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The Maximum Principle for Partially Observed Optimal Control of FBSDE Driven by Teugels Martingales and Independent Brownian Motion

S. Bougherara[1] · N. Khelfallah[1]

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Abstract The aim of this paper is to study a stochastic partially observed optimal control problem, for systems of forward backward stochastic differential equations (FBSDE for short), which are driven by both a family of Teugels martingales and an independent Brownian motion. By using Girsavov's theorem and a standard spike variational technique, we prove necessary conditions to characterize an optimal control under a partial observation, where the control domain is supposed to be convex. Moreover, under some additional convexity conditions, we prove that these partially observed necessary conditions are sufficient. In fact, compared to the existing methods, we get the last achievement in two different cases according to the linearity or the nonlinearity of the terminal condition for the backward component. As an illustration of the general theory, an application to linear quadratic control problems is also investigated.

Keywords Forward-backward stochastic differential equation · Optimal control · Maximum principle · Partially observed optimal control · Teugels martingale · Lévy process

Mathematics Subject Classification (2010) 93E20 · 60H10 · 60H30

✉ N. Khelfallah
nabilkhelfallah@yahoo.fr

S. Bougherara
bougherarasaliha@yahoo.fr

1 Laboratory of Applied Mathematics, University of Biskra, Po. Box 145, Biskra (07000), Algeria

1 Introduction

In this paper, we are interested in partially observed optimal control of systems driven by a forward backward stochastic differential equation of the type

$$\begin{cases} dx_t = b(t, x_t, u_t)\, dt + g(t, x_t, u_t)\, dW_t + \sum_{i=1}^{\infty} \sigma^{(i)}(t, x_{t-}, u_t)\, dH_t^{(i)}, \\ x_0 = x, \\ -dy_t = f(t, x_{t-}, y_{t-}, z_t, Z_t, u_t)\, dt - z_t dW_t - \sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, \\ y_T = \varphi(x_T), \end{cases} \tag{1}$$

where W_t is a brownian motion and $(H_t^{(i)})_{i=1}^{\infty}$ is a family of pairwise orthogonal martingales associated with some Lévy process L_t, which is independent from W_t. These martingales are called Teugels martingales. The control problem consists in minimizing the following cost functional

$$J(u) = \mathbb{E}\left[\int_0^T l(t, x_t, y_t, z_t, k_t, u_t)\, dt + h(x_T) + M(y_0)\right],$$

over a partially observed class of admissible controls to be specified later.

Since 1990, more precisely since the work of Pardoux and Peng [12], the theory of backward stochastic differential equations (BSDEs) and FBSDEs has found important applications and has become a powerful tool in many fields. For instance, the financial mathematics, optimal control, stochastic games, partial differential equations and homogenization, see e.g. [1, 5–7].

In the literature, optimal control problems are handled in two different approaches. One is the Bellman dynamic programming principle, and the second is the maximum principle. Our purpose in this framework is precisely to deal with the second approach in the case where the full range of information available to the controller is assumed to be partially observed. It is well-known that a partially observed stochastic optimal control of BSDEs and FBSDEs driven only by a brownian motion has been studied by many authors through several articles, such as [3, 13, 14, 16, 18] and the references therein. The case of FBSDEs driven by both a brownian motion and a Lévy process has been considered in [19], by using certain classical convex variational techniques.

A worthy and powerful motivation for studying SDEs and BSDEs driven by a brownian motion and Teugels martingales is due to the very useful representation theorem provided by Nualart and Schoutens [10]. This theorem asserts that every square integrable martingale adapted to the natural filtration of a brownian motion and an independent Lévy process, can be written as the sum of a stochastic integral with respect to the brownian motion and the sum of stochastic integrals with respect to the Teugels martingales associated to the Lévy process. In other words, this representation formula put brownian motion and Lévy processes in a unified theory of square integrable martingales. See the excellent accounts by Davis [4], Schoutens [17]. In another paper [11], the authors have proved an existence and uniqueness result for BSDEs driven by Teugels martingales, under Lipschitz conditions. Moreover, an application to Clark–Ocone and Feynman–Kac formulas for Lévy processes is presented. Their result has been extended to the locally Lipschitz property in [2].

It then becomes quite natural to investigate control problems for systems driven by this kind of equations. Let us point out that the first work in this direction has been carried out by Mitsui-Tabata [9], for the case of a linear quadratic problem. Then Meng-Tang [8] studied the stochastic maximum principle for systems driven by an Itô forward SDE, by using convex perturbations technique. Optimal control of BSDEs driven by Teugels martingales has been addressed in Tang and Zhang [16], where necessary and sufficient conditions have been established. It is worth noting that in all previous control problems the information of the control problem is assumed to be completely observed. In return, this is not always reasonable in the real world applications because the controllers can only get partial information at most cases. This gives us a motivation to study this kind of control problems.

However, up to now, there is only one literature (see Bahlali et al. [3]) dealing with a partial information control problem for a system governed by SDEs driven by a both Teugels martingales and an independent brownian motion. In this work, the control variable is allowed to enter into the both coefficient and is assumed to be adapted to subfiltration which is possibly less than the whole one. The authors investigated a partial information necessary as well as sufficient conditions by using certain classical convex variational techniques.

The main contribution of our present paper is to investigate a partially observed necessary as well as sufficient conditions satisfied by an optimal control. To obtain the optimality necessary conditions, we use the convex perturbation method and differentiate the perturbed both the state equations and the cost functional, in order to get the adjoint process, which is a solution of a backward forward SDE, driven by both a brownian motion and a family of Teugels martingales, on top of the variational inequality between the Hamiltonians. Moreover, an additional technical assumptions are required to prove that these partially observed necessary conditions are in fact sufficient.

The rest of the paper is structured as follows. A brief introduction to Teugels martingales and a precise formulation of the control problem are presented in Section 2. Section 3 consists of the proof of partially observed necessary conditions of optimality in term of classical convex variational techniques. Under some additional convexity conditions, we show that these partially observed necessary conditions of optimality are also sufficient in Section 4. Finally in Section 5, we illustrate the general results by solving an example.

2 Preliminaries and Problem Formulation

2.1 Preliminaries and Assumptions

Let $(\Omega, \mathcal{F}, P)$ be a complete filtered probability space equipped with two mutually independent standard brownian motions W and Y valued in $\mathbb{R}^d$ and $\mathbb{R}^r$, respectively and an independent $\mathbb{R}^m$-valued Lévy process $\{L_t, t \in [0, T]\}$ of the form $L_t = bt + l_t$, where l_t is a pure jump process. Assume further that the Lévy measure $\nu(dz)$ corresponding to the Lévy process L_t satisfies

i) $\int_{\mathbb{R}} \left(1 \wedge z^2\right) \nu(dz) < \infty$,
ii) $\int_{(]-\varepsilon,\varepsilon[)^c} e^{\alpha|z|} \nu(dz) < \infty$, for every $\varepsilon > 0$ and some $\alpha > 0$.

The two above settings imply that the random variable L_t have moments in all orders. We also assume that

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^L \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of the P-null set and $\mathcal{F}_t^W$, $\mathcal{F}_t^Y$ and $\mathcal{F}_t^L$ denotes the P-completed natural filtration generated by W, Y and L respectively.

Let us recall briefly the L^2 theory of Lévy processes as it is investigated in Nualart-Schoutens [10] and Schoutens [17]. A convenient basis for martingale representation is provided by the so-called Teugels martingales. This means that this family has the predictable representation property.

Denote by $\Delta L_t = L_t - L_{t-}$ and define the power jump processes $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum_{0<s\leq t} (\triangle L_s)^{(i)}$ for $i \geq 2$. If we denote $\mathbf{Y}_t^{(i)} = L_t^{(i)} - \mathbb{E}\left[L_t^{(i)}\right]$, $i \geq 1$, then the family of Teugels martingales $\left(H_t^{(i)}\right)_{i=1}^{\infty}$, is defined by $H_t^{(i)} = \sum_{j=1}^{j=i} a_{ij}\mathbf{Y}_t^{(j)}$. The coefficients a_{ij} correspond to the orthonormalization of the polynomials 1, x, x^2, ... with respect to the measure $\mu(dx) = x^2\nu(dx)$.

Then $\left(H_t^{(i)}(t)\right)_{i=1}^{\infty}$ is a family of strongly orthogonal martingales such that $\left\langle H_t^{(i)}, H^{(j)}\right\rangle_t = \delta_{ij}.t$ and that $\left[H^{(i)}, H^{(j)}\right]_t - \left\langle H^{(i)}, H^{(j)}\right\rangle_t$ is an $\mathcal{F}_t$-martingale, see [15]. We refer the reader to [2, 4, 10] for the detailed proofs.

Throughout what follows, we shall assume the following notations

l^2 : the Hilbert space of real-valued sequences $x = (x_n)_{n\geq 0}$ such that

$$\|x\| = \left(\sum_{i=1}^{\infty} x_i\right)^{\frac{1}{2}} < \infty.$$

For any integer $m \geq 1$, we define

$l^2(\mathbb{R}^m)$: the space of $\mathbb{R}^m$-valued sequences $(x_i)_{i\geq 1}$ such that

$$\left(\sum_{i=1}^{\infty} \|x_i\|_{\mathbb{R}^m}^2\right)^{\frac{1}{2}} < \infty.$$

(a, b) : the inner product in $\mathbb{R}^n$, $\forall a, b \in \mathbb{R}^n$.

$|a| = \sqrt{(a, a)}$: the norm of $\mathbb{R}^n$, $\forall a \in \mathbb{R}^n$.

(A, B) : the inner product in $\mathbb{R}^{n\times d}$, $|A| = \sqrt{(A, A)}$: the norm of $\mathbb{R}^{n\times n}$.

$l^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$: the Banach space of $l^2(\mathbb{R}^m)$- valued $\mathcal{F}_t$- predictable processes such that

$$\left(\mathbb{E}\int_0^T \sum_{i=1}^{\infty} \left\| f^{(i)}(t)\right\|_{\mathbb{R}^m}^2 dt\right)^{\frac{1}{2}} < \infty.$$

$\mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$: the Banach space of $\mathbb{R}^m$-valued $\mathcal{F}_t$-adapted process such that

$$\left(\mathbb{E}\int_0^T |f(t)|_{\mathbb{R}^m}^2 dt\right)^{\frac{1}{2}} < \infty.$$

$\mathcal{S}^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$: the Banach space of $\mathbb{R}^m$-valued $\mathcal{F}_t$-adapted and càdlàg process such that

$$\left(\mathbb{E} \sup_{0\leq t\leq T} |f(t)|^2\right)^{\frac{1}{2}} < \infty.$$

$L^2(\Omega, \mathcal{F}, P, \mathbb{R}^m)$: the Banach space of $\mathbb{R}^m$-valued square integrable random variables on $(\Omega, \mathcal{F}, P)$.

2.2 Formulation of the Control Problem

Let T be a strictly positive real number. An admissible control is an $\mathcal{F}_t^Y$- predictable process $u = (u_t)$ with values in some convex subset U of $\mathbb{R}^k$ *and satisfies* $\mathbb{E}\left[\sup_{0\le t\le T} |u_t|^2\right] < \infty$. We denote the set of all admissible controls by $\mathcal{U}$. The control u is called partially observable. Let us also assume that the coefficient of the controlled FBSDE (1) are defined as follows

$$\begin{aligned}
&b : [0, T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n,\\
&g : [0, T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d},\\
&\sigma : [0, T] \times \Omega \times \mathbb{R}^n \times U \to l^2\left(\mathbb{R}^n\right),\\
&f : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times l^2\left(\mathbb{R}^m\right) \times U \to \mathbb{R}^m,\\
&\varphi : \mathbb{R}^n \times \Omega \to \mathbb{R}^m.
\end{aligned}$$

We assume that the state processes (x, y, z, Z) cannot be observed directly, but the controllers can observe a related noisy process Y, which we call the observation process, via the following Itô process

$$dY_t = \xi\left(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, Z_t^\upsilon, \upsilon_t\right) dt + dW_t^\upsilon \ , \ Y_0 = 0, \tag{2}$$

where

$$\xi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times l^2\left(\mathbb{R}^m\right) \times U \to \mathbb{R}^n,$$

and W^υ is an $\mathbb{R}^r$-valued stochastic processes depending on the control υ. Define $dP^\upsilon = \Gamma^\upsilon dP$, where

$$\Gamma_t^\upsilon := \exp\left\{\int_0^t \left(\xi\left(s, x_s, y_s, z_s, Z_s, \upsilon_s\right), dY_s\right) - \frac{1}{2}\int_0^t \left|\xi\left(s, x_s, y_s, z_s, Z_s, \upsilon_s\right)\right|^2 ds\right\}.$$

Obviously, Γ^υ is the unique $\mathcal{F}_t^Y$-adapted solution of

$$d\Gamma_t^\upsilon = \Gamma_t^\upsilon\left(\xi\left(t, x_t, y_t, z_t, Z_t, \upsilon_t\right), dY_t\right) \ , \ \Gamma_0^\upsilon = 1. \tag{3}$$

Then Girsanov's theorem shows that

$$dW_t^\upsilon = dY_t - \int_0^t \xi\left(s, x_s^\upsilon, y_s^\upsilon, z_s^\upsilon, Z_s^\upsilon, \upsilon_s\right) ds,$$

is an $\mathbb{R}^r$-valued brownian motion and $(H_t^{(i)})_{i=1}^\infty$ is still a Teugels martingale under the probability measure P^υ.

The objective is to characterize an admissible controls which minimize the following cost functional.

$$J(u) = \mathbb{E}^u\left[h(y_0) + M(x_T) + \int_0^T l(t, x_t, y_t, z_t, Z_t, u_t)\, dt\right], \tag{4}$$

where $\mathbb{E}^u$ denotes the expectation with respect to the probability measure space P^u and

$$\begin{aligned}
M &: \mathbb{R}^n \times \Omega \to \mathbb{R},\\
h &: \mathbb{R}^m \times \Omega \to \mathbb{R},\\
l &: [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times l^2(\mathbb{R}^m) \times U \to \mathbb{R}.
\end{aligned}$$

It is obvious that the cost functional (4) can be rewritten as the following

$$J(u) = \mathbb{E}\left[h(y_0) + \Gamma_T M(x_T) + \int_0^T \Gamma_t l(t, x_t, y_t, z_t, Z_t, u_t)\, dt\right]. \tag{5}$$

Now, we can state our partially observed control problem.

Problem A Minimize (5) over $u \in \mathcal{U}$, subject to Eqs. 1 and 3.

A control is said to be partially observed if the control is a non-anticipative functional of the observation Y. A set of controls is said to be partially observed if its every element is partially observed. Hence, the set of admissible controls $\mathcal{U}$ is partially observed.

An admissible control $\hat{u}$ is called a partially observed optimal if it attains the minimum of $J(u)$ over $\mathcal{U}$. The Eqs. 1 and 2 are called respectively the state and the observation equations, and the solution $\left(\hat{x}, \hat{y}, \hat{z}, \hat{Z}\right)$ corresponding to $\hat{u}$ is called an optimal trajectory.

Throughout this paper, we shall make the following assumptions

(A_1)

- The random mappings b, g, σ and φ are measurable with $b(., 0, 0) \in \mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^n)$, $g(., 0, 0) \in \mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^n)$, $\sigma(., 0, 0) \in l^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$ and $\varphi(0) \in L^2(\Omega, \mathcal{F}, P, \mathbb{R}^m)$.
- b, g, σ and φ are continuously differentiable in (x, u). They are bounded by $(1 + |x| + |u|)$ and their derivatives in (x, u) are continuous and uniformly bounded.
- The random mapping f is measurable with $f(., 0, 0, 0, 0) \in \mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$, f is continuous and continuously differentiable with respect to (x, y, z, Z, u). Moreover it is bounded by $(1 + |x| + |y| + |z| + |Z| + |u|)$ and their derivatives are uniformly bounded.

(A_2)

- l is continuously differentiable with respect to (x, y, z, Z, u) and bounded by $\left(1 + |x|^2 + |y|^2 + |z|^2 + |Z|^2 + |u|^2\right)$. Furthermore, their derivatives are uniformly bounded.
- M is continuously differentiable in x and h is continuously differentiable in y. Moreover, for almost all $(t, \omega) \in [0, T] \times \Omega$, there exists a constant C, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$|M_x| \leq C(1 + |x|) \text{ and } \left|h_y\right| \leq C(1 + |y|).$$

(A_3) ξ is continuously differentiable in (x, y, z, Z, u) and their derivatives in (x, y, z, Z, u) are uniformly bounded.

Following [11], it holds that under assumptions (A_1), there is a unique solution

$$(x, y, z, Z) \in \mathcal{S}^2_{\mathcal{F}}\left(0, T, \mathbb{R}^n\right) \times \mathcal{S}^2_{\mathcal{F}}\left(0, T, \mathbb{R}^m\right) \times \mathcal{L}^2_{\mathcal{F}}\left(0, T, \mathbb{R}^{m \times d}\right) \times l^2_{\mathcal{F}}\left(0, T, \mathbb{R}^m\right),$$

which solves the state Eq. 1.

Let $\left(x^1_t, y^1_t, z^1_t, Z^1_t\right)$ and Γ^1_t be the solutions at time t of the following linear FBSDE and SDE, respectively,

$$\begin{cases} dx^1_t &= \left(b_x(t)\, x^1_t + b_u(t)(\upsilon_t - u_t)\right) dt + \left(g_x(t)\, x^1_t + g_u(t)(\upsilon_t - u_t)\right) dW_t \\ &\quad + \sum\limits_{i=1}^{\infty} \left(\sigma^{(i)}_x(t)\, x^1_{t-} + \sigma^{(i)}_u(t)(\upsilon_t - u_t)\right) dH^{(i)}_t, \\ -dy^1_t &= \left[f_x(t)\, x^1_t + f_y(t)\, y^1_t + f_z(t)\, z^1_t + f_Z(t)\, Z^1_t \right. \\ &\quad \left. + f_u(t)(\upsilon_t - u_t)\right] dt - z^1_t dW_t - \sum\limits_{i=1}^{\infty} Z^{(i),1}_{t-} dH^{(i)}_t, \\ x^1_0 &= 0,\ y^1_T = \varphi_x(x_T)\, x^1_T, \end{cases} \tag{6}$$

and

$$\begin{cases} d\Gamma_t^1 & = \Big[\Gamma_t^1 \xi^* (t) + \Gamma_t \left(\xi_x (t) x_t^1\right)^* + \Gamma_t \left(\xi_y (t) y_t^1\right)^* + \Gamma_t \left(\xi_z (t) z_t^1\right)^* \\ & \quad + \Gamma_t \left(\xi_Z (t) Z_t^1\right)^* + \Gamma_t \left(\xi_u (t) (\upsilon_t - u_t)\right)^* \Big] dY_t, \\ \Gamma_0^1 & = 0, \end{cases} \tag{7}$$

where

$$\begin{aligned} b_\rho (t) &= b_\rho (t, x_t, u_t) && \text{for } \rho = x, u \text{ and } b = b, g, \sigma, \\ f_\rho (t) &= f_\rho (t, x_t, y_t, z_t, Z_t, u_t) && \text{for } \rho = x, y, z, Z, u \text{ and } f = f, \xi. \end{aligned}$$

Set $\vartheta_t = \Gamma^{-1}\Gamma^1$ satisfies the following dynamics

$$\begin{cases} d\vartheta_t & = \left(\xi_x x_t^1 + \xi_y y_t^1 + \xi_z z_t^1 + \xi_Z Z_t^1 + \xi_u (\upsilon_t - u_t)\right) d\tilde{W}, \\ \vartheta_0 & = 0. \end{cases} \tag{8}$$

For any $u \in \mathcal{U}$ and the corresponding state trajectory (x, y, z, Z), we introduce the following system of forward backward SDE, called the adjoint equations,

$$\begin{cases} -dp_t & = \left(b_x^* (t) p_t + f_x^* (t) q_t + g_x^* (t) k_t + \xi_x^* (t) \Xi_t + \sum\limits_{i=1}^{\infty} \sigma_x^{(i)*} (t) Q_t + l_x (t)\right) dt \\ & \quad - k_t dW_t - \sum\limits_{i=1}^{\infty} Q_t^{(i)} dH_t^{(i)}, \\ dq_t & = \left(f_y^* (t) q_t + \xi_y^* (t) \Xi_t + l_y (t)\right) dt + \left(f_z^* (t) q_t + \xi_z^* (t) \Xi_t + l_z (t)\right) dW_t \\ & \quad + \sum\limits_{i=1}^{\infty} \left(f_{Z^{(i)}}^* (t) q_t + \xi_{Z^{(i)}}^* (t) \Xi_t + l_{Z^{(i)}} (t)\right) dH_t^{(i)}, \\ p_T & = M_x (x_T) + \varphi_x^* (x_T) q_T, \ q_0 = h_y (y_0). \end{cases} \tag{9}$$

It is clear that (p, k, Q) is the adjoint process corresponding to the forward part of our system (1) and q is corresponding to the backward part. Manifestly, the above FBSDE admit a unique solution

$$(p, k, Q, q) \in \mathcal{S}_{\mathcal{F}}^2 \left(0, T, \mathbb{R}^n\right) \times \mathcal{L}_{\mathcal{F}}^2 \left(0, T, \mathbb{R}^{n \times d}\right) \times l_{\mathcal{F}}^2 \left(0, T, \mathbb{R}^n\right) \times \mathcal{S}_{\mathcal{F}}^2 \left(0, T, \mathbb{R}^m\right).$$

under the assumptions (A_1). We further introduce the following auxiliary BSDE, which also admit a unique solution under the assumptions (A_1),

$$-dP_t = l(t, x_t, y_t, z_t, Z_t, \upsilon_t) dt - \Xi_t d\tilde{W}_t \ , \ P_T = M(x_T). \tag{10}$$

Let us now, define the Hamiltonian function

$$\begin{aligned} H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times l^2 (\mathbb{R}^m) \times \mathcal{U} \times \mathbb{R}^n \\ \times \mathbb{R}^m \times \mathbb{R}^{n \times d} \times l^2 (\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}, \end{aligned}$$

by

$$\begin{aligned} & H (t, x, y, z, Z, u, p, q, P, Q, \Xi) := (p, b (t, x, u)) + (q, f (t, x, y, z, Z, u)) + (\Xi, \xi) \\ & + \sum_{i=1}^{i=d} \left(k^{(i)}, g^{(i)} (t, x, u)\right) + \sum_{i=1}^{\infty} \left(Q^{(i)}, \sigma^{(i)} (t, x, u)\right) + l (t, x, y, z, Z, u). \end{aligned}$$

3 A Partial Information Necessary Conditions for Optimality

In this section, we derive a partially observed necessary conditions for optimality for our control problem under the previous assumptions. The main objective is to solve the problem A.

3.1 Some Auxiliary Results

Let υ be an arbitrary element of $\mathcal{U}$, then for a sufficiently small $\theta > 0$ and for each $t \in [0, T]$, we define a perturbed control as follows

$$u_t^\theta = u_t + \theta\left(\upsilon_t - u_t\right).$$

Since the action space being convex, it is clear that u_t^θ is an admissible control. Let us now, pointing out that we need the following two lemmas to state and prove the main result of this section. In fact, they play a crucial role in the sequel.

Lemma 4.1 *If the assumptions* (A_1) *and* (A_3) *hold true, then we have the following estimates*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|x_t^\theta - x_t\right|^2\right] = 0, \tag{11}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|y_t^\theta - y_t\right|^2 + \int_0^T\left(\left|z_t^\theta - z_t\right|^2 + \left\|Z_t^\theta - Z_t\right\|_{l^2(\mathbb{R}^m)}^2\right)ds\right] = 0, \tag{12}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|\Gamma_t^\theta - \Gamma_t\right|^2\right] = 0. \tag{13}$$

Proof We first prove (11). Applying Itô's formula to $\left|x_t^\theta - x_t\right|^2$, taking expectations and using the relations $\left\langle H^{(i)}, H^{(j)}\right\rangle_s = \delta_{i,j}.t$ and $\left[H^{(i)}, H^{(j)}\right]_t - \left\langle H^{(i)}, H^{(j)}\right\rangle_t$ is an $\mathcal{F}_t$-martingale together with the fact that b, σ, g are uniformly Lipschitz in (x, u), one can get

$$\begin{aligned}\mathbb{E}\left|x_t^\theta - x_t\right|^2 &\le C\mathbb{E}\textstyle\int_0^t\left|x_s^\theta - x_s\right|^2 ds + C\mathbb{E}\int_0^t\left|u_s^\theta - u_s\right|^2\\ &\le C\mathbb{E}\textstyle\int_0^t\left|x_s^\theta - x_s\right|^2 ds + C\theta^2.\end{aligned}$$

Thus (11) follows immediately, by using Gronwall's lemma and letting θ go to 0.

Let us now prove (12). Applying Itô's formula to $\left|y_t^\theta - y_t\right|^2$ and taking expectation to obtain

$$\begin{aligned}&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\textstyle\int_t^T\left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T\left\|Z_s^\theta - Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds = \mathbb{E}\left|\varphi\left(x_T^\theta\right) - \varphi\left(x_T\right)\right|^2\\ &\quad+2\mathbb{E}\textstyle\int_t^T\left(y_{s-}^\theta - y_{s-}\right)\left[f\left(s, x_s^\theta, y_{s-}^\theta, z_s^\theta, Z_s^\theta, u_s^\theta\right) - f\left(s, x_s, y_{s-}, z_s, Z_s, u_s\right)\right]ds.\end{aligned}$$

From Young's inequality, for each $\varepsilon > 0$, we have

$$\begin{aligned}&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T\left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T\left\|Z_s^\theta - Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds\\ &\le \mathbb{E}\left|\varphi\left(x_T^\theta\right) - \varphi\left(x_T\right)\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|y_s^\theta - y_s\right|^2 ds\\ &\quad+\varepsilon\mathbb{E}\int_t^T\left|f\left(s, x_s^\theta, y_{s-}^\theta, z_s^\theta, Z_s^\theta, u_s^\theta\right) - f\left(s, x_s, y_{s-}, z_s, Z_s, u_s\right)\right|^2 ds.\end{aligned}$$

Then,

$$\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T \left\|Z_s^\theta - Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds \\
&\leq \mathbb{E}\left|\varphi\left(x_T^\theta\right) - \varphi\left(x_T\right)\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds \\
&\quad + C\varepsilon\mathbb{E}\int_t^T \left|f\left(s, x_s^\theta, y_s^\theta, z_s^\theta, Z_s^\theta, u_s^\theta\right) - f\left(s, x_s, y_s, z_s, Z_s, u_s^\theta\right)\right|^2 ds \\
&\quad + C\varepsilon\mathbb{E}\int_t^T \left|f\left(s, x_s, y_s, z_s, Z_s, u_s^\theta\right) - f\left(s, x_s, y_s, z_s, Z_s, u_s\right)\right|^2 ds.
\end{aligned}$$

Due the fact that φ and f are uniformly Lipschitz with respect to x, y, z, Z and u, one can get

$$\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T \left\|Z_s^\theta - Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds \\
&\leq \left(\frac{1}{\varepsilon} + C\varepsilon\right)\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds \\
&\quad + C\varepsilon\mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left\|Z_s^\theta - Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds + \alpha_t^\theta,
\end{aligned} \tag{14}$$

where α_t^θ is given by

$$\alpha_t^\theta = \mathbb{E}\left|x_T^\theta - x_T\right|^2 + C\varepsilon\mathbb{E}\int_t^T \left|x_s^\theta - x_s\right|^2 ds + C\varepsilon\theta^2.$$

By invoking (11) and sending θ to 0, we have

$$\lim_{\theta \to 0} \alpha_t^\theta = 0. \tag{15}$$

We now pick up $\varepsilon = \frac{1}{2C}$, and replacing its value in Eq. 14 to obtain

$$\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left\|Z_s^\theta - Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds \\
&\leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + \alpha_t^\theta.
\end{aligned}$$

Consequently, we obtain the desired result (12), by using Gronwall's lemma and letting θ goes to 0.We now proceed to prove (13). Itô's formula applied to $\left|\Gamma_t^\theta - \Gamma_t^u\right|^2$ yields

$$\mathbb{E}\left|\Gamma_t^\theta - \Gamma_t^u\right|^2 \leq C\mathbb{E}\textstyle\int_0^t \left|\Gamma_s^\theta - \Gamma_s^u)\right|^2 ds + C\beta_t^\theta. \tag{16}$$

Here, β_t^θ is given by the following equality

$$\beta_t^\theta = \mathbb{E}^u\int_0^t \left|\xi\left(s, x_s^\theta, y_s^\theta, z_s^\theta, Z_s^\theta, u_s^\theta\right) - \xi\left(s, x_s, y_s, z_s, Z_s, u_s\right)\right|^2 ds.$$

Keeping in mind that ξ is continuous in (x, y, z, Z, u), it is not difficult to see that

$$\lim_{\theta \to 0} \beta_t^\theta = 0.$$

Hence, we obtain (13) by using Gronwall's lemma and by sending θ to 0. Before we state and prove the next lemma, let us introduce the following short hand notations,

$$\tilde{\rho}_t^\theta = \theta^{-1}\left(\rho_t^\theta - \rho_t\right) - \rho_t^1, \text{ for } \rho = x, y, z, Z \text{ and } \Gamma. \tag{17}$$

□

Lemma 4.2 *Assume that* (A_1) *and* (A_3) *are in force. Then, we have the following convergence results*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|\tilde{x}_t^\theta\right|^2\right] = 0, \tag{18}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|y_t^\theta\right|^2 + \int_0^T\left(\left|\tilde{z}_t^\theta\right|^2 + \left\|\tilde{Z}_t^\theta\right\|_{l^2(\mathbb{R}^m)}^2\right)dt\right] = 0, \tag{19}$$

$$\mathbb{E}\int_0^T\left|\tilde{\Gamma}_t^\theta\right|^2 dt = 0 \tag{20}$$

Proof First, we start by giving the proof of Eq. 18. By the notation (17) and the first-order expansion, it is easy to check that $\tilde{x}_t^\theta$ satisfies the following SDE

$$\begin{cases} d\tilde{x}_t^\theta &= \left(b_t^x\tilde{x}_t^\theta dt + \alpha_t^\theta\right)dt + \left(g_t^x\tilde{x}_t^\theta dt + \beta_t^\theta\right)dW_t \\ &\quad + \sum_{i=1}^{\infty}\left(\sigma_t^{(i),x}\tilde{x}_t^\theta + \gamma_t^{(i),\theta}\right)dH_t^{(i)}, \\ \tilde{x}_0^\theta &= 0, \end{cases} \tag{21}$$

where

$$b_t^x = \int_0^1 b_x\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t^\theta\right)d\lambda, \text{ for } b = b, g, \sigma.$$

$$\begin{aligned}\alpha_t^\theta &= \int_0^1\left[b_x\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - b_x\left(t, x_t, u_t\right)\right]d\lambda x_t^1 \\ &\quad + \int_0^1\left[b_u\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - b_u\left(t, x_t, u_t\right)\right]d\lambda\left(\upsilon_t - u_t\right), \\ \beta_t^\theta &= \int_0^1\left[g_x\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - g_x\left(t, x_t, u_t\right)\right]d\lambda x_t^1 \\ &\quad + \int_0^1\left[g_u\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - g_u\left(t, x_t, u_t\right)\right]d\lambda\left(\upsilon_t - u_t\right),\end{aligned}$$

and

$$\begin{aligned}\gamma_t^{(i),\theta} &= \int_0^1\left[\sigma_x^{(i)}\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - \sigma_x^{(i)}\left(t, x_t, u_t\right)\right]d\lambda x_t^1 \\ &\quad + \int_0^1\left[\sigma_u^{(i)}\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - \sigma_u^{(i)}\left(t, x_t, u_t\right)\right]d\lambda\left(\upsilon_t - u_t\right).\end{aligned}$$

Since b_x, b_u, g_x, g_u and σ_x, σ_u are continuous in (x, u), it is not difficult to see that

$$\lim_{\theta\to 0}\mathbb{E}\left(\left|\alpha_t^{\theta}\right|^2+\left|\beta_t^{\theta}\right|^2+\left|\gamma_t^{(i),\theta}\right|^2\right)=0, \tag{22}$$

Applying Itô's formula to $\left(\tilde{x}_t^{\theta}\right)^2$, we get

$$\begin{aligned}\mathbb{E}\left|\tilde{x}_t^{\theta}\right|^2 &= 2\mathbb{E}\int_0^t \tilde{x}_s^{\theta}\left(b_s^x\tilde{x}_s^{\theta}+\alpha_s^{\theta}\right)ds+\mathbb{E}\int_0^t\left|\left(g_s^x\tilde{x}_s^{\theta}+\beta_s^{\theta}\right)\right|^2 ds\\ &\quad+\sum_{i=1}^{\infty}\mathbb{E}\int_0^t\left|\sigma_s^{(i),x}\tilde{x}_s^{\theta}+\gamma_s^{(i),\theta}\right|^2 ds.\end{aligned}$$

Using the inequality $2ab \le a^2+b^2$, seeing that b_s^x, g_s^x and σ_s^x are bounded, to obtain

$$\mathbb{E}\left|\tilde{x}_t^{\theta}\right|^2\le(1+2C)\,\mathbb{E}\int_0^t\left|\tilde{x}_s^{\theta}\right|^2 ds+\mathbb{E}\int_0^t\left(\left|\alpha_s^{\theta}\right|^2+\left|\beta_s^{\theta}\right|^2+\left|\gamma_s^{(i),\theta}\right|^2\right)ds.$$

Finally, by using Gronwall's lemma and Eq. 22, we obtain (18).

We now turn out to prove (19). Again, in view of the notations (17), one can easily show that $\left(\tilde{y}_t^{\theta}, \tilde{z}_t^{\theta}, \tilde{Z}_t^{\theta}\right)$ satisfies the following BSDE

$$\begin{cases} d\tilde{y}_t^{\theta}=\left(f_t^x\tilde{x}_t^{\theta}+f_t^y\tilde{y}_t^{\theta}+f_t^z\tilde{z}_t^{\theta}+f_t^Z\tilde{Z}_t^{\theta}+\chi_t^{\theta}\right)dt+\tilde{z}_t^{\theta}dW_t+\sum\limits_{i=1}^{\infty}\tilde{Z}_t^{\theta}dH_t^{(i)},\\ \tilde{y}_T^{\theta}=\theta^{-1}\left(\varphi\left(x_T^{\theta}\right)-\varphi\left(x_T\right)\right)-\varphi_x\left(x_T\right)x_T^1,\end{cases}$$

where $\tilde{x}_t^{\theta}$ is the solution to the SDE (21) and

$$f_t^x=-\textstyle\int_0^1 f_x\left(\Lambda_t^{\theta}\left(u_t\right)\right)d\lambda,\ \text{for}\ x=x,y,z,Z,$$

$$\begin{aligned}\chi_t^{\theta} &= \int_0^1\left(f_x\left(\Lambda_s^{\theta}\left(u_t\right)\right)-f_x\left(t,x_t,y_t,z_t,Z_t,u_t\right)\right)d\lambda\ x_t^1\\ &\quad+\int_0^1\left(f_y\left(\Lambda_s^{\theta}\left(u_t\right)\right)-f_y\left(t,x_t,y_t,z_t,Z_t,u_t\right)\right)d\lambda\ y_t^1\\ &\quad+\int_0^1\left(f_z\left(\Lambda_s^{\theta}\left(u_t\right)\right)-f_z\left(t,x_t,y_t,z_t,Z_t,u_t\right)\right)d\lambda\ z_t^1\\ &\quad+\int_0^1\left(f_Z\left(\Lambda_s^{\theta}\left(u_t\right)\right)-f_Z\left(t,x_t,y_t,z_t,Z_t,u_t\right)\right)d\lambda\ Z_t^1\\ &\quad+\int_0^1\left(f_u\left(\Lambda_s^{\theta}\left(u_t\right)\right)-f_u\left(t,x_t,y_t,z_t,Z_t,u_t\right)\right)d\lambda\ \left(\upsilon_t-u_t\right),\end{aligned}$$

and

$$\begin{aligned}\Lambda_t^{\theta}(u) &= \left(t,x_t+\lambda\theta\left(\tilde{x}_t^{\theta}+x_t^1\right),y_t+\lambda\theta\left(\tilde{y}_t^{\theta}+y_t^1\right),\right.\\ &\qquad\left.z_t+\lambda\theta\left(\tilde{z}_t^{\theta}+z_t^1\right),Z_t+\lambda\theta\left(\tilde{Z}_t^{\theta}+Z_t^1\right),u_t+\lambda\theta\left(\upsilon_t-u_t\right)\right).\end{aligned}$$

Due the fact that f_x, f_y, f_z and f_Z are continuous, we have

$$\lim_{\theta\to 0}\mathbb{E}\left|\chi_t^\theta\right|^2 = 0. \tag{23}$$

Again, Itô's formula applied to $\left|\tilde{y}_t^\theta\right|^2$ leads to the following equality

$$\begin{aligned}&\mathbb{E}\left|\tilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|^2_{l^2(\mathbb{R}^m)}\\ &= \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + 2\mathbb{E}\int_t^T \tilde{y}_s^\theta\left(f_s^x\tilde{x}_s^\theta + f_s^y\tilde{y}_s^\theta + f_s^z\tilde{z}_s^\theta + f_s^Z\tilde{Z}_s^\theta + \chi_s^\theta\right)ds.\end{aligned}$$

By using Young's inequality, for each $\varepsilon > 0$, we obtain

$$\begin{aligned}&\mathbb{E}\left|\tilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|^2_{l^2(\mathbb{R}^m)} ds\\ &\leq \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\tilde{y}_s^\theta\right|^2 ds + \varepsilon\mathbb{E}\int_t^T \left|\left(f_s^x\tilde{x}_s^\theta + f_s^y\tilde{y}_s^\theta + f_s^z\tilde{z}_s^\theta + f_s^Z\tilde{Z}_s^\theta + \chi_s^\theta\right)\right|^2 ds\\ &\leq \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\tilde{y}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|f_s^x\tilde{x}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|f_s^y\tilde{y}_s^\theta\right|^2 ds\\ &\quad + C\varepsilon\mathbb{E}\int_t^T \left|f_s^z\tilde{z}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|f_s^Z\tilde{Z}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|\chi_s^\theta\right|^2 ds.\end{aligned}$$

It follows that, in view of the boundedness of f_t^x, f_t^y, f_t^z and f_t^Z,

$$\begin{aligned}&\mathbb{E}\left|\tilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|^2_{l^2(\mathbb{R}^m)} ds\\ &\leq \left(\frac{1}{\varepsilon} + C\varepsilon\right)\mathbb{E}\int_t^T \left|\tilde{y}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds\\ &\quad + C\varepsilon\mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|^2_{l^2(\mathbb{R}^m)} ds + \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + C\varepsilon\eta_t^\theta,\end{aligned}$$

where

$$\eta_t^\theta = \mathbb{E}\int_t^T \left|f_s^x\tilde{x}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left|\chi_s^\theta\right|^2 ds.$$

Hence, in view of Eq. 18, the fact that φ_x, f_s^x are continuous and bounded, we get

$$\lim_{\theta\to 0}\mathbb{E}\left|\tilde{y}_T^\theta\right|^2 = 0. \tag{24}$$

and

$$\lim_{\theta\to 0}\mathbb{E}\int_t^T \left|f_s^x\tilde{x}_s^\theta\right|^2 ds = 0. \tag{25}$$

Furthermore, From Eqs. 23 and 25, we deduce that

$$\lim_{\theta\to 0}\eta_t^\theta = 0. \tag{26}$$

If we choose $\varepsilon = \frac{1}{2C}$, it holds that,

$$\begin{aligned}&\mathbb{E}\left|\tilde{y}_t^\theta\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|^2_{l^2(\mathbb{R}^m)} ds\\ &\leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|\tilde{y}_s^\theta\right|^2 ds + \frac{1}{2}\eta_t^\theta.\end{aligned}$$

The estimates (19) follow from an application of Gronwall's lemma together with Eqs. 23 and 26.

Now we proceed to prove (20). From Eq. 17, it is plain to check that $\tilde{\Gamma}^{\theta}$ satisfies the following equality,

$$d\tilde{\Gamma}^{\theta} = \left[\tilde{\Gamma}^{\theta}_t \xi\left(t, x^{\theta}_t, y^{\theta}_t, z^{\theta}_t, Z^{\theta}_t, u^{\theta}_t\right) + \bar{\chi}^{\theta}_t\right] dY_t$$
$$+\Gamma_t \left\{\xi^x_t \tilde{x}^{\theta}_t + \xi^y_t \tilde{y}^{\theta}_t + \xi^z_t \tilde{z}^{\theta}_t + \xi^Z_t \tilde{Z}^{\theta}_t\right\} dY_t,$$

where

$$\xi^x_t = \int_0^1 \xi_x\left(\Lambda^{\theta}_t(u_t)\right) d\lambda, \text{ for } x = x, y, z, Z,$$

and $\bar{\chi}^{\theta}_t$ is given by

$$\bar{\chi}^{\theta}_t = \Gamma_t \left[\int_0^1 \left(\xi_x\left(\Lambda^{\theta}_s(u_t)\right) - \xi_x\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ x^1_t\right.$$
$$+\int_0^1 \left(\xi_y\left(\Lambda^{\theta}_s(u_t)\right) - \xi_y\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ y^1_t$$
$$+\int_0^1 \left(\xi_z\left(\Lambda^{\theta}_s(u_t)\right) - \xi_z\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ z^1_t$$
$$+\int_0^1 \left(\xi_Z\left(\Lambda^{\theta}_s(u_t)\right) - \xi_Z\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ Z^1_t$$
$$\left.+\int_0^1 \left(\xi_u\left(\Lambda^{\theta}_s(u_t)\right) - \xi_u\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ (\upsilon_t - u_t)\right]$$
$$+\Gamma^1_t\left[\xi\left(t, x^{\theta}_t, y^{\theta}_t, z^{\theta}_t, Z^{\theta}_t, u^{\theta}_t\right) - \xi(t)\right],$$

We deduce, taking into account the fact that ξ_x, ξ_y, ξ_z and ξ_Z are continuous,

$$\lim_{\theta\to 0} \mathbb{E}\left|\bar{\chi}^{\theta}_t\right|^2 = 0. \tag{27}$$

Applying Itô's formula to $\left|\tilde{\Gamma}^{\theta}_t\right|^2$, taking expectation, and using the fact that ξ, ξ^x_t, ξ^y_t, ξ^z_t and ξ^Z_t are bounded, to obtain

$$\mathbb{E}\left|\tilde{\Gamma}^{\theta}_t\right|^2 \le C\mathbb{E}\int_0^T \left|\tilde{\Gamma}^{\theta}_t\right|^2 dt + C\mathbb{E}\int_0^T \left|\tilde{x}^{\theta}_t\right|^2 dt$$
$$+C\mathbb{E}\int_0^T \left|\tilde{y}^{\theta}_t\right|^2 dt + C\mathbb{E}\int_0^T \left|\tilde{z}^{\theta}_t\right|^2 dt$$
$$+C\mathbb{E}\int_0^T \left\|\tilde{Z}^{\theta}_t\right\|^2 dt + C\mathbb{E}\int_0^T \left|\bar{\chi}^{\theta}_t\right|^2 dt.$$

Keeping in mind the relations (18) and (27), we deduce that, using Gronwall's inequality,

$$\lim_{\theta\to 0} \sup_{0\le t\le T} \mathbb{E}\left|\tilde{\Gamma}^{\theta}_t\right|^2 = 0.$$

□

3.2 Variational Inequality and Optimality Necessary Conditions

Since u is an optimal control, then, with the fact that $\theta^{-1}\left[J\left(u_t^\theta\right) - J\left(u_t\right)\right] \geq 0$, we have the following lemma.

Lemma 4.3 *Suppose that the assumptions* (A_1), (A_2) *and* (A_3) *are satisfied. Then the following variational inequality holds*

$$\begin{aligned}
0 \leq\ & \mathbb{E}\left[\Gamma_T M_x\left(x_T\right) x_T^1 + h_y\left(y_0\right) y_0^1 + \Gamma_T^1 M(x_T)\right] \\
& + \mathbb{E}\int_0^T \left(\Gamma_t^1 l(t) + \Gamma_t\left(l_x(t)\, x_t^1 + l_y(t)\, y_t^1 + l_z(t)\, z_t^1 + l_Z(t)\, Z_t^1 + l_u(t)\left(\upsilon_t - u_t\right)\right)\right) dt,
\end{aligned} \tag{28}$$

where $l_\rho(t) = l_\rho\left(t, x_t, y_t, z_t, Z_t, u_t\right)$ *for* $\rho = x, y, z, Z$.

Proof From the definition of the cost functional and by using the first order development, one can get

$$\begin{aligned}
0 \leq \theta^{-1}\left[J\left(u_t^\theta\right) - J\left(u_t\right)\right] =\ & \theta^{-1}\mathbb{E}\left[\left(\Gamma_T^\theta - \Gamma_T\right) M\left(x_T^\theta\right)\right] \\
& + \theta^{-1}\mathbb{E}\left[\Gamma_T \int_0^1 M_x\left(x_T + \lambda\left(x_T^\theta - x_T\right)\right)\left(x_T^\theta - x_T\right) d\lambda\right] \\
& + \theta^{-1}\mathbb{E}\left[\int_0^1 h_y\left(y_0 + \lambda\left(y_0^\theta - y_0\right)\right)\left(y_0^\theta - y_0\right) d\lambda\right] \\
& + \theta^{-1}\mathbb{E}\left[\int_0^T l\left(t, x_t^\theta, y_t^\theta, z_t^\theta, Z_t^\theta, u_t^\theta\right)\left(\Gamma_t^\theta - \Gamma_t\right) dt\right] \\
& + \theta^{-1}\mathbb{E}\left[\int_0^T \Gamma_t\left(\int_0^1 \left(l_x\left(\Lambda_t^\theta(u)\right)\left(x_t^\theta - x_t\right)\right.\right.\right. \\
& + l_y\left(\Lambda_t^\theta(u)\right)\left(y_t^\theta - y_t\right) + l_z\left(\Lambda_t^\theta(u)\right)\left(z_t^\theta - z_t\right) \\
& \left.\left.\left. + l_Z\left(\Lambda_t^\theta(u)\right)\left(Z_t^\theta - Z_t\right) + l_u\left(\Lambda_t^\theta(u)\right)\left(u_t^\theta - u_t\right)\right) d\lambda\right) dt\right].
\end{aligned}$$

Finally by using Eqs. 18, 19, 20 and letting θ go to 0, we obtain (28).
In view of Eq. 8, the variational inequality (28) can be rewritten as

$$\begin{aligned}
0 \leq\ & \mathbb{E}^u\left[M_x\left(x_T\right) x_T^1\right] + \mathbb{E}^u\left[h_y\left(y_0\right) y_0^1\right] + \mathbb{E}^u\left[\vartheta_T M(x_T)\right] \\
& + \mathbb{E}^u\int_0^T \left(\vartheta_t l(t) + \left(l_x(t)\, x_t^1 + l_y(t)\, y_t^1 + l_z(t)\, z_t^1 + l_Z(t)\, Z_t^1 + l_u(t)\left(\upsilon_t - u_t\right)\right)\right) dt,
\end{aligned} \tag{29}$$

□

The main result of this section can be stated us follows.

Theorem 4.4 (Partial information maximum principle) *Suppose* (A_1), (A_2), *and* (A_3) *hold. Let* (x, y, z, Z, u) *be an optimal solution of the control problem A. There are*

4-type (p, q, k, Q) *and a pair* (P, Ξ) *of* $\mathcal{F}_t$*-adapted processes which satisfy* (9) *and* (10) *respectively, such that the following maximum principle holds true,*

$$\mathbb{E}^u\left[\left(H_v\left(t, x_t, y_t, z_t, Z_t, u_t, p_t, q_t, k_t, Q_t, \Xi_t\right), \left(\upsilon_t - u_t\right)\right) \mid \mathcal{F}_t^Y\right] \geq 0,\ \forall \upsilon \in \mathcal{U},\ a.\mathbb{E},\ a.s. \tag{30}$$

Proof By applying Itô's formula to $\left(p_t, x_t^1\right)$ and $\left(q_t, y_t^1\right)$ and using the fact that $q_0 = h_y(y_0)$ and $p_T = M_x\left(x_T\right) + \varphi_x\left(x_T\right) q_T$, we have

$$\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T\right) x_T^1\right] + \mathbb{E}^u\left[\varphi_x\left(x_T\right) q_T x_T^1\right] \\
&= -\mathbb{E}^u \int_0^T f_x\left(t, x_t, y_t, z_t, Z_t, u_t\right) q_t x_t^1 dt \\
&\quad - \mathbb{E}^u \int_0^T l_x\left(t, x_t, y_t, z_t, Z_t, u_t\right) x_t^1 dt \\
&\quad - \mathbb{E}^u \int_0^T \xi_x\left(t, x_t, y_t, z_t, Z_t, u_t\right) \Xi_t x_t^1 dt \\
&\quad + \mathbb{E}^u \int_0^T b_u\left(t, x_t, u_t\right)\left(\upsilon_t - u_t\right) p_t dt \\
&\quad + \mathbb{E}^u \int_0^T g_u\left(t, x_t, u_t\right)\left(\upsilon_t - u_t\right) k_t dt \\
&\quad + \sum_{i=1}^{\infty} \mathbb{E}^u \int_0^T \sigma_u^{(i)}\left(t, x_t, u_t\right)\left(\upsilon_t - u_t\right) Q_t^{(i)} dt,
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
&- \mathbb{E}^u\left[\varphi_x\left(x_T\right) q_T x_T^1\right] + \mathbb{E}^u\left[h_y(y_0) y_0^1\right] \\
&= \mathbb{E}^u \int_0^T f_x\left(t, x_t, y_t, z_t, Z_t, u_t\right) q_t x_t^1 dt \\
&\quad + \mathbb{E}^u \int_0^T f_v\left(t, x_t, y_t, z_t, Z_t, u_t\right)\left(\upsilon_t - u_t\right) q_t dt \\
&\quad - \mathbb{E}^u \int_0^T \left(l_y(t)\, y_t^1 + l_z(t)\, z_t^1 + \sum_{i=1}^{\infty} l_{Z^{(i)}}(t)\, Z_t^{(i)1}\right) dt \\
&\quad - \mathbb{E}^u \int_0^T \Xi_t \left(\xi_y(t)\, y_t^1 + \xi_z(t)\, z_t^1 + \sum_{i=1}^{\infty} \xi_{Z^{(i)}}(t)\, Z_t^{(i)1}\right) dt
\end{aligned} \tag{32}$$

On the other hand, Itô's formula applied to (ϑ_t, P_t), gives us

$$\begin{aligned}
&\mathbb{E}^u\left(\vartheta_T M\left(x_T\right)\right) \\
&= -\mathbb{E}^u \int_0^T \vartheta_t l(t) dt + \mathbb{E}^u \int_0^T \Xi_t \left(\xi_x x_t^1 + \xi_y y_t^1 + \xi_z z_t^1 + \xi_Z Z_t^1 + \xi_v\left(\upsilon_t - u_t\right)\right) dt.
\end{aligned} \tag{33}$$

Consequently, From Eqs. 31, 32, and 33, we infer that

$$\mathbb{E}^u\left[M_x(x_T)x_T^1\right]+\mathbb{E}^u\left[h_y(y_0)y_0^1\right]+\mathbb{E}^u\left[\vartheta_T M(x_T)\right]$$
$$=\mathbb{E}^u\int_0^T(b_\upsilon(t,x_t,u_t)p_t(\upsilon_t-u_t)+g\upsilon(t,x_t,u_t)k_t(\upsilon_t-u_t)+l_\upsilon(t)(\upsilon_t-u_t)$$
$$+f_\upsilon(t)(\upsilon_t-u_t)q_t+\Xi_t\xi_\upsilon(\upsilon_t-u_t))+\sum_{i=1}^{\infty}\sigma_u^{(i)}(t,x_t,u_t)\,Q_t^{(i)}(\upsilon_t-u_t)\Big)dt$$
$$-\mathbb{E}^u\int_0^T\left(\vartheta_t l(t)+l_x(t)x_t^1+l_y(t)y_t^1+l_z(t)z_t^1+\sum_{i=1}^{\infty}l_{Z^{(i)}}(t)Z_t^{(i)1}+l_\upsilon(t)(\upsilon_t-u_t)\right)dt,$$

thus

$$\mathbb{E}^u\left[M_x(x_T)x_T^1\right]+\mathbb{E}^u\left[h_y(y_0)y_0^1\right]+\mathbb{E}^u\left[\vartheta_T M(x_T)\right]$$
$$=\mathbb{E}^u\int_0^T H_\upsilon(t,x_t,y_t,z_t,Z_t,u_t,p_t,q_t,k_t,Q_t)(\upsilon_t-u_t)\,dt$$
$$-\mathbb{E}^u\int_0^T\left(l_x(t)x_t^1+l_y(t)x_y^1+l_z(t)z_t^1+\sum_{i=1}^{\infty}l_{Z^{(i)}}(t)Z_t^{(i)1}+l_u(t)(\upsilon_t-u_t)\right)dt,$$

This together with the variational inequality (29) imply (30), which achieve the proof. □

4 Partial Information Sufficient Conditions of Optimality

In this section, we will prove that the partial information maximum principle condition for the Hamiltonian function is in fact sufficient under additional convexity assumptions. It should be noted that we shall prove our result in two different cases. In the first case, we are going to prove the sufficient condition without assuming the linearity of the terminal condition for the backward part of the state equation. To this end, we restrict ourselves to the one dimensional case $n=m=1$ and we state now the main result of this section.

Theorem 5.5 *Suppose* (A_1)*,* (A_2)*, and* (A_3) *hold. Assume further that the functions* φ*,* M *and* $H(t,.,.,.,.,p_t,q_t,k_t,Q_t,\Xi_t)$ *are convex, h is convex function and increasing. If the following maximum condition holds*

$$\mathbb{E}^u\left(H_\upsilon(t,x_t,y_t,z_t,Z_t,u_t,p_t,q_t,k_t,Q_t,\Xi_t),(\upsilon_t-u_t)\mid\mathcal{F}_t^Y\right)\geq 0,\tag{34}$$

$\forall\upsilon_t\in\mathcal{U}$*, a.e, a.s, then u is an optimal control in the sense that* $J(u)\leq\inf_{\upsilon\in\mathcal{U}}J(\upsilon)$*.*

Proof Let u be an arbitrary element of $\mathcal{U}$ (candidate to be optimal) and $(x^u,y^u,z^u,\ Z^u)$ is the corresponding trajectory. For any $\upsilon\in\mathcal{U}$ and its corresponding trajectory $(x^\upsilon,y^\upsilon,z^\upsilon,Z^\upsilon)$, by the definition of the cost function (5), one can write

$$J(\upsilon)-J(u)=\mathbb{E}\left[\Gamma_T^\upsilon M\left(x_T^\upsilon\right)-\Gamma_T^u M\left(x_T^u\right)\right]+\mathbb{E}\left[h\left(y_0^\upsilon\right)h\left(y_0^u\right)-h\left(y_0^u\right)\right]$$
$$+\mathbb{E}\int_0^T\left(\Gamma_t^\upsilon l\left(t,x_t^\upsilon,y_t^\upsilon,z_t^\upsilon,Z_t^\upsilon,\upsilon_t\right)-\Gamma_t^u l\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t\right)\right)dt.$$

Since h and M are convex

$$\mathbb{E}\left[h\left(y_0^{\upsilon}\right)-h\left(y_0^{u}\right)\right]\geq\mathbb{E}\left(h_y\left(y_0^{u}\right)\left(y_0^{\upsilon}-y_0^{u}\right)\right),$$
and
$$\begin{aligned}&\mathbb{E}\left(\Gamma_T^{\upsilon}M\left(x_T^{\upsilon}\right)-\Gamma_T^{u}M\left(x_T^{u}\right)\right)\geq\\&\mathbb{E}\left[\left(\Gamma_T^{\upsilon}-\Gamma_T^{u}\right)M\left(x_T^{u}\right)\right]+\mathbb{E}^u\left[M_x\left(x_T^{u}\right)\left(x_T^{\upsilon}-x_T^{u}\right)\right].\end{aligned}\tag{35}$$

And

$$\begin{aligned}&\mathbb{E}\int_0^T\left(\Gamma_t^{\upsilon}l\left(t,x_t^{\upsilon},y_t^{\upsilon},z_t^{\upsilon},Z_t^{\upsilon},\upsilon_t\right)-\Gamma_t^{u}l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)\right)dt\\&=\mathbb{E}\int_0^T\Gamma_t^{\upsilon}\left(l\left(t,x_t^{\upsilon},y_t^{\upsilon},z_t^{\upsilon},Z_t^{\upsilon},\upsilon_t\right)-l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)\right)dt\\&\quad+\mathbb{E}\int_0^T\left(\Gamma_t^{\upsilon}-\Gamma_t^{u}\right)l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)dt.\end{aligned}\tag{36}$$

Thus

$$\begin{aligned}&J(\upsilon)-J(u)\\&\geq\mathbb{E}^u\left[M_x\left(x_T^{u}\right)\left(x_T^{\upsilon}-x_T^{u}\right)\right]+\mathbb{E}\left[h_y\left(y_0^{u}\right)\left(y_0^{\upsilon}-y_0^{u}\right)\right]\\&\quad+\mathbb{E}^u\int_0^T\left(l\left(t,x_t^{\upsilon},y_t^{\upsilon},z_t^{\upsilon},Z_t^{\upsilon},\upsilon_t\right)-l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)\right)dt\\&\quad+\mathbb{E}\left[\left(\Gamma_T^{\upsilon}-\Gamma_T^{u}\right)\left(\int_0^T l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)dt+M\left(x_T^{u}\right)\right)\right].\end{aligned}$$

Noting that

$$\begin{aligned}&p_T=M_x\left(x_T\right)+\varphi_x^*\left(x_T\right)q_T,\\&q_0=h_y\left(y_0\right),\end{aligned}$$

then, we have

$$\begin{aligned}&J(\upsilon)-J(u)\geq\mathbb{E}^u\left[p_T^u\left(x_T^{\upsilon}-x_T^{u}\right)\right]-\mathbb{E}^u\left[q_T^u\varphi_x\left(x_T\right)\left(x_T^{\upsilon}-x_T^{u}\right)\right]+\mathbb{E}\left[q_0^u\left(y_0^{\upsilon}-y_0^{u}\right)\right]\\&+\mathbb{E}^u\int_0^T\left(l\left(t,x_t^{\upsilon},y_t^{\upsilon},z_t^{\upsilon},Z_t^{\upsilon},\upsilon_t\right)-l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)\right)dt\\&+\mathbb{E}\left[\left(\Gamma_T^{\upsilon}-\Gamma_T^{u}\right)\left(\int_0^T l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)dt+M\left(x_T^{u}\right)\right)\right],\end{aligned}$$

by using the fact that h is convex function and increasing, we can write

$$\begin{aligned}&J(\upsilon)-J(u)\geq\mathbb{E}^u\left[p_T^u\left(x_T^{\upsilon}-x_T^{u}\right)\right]-\mathbb{E}^u\left[q_T^u\left(y_T^{\upsilon}-y_T^{u}\right)\right]+\mathbb{E}\left[q_0^u\left(y_0^{\upsilon}-y_0^{u}\right)\right]\\&+\mathbb{E}^u\int_0^T\left(l\left(t,x_t^{\upsilon},y_t^{\upsilon},z_t^{\upsilon},Z_t^{\upsilon},\upsilon_t\right)-l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)\right)dt\\&+\mathbb{E}\left[\left(\Gamma_T^{\upsilon}-\Gamma_T^{u}\right)\left(\int_0^T l\left(t,x_t^{u},y_t^{u},z_t^{u},Z_t^{u},u_t\right)dt+M\left(x_T^{u}\right)\right)\right].\end{aligned}$$

On other hand, by applying Ito's formula respectively to $p_t^u\left(x_t^\upsilon - x_t^u\right)$, $q_t^u\left(y_t^\upsilon - y_t^u\right)$ and $P_t^u\left(\Gamma_t^\upsilon - \Gamma_t^u\right)$, and by taking expectations to the previous inequality, we get

$$\begin{aligned}
J(\upsilon)-J(u) \geq \mathbb{E}^u\int_0^T &\left(H\left(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, Z_t^\upsilon, \upsilon_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right.\\
&\left.-H\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right) dt\\
&-\mathbb{E}^u\int_0^T H_x\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(x^\upsilon - x^u\right) dt\\
&-\mathbb{E}^u\int_0^T H_y\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(y^\upsilon - y^u\right) dt\\
&-\mathbb{E}^u\int_0^T H_z\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(z^\upsilon - z^u\right) dt\\
&-\sum_{i=0}^{+\infty}\mathbb{E}^u\int_0^T H_{Z^{(i)}}\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(Z^{(i)\upsilon}-Z^{(i)u}\right) dt.
\end{aligned} \tag{37}$$

By using the fact H is convex in (x, y, z, Z, u), we get

$$\begin{aligned}
&\left(H\left(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, Z_t^\upsilon, \upsilon_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right.\\
&\left.-H\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right)\\
&\geq H_x\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(x_t^\upsilon - x_t^u\right)\\
&\quad+H_y\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(y_t^\upsilon - y_t^u\right)\\
&\quad+H_z\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(z_t^\upsilon - z_t^u\right)\\
&\quad+H_Z\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(Z_t^\upsilon - Z_t^u\right)\\
&\quad+H_\upsilon\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(\upsilon_t - u_t\right).
\end{aligned} \tag{38}$$

Substituting (38) into (37), we have

$$J(\upsilon) - J(u) \geq \mathbb{E}^u \int_0^T H_\upsilon\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(\upsilon_t - u_t\right) dt,$$

and thus

$$J(\upsilon)-J(u) \geq \mathbb{E}\int_0^T \Gamma_t^u \mathbb{E}\left[H_\upsilon\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k, Q_t^u, \Xi_t^u\right)\left(u_t-\upsilon_t\right) \mid \mathcal{F}_t^Y\right] dt,$$

in view of the condition (34) above and keeping in mind that $\Gamma_t^\upsilon > 0$, one can get $J(u) - J(\upsilon) \leq 0$, which achieve the proof. □

Before we treat the second result of this section, it is worth to pointing out that we can prove a partial observed sufficient conditions of optimality without assuming neither that x and y need to be in the dimension one, nor that the function φ needs to be negative and decreasing.

Assume that $\varphi(x) = Nx$, where N is a nonzero constant matrix with order $m \times n$. Then, by using similar arguments developed above, we can easily state and prove the following theorem which illustrate the second case.

Theorem 5.6 *Assume that* (A_1), (A_2) *and* (A_3) *are in force. Assume that the functions* $h(.)$, $M(.)$ *and* $H(t,.,.,.,.,p_t,q_t,k_t,Q_t,\Xi_t)$ *are convex with* $\varphi(x)=Nx$. *If further the maximum condition* (34) *holds true, then* u *is an optimal control in the sense that*

$$J(u)\leq\inf_{\upsilon\in\mathcal{U}}J(\upsilon). \tag{39}$$

Proof Let υ be an arbitrary element of $\mathcal{U}$ and $(x^\upsilon,y^\upsilon,z^\upsilon,Z^\upsilon)$ is its corresponding trajectory. By using the definition of cost functional (5), taking under consideration the convexity property of h and M, a simple computation gives us

$$\begin{aligned}J(\upsilon)-J(u)\geq\;&\mathbb{E}^u\left[M_x\left(x_T^u\right)\left(x_T^\upsilon-x_T^u\right)\right]+\mathbb{E}\left[h_y\left(y_0^u\right)\left(y_0^\upsilon-y_0^u\right)\right]\\&+\mathbb{E}^u\int_0^T\left(l\left(t,x_t^\upsilon,y_t^\upsilon,z_t^\upsilon,Z_t^\upsilon,\upsilon_t\right)-l\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t\right)\right)dt\\&+\mathbb{E}\left[\left(\Gamma_T^\upsilon-\Gamma_T^u\right)\left(\int_0^T l\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t\right)dt+M\left(x_T^u\right)\right)\right].\end{aligned}$$

On the other hand, in view of $\varphi(x)=Nx$, we apply Ito's formula to $p_t^u\left(x_t^\upsilon-x_t^u\right)$, $q_t^u\left(y_t^\upsilon-y_t^u\right)$ and $P_t^u\left(\Gamma_t^\upsilon-\Gamma_t^u\right)$, respectively, then by combining their results together with the above inequality one can get

$$\begin{aligned}J(\upsilon)-J(u)\geq\;&\mathbb{E}^u\int_0^T\left(H\left(t,x_t^\upsilon,y_t^\upsilon,z_t^\upsilon,Z_t^\upsilon,\upsilon_t,p_t^u,q_t^u,k_t^u,Q_t^u,\Xi_t^u\right)\right.\\&\left.-H\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t,p_t^u,q_t^u,k_t^u,Q_t^u,\Xi_t^u\right)\right)dt\\&-\mathbb{E}^u\int_0^T H_x\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t,p_t^u,q_t^u,k_t^u,Q_t^u,\Xi_t^u\right)\left(x^\upsilon-x^u\right)dt\\&-\mathbb{E}^u\int_0^T H_y\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t,p_t^u,q_t^u,k_t^u,Q_t^u,\Xi_t^u\right)\left(y^\upsilon-y^u\right)dt\\&-\mathbb{E}^u\int_0^T H_z\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t,p_t^u,q_t^u,k_t^u,Q_t^u,\Xi_t^u\right)\left(z^\upsilon-z^u\right)dt\\&-\sum_{i=0}^{+\infty}\mathbb{E}^u\int_0^T H_{Z^{(i)}}\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t,p_t^u,q_t^u,k_t^u,Q_t^u,\Xi_t^u\right)\left(Z^{(i)\upsilon}-Z^{(i)u}\right)dt.\end{aligned}$$

Since H is convex with respect to (x,y,z,Z,u) for almost all $(t,w)\in[0,T]\times\Omega$,

$$J(u)-J(\upsilon)\leq-\mathbb{E}^u\int_0^T H_\upsilon\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t,p_t^u,q_t^u,k_t^u,Q_t^u,\Xi_t^u\right)\left(\upsilon_t-u_t\right)dt,.$$

It turns out, using the condition (34) taking into account the fact that $\Gamma_t^\upsilon>0$,

$$J(u)-J(\upsilon)\leq 0$$

This means that u is an optimal partially observed control process and (x^u,y^u,z^u,Z^u) is an optimal 4-type. The proof is complete. □

5 Application

In this section, we consider a partial observed linear quadratic control problem as a particular case of our control problem A. We find an explicit expression of the corresponding optimal control by applying the necessary and sufficient conditions of optimality. Consider the following control problem,

Minimize the expected quadratic cost function

$$\begin{aligned} J(u) := \; & \mathbb{E}^{\upsilon}\left[M_1(x_T, x_T) + M_2(y_0, y_0)\right] \\ & + \mathbb{E}^{\upsilon}\int_0^T \left[K_t(x_t, x_t) + L_t(y_t, y_t) + \digamma_t(z_t, z_t)\right. \\ & \left. + \sum_{i=1}^{\infty} G_t\left(Z_t^{(i)}, Z_t^{(i)}\right) + R_t(u_t, u_t)\right] dt, \end{aligned} \tag{40}$$

subject to

$$\begin{cases} dx_t &= \left[\left(A_t^1, x_t\right) + \left(A_t^2, u_t\right)\right] dt + \left[\left(A_t^3, x_t\right) + \left(A_t^4, u_t\right)\right] dW_t \\ & \quad + \sum_{i=1}^{\infty}\left[\left(A_t^{5,(i)}, x_t\right) + \left(A_t^{6,(i)}, u_t\right)\right] dH_t^{(i)}, \\ dy_t &= -\left[\left(B_t^1, x_t\right) + \left(B_t^2, y_t\right) + \left(B_t^3, z_t\right) + \sum_{i=1}^{\infty}\left(B_t^{4,(i)}, Z_t^{(i)}\right) + \left(B_t^5, u_t\right)\right] dt \\ & \quad + z_t dW_t + \sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, \\ x_0 &= 0, \; y_T = \zeta, \end{cases} \tag{41}$$

where the observation state is given by the following SDE,

$$dY_t = \Lambda_t dt + d\tilde{W}_t \; , Y_0 = 0. \tag{42}$$

Define $dP^{\upsilon} = \Gamma^{\upsilon} dP$ and we denote by Γ^{υ} the unique $\mathcal{F}_t^Y$ adapted solution of

$$d\Gamma_t^{\upsilon} = \Gamma_t^{\upsilon}\left(D(t), dY_t\right) \; , \; \Gamma_0^{\upsilon} = 1. \tag{43}$$

Here, $K(.) > 0$, $L(.) > 0$, $\digamma(.) > 0$, $G(.) > 0$, $R(.) > 0$, $M_1 \geq 0$, $M_2 \geq 0$, $A^i(.)$, $B^j(.)$ and $D(.)$ are bounded and deterministic, for $i = 1, ..., 6$, and $j = 1, ..., 5$.

To overcome this problem, we first write down the Hamiltonian function

$$\begin{aligned} & H(t, x, y, z, Z, u, p, q, k, Q, \Xi) \\ & := \left(p_t, \left(A_t^1, x_t\right) + \left(A_t^2, u_t\right)\right) + \left(k_t, \left(A_t^3, x_t\right) + \left(A_t^4, u_t\right)\right) + \Lambda_t \Xi_t \\ & \quad + \left(q_t, \left(B_t^1, x_t\right) + \left(B_t^2, y_t\right) + \left(B_t^3, z_t\right) + \sum_{i=1}^{\infty}\left(B_t^{4,(i)}, Z_t^{(i)}\right) + \left(B_t^5, u_t\right)\right) \\ & \quad + \sum_{i=1}^{\infty}\left(Q_t^{(i)}, \left(A_t^{5,(i)}, x_t\right) + \left(A_t^{6,(i)}, u_t\right)\right) + \left[K_t(x_t, x_t) + L_t(y_t, y_t)\right. \\ & \quad \left. + \digamma_t(z_t, z_t) + \sum_{i=1}^{\infty} G_t\left(Z_t^{(i)}, Z_t^{(i)}\right) + R_t(u_t, u_t)\right], \end{aligned} \tag{44}$$

and the adjoint equations associated to the system (41)–(43) are given by

$$\begin{cases} -dp_t &= \left[\left(p, A_t^1\right) + \left(k_t, A_t^3\right) + \left(q, B_t^1\right) + \sum_{i=1}^{\infty} \left(Q_t^{(i)}, A_t^{5,(i)} \right) \right. \\ & \left. + 2x_t K_t \right] dt - k_t dW_t - \sum_{i=1}^{\infty} Q_t^{(i)} dH_t^{(i)}, \\ p_T &= 2M_1 x_T, \\ dq_t &= \left[\left(q_t, B_t^2\right) + 2L_t y_t \right] dt + \left[\left(q_t, B_t^3\right) + 2F_t z_t \right] dW_t \\ & + \sum_{i=1}^{\infty} \left[\left(q_t, B_t^{4,(i)} \right) + 2G_t Z_t^{(i)} \right] dH_t^{(i)}, \\ q_0 &= 2M_2 y_0. \end{cases} \tag{45}$$

and

$$\begin{cases} -dP_t = \left(K_t\left(x_t, x_t\right) + L_t\left(y_t, y_t\right) + F_t\left(z_t, z_t\right) \right. \\ \left. + \sum_{i=1}^{\infty} G_t \left(Z_t^{(i)}, Z_t^{(i)} \right) + R_t\left(u_t, u_t\right) \right) dt - \Xi_t d\tilde{W}_t \\ P_T = M_1\left(x_T, x_T\right). \end{cases} \tag{46}$$

According to Theorem (4.4), if $\hat{u}$ is a partial observed optimal control, then it satisfies

$$\begin{aligned} 2\hat{u}_t = R_t^{-1} & \left(-A_t^2 \mathbb{E}\left[\hat{p}_t \mid \mathcal{F}_t^Y \right] - B_t^5 \mathbb{E}\left[\hat{q}_t \mid \mathcal{F}_t^Y \right] \right. \\ & \left. - A_t^4 \mathbb{E}\left[\hat{k}_t \mid \mathcal{F}_t^Y \right] - \sum_{i=1}^{\infty} A_t^{6,(i)} \mathbb{E}\left[\hat{Q}_t^{(i)} \mid \mathcal{F}_t^Y \right] \right). \end{aligned} \tag{47}$$

• Conversely, for the sufficient part, let $\hat{u} \in \mathcal{U}$ be a candidate to be optimal control and let $\left(\hat{x}, \hat{y}, \hat{z}, \hat{Z}\right)$ be the solution to the FBSDE (41) corresponding to $\hat{u}$ and (p, k, Q, q), (P, Ξ) are the solution to the corresponding solution to Eqs. 45 and 46. It is straight forward to check that the functional H is convex in (x, y, z, Z, u). Thus, If $\hat{u}$ satisfies (47) and the partially observed maximum principle condition (30) above. Then by applying Theorem (5.6), one can easily check that $\hat{u}$ is an optimal control of our partially observed control problem.

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