proved that the monotonicity and continuity of g guarantee the existence of the function G defined in (H2).

For completeness purpose we state, without proof, the existence and regularity result of system (1.1). First, we introduce the following spaces:

$$\mathcal{H} = H^1_\star(0, 1) \times L^2_\star(0, 1) \times H^1_0(0, 1) \times L^2(0, 1),$$

and

$$\widetilde{\mathcal{H}} = \Phi_0 \in \left[H^2_\star(0, 1) \cap H^1_\star(0, 1)\right] \times H^1_\star(0, 1) \times \left[H^2(0, 1) \cap H^1_0(0, 1)\right] \times H^1_0(0, 1),$$

where

$$L^2_\star(0, 1) = \{\psi \in L^2(0, 1) \ : \int_0^1 \psi(x)dx = 0\}, \quad H^1_\star(0, 1) = H^1(0, 1) \cap L^2_\star(0, 1),$$
$$H^2_\star(0, 1) = \{\psi \in H^2(0, 1) \ : \psi_x(0) = \psi_x(1) = 0\}.$$

For $\Phi = (u, u_t, \phi, \phi_t)$, we have the following existence and regularity result:

Proposition 2.2 *Assume that (H1) and (H2) are satisfied. Then for all $\Phi_0 \in \mathcal{H}$, the system (1.1) has a unique global (weak) solution*

$$u \in C(R_+; H^1_\star(0, 1)) \cap C^1(R_+; L^2_\star(0, 1)), \quad \phi \in C(R_+; H^1_0(0, 1)) \cap C^1(R_+; L^2(0, 1)).$$

Moreover, if $\Phi_0 \in \widetilde{\mathcal{H}}$*, then the solution satisfies*

$$u \in L^\infty(R_+; H^2_\star(0,1) \cap H^1_\star(0,1)) \cap W^{1,\infty}(R_+; H^1_\star(0,1)) \cap W^{2,\infty}(R_+; L^2_\star(0,1))$$
$$\phi \in L^\infty(R_+; H^2(0,1) \cap H^1_0(0,1)) \cap W^{1,\infty}(R_+; H^1_0(0,1)) \cap W^{2,\infty}(R_+; L^2(0,1)).$$

Remark 2.3 This result can be proved using the theory of maximal nonlinear monotone operators (see [24]).

3 Technical Lemmas

In this section, we state and prove several lemmas needed for the proof of our stability result.

Lemma 3.1 *Let* $\Phi_0 \in \mathcal{H}$*. Then the energy functional* E*, defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right] dx, \tag{3.1}$$

satisfies

$$E'(t) = -\alpha(t)\int_0^1 \phi_t g(\phi_t) dx \le 0. \tag{3.2}$$

Proof Equation 3.2 follows by multiplying the first and the second equation of Eq. 1.1 by u_t and ϕ_t, respectively, integrating by parts over (0, 1) and using the boundary conditions. □

Remark 3.2 The energy functional $E(t)$ defined by Eq. 3.1 is nonnegative. In fact, it can easily be verified that

$$\mu u_x^2 + 2\mathrm{bu}_x\phi + \xi\phi^2 = \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\phi\right)^2 + \xi\left(\phi + \frac{b}{\xi}u_x\right)^2 + \left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\phi^2\right].$$

So, using the fact that $\mu\xi > b^2$, we obtain

$$\mu u_x^2 + 2\mathrm{bu}_x\phi + \xi\phi^2 > \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\phi^2\right] > 0.$$

Consequently,

$$E(t) > \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu_1 u_x^2 + \delta\phi_x^2 + \xi_1\phi^2 + J\phi_t^2\right] \mathrm{dx}, \tag{3.3}$$

where $2\mu_1 = \mu - \frac{b^2}{\xi} > 0$ and $2\xi_1 = \xi - \frac{b^2}{\mu} > 0$.

Lemma 3.3 *Assume that (H1) and (H2) hold. Then, for all* $\Phi_0 \in \mathcal{H}$*, the functional*

$$F_1(t) := J\int_0^1 \phi_t\phi dx + \frac{b\rho}{\mu}\int_0^1 \phi\int_0^x u_t(y)dydx$$

satisfies for any $\varepsilon_1 > 0$,

$$F_1'(t) \le -\delta \int_0^1 \phi_x^2 dx - \xi_1 \int_0^1 \phi^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \phi_t^2 dx \\ + \varepsilon_1 \int_0^1 u_t^2 dx + c \int_0^1 g^2(\phi_t) dx. \tag{3.4}$$

Proof Direct computations using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yield

$$\begin{aligned} F_1'(t) &= -\delta \int_0^1 \phi_x^2 \mathrm{dx} - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 \mathrm{dx} + J \int_0^1 \phi_t^2 \mathrm{dx} \\ &\quad + \frac{b\rho}{\mu} \int_0^1 \phi_t \int_0^x u_t(y) \mathrm{dy dx} - \alpha(t) \int_0^1 \phi g(\phi_t) \mathrm{dx} \\ &\le -\delta \int_0^1 \phi_x^2 \mathrm{dx} - \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 \mathrm{dx} + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \phi_t^2 \mathrm{dx} \\ &\quad + \varepsilon_1 \int_0^1 \left(\int_0^x u_t(y) \mathrm{dy}\right)^2 \mathrm{dx} + c \int_0^1 g^2(\phi_t) \mathrm{dx}. \end{aligned} \tag{3.5}$$

By Cauchy-Schwarz inequality, it is clear that

$$\int_0^1 \left(\int_0^x u_t(y) \mathrm{dy}\right)^2 \mathrm{dx} \le \int_0^1 \left(\int_0^1 u_t \mathrm{dx}\right)^2 \mathrm{dx} \le \int_0^1 u_t^2 \mathrm{dx}. \tag{3.6}$$

Estimate (3.4) follows by substituting (3.6) into Eq. 3.5 and recalling that $2\xi_1 = \xi - \dfrac{b^2}{\mu} > 0$. □

Lemma 3.4 *Assume (H1), (H2), and* Eq. 1.2 *hold. Then, for all* $\Phi_0 \in \mathcal{H}$, *the functional*

$$F_2(t) := b \int_0^1 \phi_x u_t dx + b \int_0^1 u_x \phi_t dx$$

satisfies

$$F_2'(t) \le -\frac{b^2}{2J} \int_0^1 u_x^2 dx + c \int_0^1 \phi_x^2 dx + c \int_0^1 g^2(\phi_t) dx. \tag{3.7}$$

Proof By differentiating F_2, using (1.1), and then integrating by parts, we obtain

$$\begin{aligned} F_2'(t) &= -\frac{b^2}{J} \int_0^1 u_x^2 \mathrm{dx} - \frac{b\xi}{J} \int_0^1 u_x \phi \mathrm{dx} + \frac{b^2}{\rho} \int_0^1 \phi_x^2 \mathrm{dx} - \frac{b\alpha(t)}{J} \int_0^1 u_x g(\phi_t) \mathrm{dx} \\ &\quad + b\left(\frac{\mu}{\rho} - \frac{\delta}{J}\right) \int_0^1 u_{\mathrm{xx}} \phi_x \mathrm{dx}. \end{aligned}$$

The use of Young's and Poincaré's inequalities, bearing in mind (1.2), yields estimate (3.7). □

Lemma 3.5 *Let* $\Phi_0 \in \mathcal{H}$. *Then the functional defined by*

$$F_3(t) := -\rho \int_0^1 u_t u dx$$

satisfies

$$F_3'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx + c \int_0^1 \phi^2 dx. \tag{3.8}$$

Proof Direct computations give

$$F_3'(t) = -\rho \int_0^1 u_t^2 \mathrm{dx} + \mu \int_0^1 u_x^2 \mathrm{dx} + b \int_0^1 u_x \phi \mathrm{dx}.$$

Estimate (3.8) easily follows by using Young's inequality. □

Lemma 3.6 *Suppose (H1), (H2), and* Eq. 1.2 *hold. Let* $\Phi_0 \in \mathcal{H}$*. Then, for* $N, N_1, N_2 > 0$ *sufficiently large, the Lyapunov functional defined by*

$$\mathcal{L}(t) := NE(t) + N_1 F_1(t) + N_2 F_2(t) + F_3(t)$$

satisfies, for some positive constants d_1, d_2, k_1*,*

$$d_1 E(t) \leq \mathcal{L}(t) \leq d_2 E(t), \qquad \forall t \geq 0 \tag{3.9}$$

and

$$\mathcal{L}'(t) \leq -k_1 E(t) + c \int_0^1 (\phi_t^2 + g^2(\phi_t)) dx, \quad \forall t \geq 0. \tag{3.10}$$

Proof It follows that

$$|\mathcal{L}(t) - \mathrm{NE}(t)| \leq N_1 \int_0^1 |J \phi_t \phi| \, \mathrm{dx} + \frac{b\rho N_1}{\mu} \int_0^1 \left| \phi \int_0^x u_t(y) \mathrm{dy} \right| \mathrm{dx} + \rho \int_0^1 |u_t u| \mathrm{dx}$$
$$+ \mathrm{bN}_2 \int_0^1 |\phi_x u_t + u_x \phi_t| \, \mathrm{dx}.$$

Exploiting Young's, Cauchy-Schwarz, and Poincaré inequalities, we obtain

$$|\mathcal{L}(t) - \mathrm{NE}(t)| \leq c \int_0^1 \left(u_t^2 + \phi_x^2 + u_x^2 + \phi^2 + \phi_t^2 \right) \mathrm{dx}.$$

Using (3.3), we obtain

$$|\mathcal{L}(t) - \mathrm{NE}(t)| \leq c E(t),$$

that is

$$(N - c) E(t) \leq \mathcal{L}(t) \leq (N + c) E(t).$$

Now, by choosing N(depending on N_1 and N_2) sufficiently large we obtain (3.9).

The proof of Eq. 3.10 is as follows: We let $\varepsilon_1 = \frac{\rho}{2N_1}$ and then combine Eqs. 3.2, 3.4, 3.7, and 3.8, to obtain

$$\mathcal{L}'(t) \leq -\frac{\rho}{2} \int_0^1 u_t^2 \mathrm{dx} - \left[\frac{b^2}{2J} N_2 - \frac{3\mu}{2} \right] \int_0^1 u_x^2 \mathrm{dx} - [\delta N_1 - c N_2] \int_0^1 \phi_x^2 \mathrm{dx}$$
$$- [\xi_1 N_1 - c] \int_0^1 \phi^2 \mathrm{dx} + \mathrm{cN}_1 (1 + N_1) \int_0^1 \phi_t^2 \mathrm{dx} + c(N_1 + N_2) \int_0^1 g(\phi_t^2) \mathrm{dx}.$$

We choose N_2 so large that

$$\alpha_1 = \frac{b^2}{2J} N_2 - \frac{3\mu}{2} > 0\,,$$

then we choose N_1 so large that

$$\alpha_2 = \delta N_1 - cN_2 > 0 \text{ and } \alpha_3 = \xi_1 N_1 - c > 0.$$

So, we end up with

$$\begin{aligned}\mathcal{L}'(t) \leq & -\frac{\rho}{2}\int_0^1 u_t^2 \mathrm{dx} - \alpha_1 \int_0^1 u_x^2 \mathrm{dx} - \alpha_2 \int_0^1 \phi_x^2 \mathrm{dx} - \alpha_3 \int_0^1 \phi^2 \mathrm{dx} \\ & + c\int_0^1 (\phi_t^2 + g^2(\phi_t))\mathrm{dx}.\end{aligned} \tag{3.11}$$

On the other hand, from Eq. 3.1, using Young's inequality, we obtain

$$\begin{aligned}E(t) &\leq \frac{1}{2}\int_0^1 \left[\rho u_t^2 + (\mu + b)u_x^2 + \delta\phi_x^2 + (\xi + b)\phi^2 + J\phi_t^2\right] \mathrm{dx} \\ &\leq c\int_0^1 \left[u_t^2 + u_x^2 + \phi_x^2 + \phi^2 + \phi_t^2\right] \mathrm{dx}\end{aligned}$$

which implies that

$$-\int_0^1 \left[u_t^2 + u_x^2 + \phi_x^2 + \phi^2\right] \mathrm{dx} \leq -c' E(t) + c'' \phi_t^2. \tag{3.12}$$

The combination of Eqs. 3.11 and 3.12 gives (3.10). □

4 Stability Result

In this section, we state and prove our stability result.

Theorem 4.1 *Suppose (H1), (H2), and* Eq. 1.2 *hold. Let* $\Phi_0 \in \mathcal{H}$. *Then there exist positive constants* a_1, a_2, a_3, *and* ϵ_0 *such that the solution of* Eq. 1.1 *satisfies*

$$E(t) \leq a_1 G_1^{-1}\left(a_2\int_0^t \alpha(s)ds + a_3\right), \quad t \geq 0, \tag{4.1}$$

where

$$G_1(t) = \int_t^1 \frac{1}{G_0(s)} ds \text{ and } G_0(t) = tG'(\epsilon_0 t).$$

Remark 4.2 G_1 strictly decreases and is convex on (0, 1] and $\lim_{t\to 0} G_1(t) = +\infty$.

Proof We multiply (3.10) by $\alpha(t)$ to get

$$\alpha(t)\mathcal{L}'(t) \leq -k_1\alpha(t)E(t) + c\alpha(t)\int_0^1 (\phi_t^2 + g^2(\phi_t))\mathrm{dx}. \tag{4.2}$$

Now, we discuss two cases:

Case I: G is linear on $[0, \epsilon]$. In this case, using (H2) and Eq. 3.2, we deduce that

$$\alpha(t)\mathcal{L}'(t) \leq -k_1\alpha(t)E(t) + c\alpha(t)\int_0^1 \phi_t g(\phi_t)\mathrm{dx} = -k_1\alpha(t)E(t) - \mathrm{cE}'(t),$$

which can be rewritten as

$$(\alpha(t)\mathcal{L}(t) + cE(t))' - \alpha'(t)\mathcal{L}(t) \leq -k_1\alpha(t)E(t).$$

Using (H1), we obtain

$$(\alpha(t)\mathcal{L}(t) + \mathrm{cE}(t))' \leq -k_1\alpha(t)E(t).$$

By exploiting (3.9), it can easily be shown that

$$\mathcal{R}_0(t) := \alpha(t)\mathcal{L}(t) + \mathrm{cE}(t) \sim E(t). \tag{4.3}$$

So, for some positive constant λ_1, we obtain

$$\mathcal{R}_0'(t) + \lambda_1\alpha(t)\mathcal{R}_0(t) \leq 0, \quad \forall t \geq 0. \tag{4.4}$$

The combination of Eq. 4.4 and 4.3, gives

$$E(t) \leq E(0)e^{-\lambda_1\int_0^t \alpha(s)ds} = E(0)G_1^{-1}\left(\lambda_1\int_0^t \alpha(s)\mathrm{ds}\right). \tag{4.5}$$

Case II: G is nonlinear on $[0, \epsilon]$. In this case, we first choose $0 < \epsilon_1 \leq \epsilon$ such that

$$sg(s) \leq \min\{\epsilon, G(\epsilon)\}, \quad \forall |s| \leq \epsilon_1. \tag{4.6}$$

Using (H2) along with fact that g is continuous and $|g(s)| > 0$, for $s \neq 0$, it follows that

$$\begin{cases} s^2 + g^2(s) \leq G^{-1}(\mathrm{sg}(s)), & \forall |s| \leq \epsilon_1, \\ c_1|s| \leq |g(s)| \leq c_2|s|, & \forall |s| \geq \epsilon_1. \end{cases} \tag{4.7}$$

To estimate the last integral in Eq. 4.2, we consider the following partition of (0, 1):

$$I_1 = \{x \in (0,1) : |\phi_t| \leq \epsilon_1\}, \qquad I_2 = \{x \in (0,1) : |\phi_t| > \epsilon_1\}.$$

Now, with $I(t)$ defined by

$$I(t) = \int_{I_1} \phi_t g(\phi_t)\mathrm{dx},$$

we have, using Jensen inequality (note that G^{-1} is concave and recall (4.6))

$$G^{-1}(I(t)) \geq c\int_{I_1} G^{-1}(\phi_t g(\phi_t))\,\mathrm{dx}. \tag{4.8}$$

The combination of Eq. 4.7 and 4.8 yields

$$\begin{aligned} \alpha(t)\int_0^1 \left(\phi_t^2 + g^2(\phi_t)\right)\mathrm{dx} &= \alpha(t)\int_{I_1}\left(\phi_t^2 + g^2(\phi_t)\right)\mathrm{dx} + \alpha(t)\int_{I_2}\left(\phi_t^2 + g^2(\phi_t)\right)\mathrm{dx} \\ &\leq \alpha(t)\int_{I_1} G^{-1}(\phi_t g(\phi_t))\,\mathrm{dx} + c\alpha(t)\int_{I_2}\phi_t g(\phi_t)\mathrm{dx} \\ &\leq c\alpha(t)G^{-1}(I(t)) - \mathrm{cE}'(t). \end{aligned} \tag{4.9}$$

So, by substituting (4.9) into (4.2) and using (4.3) and (H1), we have

$$\mathcal{R}_0'(t) \leq -k_1\alpha(t)E(t) + c\alpha(t)G^{-1}(I(t)). \tag{4.10}$$

Now, for $\epsilon_0 < \epsilon$ and $\delta_0 > 0$, using (4.10) and the fact that $E' \leq 0$, $G' > 0$, $G'' > 0$ on $(0, \epsilon]$, we find that the functional $\mathcal{R}_1$, defined by

$$\mathcal{R}_1(t) := G'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathcal{R}_0(t) + \delta_0 E(t),$$

satisfies, for some $b_1, b_2 > 0$,

$$b_1\mathcal{R}_1(t) \leq E(t) \leq b_2\mathcal{R}_1(t) \tag{4.11}$$

and

$$\mathcal{R}_1'(t) := \epsilon_0 \frac{E'(t)}{E(0)} G''\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{R}_0(t) + G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{R}_0'(t) + \delta_0 E'(t)$$
$$\leq -k_1 \alpha(t) E(t) G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\alpha(t) G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) G^{-1}(I(t)) + \delta_0 E'(t). \quad (4.12)$$

Let G^* be the convex conjugate of G defined by

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \qquad \text{if } s \in (0, G'(\epsilon)],$$

satisfying the following general Young's inequality

$$\text{AB} \leq G^*(A) + G(B), \qquad \text{if } A \in (0, G'(\epsilon)], B \in (0, \epsilon].$$

With $A = G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = G^{-1}(I(t))$, using (4.6), we obtain

$$c\alpha(t) G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) G^{-1}(I(t)) \leq c\alpha(t) G^*\left(G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\alpha(t) I(t).$$

By exploiting (3.2) and the fact that $G^\star(s) \leq s(G')^{-1}(s)$, we get

$$c\alpha(t) G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) G^{-1}(I(t)) \leq c\epsilon_0 \alpha(t) \frac{E(t)}{E(0)} G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t). \quad (4.13)$$

By substituting (4.13) into Eq. 4.12, we obtain

$$\mathcal{R}_1'(t) \leq -k_1 \alpha(t) E(t) G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\epsilon_0 \alpha(t) \frac{E(t)}{E(0)} G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) + \delta_0 E'(t)$$
$$\leq -\left(k_1 E(0) - c\epsilon_0\right) \alpha(t) \frac{E(t)}{E(0)} G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + (\delta_0 - c) E'(t).$$

Letting $\epsilon_0 = \frac{k_1}{2c} E(0)$, $\delta_0 = 2c$, and recall that $E'(t) \leq 0$, we end up with

$$\mathcal{R}_1'(t) \leq -k\alpha(t) \frac{E(t)}{E(0)} G'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -k\alpha(t) G_0\left(\frac{E(t)}{E(0)}\right), \quad (4.14)$$

where $k > 0$ and $G_0(t) = tG'(\epsilon_0 t)$.

Note that

$$G_0'(t) = G'(\epsilon_0 t) + \epsilon_0 t G''(\epsilon_0 t).$$

So, using the strict convexity of G on $(0, \epsilon]$, we find that $G_0(t), G_0'(t) > 0$ on $(0, 1]$. With $\mathcal{R}(t) := \frac{b_1 \mathcal{R}_1(t)}{E(0)}$ it is obvious that $\mathcal{R}(t) \leq \frac{E(t)}{E(0)} \leq 1$. Now, using Eqs. 4.11 and 4.14, we have

$$\mathcal{R}(t) \sim E(t) \quad (4.15)$$

and, for some $a_2 > 0$,

$$\mathcal{R}'(t) \leq -a_2 \alpha(t) G_0(\mathcal{R}(t)). \quad (4.16)$$

Inequality (4.16) implies that $\frac{d}{dt}\left[G_1(\mathcal{R}(t))\right] \geq a_2 \alpha(t)$, where

$$G_1(t) = \int_t^1 \frac{1}{G_0(s)} ds.$$

Thus, by integrating over $[0, t]$, we obtain, for some $a_3 > 0$,

$$\mathcal{R}(t) \leq G_1^{-1}\left(a_2 \int_0^t \alpha(s) \mathrm{ds} + a_3\right). \quad (4.17)$$

Here, we used, based on the properties of G_0, the fact that G_1 is strictly decreasing on (0, 1]. Finally, using (4.15) and (4.17), we obtain (4.1). □

Examples We give some examples to illustrate the energy decay rates given by Theorem 4.1.

(1) If g satisfies

$$k_2 \min\left\{|s|, |s|^q\right\} \le |g(s)| \le k_3 \max\left\{|s|, |s|^{1/q}\right\}$$

for some $k_2, k_3 > 0$ and $q \ge 1$, then $G(s) = cs^q$ satisfies (H2). Then, we end up with the following energy decay rate:

$$E(t) \le \begin{cases} c \exp\left(-k_4 \int_0^t \alpha(s)\mathrm{d}s\right) & \text{if } q = 1, \\ c\left(k_4 \int_0^t \alpha(s)\mathrm{d}s + k_5\right)^{-\frac{1}{q-1}} & \text{if } q > 1. \end{cases}$$

(2) If $G(s) = e^{-1/s}$ near zero, then we have the following energy decay rate:

$$E(t) \le \frac{c}{\ln\left(k_4 \int_0^t \alpha(s)\mathrm{d}s + k_5\right)}.$$

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