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Mild Solution and Constrained Local Controllability of Semilinear Boundary Control Systems

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Abstract The existence of mild solution and the constrained local controllability of a retarded boundary control system with nonlocal delay condition have been established. The theory of extrapolation spaces is applied to derive the mild solution. Then, the constrained local controllability is established using the generalized open mapping theorem. In the last section, application of the result is shown through examples of control systems represented by hyperbolic partial differential equations.

Keywords Boundary control systems · Constrained controllability · Delay differential systems

Mathematics Subject Classification (2010) 93B05 · 35F30 · 34B10

1 Introduction

During last four decades, engineering mathematics has made a vast global impact on the controllability of abstract semilinear control systems in infinite dimensional spaces, for example, see articles [1–3] and references therein. The boundary control problem arises in a number of physical, chemical, and biological phenomena, some examples can be seen in [4–7]. For these reasons, study of abstract boundary control problems is one of the most exciting areas in applied mathematics. Lasiecka [8] first established the semigroup approach for abstract parabolic boundary problems and then developed theory for boundary control problems jointly with Triggiani [9, 10]. Numerical treatment for the exact and approximate controllability of infinite dimensional control problems is explicated by Glowinski et al. [11]. In 2006, Boulite et al. [12] have proved the existence of solution and the exact

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controllability of the following semilinear nonautonomous boundary control problem with nonlocal initial condition:

$$\begin{aligned} \dot{x}(t) &= A_{\max}(t)x(t), \quad 0 \le t \le T \\ L(t)x(t) &= f(t, x(t)) + B(t)u(t), \quad 0 \le t \le T \text{ on } \partial X \\ & x(0) + g(t_1, t_2, ..., t_p, x) = x_0 \qquad \qquad ((\mathrm{BP})_{\mathrm{f,g}}) \end{aligned}$$

where $0 \le t_1 \le t_2 \le ..... \le t_p \le T$, $f : [0, T] \times X \to \partial X$ and $g : [0, T]^p \times C([0, T]; X) \to X$.

Engel et al. [13] have developed semigroup approach for classical solutions to abstract linear time invariant Cauchy problem ($(\mathrm{BP})_{0,0}$). They have also given the maximal reachability space for ($(\mathrm{BP})_{0,0}$). Kumpf and Nickel [14] have presented the application of the theory of "one-sided coupled" operator matrices, developed by Engel [15], through example of one-dimensional heat equation with dynamic boundary conditions and boundary control.

Nonlocal conditions represent more practical aspect for the physical measurements as compared to the classical initial conditions. The importance of nonlocal conditions was introduced first time by Byszewski [16] and subsequently developed by many authors (few recent references are [17–20]).

In this paper, we consider the following nonautonomous semilinear boundary control system with nonlocal delay condition:

$$\begin{cases} \dot{x}(t) = A_{\max}(t)x(t), \; 0 \le t \le T \\ L(t)x(t) = B(t)u(t) + f(t, x(t), x(b(t)), u(t)), \; 0 \le t \le T \\ h(x) = \phi \text{ on } [-\tau, 0] \end{cases} \tag{1}$$

where τ represents the delay factor, $x(t) \in X$ is called the state of the system, and X is the state space. The control variable $u(t) \in U$, where U is the control space. The control operators $B(t) : U \to \partial X$, for all $t \in [0, T]$, are bounded linear operators such that $B(\cdot) \in L^{\infty}([0, T]; \mathscr{L}(U; \partial X))$ and $f : [0, T] \times X \times X \times U \to \partial X$ is nonlinear map, where ∂X is the boundary space. The nonlocal delay condition is given by h and ϕ. A few expressions for h are given in the last section for application of the theory.

Throughout this paper X, ∂X, and U represent the Banach spaces with norms $||\cdot||$, $||\cdot||_{\partial X}$, and $||\cdot||_U$, respectively. The set $\mathscr{L}(X; U)$ stands for the Banach space of bounded linear operators from X into U. Other details and standing assumptions on the system operators are given in the next section.

2 Existence of Mild Solution

Let $\mathscr{C}_t = C([-\tau, t]; X)$, $\tau > 0$, $0 \le t \le T < \infty$ be a Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the norm $||\phi||_{\mathscr{C}_t} = \sup_{-\tau \le \eta \le t} ||\phi(\eta)||_X$. This section explains the existence and uniqueness of mild solution of the following system:

$$\begin{cases} \dot{x}(t) = A_{\max}(t)x(t), \; 0 \le t \le T \\ L(t)x(t) = f(t, x(t), x(b(t))), \; 0 \le t \le T \\ h(x) = \phi \text{ on } [-\tau, 0]. \end{cases} \tag{2}$$

under the following assumptions:

(A1) D is a dense linear subspace of X. Moreover, D is equipped with the norm $|\cdot|$ which is finer (stronger) than $||\cdot||_X$ such that $(D, |\cdot|)$ is complete.
(A2) For all $t \in [0, T]$, $A_{\max}(t) : (D, |\cdot|) \to X$ and $L(t) : (D, |\cdot|) \to \partial X$ are bounded linear operators and $L(t)$ is surjective.
(A3) The family of operators, $A(t) := A_{\max}(t)|_{\ker L(t)}$, $t \in [0, T]$, generates an exponentially bounded evolution family $(S(t, s))_{(s,t)\in\Delta T}$, $\Delta T := \{(a, b) : 0 \le a \le b \le T\}$.
(A4) $h : \mathscr{C}_0 \to \mathscr{C}_0$ and there exists $\chi \in \mathscr{C}_0$ such that $h(\chi) = \phi$.
(A5) The map $b : [0, T] \to [-\tau, T]$ is nondecreasing and non-expansive such that it satisfies the delay property, i.e., $b(t) \le t$ for $t \in [0, T]$.
(A6) The nonlinear map $f : [0, T] \times X \times X \to \partial X$ is uniformly continuous in t and locally Lipschitz in X, i.e., for every $t_0 \in [0, T]$ and constant $R \ge 0$, there is a constant $N_{(R,t_0)}$ such that: $||f(t, x(t), x(b(t))) - f(t, y(t), y(b(t)))|| \le N_{(R,t_0)} [||x(t) - y(t)|| + ||x(b(t)) - y(b(t))||]$ for all $(x(t), y(t)), (x(b(t)), y(b(t))) \in B_R(X^2; (\chi(0), \chi(0)))$, where

$$B_R(X^2; (\chi(0), \chi(0))) := \{(x_1(t), x_2(t)) \in X^2 : \sum_{i=1}^{2} ||x_i(t)-\chi(0)||_X \le R, t \in [-\tau, t_0]\}$$

is a ball in $X \times X$.

To prove the main abstract results, we take the setting of Griener [21] and others [12, 13] who have used the semigroup approach to study the well-posedness of systems when operator $A_{\max}(t)$ is the perturbation of operator $A(t)$ which is a generator of the evolution family and the perturbation is due to the change in its domain.

Under assumptions $(A1) - (A3)$, the proof of the following properties can be seen in [21]. These properties play key role for the well-posedness of the system (2).

Lemma 2.1 [21] *For each $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, $t \in [0, T]$, the following assertions hold:*

(i) $D = D(A(t)) \oplus \ker(\lambda I - A_{\max}(t))$,
(ii) $L(t)|_{\ker(\lambda I - A_{\max}(t))}$ *is an isomorphism from* $\ker(\lambda I - A_{\max}(t))$ *onto* ∂X *and its inverse* $L_{\lambda,t} := [L(t)|_{\ker(\lambda - A_{\max}(t))}]^{-1} : \partial X \to \ker(\lambda I - A_{\max}(t))$ *is bounded,*
(iii) $Q_\lambda(t) := L_{\lambda,t}L(t)$ *is a projection in D onto* $\ker(\lambda I - A_{\max}(t))$ *along* $D(A(t))$.

From Lemma 2.1, $\text{ran}(Q_\lambda(t)) = \ker(\lambda I - A_{\max}(t))$ and $\ker(Q_\lambda(t)) = \text{ran}(I - Q_\lambda(t)) = D(A(t))$, and for each $x(t) \in D(A_{\max}(t))$,

$$x(t) = (I - Q_\lambda(t))x(t) + Q_\lambda(t)x(t).$$

Therefore

$$\begin{aligned}
\dot{x}(t) &= A_{\max}(t)x(t) = A_{\max}(t)((I - Q_\lambda(t))x(t) + Q_\lambda(t)x(t)) \\
&= A_{\max}(t)(I - L_{\lambda,t}L(t))x(t) + A_{\max}(t)L_{\lambda,t}L(t)x(t) \\
&= A_{\max}(t)(x(t) - L_{\lambda,t}L(t)x(t)) + \lambda L_{\lambda,t} f(t, x(t), x(b(t))) \\
&= A(t)(x(t) - L_{\lambda,t}L(t)x(t)) + \lambda L_{\lambda,t} f(t, x(t), x(b(t))).
\end{aligned}$$

These observations lead us towards the following result, the detailed proof can be written as the proof of Proposition 2.6 in [13].

Lemma 2.2 [13] *Let assumptions* $(A1) - (A3)$ *be satisfied and let* $x_0 \in X$, $\lambda \in \rho(A(t))$, $t \in [0, T]$. *Then, system* (2) *is equivalent to the following system:*

$$\begin{cases} \dot{x}(t) &= A(t)(x(t) - L_{\lambda,t} f(t, x(t), x(b(t)))) + \lambda L_{\lambda,t} f(t, x(t), x(b(t))), \\ h(x) &= \phi \text{ on } [-\tau, 0], \end{cases} \tag{3}$$

i.e., $x(\cdot)$ *is the solution of system* (2) *iff it is the solution of system* (3).

In (3), $(x(t) - L_{\lambda,t} f(t, x(t), x(b(t)))) \in D(A(t))$, $L_{\lambda,t} f(t, x(t), x(b(t))) \in \ker(\lambda I - A_{\max}(t))$, and $A(t)(x(t) - L_{\lambda,t} f(t, x(t), x(b(t)))) \in X$. Using the theory of extrapolation spaces [22], (3) can be written into standard Cauchy problem in a bigger state space X_{-1} as follows:

$$\begin{cases} \dot{x}(t) &= A_{-1}(t)x(t) + (\lambda I - A_{-1}(t))L_{\lambda,t} f(t, x(t), x(b(t))), \\ h(x) &= \phi \text{ on } [-\tau, 0], \end{cases} \tag{4}$$

where $A_{-1}(t) : D(A_{-1}(t)) = X \to X_{-1}$ is the continuous extension of $A(t)$. The bigger state space X_{-1} is the extrapolated space of X which satisfies the following:

(1) X_{-1} is a Banach space containing X as a dense subspace.
(2) $A_{-1}(t)$ generates an extrapolated evolution family $S_{-1}(t, s)$ on X_{-1}.

Applying the variation of parameters, the solution of (4) is given by

$$x(t) = \begin{cases} \chi(t), \ t \in [-\tau, 0], \\ S_{-1}(t, 0)\chi(0) + \int_0^t S_{-1}(t, s)(\lambda I - A_{-1}(t))L_{\lambda,s} f(s, x(s), x(b(s)))ds. \end{cases} \tag{5}$$

Since $S(t, s) = S_{-1}(t, s)|_X$, using $\lim_{\lambda\to+\infty} \lambda R(\lambda, A_{-1})x = x$, we obtain

$$\begin{aligned} \lim_{\lambda\to+\infty} \quad & \int_0^t S(t, s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)))ds \\ &= \int_0^t S_{-1}(t, s)(\lambda I - A_{-1}(t))L_{\lambda,s} f(s, x(s), x(b(s)))ds. \end{aligned}$$

Thus Eq. 5 can be written as follows:

$$x(t) = \begin{cases} \chi(t), \ t \in [-\tau, 0], \\ S(t, 0)\chi(0) + \lim_{\lambda\to+\infty} \int_0^t S(t, s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)))ds. \end{cases} \tag{6}$$

Definition 2.1 Let $f \in L^1([0, T] \times X \times X)$. A function $x \in C([-\tau, T]; X)$ is said to be a mild solution of the system (3) if it satisfies Eq. 5.

Now, we prove the main theorem of this section for the existence of mild solution. The theme of the proof is based on [17, 23].

Theorem 2.3 *Let assumptions* $(A1) - (A6)$ *be satisfied and* $||\lambda L_{\lambda,t}|| \leq \gamma$ *as* $\lambda \to \infty$, *then the boundary value problem* (2) *has a mild solution* $x(\cdot)$ *on* $[-\tau, T]$ *for some* $0 < T < \infty$.

Moreover, the mapping $\phi \mapsto x$ from $\mathscr{C}_0$ into $\mathscr{C}_T$ is Lipschitz and induces the uniqueness of the mild solution.

Proof We show that the system (2) has a mild solution on the interval $[0, T]$, where $0 < T < \infty$. Following the assumption $(A4)$, let us define $\bar{\chi} \in \mathscr{C}_T$ by

$$\bar{\chi}(t) = \begin{cases} \chi(t), & t \in [-\tau, 0] \\ \chi(0), & t \in [0, T]. \end{cases} \tag{7}$$

From $(A3)$, let $M \geq 1$ and $\omega \geq 0$ be such that $||S(t,s)||_X \leq Me^{\omega(t-s)}$. Let $R > 0$ be fixed such that $\sup_{t\in[0,T]} ||(S(t,0) - I)\chi(0)||_X \leq R/2$, and the following hold:

$$\gamma M e^{\omega T} T[N_{(R',T)}(2R + 2||\bar{\chi}||_T) + N_f] \leq R/2$$

and

$$\gamma T N_{(R',T)} M e^{\omega T} < 1/2,$$

where $N_f = \sup\{||f(t, \chi(0), \chi(b(0)))||_{\partial X} : t \in [0, T]\}$ and $R' = 2R + 2||\bar{\chi}||_T$. Define a map $F : \mathscr{C}_T \to \mathscr{C}_T$ by

$$(Fx)(t) = \begin{cases} \chi(t), \text{ when } t \in [-\tau, 0], \\ S(t,0)\chi(0) + \lim_{\lambda\to\infty} \int_0^t S(t,s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)))ds, \, t \in [0, T]. \end{cases} \tag{8}$$

From assumption $(A4)$, we know that there exists $\chi \in \mathscr{C}_0$ such that

$$(F\chi)(t) = \chi(t) \text{ on } [-\tau, 0].$$

Therefore, it is sufficient to show that F has a fixed point on $[0, T]$. In the following, we show that F maps $B_R(\mathscr{C}_T; \bar{\chi})$ into itself.

$$\begin{aligned}
||(Fx)(t) - \bar{\chi}(t)||_X &= ||S(t,0)\chi(0) + \lim_{\lambda\to\infty} \int_0^t S(t,s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)))ds - \bar{\chi}(t)||_X \\
&= ||S(t,0)\chi(0) - \chi(0) + \lim_{\lambda\to\infty} \int_0^t S(t,s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)))ds||_X \\
&\leq ||(S(t,0) - I)\chi(0)|| + ||\lim_{\lambda\to\infty} \int_0^t S(t,s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)))ds||_X \\
&\leq R/2 + \gamma M \int_0^t ||e^{\omega(t-s)} f(s, x(s), x(b(s)))||_{\partial X} ds \\
&\leq R/2 + \gamma M e^{\omega T} \int_0^t (||f(s, x(s), x(b(s))) - f(s, \chi(0), \chi(b(0))) \\
&\quad + f(s, \chi(0), \chi(b(0)))||_{\partial X}) ds \\
&\leq R/2 + \gamma M e^{\omega T} [\int_0^t ||f(s, x(s), x(b(s))) - f(s, \chi(0), \chi(b(0)))||_{\partial X} ds \\
&\quad + \int_0^t ||f(s, \chi(0), \chi(b(0)))||_{\partial X} ds] \\
&\leq R/2 + \gamma M e^{\omega T} [\int_0^t ||f(s, x(s), x(b(s))) - f(s, \chi(0), \chi(b(0)))||_{\partial X} ds + T N_f] \\
&\leq R/2 + \gamma M e^{\omega T} T[N_{(R',T)}(2R + 2||\bar{\chi}||_T) + N_f] \\
&\leq R/2 + R/2 = R.
\end{aligned}$$

Thus, $Fx \in B_R(\mathscr{C}_T; \bar{\chi})$ for $t \in [0, T]$ implying F maps the ball $B_R(\mathscr{C}_T; \bar{\chi})$ into itself. Further, F satisfies local Lipschitz condition in this ball with constant $N = N_{(R',T)}$. Suppose $x, y \in B_R(\mathscr{C}_T; \bar{\chi})$, then

$$\begin{aligned} ||(Fx)(t) - (Fy)(t)||_X &= ||\lim_{\lambda\to\infty}\int_0^t S(t,s)\lambda L_{\lambda,s}(f(s,x(s),x(b(s))) - f(s,y(s),y(b(s))))ds||_X \\ &\leq \gamma M e^{\omega T}\int_0^t ||f(s,x(s),x(b(s))) - f(s,y(s),y(b(s)))||_{\partial X} ds \\ &\leq \gamma M e^{\omega T}\int_0^t N_{(R',T)}(||x(s)-y(s)||_X + ||x(b(s)) - y(b(s))||_X)ds \\ &\leq \gamma M e^{\omega T}\int_0^t N_{(R',T)}(||x-y||_T + ||x-y||_T)ds \\ &\leq 2\gamma M e^{\omega T} T N_{(R',T)}||x-y||_T \end{aligned}$$

Since $2\gamma M e^{\omega T} T N_{(R',T)} < 1$, by Banach contraction principle, F has a unique fixed point in $B_R(\mathscr{C}_T; \bar{\chi})$ for $t \in [0, T]$. This fixed point is the required mild solution of (2) on $[0, T]$.

From the above discussion, we conclude that a mild solution $x(\cdot)$ of (2) on the interval $[0, T]$ can be extended to the interval $[0, 2T]$ by defining $x(t) = y(t)$ on $[T, 2T]$, which is as follows: if $0 < \tau < T$, then

$$y(t) = S(t,T)x(T) + \lim_{\lambda\to\infty}\int_T^t S(t,s)f(s,y(s),y(b(s)))ds,$$

and if $0 < T < \tau$, then $T - \tau < 0$ and hence

$$y(t) = \begin{cases} \eta(t), \text{ when } t \in [T-\tau, 0], \\ S(t,0)\eta(0) + \lim_{\lambda\to\infty}\int_0^t S(t,s)\lambda L_{\lambda,s} f(s,x(s),x(b(s)))ds, t \in [0,T], \\ S(t,T)x(T) + \lim_{\lambda\to\infty}\int_T^t S(t,s)\lambda L_{\lambda,s} f(s,y(s),y(b(s)))ds, t \in [T,2T]. \end{cases}$$

Here, $\eta(t)$ is the restriction of $\chi(t)$ on the subinterval $[T-\tau, 0]$ and satisfies assumption $(A4)$. In this way, the interval of solution can be extended up to $T_{\max} < \infty$ and in such case $\lim_{t\uparrow T_{\max}} ||x(t)|| = \infty$ for which the proof replicates the explanation of Theorem 6.1.4 in [23].

Let $x_1(\cdot), x_2(\cdot) \in \mathscr{C}_T$ be two mild solutions corresponding to the nonlocal delay functions $\phi_1, \phi_2 \in \mathscr{C}_0$, respectively. From assumption $(A4)$, we get $\chi_1, \chi_2 \in \mathscr{C}_0$ such that $h(\chi_i) = \phi_i$ for $i = 1, 2$. From the above discussion, we have

$$x_i(t) = \begin{cases} \chi_i(t), \text{ when } t \in [-\tau, 0], \\ S(t,0)\chi_i(0) + \lim_{\lambda\to\infty}\int_0^t S(t,s)\lambda L_{\lambda,s} f(s,x_i(s),x_i(b(s)))ds, t \in [0,T]. \end{cases} \tag{9}$$

Therefore, it is sufficient to prove that the mapping $\chi \mapsto x$ is Lipschitz which renders the uniqueness of mild solution.

For $t \in [-\tau, 0]$, clearly, with the constant 1, the mapping is Lipschitz. Now, we consider the case $t \in [0, T]$.

$$\begin{aligned} ||x_1(t) - x_2(t)|| &\leq ||S(t,0)(\chi_1(0) - \chi_2(0))|| \\ &\quad + \gamma M e^{\omega T}\int_0^t ||f(s,x_1(s),x_1(b(s))) - f(s,x_2(s),x_2(b(s)))||_{\partial X} ds \\ &\leq M e^{\omega T}||\chi_1(0) - \chi_2(0)|| \\ &\quad + \gamma M e^{\omega T} N_{(R,T)}\int_0^t (||x_1(s) - x_2(s)|| + ||x_1(b(s)) - x_2(b(s))||)ds. \end{aligned} \tag{10}$$

Let $t_1 \in [0, t]$ be such that $b(s) \le 0$ for $s \in [0, t_1]$. Then $x_i(b(s)) = \chi_i(b(s))$; $i = 1, 2$; and hence from (10), we get

$$
\begin{aligned}
||x_1(t) - x_2(t)|| \le\ & Me^{\omega T}||\chi_1(0) - \chi_2(0)|| \\
& +\gamma Me^{\omega T}N_{(R,T)}\left(\int_0^t ||x_1(s) - x_2(s)||ds \right.\\
& \left. + \int_0^{t_1} ||\chi_1(b(s)) - \chi_2(b(s))||ds + \int_{t_1}^t ||x_1(b(s)) - x_2(b(s))||ds\right) \\
\le\ & Me^{\omega T}||\chi_1 - \chi_2||_0 + \gamma Me^{\omega T}N_{(R,T)}\int_0^{t_1} ||\chi_1 - \chi_2||_0 ds \\
& +2\gamma Me^{\omega T}N_{(R,T)}\int_0^t ||x_1(s) - x_2(s)||ds \\
\le\ & Me^{\omega T}\left(1 + \gamma TN_{(R,T)}\right)||\chi_1 - \chi_2||_0 \\
& +2\gamma Me^{\omega T}TN_{(R,T)}\int_0^t ||x_1(s) - x_2(s)||ds
\end{aligned}
$$

which by Gronwall's inequality implies that

$$||x_1(t) - x_2(t)|| \le M\left(1 + \gamma TN_{(R,T)}\right)e^{(\omega T + 2\gamma Me^{\omega T}TN_{(R,T)})}||\chi_1 - \chi_2||_0$$

which yields the Lipschitz continuity of the map $\chi \mapsto x(\cdot)$ and the uniqueness of the mild solution. □

3 Controllability Result

In this section, we shall discuss the constrained controllability of semilinear boundary control system. We provide sufficient conditions for the constrained exact local controllability on $[0, T]$ assuming that control functions take values in a closed convex cone with vertex at zero. There are several research papers by Klamka [24, 25] on the constrained controllability of abstract control systems with initial condition. Azzouzi et al. [26] have investigated the constrained approximate controllability of boundary control systems for controls in a convex cone. We have extended the work by Chukwu and Lenhart [27] and Klamka [24] to boundary control systems with nonlocal delay condition.

Let $U_0 \subset U$ be a closed convex cone with nonempty interior and take the set of admissible controls for the semilinear boundary control systems:

$$U_{ad} = L^\infty([0, T], U_0) \subset V = L^\infty([0, T], U).$$

Definition 3.1 The constrained attainable set at time $T > 0$, denoted by $K_T(U_0)$, is defined as:

$$K_T(U_0) = \{x \in X : x = x(T, \chi(0), u),\ u(t) \in U_0 \text{ a. e. on } [0, T]\},$$

where $x(t, \chi(0), u)$ is a solution of system (1).

The extrapolated semilinear boundary control system is

$$
\begin{aligned}
\dot{x}(t) &= A_{-1}(t)x(t) + (\lambda I - A_{-1}(t))L_{\lambda,t}B(t)u(t) \\
&\quad +(\lambda I - A_{-1}(t))L_{\lambda,t}f(t, x(t), x(b(t)), u(t)) \\
\text{with} \quad x(0) &= \chi(0).
\end{aligned} \tag{11}
$$

Following the discussion in Section 2, the mild solution of (11) can be written as

$$x(t, \chi(0), u) = S(t,0)\chi(0) + \lim_{\lambda\to\infty}\int_0^t S(t,s)\lambda L_{\lambda,s}B(s)u(s)ds + \lim_{\lambda\to\infty}\int_0^t S(t,s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)), u(s))ds. \quad (12)$$

Let us consider the associated linear boundary control system:

$$\dot{y}(t) = A_{-1}(t)y(t) + (\lambda I - A_{-1}(t))L_{\lambda,t}B(t)v(t)$$
$$\text{with initial condition } \; y(0) = 0 \quad (13)$$

which possesses the mild solution

$$y(t, 0, v) = y(t) = \lim_{\lambda\to\infty}\int_0^t S(t,s)\lambda L_{\lambda,s}B(s)v(s)ds.$$

For $\lambda \in \rho(A(t))$, we define the following operators:

(B) $\mathcal{B}_\lambda : U \to C([0, T]; X)$ by

$$\mathcal{B}_\lambda u = \int_0^{\cdot} S(\cdot, s)\lambda L_{\lambda,s}B(s)u(s)ds.$$

(F) $\mathcal{F}_\lambda : X \times U = Z \to C([0, T]; X)$ by

$$\mathcal{F}_\lambda(x, u) = \int_0^{\cdot} S(\cdot, s)\lambda L_{\lambda,s} f(s, x(s), x(b(s)), u(s))ds.$$

Here, $Z = X \times U$ is a product space which is a Banach space under the norm

$$||(x, u)||_Z = ||x||_X + ||u||_U.$$

We impose the following hypotheses:

(F1) the nonlinear map f satisfies the local Lipschitz condition in $X \times X \times U$:

$$||f(t, x_1(t), x_1(b(t)), u_1(t)) - f(t, x_2(t), x_2(b(t)), u_2(t))||$$
$$\leq N_{(2R,t_0)}(||x_1(t) - x_2(t)|| + ||x_1(b(t)) - x_2(b(t))|| + ||u_1(t) - u_2(t)||),$$

(F2) the nonlinear map f is Frechet differentiable in the argument spaces X and U, and satisfies $f(t, x(t), x(b(t)), u(t))|_{u=0} = 0$,

$$D_x f(t, x(t), x(b(t)), u(t))|_{u=0} = 0 \;\text{ and }\; D_u f(t, x(t), x(b(t)), u)|_{u=0} = 0,$$

(F3) $\mathcal{B}_\lambda$ and $\mathcal{F}_\lambda$ are continuously differentiable on U and their derivatives $\{D_u\mathcal{B}_\lambda\}$ and $\{D_u\mathcal{F}_\lambda\}$ converge uniformly on U and Z, respectively, as $\lambda \to \infty$.

Definition 3.2 The system (1) is said to be U_0-exactly locally controllable on $[0, T]$ if the attainable set $K_T(U_0)$ contains a neighborhood of $x(0) = \chi(0) \in X$ in the space X.

Definition 3.3 The system (1) is said to be U_0-exactly globally controllable on $[0, T]$ if $K_T(U_0) = X$.

Definition 3.4 The corresponding linear system (13) is U_0-exactly globally controllable on $[0, T]$ if $ran(\mathcal{B}_\lambda(U_0)(T)) = X$.

Before stating the main result, let us recall the generalized open mapping theorem.

Lemma 3.1 [28] *Let X, Y be Banach spaces and $F : B_r(x_0) \subset X \to Y$ such that*

$$||Fx - F\bar{x} - T(x - \bar{x})|| \leq k||x - \bar{x}|| \quad on \quad B_r(x_0) \times B_r(x_0)$$

for some $k > 0$ and $T \in \mathscr{L}(X; Y)$ with $ran(T) = Y$. Then, $B_\rho(Fx_0) \subset FB_r(x_0)$ for some $\rho > 0$ provided that k is sufficiently small.

Now, we shall proceed to the main result.

Theorem 3.2 *Let us suppose that $(A1)-(A5)$ and $(F1)-(F3)$ hold and the linear boundary control system (13) is U_0-exactly globally controllable on $[0, T]$. Then, the semilinear boundary control system (1) is U_0-exactly locally controllable on $[0, T]$.*

Proof Define the control to state operator $\mathcal{G} : U_{ad} \to X$ by $\mathcal{G}(u) = x(T, \chi(0), u)$. Then, from the integral equation (12), we have

$$\mathcal{G}(u) = S(T, 0)\chi(0) + \lim_{\lambda\to\infty} \mathcal{B}_\lambda(u)(T) + \lim_{\lambda\to\infty} \mathcal{F}_\lambda(x, u)(T). \tag{14}$$

Using the assumption $(F3)$ and differentiating (14), we get

$$D_u\mathcal{G}(u) = \lim_{\lambda\to\infty} D_u(\mathcal{B}_\lambda(u)(T)) + \lim_{\lambda\to\infty} D_u(\mathcal{F}_\lambda(x, u)(T)), \tag{15}$$

where

$$\begin{aligned}
&D_u(\mathcal{B}_\lambda(u)(T)) = \int_0^T S(T, s)\lambda L_{\lambda,s}B(s)ds, \quad \text{and}\\
&D_u(\mathcal{F}_\lambda(x, u)(T))\\
&= \int_0^T S(T, s)\lambda L_{\lambda,s}D_u f(s, x(s, \chi(0), u(s)), x(b(s), \chi(0), u(s)), u(s))ds\\
&\quad + \int_0^T S(T, s)\lambda L_{\lambda,s}D_{x(t)} f(s, x(s, \chi(0), u(s)), x(b(s), \chi(0), u(s)), u(s))\\
&\qquad\qquad D_u x(s, \chi(0), u(s))ds\\
&\quad + \int_0^T S(T, s)\lambda L_{\lambda,s}D_{x(b(t))} f(s, x(s, \chi(0), u(s)), x(b(s), \chi(0), u(s)), u(s))\\
&\qquad\qquad D_u x(b(s), \chi(0), u(s))ds.
\end{aligned}$$

Let $t_1 \in [0, T]$ be such that $b(s) \leq 0$ for $s \in [0, t_1]$. Then,

$$\begin{aligned}
&D_u(\mathcal{F}_\lambda(x, u)(T))\\
&= \int_0^T S(T, s)\lambda L_{\lambda,s}D_u f(s, x(s, \chi(0), u(s)), x(b(s), \chi(0), u(s)), u(s))ds\\
&\quad + \int_0^{t_1} S(T, s)\lambda L_{\lambda,s}D_x f(s, x(s, \chi(0), u(s)), \chi(b(s)), u(s))\\
&\qquad\qquad D_u x(s, \chi(0), u(s))ds\\
&\quad + \int_{t_1}^T S(T, s)\lambda L_{\lambda,s}D_{x(t)} f(s, x(s, \chi(0), u(s)), x(b(s), \chi(0), u(s)), u(s))\\
&\qquad\qquad D_u x(s, \chi(0), u(s))ds\\
&\quad + \int_{t_1}^T S(T, s)\lambda L_{\lambda,s}D_{x(b(t))} f(s, x(s, \chi(0), u(s)), x(b(s), \chi(0), u(s)), u(s))\\
&\qquad\qquad D_u x(b(s), \chi(0), u(s))ds.
\end{aligned}$$

Now, using assumption $(F2)$ in Eq. 15, we get

$$D_u\mathcal{G}(u)|_{u=0}v = \lim_{\lambda\to\infty}\int_0^T S(T,s)\lambda L_{\lambda,s}B(s)v(s)ds = y(T,0,v).$$

Since the linear system (13) is U_0-exactly globally controllable, therefore $D_u\mathcal{G}(u)|_{u=0}$, mapping $v \mapsto y(T,0,v)$ is a surjective map with $D_u\mathcal{G}(0)(U_{ad}) = X$. Further, let u_1 and $u_2 \in U_{ad}$ for which the corresponding trajectories are $x_1(t) = x(t,\chi(0),u_1)$ and $x_2(t) = x(t,\chi(0),u_2)$, respectively. Then,

$$\begin{aligned}
||x_1(t)-x_2(t)|| \leq\ & ||\lim_{\lambda\to\infty}\int_0^t S(T,s)\lambda L_{\lambda,s}B(s)(u_1(s)-u_2(s))ds|| \\
& +||\lim_{\lambda\to\infty}\int_0^t S(T,s)\lambda L_{\lambda,s}\left[f(s,x_1(s),x_1(b(s)),u_1(s))\right. \\
& \left. - f(s,x_2(s),x_2(b(s)),u_2(s))\right]ds|| \\
\leq\ & \gamma Me^{\omega T}||B(\cdot)||_\infty\int_0^t ||u_1(s)-u_2(s)||ds \\
& +\gamma Me^{\omega T}N_{(2R,T)}\int_0^t (||x_1(s)-x_2(s)|| \\
& + ||x_1(b(s))-x_2(b(s))|| + ||u_1(s)-u_2(s)||)\,ds \\
\leq\ & \gamma Me^{\omega T}\left(||B(\cdot)||_\infty + N_{(2R,T)}\right)\int_0^t ||u_1(s)-u_2(s)||ds \\
& +2\gamma Me^{\omega T}N_{(2R,T)}\int_0^t ||x_1(s)-x_2(s)||ds \\
\leq\ & \gamma MTe^{\omega T}\left(||B(\cdot)||_\infty + N_{(2R,T)}\right)||u_1-u_2||_\infty \\
& +2\gamma Me^{\omega T}N_{(2R,T)}\int_0^t ||x_1(s)-x_2(s)||ds \ \ \forall\, t\in[0,T]
\end{aligned}$$

which by Gronwall's inequality, implies that

$$||x_1(t)-x_2(t)|| \leq \gamma MT(||B(\cdot)||_\infty + N_{(2R,T)})e^{(2\gamma Me^{\omega T}N_{(2R,T)}+\omega T)}||u_1-u_2||_\infty$$

for all $t\in[0,T]$. Therefore,

$$\begin{aligned}
||\mathcal{G}(u_1)-\mathcal{G}(u_2)|| \leq\ & ||x_1(T)-x_2(T)|| \\
\leq\ & \gamma MT(||B(\cdot)||_\infty + N_{(2R,T)})e^{(2\gamma Me^{\omega T}N_{(2R,T)}+\omega T)}||u_1-u_2||_\infty
\end{aligned}$$

and

$$\begin{aligned}
||D_u\mathcal{G}(0)(u_1-u_2)|| =\ & ||\lim_{\lambda\to\infty}\int_0^T S(T,s)\lambda L_{\lambda,s}B(s)(u_1(s)-u_2(s))ds|| \\
\leq\ & \gamma Me^{\omega T}||B(\cdot)||_\infty\int_0^T ||(u_1(s)-u_2(s))||ds \\
\leq\ & \gamma MTe^{\omega T}||B(\cdot)||_\infty||u_1-u_2||_\infty.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&||\mathcal{G}(u_1) - \mathcal{G}(u_2) - D_u\mathcal{G}(0)(u_1 - u_2)|| \leq ||\mathcal{G}(u_1) - \mathcal{G}(u_2)|| + ||D_u\mathcal{G}(0)(u_1 - u_2)|| \\
\leq\ & \gamma MT(||B(\cdot)||_\infty + N_{(2R,T)})e^{(2\gamma Me^{\omega T}N_{(2R,T)}+\omega T)}||u_1 - u_2||_\infty \\
&+\gamma MTe^{\omega T}||B(\cdot)||_\infty||u_1 - u_2||_\infty \\
\leq\ & \gamma MTe^{\omega T}\left[(||B(\cdot)||_\infty + N_{(2R,T)})e^{(2\gamma Me^{\omega T}N_{(2R,T)})} + ||B(\cdot)||_\infty\right]||u_1 - u_2||_\infty.
\end{aligned}
$$

By Lemma 3.1, the operator $\mathcal{G}$ transforms a neighborhood of zero in the space U_{ad} onto a neighborhood of $\mathcal{G}(0) = S(T,0)\chi(0)$ in the space X. Hence, by Definition 3.2, the semilinear system (1) is U_0-exactly locally controllable. □

4 Application

Example 1 Let us consider the semilinear wave equation of a semi-infinite string ($0 \leq x < \infty$) with boundary control action:

$$
\begin{cases}
\dfrac{\partial^2 y}{\partial t^2}(t,x) = \alpha(t,x)\dfrac{\partial^2 y}{\partial x^2}(t,x) + \left(1 + \dfrac{\partial \alpha}{\partial x}(t,x)\right)\dfrac{\partial y}{\partial x}(t,x), \quad x \geq 0,\ t \in [0,T] \\
\dfrac{\partial y}{\partial t}(t,0) = \xi(t,0)u(t), \quad t \in [0,T] \\
\alpha(t,0)\dfrac{\partial y}{\partial x}(t,0) + y(t,0) = \beta(t,0)u(t) \\
\qquad\qquad + \displaystyle\int_0^\infty \mu(t)\left(y(t,x) - y(t-\tau,x)\right)u^2(t)dx \\
y(\theta,x) = \phi_1(\theta,x), \quad \dfrac{\partial y}{\partial t}(\theta,x) = \phi_2(\theta,x), \quad \theta \in [-\tau,0],\ x \geq 0,
\end{cases}
\tag{16}
$$

where α, β, ξ, andμ are coefficient functions satisfying the following conditions:

(a) $\alpha \in C^1([0,T]; W^{1,\infty}[0,\infty)),\ 0 < \alpha_1 \leq \alpha(t,x) < \infty,$
(b) $\beta, \xi \in L^\infty([0,T] \times \{0\}):\ 0 < \nu_1 \leq \beta(t,0) < \infty$ and $0 < \nu_2 \leq \xi(t,0) < \infty,$
(c) $\mu \in L^\infty([0,T]),\ 0 < \mu_1 \leq \mu(t) \leq \mu_2 < \infty.$

The boundary control wave (16) can be reformulated as abstract boundary control problem with nonlocal delay condition. In this connection, let us consider the following:

(i) The state space $X = L^1[0,\infty)$, $\partial X = \mathbb{R} = U$ and $D(\mathcal{A}_{\max}(t)) = W^{2,1}[0,\infty)$.
(ii) The linear operator

$$\mathcal{A}_{\max}(t)\psi = \alpha(t,\cdot)\frac{\partial^2}{\partial x^2}\psi + \left(1 + \frac{\partial}{\partial x}\alpha(t,\cdot)\right)\frac{\partial}{\partial x}\psi.$$

(iii) The boundary operator

$$\mathcal{L}(t)\psi = \alpha(t,0)\frac{\partial \psi}{\partial x}(0) + \psi(0)$$

with domain

$$D(\mathcal{L}(t)) = D(\mathcal{A}_{\max}(t))$$

and

$$\ker(\mathcal{L}(t)) = \{\psi \in X : \frac{\partial \psi}{\partial x}(0) = 0 \text{ and } \psi(0) = 0\}.$$

(iv) The control evaluation operator

$$E(t)\psi = \frac{\partial \psi}{\partial t}(t,0), \quad \text{with} \quad \ker(E(t)) = \{\psi \in X : \frac{\partial \psi}{\partial t}(t,0) = 0\}.$$

(v) The control operators

$$B_1(t)u(t) = \xi(t,0)u(t), \quad B_2(t)u(t) = \beta(t,0)u(t).$$

(vi) $b(t) = t - \tau$ for $t \in [0, T]$ satisfies assumption $(A6)$.

(vii) The nonlinear map

$$\begin{aligned}\mathcal{F}(t, \psi(t), \psi(b(t)), u(t)) &= \mathcal{F}(t, \psi(t, \cdot), \psi(b(t), \cdot), u(t)) \\ &= \int_0^1 \mu(t)\left(\psi(t,x) - \psi(t-\tau, x)\right) u^2(t)dx.\end{aligned}$$

The construction of operators to transform into the first order abstract Cauchy system proceeds further. Let us define a Banach space $Z = W^{1,1}[0,\infty) \otimes L^1[0,\infty)$ equipped with the norm $\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_Z = ||\psi_1||_{W^{1,1}[0,\infty)} + ||\psi_2||_{L^1[0,\infty)}$. Now,

(vi) the system operator

$$A_{\max}(t)z(t) = \begin{pmatrix} 0 & I \\ \mathcal{A}_{\max}(t) & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \dot{y}(t) \\ \frac{\partial}{\partial x}\left(\alpha(t,\cdot)\frac{\partial y}{\partial x} + y(t,\cdot)\right) \end{pmatrix}$$

with domain

$$D = D(A_{\max}(t)) = W^{2,1}[0,\infty) \otimes W^{1,1}[0,\infty),$$

(vii) the boundary operator

$$L(t)z(t) = \begin{pmatrix} 0 & E(t) \\ \mathcal{L}(t) & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \dot{y}(t,0) \\ \alpha(t,0)\frac{\partial y}{\partial x}(t,0) + y(t,0) \end{pmatrix}$$

with domain

$$D(L(t)) = D \;\text{ and }\; \ker(L(t)) = \ker(\mathcal{L}(t)) \otimes \ker(E(t)),$$

(viii) the boundary control operator

$$B(t)u(t) = \begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix} u(t) = \begin{pmatrix} \xi(t,0) \\ \beta(t,0) \end{pmatrix} u(t),$$

(ix) the nonlinear map

$$f(t, \psi(t), \psi(b(t)), u(t)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathcal{F}(t, \psi(t,\cdot), \psi(b(t),\cdot), u(t)),$$

(x) and the nonlocal delay condition is represented by

$$\hbar(z) = \phi, \;\; i.e. \; \begin{pmatrix} y(\theta,\cdot) \\ \dot{y}(\theta,\cdot) \end{pmatrix} = \begin{pmatrix} \phi_1(\theta,\cdot) \\ \phi_2(\theta,\cdot) \end{pmatrix}, \;\; \theta \in [-\tau, 0].$$

We know that $A_{\max}(t)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\psi_2\\ \frac{\partial}{\partial x}\left(\alpha(t,x)\frac{\partial\psi_1}{\partial x}+\psi_1\right)\end{pmatrix}\in Z$. Then, to verify that D is densely imbedded in Z of assumption $(A1)$, we obtain the following inequality:

$$\begin{aligned}
||\psi||_D &= ||\psi_1||_{W^{2,1}[0,\infty)}+||\psi_2||_{W^{1,1}[0,\infty)}\\
&= \int_0^\infty (|\psi_1(x)|+|\psi_1'(x)|+|\psi_1''(x)|)dx+||\psi_2||_{W^{1,1}[0,\infty)}\\
&= ||\psi_1||_{L^1[0,\infty)}+||\psi_1'||_{L^1[0,\infty)}+||\psi_2||_{W^{1,1}[0,\infty)}+\int_0^\infty |\psi_1''(x)|dx\\
&\le ||\psi_1||_{L^1[0,\infty)}+||\psi_1'||_{L^1[0,\infty)}+||\psi_2||_{W^{1,1}[0,\infty)}\\
&\quad+\int_0^\infty \left|\frac{1+\frac{\partial\alpha}{\partial x}(t,x)}{\alpha(t,x)}\psi_1'(x)\right|dx+\int_0^\infty\left|\psi_1''(x)+\frac{1+\frac{\partial\alpha}{\partial x}(t,x)}{\alpha(t,x)}\psi_1'(x)\right|dx\\
&\le ||\psi_1||_{L^1[0,\infty)}+||\psi_1'||_{L^1[0,\infty)}+\sup_{t\in[0,T]}\frac{1+||\alpha'(t,\cdot)||_\infty}{\alpha_1}||\psi_1'||_{L^1[0,\infty)}\\
&\quad+||\psi_2||_{W^{1,1}[0,\infty)}+\frac{1}{\alpha_1}\int_0^\infty\left|\alpha(t,x)\psi_1''(x)+\left(1+\frac{\partial\alpha}{\partial x}(t,x)\right)\psi_1'(x)\right|dx\\
&\le ||\psi_1||_{L^1[0,\infty)}+\max\left\{1,\ \sup_{t\in[0,T]}\frac{1+||\alpha'(t,\cdot)||_\infty}{\alpha_1}\right\}||\psi_1'||_{L^1[0,\infty)}\\
&\quad+||\psi_2||_{L^1[0,\infty)}+||\psi_2||_{W^{1,1}[0,\infty)}+\frac{1}{\alpha_1}\int_0^\infty|\mathcal{A}_{\max}(t)\psi_1(x)|dx\\
&\le \max\left\{1,\ \sup_{t\in[0,T]}\frac{1+||\alpha'(t,\cdot)||_\infty}{\alpha_1}\right\}\left(||\psi_1||_{W^{1,1}[0,\infty)}+||\psi_2||_{L^1[0,\infty)}\right)\\
&\quad+\max\left\{1,\frac{1}{\alpha_1}\right\}\left(||\psi_2||_{W^{1,1}[0,\infty)}+||\mathcal{A}_{\max}(t)\psi_1||_{L^1[0,\infty)}\right)\\
&\le \max\left\{1,\frac{1}{\alpha_1},\ \sup_{t\in[0,T]}\frac{1+||\alpha'(t,\cdot)||_\infty}{\alpha_1}\right\}(||\psi||_Z+||A_{\max}(t)\psi||_Z).
\end{aligned}$$

This implies that $||\cdot||_D$ is weaker than the graph norm of the system operator $A_{\max}(t)$. Hence, the domain D equipped with the norm $||\cdot||_D$ is densely imbedded in the Banach space Z. Let us write $A_{\max}(t)=\mathscr{A}_1(t)+\mathscr{A}_2(t)+\mathscr{A}_3$, where

$$\mathscr{A}_1(t)=\begin{pmatrix}0&0\\A_1(t)&0\end{pmatrix},\ \mathscr{A}_2(t)=\begin{pmatrix}0&0\\A_2(t)&0\end{pmatrix},\ \mathscr{A}_3=\begin{pmatrix}0&I\\0&0\end{pmatrix}$$

and

$$A_1(t)\psi=\alpha(t,x)\frac{\partial^2\psi}{\partial x^2},\quad A_2(t)\psi=\left(1+\frac{\partial\alpha}{\partial x}(t,x)\right)\frac{\partial\psi}{\partial x}.$$

Let us consider the operator $A_1\psi=\frac{\partial^2\psi}{\partial x^2}$ with domain

$$\begin{aligned}
D(A_1) &= \left\{\psi\in X:\psi,\frac{\partial\psi}{\partial x}\text{ are absolutely continuous and }\frac{\partial^2\psi}{\partial x^2}\in X\right\}\cap\ker(\mathcal{L}(t))\\
&= D(A_1(t))\cap\ker(\mathcal{L}(t)).
\end{aligned}$$

Then, from the semigroup theory, A_1 is the infinitesimal generator of a C_0- semigroup on X. Similarly, $A_2 : D(A_2) \subset X \to X$ with domain

$$D(A_2) = \{\psi \in X : \psi \text{ is absolutely continuous}, \frac{\partial \psi}{\partial x} \in X\} \cap \ker(\mathcal{L}(t))$$

is the infinitesimal generator of a C_0-semigroup on X. From (a), we conclude that the associated operators $A_1(t)$ and $A_2(t)$ are infinitesimal generators of C_0-semigroups on X for all $t \in [0, T]$. Hence, the matrix operators $\mathscr{A}_1(t)$ and $\mathscr{A}_2(t)$ for all $t \in [0, T]$ are infinitesimal generators of C_0-semigroups on Z. Further, $\mathscr{A}_3$ is bounded linear operator, therefore by Bounded Perturbation Theorem, it implies that $A(t) = A_{\max}(t)|_{\ker L(t)}$ generates a C_0-semigroup on Z for every $t \in [0, T]$. The null space $\ker(L(t)),\ t \in [0, T]$ of $L(t)$ is defined as follows

$$\begin{aligned}\ker L(t) &= \left\{\begin{pmatrix} y \\ \dot{y} \end{pmatrix} \in Z : \frac{\partial y}{\partial x}(t,0) = 0 = y(t,0) \text{ and } \frac{\partial y}{\partial t}(t,0) = 0\ \forall\, t \in [0,T]\right\} \\ &= \ker(\mathcal{L}(t)) \otimes \ker(E(t)).\end{aligned}$$

Further discussion will employ the relative boundedness of two operators. We say that an operator A_2 is A_1-bounded if there exist constants $a > 0$ and $b > 0$ such that

$$||A_2 y|| \le a||A_1 y|| + b||y|| \ \ \forall\ \ y \in D(A_1).$$

For $\psi_1 \in \ker(\lambda I - \mathcal{A}_{\max}(t))$, we have

$$\begin{aligned}\left\| L(t)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right\|_{\mathbb{R}\otimes\mathbb{R}} &= \left\|\begin{pmatrix} \psi_2(0) \\ \alpha(t,0)\frac{\partial \psi_1}{\partial x}(0) + \psi_1(0) \end{pmatrix}\right\|_{\mathbb{R}\otimes\mathbb{R}} \\ &= |\psi_2(0)| + \left|\alpha(t,0)\frac{\partial \psi_1}{\partial x}(0) + \psi_1(0)\right| \\ &= \int_0^\infty \left|\frac{\partial}{\partial x}\,(\psi_2(x))\right| dx \\ &\quad + \int_0^\infty \left|\frac{\partial}{\partial x}\left(\alpha(t,x)\frac{\partial \psi_1}{\partial x}(x) + \psi_1(x)\right)\right| dx \\ &= ||\psi_2'||_{L^1[0,\infty)} + ||\mathcal{A}_{\max}(t)\psi_1||_{L^1[0,\infty)} \\ &\ge \eta||\psi_2||_{L^1[0,\infty)} + ||\mathcal{A}_{\max}(t)\psi_1||_{L^1[0,\infty)},\ \eta > 0 \\ &\ge \min\{\eta, 1\}\left\|\begin{pmatrix} \psi_2 \\ \mathcal{A}_{\max}(t)\psi_1 \end{pmatrix}\right\|_Z \\ &= \min\{\eta, 1\}\left\|\lambda\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right\|_Z \\ &= \min\{\eta, 1\}\lambda||\psi||_Z.\end{aligned}$$

This implies that the conditions of Lemma 2.1 are satisfied. Therefore, the evolution family $S(t,s)_{(s,t)\in\Delta}$ associated to the boundary control wave Eq. (16) exists and is exponentially stable $||S(t,s)|| \le Me^{\omega t}$. The nonlinear function $\mathcal{F}$ satisfies assumptions $(F1) - (F2)$ and correspondingly the matrix operator map f. The set of admissible controls is $U_{ad} = \{u \in L^\infty([0,T];\mathbb{R}) : u(t) \ge 0\}$, i.e., U_0 is the nonnegative real interval $[0, \infty)$. The U_0-exact global controllability of the linear control system associated to the system (16) is well explained in the pioneer work of Klamka [24] and Son [29]. Hence, from Theorem 3.2, the semilinear wave Eq. (16) is U_0-exactly locally controllable.

Example 2 Consider the following boundary control system:

$$\frac{\partial p}{\partial t}(t,x)+\frac{\partial}{\partial x}(\alpha(t,x)p(t,x)) = -\mu(t,x)p(t,x),\ x\geq 0,\ t\in[0,T], \tag{17a}$$

$$\alpha(t,0)p(t,0) = c(t)u(t)+\int_0^\infty \beta(t,x)\,(p(t,x) + p(t-\tau,x))\,u^2(t)dx, \tag{17b}$$

$$\sum_{i=1}^{n} k_i p(t_i,x) = p_0(x),\ x\geq 0,\ t_i\in[-\tau,0], \tag{17c}$$

where $p(t,x)$ is the density of a population of size x at time t; α is the growth rate depending upon the size x and time t; μ is the aging function; β is the birth function. The control function $u(t)$ represents the inflow of zero-size individuals from an external source.

The coefficients satisfy the following conditions:

(a) $\alpha\in C([0,T];W^{1,\infty}[0,\infty)):\ 0<\alpha_1\leq\alpha(t,x)\leq\alpha_2<\infty$,
(b) $\beta\in L^\infty([0,T]\times\mathbb{R}^+):\ 0<\beta_1\leq\beta(t,x)\leq\beta_2<\infty$,
(c) $\mu\in L^\infty([0,T]\times\mathbb{R}^+)$: $0<\mu_1\leq\mu(t,x)\leq\mu_2<\infty$.
(d) $c\in L^\infty([0,T];\mathbb{R}):\ c(t)>0$.

For $-\tau\leq t_1<\ldots<t_n\leq 0$, the Eq. (17c) represents the nonlocal delay condition and p_0 is the initial population of individual of size x.

The population model (17) resembles the form of abstract boundary control system if we consider the following:

(i) The state space $X=L^1([0,\infty))$ and $D=D(A_{\max}(t))=W^{1,1}([0,\infty))$, where the operator $A_{\max}(t),\ t\in[0,T]$, is defined as

$$A_{\max}(t)\psi = -\alpha(t,\cdot)\frac{\partial}{\partial x}\psi-\frac{\partial}{\partial x}\alpha(t,\cdot)\psi-\mu(t,\cdot)\psi.$$

(ii) The boundary $\partial X=\mathbb{R}$ and the boundary operator $L(t),\ t\in[0,T]$, is given by

$$L(t)\psi=\alpha(t,0)\psi(0).$$

(iii) The control space $U=\mathbb{R}$ and the control function $u\in L^\infty([0,T];U)$ and the admissible control set is $U_{ad}=\{u\in L^\infty([0,T];U):u(t)\geq 0\}$, i.e. $U_0=\mathbb{R}^+=[0,\infty)$. The boundary control operator $B(t),\ t\in[0,T]$, is defined as $B(t)u(t)=c(t)u(t)$ for all $t\in[0,T]$.
(iv) The delay function $b(t)=t-\tau, t\in[0,T]$, which satisfies assumption $(A6)$.
(v) The nonlinear map

$$f(t,\psi(t),\psi(b(t)),u(t))=\int_0^\infty \beta(t,x)\,(\psi(t,x)+\psi(b(t),x))\,u^2(t)dx.$$

(vi) The nonlocal condition is $h(p)=\sum_{i=1}^n k_i p(t_i,\cdot)=p_0=\phi$.

The verification of assumptions $(A1)-(A6)$ are well explained in the work of Boulite et al. [12]. The nonlinear map f satisfies assumptions $(F1)-(F2)$. The assumption $(F3)$ follows from [30]. Further, since $c(t)>0$ and $u(t)\in U_0$ for $t\in[0,T]$, therefore the polar cone $(B(t)U_0)^o=\{0\}$. The condition in Theorem 2.1 of Son [29] and Corollary 4.1 of Klamka [24] are required for the U_0-exact global controllability of the linear system corresponding to (17) is satisfied. Hence, by the theorem on controllability of boundary control system, the population model (17) is constrained controllable.

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