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Switching in Time-Optimal Problem: the 3D Case with 2D Control

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Abstract We study local structure of time-optimal controls and trajectories for a 3D control-affine system with a 2D control parameter with values in the disk. In particular, we give sufficient conditions, in terms of Lie bracket relations, for optimal controls to be smooth or to have only isolated jump discontinuities.

Keywords Optimal control · Lie brackets · Switching

Mathematics Subject Classifications (2010) 49K15

1 Introduction

This paper is a one more step towards the understanding of the structure of time-optimal controls and trajectories for control affine systems of the following form:

$$\dot{q} = f_0(q) + \sum_{i=1}^{k} u_i f_i(q), \quad q \in M, \ u \in U \tag{1}$$

where M is an n-dimensional manifold, $U = \{(u_1, \ldots, u_k) : \sum_{i=1}^{k} u_i^2 \leq 1\}$ is a k-dimensional ball, and $f_0, f_1, \ldots, f_k$ are smooth vector fields. We also assume that $f_1(q), \ldots, f_k(q)$ are linearly independent in the domain under consideration.

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The case $k = n$ is the Zermelo navigation problem: optimal controls are smooth in this case (see Remark 2.27). In more general situations, discontinuous controls are unavoidable and, in principle, any measurable function can be an optimal control (see [8]). Therefore, it is reasonable to focus on generic ensembles of vector fields $f_0, f_1, \dots, f_k$.

If $k = 1$, $n = 2$, then, for a generic pair of vector fields f_0, f_1, any optimal control is piecewise smooth; moreover, any point in M has a neighbourhood such that all optimal trajectories contained in the neighbourhood are concatenations of at most 2 smooth pieces (i.e. they have at most one switching in the control-theoretic terminology), see [4] and [12]. The complexity of optimal controls grows fast with n. For $k = 1$, $n = 3$, the generic situation is only partially studied (see [7, 13] and [3, 10, 11]): we know that any point out of a 1D Whitney-stratified subset of "bad points" has a small neighbourhood that contains only optimal trajectories with at most three switchings. We still do not know if there is any bound on the number of switchings in the points of the "bad" 1D subset. We know however that the chattering phenomenon (a Pontryagin extremal with convergent sequences of switching points) is unavoidable for $k = 1$ and sufficiently big n, see [6] and [14].

In this paper, we study the case $k = 2$, $n = 3$. In particular, for a generic triple (f_0, f_1, f_2), we obtain that any point out of a discrete subset of bad points in M has a neighbourhood such that any optimal trajectory contained in the neighbourhood has at most one switching.

Actually, we have much more precise results about the structure of optimal controls formulated in Theorems 3.1 and 3.5. In particular, we compute the right-hand and the left-hand limits of the control in the switching point in terms of the Lie bracket relations. Moreover, we expect that the techniques developed here are efficient also in the case $k = n-1$ with an arbitrary n and that, in general, complexity of the switchings depends much more on $n - k$ than on n.

2 Preliminaries

In this section, we recall some basic definitions in geometric control theory. For a more detailed introduction, see [2].

Definition 2.1 Given an n-dimensional manifold M, we call $\mathrm{Vec}(M)$ the *set of the smooth vector fields* on M, i.e. each $f \in \mathrm{Vec}(M)$ is a map of class $\mathcal{C}^2$ with respect to $q \in M$

$$f : M \longrightarrow TM$$

such that if $q \in M$, then $f(q) \in T_q M$.

Definition 2.2 A *smooth dynamical system* on M is defined by an ordinary differential equation

$$\dot{q} = f(q) \tag{2.1}$$

where $q \in M$, $f \in \mathrm{Vec}(M)$.

A *solution* of Eq. 2.1 is a map

$$q : I \longrightarrow M$$

with $I \subseteq \mathbb{R}$ interval, such

$$\frac{d}{dt} q(t) = f(q(t))$$

for every $t \in I$.

Theorem 2.3 *Given an n-dimensional manifold M and* Eq. 2.1 *a smooth dynamical system on M, for each initial point $q_0 \in M$ there exists a unique solution $q(t, q_0)$ on M, defined in an interval $I \subseteq \mathbb{R}$ small enough, such that $q(0, q_0) = q_0$.*

Definition 2.4 $f \in \mathrm{Vec}(M)$ is a *complete vector field* if for each $q_0 \in M$, the maximal solution $q(t, q_0)$ of Eq. 2.1 is defined for every $t \in \mathbb{R}$.

Remark 2.5 $f \in \mathrm{Vec}(M)$ with a compact support is a complete vector field.

Remark 2.6 Since we are interested in the local behaviour of trajectories, during all this work, we consider directly complete vector fields.

Definition 2.7 Given a manifold M and a set $U \subseteq \mathbb{R}^m$, a *control system* is a family of dynamical systems

$$\dot{q} = f_u(q)$$

where $q \in M$ and $\{f_u\}_{u \in U} \subseteq \mathrm{Vec}(M)$ are a family of vector fields on M parametrized by $u \in U$.

U is called *space of control parameters*.

We are interested in time-dependent controls.

Definition 2.8 An *admissible control* is a measurable, essentially bounded map

$$\begin{aligned} u : (t_1, t_2) &\longrightarrow U \\ t &\longmapsto u(t), \end{aligned}$$

from a time interval (t_1, t_2) to U.

We call $\mathcal{U}$ the *set of admissible controls*.

Therefore, we consider the following control system in M

$$\dot{q} = f_u(q) \tag{2.2}$$

where $q \in M$ and $\{f_u\}_{u \in \mathcal{U}} \subseteq \mathrm{Vec}(M)$, with u admissible control.

With the following theorem, we want to show that for any admissible control and for any admissible control and for every initial point, existence and local uniqueness of the solution of the associated control system are guaranteed.

Theorem 2.9 *Fixed an admissible control $u \in \mathcal{U}$,* Eq. 2.2 *is a non-autonomous ordinary differential equation, where the right-hand side is smooth with respect to q, and measurable essentially bounded with respect to t, then, for each $q_0 \in M$, there exists a local unique solution $q_u(t, q_0)$, depending on $u \in \mathcal{U}$, such that $q_u(0, q_0) = q_0$ and it is Lipschitz continuous with respect to t.*

Definition 2.10 We denote $q_u(t, q_0)$ the *admissible trajectory* solution of Eq. 2.2, chosen $u \in \mathcal{U}$, and

$$A_{q_0} = \{q_u(t, q_0) : t \geq 0, u \in \mathcal{U}\}$$

the *attainable set* from q_0.

Moreover, we will write $q_u(t) = q_u(t, q_0)$ if we do not need to stress that the initial position is q_0.

In this paper, we are going to study an affine control system.

Definition 2.11 An *affine control system* is a control system with the following form:

$$\dot{q} = f_0(q) + u_1 f_1(q) + \ldots + u_k f_k(q) \tag{2.3}$$

where $f_0 \ldots f_k \in \mathrm{Vec}(M)$ and $u = (u_1, \ldots, u_k) \in \mathcal{U}$ are an admissible control which takes value in the set $U \subseteq \mathbb{R}^k$. The uncontrollable term f_0 is called *drift*.

Moreover, we can consider the $n \times k$ matrix

$$f(q) = \big(f_1(q), \ldots, f_k(q) \big)$$

and rewrite system (2.3) as

$$\dot{q} = f_0(q) + f(q)u.$$

2.1 Time-Optimal Problem

Let us introduce the time-optimal problem.

Definition 2.12 Given the control system (2.2), $q_0 \in M$ and $q_1 \in A_{q_0}$, the *time-optimal problem* consists in minimizing the time of motion from q_0 to q_1 via admissible trajectories:

$$\begin{cases} \dot{q} = f_u(q) \qquad u \in \mathcal{U} \\ q_u(0, q_0) = q_0 \\ q_u(t_1, q_0) = q_1 \\ t_1 \to \min \end{cases} \tag{2.4}$$

We call these minimizing trajectories *time-optimal trajectories*, and *time-optimal controls* the correspondent controls.

2.1.1 Existence of Time-Optimal Trajectories

Classical Filippov's theorem (see [2]) guarantees the existence of a time-optimal control for the affine control system if U is a convex and compact and q_0 is sufficiently close to q_1.

2.2 First- and Second-Order Necessary Optimality Condition

Now, we need to introduce basic notions about Lie brackets, Hamiltonian systems and Poisson brackets, so that we can present the first- and second-order necessary conditions for optimal trajectories: Pontryagin maximum principle and Goh condition.

Definition 2.13 Let $f, g \in \mathrm{Vec}(M)$, we define their *Lie brackets* the following vector field

$$[f, g](q) = \frac{\partial}{\partial t}_{|t=0} e_*^{-tf} g(q), \quad \forall q \in M$$

where e_*^{-tf} is the push forward of the flow e^{-tf}, defined by f.

An equivalent definition of Lie brackets, which helps to understand their geometric meaning is the following:

$$[f, g](q) = \frac{\partial}{\partial t}_{|t=0} e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q), \quad \forall q \in M.$$

Definition 2.14 An *Hamiltonian* is a smooth function on the cotangent bundle

$$h \in \mathcal{C}^\infty(T^*M).$$

The *Hamiltonian vector field* is the vector field associated with h via the canonical symplectic form σ

$$\sigma_\lambda(\cdot, \overrightarrow{h}) = d_\lambda h.$$

Let $(x_1, \ldots, x_n)$ be local coordinates in M and $(\xi_1, \ldots, \xi_n, x_1, \ldots, x_n)$ induced coordinates in T^*M, $\lambda = \sum_{i=1}^n \xi_i dx_i$. The *symplectic form* has the expression $\sigma = \sum_{i=1}^n d\xi_i \wedge dx_i$. Thus, in canonical coordinates, the Hamiltonian vector field has the following form:

$$\overrightarrow{h} = \sum_{i=1}^n \left(\frac{\partial h}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$

The *Hamiltonian system*, which corresponds to h, is

$$\dot{\lambda} = \overrightarrow{h}(\lambda), \quad \lambda \in T^*M,$$

in canonical coordinates is thus given by

$$\begin{cases} \dot{x}_i = \frac{\partial h}{\partial \xi_i} \\ \dot{\xi}_i = -\frac{\partial h}{\partial x_i} \end{cases}$$

for $i = 1, \ldots, n$.

Definition 2.15 The *Poisson brackets* $\{a, b\} \in \mathcal{C}^\infty(T^*M)$ of two Hamiltonians $a, b \in \mathcal{C}^\infty(T^*M)$ are defined as follows: $\{a, b\} = \sigma(\vec{a}, \vec{b})$; the coordinate expression is as follows:

$$\{a, b\} = \sum_{k=1}^n \left(\frac{\partial a}{\partial \xi_k} \frac{\partial b}{\partial x_k} - \frac{\partial a}{\partial x_k} \frac{\partial b}{\partial \xi_k} \right).$$

Remark 2.16 Let us recall that, given g_1 and g_2 vector fields in M, considering the Hamiltonians $a_1(\xi, x) = \langle \xi, g_1(x) \rangle$ and $a_2(\xi, x) = \langle \xi, g_2(x) \rangle$, it holds

$$\{a_1, a_2\}(\xi, x) = \langle \xi, [g_1, g_2](x) \rangle.$$

Remark 2.17 Given a smooth function Φ in $\mathcal{C}^\infty(T^*M)$, and $\lambda(t)$ solution of the Hamiltonian system $\dot{\lambda} = \overrightarrow{h}(\lambda)$, the derivative of $\Phi(\lambda(t))$ with respect to t is the following:

$$\frac{d}{dt}\Phi(\lambda(t)) = \{h, \Phi\}(\lambda(t)).$$

2.2.1 Pontryagin Maximum Principle

Now, we give the statement of the Pontryagin maximum principle for the time-optimal problem.

Theorem 2.18 (Pontryagin maximum principle) *Let an admissible control $\tilde{u}$, defined in the interval $t \in [0, \tau_1]$, be time-optimal for system (2.2), and let the Hamiltonian associated with this control system be the action on $f_u(q) \in T_q^*M$ of a covector $\lambda \in T_q^*M$:*

$$\mathcal{H}_u(\lambda) = \langle \lambda, f_u(q) \rangle.$$

Then there exists $\lambda(t) \in T^*_{q_{\tilde{u}}(t)}M$*, for* $t \in [0, \tau_1]$*, never null and Lipschitz continuous, such that for almost all* $t \in [0, \tau_1]$*, the following conditions hold:*

(1) $\dot{\lambda}(t) = \vec{\mathcal{H}}_{\tilde{u}}(\lambda(t))$
(2) $\mathcal{H}_{\tilde{u}}(\lambda(t)) = \max_{u \in U} \mathcal{H}_u(\lambda(t))$
(3) $\mathcal{H}_{\tilde{u}}(\lambda(t)) \geq 0.$

Moreover, the second condition is called maximality condition, and $\lambda(t)$ *is called extremal.*

Remark 2.19 Given the canonical projection $\pi : TM \to M$, we denote $q(t) = \pi(\lambda(t))$ the *extremal trajectory*.

2.2.2 Goh Condition

Finally, we present the Goh condition, on the singular arcs of the extremal trajectory, in which we do not have information from the maximality condition of the Pontryagin maximum principle. We state the Goh condition only for affine control systems (2.3).

Theorem 2.20 (*Goh condition*) *Let* $\tilde{q}(t),\ t \in [0, t_1]$ *be a time-optimal trajectory corresponding to a control* $\tilde{u}$*. If* $\tilde{u}(t) \in \mathrm{int}U$ *for any* $t \in (\tau_1, \tau_2)$*, then there exists an extremal* $\lambda(t) \in T^*_{q(t)}M$ *such that*

$$\left\langle \lambda(t), [f_i, f_j](q(t)) \right\rangle = 0, \qquad t \in (\tau_1, \tau_2),\ i, j = 1, \ldots, m. \tag{2.5}$$

2.3 Consequence of the Optimality Conditions: the 3D Case with 2D Control

In this paper, we are going to investigate the local regularity of time-optimal trajectories for the following 3D affine control system with a 2D control:

$$\dot{q} = f_0(q) + u_1 f_1(q) + u_2 f_2(q) \tag{2.6}$$

where the space of control parameter U is the 2D disk.

By Pontryagin maximum principle, every time-optimal trajectory of our system has an extremal that is a lift to the cotangent bundle T^*M. The extremal satisfies a Hamiltonian system, given by the Hamiltonian defined from the maximality condition.

Notation 2.21 Let us call $h_i(\lambda) = \langle \lambda, f_i(q) \rangle$, $f_{ij}(q) = [f_i, f_j](q)$, $f_{ijk}(q) = [f_i, [f_j, f_k]](q)$, $h_{ij}(\lambda) = \left\langle \lambda, f_{ij}(q) \right\rangle$, and $h_{ijk}(\lambda) = \left\langle \lambda, f_{ijk}(q) \right\rangle$, with $\lambda \in T^*_q M$ and $i, j, k \in \{0, 1, 2\}$.

Definition 2.22 The *singular locus* $\Lambda \subseteq T^*M$ is defined as follows:

$$\Lambda = \{\lambda \in T^*M : h_1(\lambda) = h_2(\lambda) = 0\}.$$

The following proposition is an immediate corollary of the Pontryagin maximum principle.

Proposition 2.23 *If an extremal* $\lambda(t),\ t \in [0, t_1]$ *does not intersect the singular locus* Λ*, then,*

$$\tilde{u}(t) = \begin{pmatrix} \frac{h_1(\lambda(t))}{(h_1^2(\lambda(t)) + h_2^2(\lambda(t)))^{1/2}} \\ \frac{h_2(\lambda(t))}{(h_1^2(\lambda(t)) + h_2^2(\lambda(t)))^{1/2}} \end{pmatrix}. \tag{2.7}$$

Moreover, this extremal is a solution of the Hamiltonian system defined by the Hamiltonian $\mathcal{H}(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + h_2^2(\lambda)}$. *Thus, it is smooth.*

Definition 2.24 We will call *bang arc* any smooth arc of a time-optimal trajectory $q(t)$, whose correspondent time-optimal control $\tilde{u}$ lies in the boundary of the space of control parameters, $\tilde{u}(t) \in \partial U$.

Corollary 2.25 *An arc of a time-optimal trajectory, whose extremal is out of the singular locus, is a bang arc.*

Proof From Proposition 2.23, given an arc of a time-optimal trajectory $q(t)$, whose extremal $\lambda(t)$ does not intersect the singular locus, its control $\tilde{u}(t)$ satisfies Eq. 2.7, as a consequence, the arc is smooth with respect to the time. Moreover, the time-optimal control belongs to the boundary of U. Hence, the arc of $q(t)$ that we are considering is a bang arc. □

From Corollary 2.25, we already have an answer about the regularity of time-optimal trajectories: every time-optimal trajectory, whose extremal lies out of the singular locus, is smooth.

However, we do not know what happens if an extremal touches the singular locus, optimal controls can be not always smooth; hence, let us give the following definitions.

Definition 2.26 A *switching* is a discontinuity of an optimal control.

Given $u(t)$ an optimal control, $\bar{t}$ is a *switching time* if $u(t)$ is discontinuous at $\bar{t}$.

Moreover, given $q_u(t)$ the admissible trajectory, $\bar{q} = q_u(\bar{t})$ is a *switching point* if $\bar{t}$ is a switching time for $u(t)$.

A concatenation of bang arcs is called *bang-bang trajectory*.

An arc of an optimal trajectory that admits an extremal totally contained in the singular locus Λ is called *singular arc*.

Let us give the last Remark that shows the smoothness of optimal trajectories if $k = n$.

Remark 2.27 If $k = n$, Pontryagin maximum principle implies that

$$\Lambda = \{\lambda \in T^*M \,:\, \langle \lambda, f_i(q)\rangle = 0,\ \forall i \in \{1, \ldots, n\}\} = \{\lambda = 0\},$$

since by assumption $f_1, \ldots, f_n$ are everywhere linear independent.

Then, every optimal trajectory will be a smooth bang arc, because its extremal $\lambda(t) \neq 0$ and must remain out of the singular locus.

3 Statement of the Result

In the rest of the paper, we always assume that $\dim M = 3$ and study the time-optimal problem for the system

$$\dot{q} = f_0(q) + u_1 f_1(q) + u_2 f_2(q), \quad (u_1, u_2) \in U, \tag{3.1}$$

where f_0, f_1 and f_2 are smooth vector fields, $U = \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$; we also assume that f_1 and f_2 are everywhere linearly independent, and we denote $f_{ij} = [f_i, f_j]$ with $i, j \in \{0, 1, 2\}$.

Theorem 3.1 *Let $\bar{q} \in M$; if*

$$\operatorname{rank}\{f_1(\bar{q}), f_2(\bar{q}), f_{01}(\bar{q}), f_{02}(\bar{q}), f_{12}(\bar{q})\} = 3, \tag{3.2}$$

then there exists a neighbourhood $O_{\bar{q}}$ of $\bar{q}$ in M such that any time-optimal trajectory contained in $O_{\bar{q}}$ is bang-bang, with no more than one switching.

From Corollary 2.25, we already know that every arc of a time-optimal trajectory, whose extremal lies out of Λ, is bang, and so smooth.

Thus, we are interested to study arcs of a time-optimal trajectory, whose extremal passes through Λ or lies in Λ.

We are going to study directly the behaviour of extremals in the cotangent bundle in the neighbourhood of $\bar{\lambda}$, which is any lift of $\bar{q}$ in $\Lambda_{\bar{q}} \subseteq T^*_{\bar{q}}M$, not null.

Let us give an equivalent condition to Eq. 3.2 at the point $\bar{\lambda}$.

Claim 3.2. *Given $\bar{\lambda} \in \Lambda_{\bar{q}} \subseteq T^*_{\bar{q}}M$, $\bar{\lambda} \neq 0$,* Eq. 3.2 *is equivalent to*

$$h_{01}^2(\bar{\lambda}) + h_{02}^2(\bar{\lambda}) + h_{12}^2(\bar{\lambda}) \neq 0. \tag{3.3}$$

Due to the homogeneity of any h_{ij} with respect to λ, inequality (3.3) *does not depend on the choice of $\bar{\lambda} \in \Lambda_{\bar{q}}$.*

Proof Since by construction $\bar{\lambda}$ is orthogonal to $f_1(\bar{q})$ and $f_2(\bar{q})$, Eq. 3.2 will be true if and only if the values $h_{01}(\bar{\lambda})$ $h_{02}(\bar{\lambda})$ and $h_{12}(\bar{\lambda})$ cannot be all null. □

In this paper, we are going to present exactly in which cases there could appear switchings, with respect to the choice of the triples $(f_0, f_1, f_2) \in (\mathrm{Vec}(M))^3$.

Let us give the following notation.

Notation 3.3 Let $\bar{\lambda} = f_1(\bar{q}) \times f_2(\bar{q}) \in \Lambda|_{\bar{q}}$ and introduce the following abbreviated notations: $r := (h_{01}^2(\bar{\lambda}) + h_{02}^2(\bar{\lambda}))^{1/2}$, $h_{12} := h_{12}(\bar{\lambda})$.

The first step is to investigate if our system admits singular arcs.

Proposition 3.4 *Assuming condition* (3.3)*, if $r^2 \neq h_{12}^2$, there are no optimal extremals in $O_{\bar{\lambda}}$ that lie in the singular locus Λ for a time interval. On the other hand, if $r^2 = h_{12}^2$, there might be arcs of optimal extremal contained in Λ.*

Thanks to Proposition 3.4, if $r^2 \neq h_{12}^2$, every optimal extremal could either remain out of the singular locus or intersect it transversally. Consequently, in a neighbourhood of $\bar{\lambda}$, we are allowed to study the solutions of the Hamiltonian system, defined by $\mathcal{H}(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + h_2^2(\lambda)}$, which has a discontinuous right-hand side at $\bar{\lambda}$.

With this approach, we proved the following result.

Theorem 3.5 *Assume that condition (3.3) is satisfied, and suppose that $r^2 \neq h_{12}^2$.*
If

$$r^2 > h_{12}^2, \tag{3.4}$$

then there exist a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ and an interval (α, β), $\alpha < 0 < \beta$, such that for any $z \in O_{\bar{\lambda}}$, there exists an unique extremal $t \mapsto \lambda(t; z)$ with the initial condition $\lambda(0; z) = z$ defined on the interval $t \in (\alpha + \hat{t}, \beta + \hat{t})$, with $\hat{t} \in (-\beta, -\alpha)$. Moreover, $\lambda(t; z)$ continuously depends on $(t, z) \in (\alpha, \beta) \times O_{\bar{\lambda}}$, and every extremal in $O_{\bar{\lambda}}$ that passes through the singular locus is piecewise smooth with only one switching. Besides that, we have:

$$u(\bar{t} \pm 0) = \frac{1}{r^2}\left(-h_{02}h_{12} \pm h_{01}(r^2 - h_{12}^2)^{\frac{1}{2}}, h_{01}h_{12} \pm h_{02}(r^2 - h_{12}^2)^{\frac{1}{2}}\right) \tag{3.5}$$

where u is the control correspondent to the extremal that passes through $\bar{\lambda}$, and $\bar{t}$ is its switching time. If

$$r^2 < h_{12}^2, \tag{3.6}$$

then there exists a neighbourhood $O_{\bar{\lambda}} \subset T^*M$ such that every optimal extremal does not intersect the singular locus in $O_{\bar{\lambda}}$; all the optimal trajectories which are close to $\bar{q}$ are smooth bang arcs.

Remark 3.6 We would like to stress the fact that formula (3.5) explicitly describes the jump of the time-optimal control at the switching point in terms of Lie bracket relations.

If the value h_{12} equals zero at the jump point, then the control reaches the antipodal point of the boundary of the disk. This happen at points where f_1, f_2 and f_{12} are linearly dependent.

Moreover, if the inequality $r^2 > h_{12}^2$ is close to being an equality, the jump will be smaller and smaller.

Remark 3.7 In general, the flow of switching extremals from Theorem 3.5 is not locally Lipschitz continuous with respect to the initial value. A straightforward calculation shows that it is not locally Lipschitz already in the following simple example:

$$\dot{x} = \begin{pmatrix} 0 \\ 0 \\ \alpha x_1 \end{pmatrix} + u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}$$

with $\alpha > 1$.

Since Pontryagin maximum principle is a necessary but not sufficient condition of optimality, even if we have found extremals that pass through the singular locus, we cannot guarantee that they are all optimal, namely that their projections in M are time-optimal trajectories. In some cases, they are certainly optimal, in particular, for linear systems with an equilibrium target, where to be an extremal is sufficient for optimality. We plan to study the general case in a forthcoming paper.

In the limit case $r^2 = h_{12}^2$, we have the following result:

Proposition 3.8 *If*

$$r^2 = h_{12}^2, \tag{3.7}$$

there exists a neighbourhood of $\bar{q}$ such that any time-optimal trajectory that contains $\bar{q}$ and is contained in the neighbourhood is a bang arc. The correspondent extremal either remains out of the singular locus Λ or lies in

$$\Lambda \cap \{\lambda \mid h_{01}^2(\lambda) + h_{02}^2(\lambda) = h_{12}^2(\lambda)\}. \tag{3.8}$$

Anyway, the correspondent optimal control will be smooth without any switching, taking values on the boundary of U, in both cases.

Remark 3.9 One can notice that the case, in which an extremal $\lambda(t)$ lies in Eq. 3.8 for a time interval, is very rare. Indeed, necessarily along the curve, the following conditions (P_k) on (f_0, f_1, f_2) hold, i.e. tag the equalities as (P_k)

$$\frac{d^k}{dt^k}\left(h_{01}^2(\lambda(t)) + h_{02}^2(\lambda(t)) - h_{12}^2(\lambda(t))\right) = 0, \quad k \in \mathbb{N},$$

and it is easy to see that at least conditions (P_0), (P_1) and (P_2) are distinct and independent.

4 Proof

In this section, we are going to present at first the proof of Theorem 3.5, secondly, we are going to prove Proposition 3.4 and finally Proposition 3.8. All together, these statements contain Theorem 3.1.

4.1 Proof of Theorem 3.5

Let us present the blow up technique, in order to analyze the discontinuous right-hand side Hamiltonian system, defined by

$$\mathcal{H}(\lambda) = h_0(\lambda) + \sqrt{h_1^2(\lambda) + h_2^2(\lambda)}, \tag{4.1}$$

in a neighbourhood $O_{\bar{\lambda}}$ of $\bar{\lambda}$. Secondly, we are going to show the proof of the Theorem if $r^2 < h_{12}^2$, and finally, we prove it if $r^2 > h_{12}^2$.

4.1.1 Blow Up Technique

In view of the fact that this is a local problem in $O_{\bar{\lambda}} \subseteq T^*M$, it is very natural to consider directly its local coordinates $(\xi, x) \in \mathbb{R}^{3*} \times \mathbb{R}^3$, such that $\bar{\lambda}$ corresponds to $(\bar{\xi}, \bar{x})$ with $\bar{x} = 0$. Hence,

$$\mathcal{H}(\xi, x) = h_0(\xi, x) + \sqrt{h_1^2(\xi, x) + h_2^2(\xi, x)}. \tag{4.2}$$

Since f_1 and f_2 are linearly independent everywhere, we can define the never null vector field f_3, such that

$$f_3(x) = f_1(x) \times f_2(x),$$

with the correspondent Hamiltonian $h_3(\xi, x) = \langle \xi, f_3(x)\rangle$. Therefore, we are allowed to consider the following smooth change of variables

$$\Phi : (\xi, x) \longrightarrow ((h_1, h_2, h_3), x),$$

so the singular locus becomes the subspace

$$\Lambda = \{((h_1, h_2, h_3), x) \mid h_1 = h_2 = 0\}.$$

Notation 4.1 In order not to do notations even more complicated, we call λ any point defined with respect to the new coordinates $((h_1, h_2, h_3), x)$, and $\bar{\lambda}$ what corresponds to the singular point that we fixed at Notation 3.3.

Definition 4.2 The *blow-up* technique is defined in the following way:

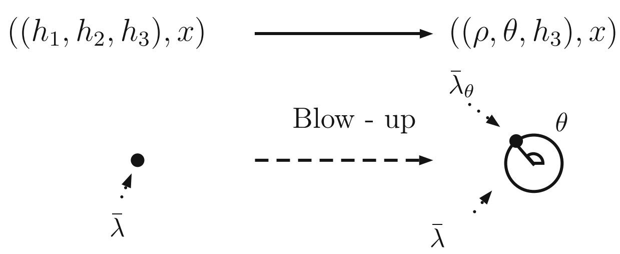

Fig. 1 Blow up technique

We make a change of variables: $(h_1, h_2) = (\rho\cos\theta, \rho\sin\theta)$. Instead of considering the components h_1 and h_2 of the singular point $\bar{\lambda}$ in Λ, as the point (0, 0) in the 2D plane, we will consider it as a circle $\{\rho = 0\}$, and we denote every point of this circle $\bar{\lambda}_\theta$, with respect to the angle (Fig. 1).

In order to write explicitly the Hamiltonian system of Eq. 4.1 out of Λ with this new formulation, let us notice the following aspects.

As it is already known from Proposition 2.23, every optimal control $\tilde{u}$ correspondent to an extremal $\lambda(t)$ that lies out of Λ satisfies formula (2.7); therefore, in this new notation, it holds

$$\tilde{u}(t) = (\cos(\theta(t)), \sin(\theta(t))),$$

where $\theta(t)$ is the θ-component of $\lambda(t)$.

Consequently, it is useful to denote

$$f_\theta(x) = \cos(\theta) f_1(x) + \sin(\theta) f_2(x)$$

and $h_\theta(\lambda) = \langle \xi, f_\theta(x)\rangle$.

Finally, we can see that

$$h_\theta(\lambda) = \sqrt{h_1^2 + h_2^2},$$

namely $h_\theta(\lambda) = \rho$, because $h_\theta(\lambda) = \cos(\theta)h_1 + \sin(\theta)h_2$, $\cos(\theta) = \frac{h_1}{\sqrt{h_1^2+h_2^2}}$ and $\sin(\theta) = \frac{h_2}{\sqrt{h_1^2+h_2^2}}$.

Hence, with this new formulation, the maximized Hamiltonian becomes

$$\mathcal{H}(\lambda) = h_0(\lambda) + h_\theta(\lambda), \tag{4.3}$$

and, thanks to Remarks 2.17 and 2.16, the Hamiltonian system has the following form:

$$\begin{cases} \dot{x} = f_0(x) + f_\theta(x) \\ \dot{\rho} = h_{0\theta}(\lambda) \\ \dot{\theta} = \frac{1}{\rho}\left(h_{12}(\lambda) + \partial_\theta h_{0\theta}(\lambda)\right) \\ \dot{h}_3 = h_{03}(\lambda) + h_{\theta 3}(\lambda) \end{cases} \tag{4.4}$$

where $h_{0\theta}(\lambda) = \cos(\theta)h_{01}(\lambda) + \sin(\theta)h_{02}(\lambda)$, and $\partial_\theta h_{0\theta}(\lambda) = \cos(\theta)h_{02}(\lambda) - \sin(\theta)h_{01}(\lambda)$.

Claim 4.3. *At the singular point $\bar{\lambda}$, the function $\theta \mapsto h_{12} + \cos(\theta)h_{02} - \sin(\theta)h_{01}$ has two, one or no zeros, if $r^2 > h_{12}^2$, $r^2 = h_{12}^2$ or $r^2 < h_{12}^2$ correspondently.*

Proof We set $(h_{01}, h_{02}) = r(\cos(\phi), \sin(\phi))$; then $h_{12} + \cos(\theta)h_{02} - \sin(\theta)h_{01} = 0$ if and only if $\sin(\theta - \phi) = \frac{h_{12}}{r}$. □

Lemma 4.4 *Given the singular point* $\bar\lambda$ *and the points* $\bar\lambda_{\theta_i}$ *such that it holds* $h_{12} + \cos(\theta_i)h_{02} - \sin(\theta_i)h_{01} = 0$. *We consider* $O_{\bar\lambda_{\theta_i}}$ *disjointed neighbourhoods of* $\bar\lambda_{\theta_i}$, *and a neighbourhood* $O_{\bar\lambda}$ *small enough such that* $\forall \hat\lambda_{\hat\theta} \in \overline{O_{\bar\lambda} \setminus \overline{\cup_i O_{\bar\lambda_{\theta_i}}}}$ *it holds* $h_{12}(\hat\lambda) + \cos(\hat\theta)h_{02}(\hat\lambda) - \sin(\hat\theta)h_{01}(\hat\lambda) \neq 0$. *For each connected component* O *of* $\overline{O_{\bar\lambda} \setminus \overline{\cup_i O_{\bar\lambda_{\theta_i}}}}$ *there exist constants* $c > 0$ *and* $\alpha > 0$ *such that if an extremal* $\lambda(t)$ *lies in* O *for a time interval* $I = (0, \tau_1)$, *with* $\lambda(0) \notin \Lambda$, *then it holds the following inequality:* $\rho(t) \geq ce^{-\alpha t}\rho(0)$, *for* $t \in I$.

Proof Without loss of generality, let us study a connected component O of $\overline{O_{\bar\lambda} \setminus \overline{\cup_i O_{\bar\lambda_{\theta_i}}}}$ where

$$h_{12}(\lambda) + \partial_\theta h_{0\theta}(\lambda) > 0.$$

Since in O the map $\lambda \to h_{12}(\lambda) + \partial_\theta h_{0\theta}(\lambda)$ is continuous and not null, it is bounded, then there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \geq h_{12}(\lambda) + \partial_\theta h_{0\theta}(\lambda) \geq c_2 > 0.$$

Given the extremal $\lambda(t)$ in O, we can observe that

$$\frac{d}{dt}\left[\rho(t)\left[h_{12}(\lambda(t)) + \partial_\theta h_{0\theta}(\lambda(t))\right]\right] = \rho(t)A(\lambda(t))$$

where

$$A(\lambda(t)) = \dot h_{12}(\lambda(t)) + \cos(\theta(t))\dot h_{02}(\lambda(t)) - \sin(\theta(t))\dot h_{01}(\lambda(t)).$$

Moreover, we can claim that $A_{|O}$ is bounded from below by a negative constant C

$$A_{|O} \geq C,$$

due to the facts that, by Remark 2.17,

$$\dot h_{ij}(\lambda(t)) = h_{0ij}(\lambda(t)) + \cos(\theta(t))h_{1ij}(\lambda(t)) + \sin(\theta(t))h_{2ij}(\lambda(t)),$$

and any function $h_{kij}(\lambda)$ is continuous in $\bar\lambda$, for each indexes $i, j, k \in \{0, 1, 2\}$.

Finally, we can see that

$$\frac{d}{dt}\left[\frac{\rho(t)\left[h_{12}(\lambda(t)) + \partial_\theta h_{0\theta}(\lambda(t))\right]}{\exp\left(\int_0^t C\left[h_{12}(\lambda(s)) + \partial_\theta h_{0\theta}(\lambda(s))\right]^{-1} ds\right)}\right] \geq 0;$$

hence, for each $t \geq 0$, by the monotonicity:

$$\begin{aligned}\rho(t) &\geq \rho(0)\, \tfrac{h_{12}(\lambda(0)) + \partial_\theta h_{0\theta}(\lambda(0))}{h_{12}(\lambda(t)) + \partial_\theta h_{0\theta}(\lambda(t))}\, \exp\left(\int_0^t C\left[h_{12}(\lambda(s)) + \partial_\theta h_{0\theta}(\lambda(s))\right]^{-1} ds\right)\\ &\geq \rho(0)\, \tfrac{c_2}{c_1}\, \exp\left(\tfrac{C}{c_2}t\right).\end{aligned}$$

Denoting $c := \frac{c_2}{c_1}$ and $\alpha := -\frac{C}{c_2}$, the thesis follows. □

4.1.2 The $r^2 < h_{12}^2$ Case

Lemma 4.4 and Claim 4.3 immediately imply the following Corollary:

Corollary 4.5 *If we assume conditions* (3.3) *and* $r^2 < h_{12}^2$*, given* $O_{\bar{\lambda}}$ *small enough, there exist two constants* $c > 0$ *and* $\alpha > 0$ *such that every extremal that lies for a time interval* I *in* $O_{\bar{\lambda}}$ *satisfies the following inequality:* $\rho(t) \geq ce^{-\alpha t}\rho(0)$*, for* $t \in I$.

This Corollary proves the $r^2 < h_{12}^2$ case of Theorem 3.5, because it shows that, given this condition, every optimal extremal in $O_{\bar{\lambda}}$ does not intersect the singular locus in finite time, and forms a smooth local flow.

4.1.3 The $r^2 > h_{12}^2$ *Case*

Proposition 4.6 *Assuming conditions* (3.3) *and* (3.4)*, there exists a unique extremal that passes through* $\bar{\lambda}$ *in finite time.*

Proof Let us prove that there is a unique solution of system (4.4) passing through its point of discontinuity $\bar{\lambda}$ in finite time.

In order to detect solutions that go through $\bar{\lambda}$, we rescale the time considering the time $t(s)$ such that $\frac{d}{ds}t(s) = \rho(s)$ and we obtain the following system:

$$\begin{cases} x' = \rho\,(f_0(x) + f_\theta(x)) \\ \rho' = \rho h_{0\theta} \\ \theta' = h_{12} + \frac{\partial}{\partial\theta}h_{0\theta} \\ h_3' = \rho\,(h_{03} + h_{\theta 3})\,, \end{cases} \tag{4.5}$$

with a smooth right-hand side.

This system has an invariant subset $\{\rho = 0\}$ in which only the θ component is moving. Moreover, as we saw from Claim 4.3, at $\bar{\lambda}$, there are two equilibria $\bar{\lambda}_{\theta_-} = ((0, \theta_-, 1), \bar{x})$ and $\bar{\lambda}_{\theta_+} = ((0, \theta_+, 1), \bar{x})$, such that $\sin(\theta_\pm - \phi) = \frac{h_{12}}{r}$ and $\cos(\theta_\pm - \phi) = \pm\frac{\sqrt{r^2 - h_{12}^2}}{r}$.

We now present Shoshitaishvili's theorem [9] which explains the behaviour of the solutions in $\bar{\lambda}_{\theta_-}$ and $\bar{\lambda}_{\theta_+}$.

Theorem 4.7 (Shoshitaishvili's theorem) *In* $\mathbb{R}^n$*, let the* C^k*-germ,* $2 \leq k < \infty$*, of the family*

$$\begin{cases} \dot{z} = Bz + r(z, \varepsilon), \\ \dot{\varepsilon} = 0, z \in \mathbb{R}^n, \varepsilon \in \mathbb{R}^l \end{cases} \tag{4.6}$$

be given, where $r \in C^k(\mathbb{R}^n \times \mathbb{R}^l)$*,* $r(0, 0) = 0$*,* $\partial_z r_{|(0,0)} = 0$ *and* $B : \mathbb{R}^n \to \mathbb{R}^n$ *are linear operators whose eigenvalues are divided into three groups:*

$$\begin{aligned} \text{I} &= \{\lambda_i, 1 \leq i \leq k^0 \,|\, \text{Re}\lambda_i = 0\} \\ \text{II} &= \{\lambda_i, k^0 + 1 \leq i \leq k^0 + k^- \,|\, \text{Re}\lambda_i < 0\} \\ \text{III} &= \{\lambda_i, k^0 + k^- + 1 \leq i \leq k^0 + k^- + k^+ \,|\, \text{Re}\lambda_i > 0\} \end{aligned}$$

$$k^0 + k^- + k^+ = n.$$

Let the subspaces of $\mathbb{R}^n$*, which are invariant with respect to* B *and which correspond to these groups be denoted by* X*,* Y^- *and* Y^+*, respectively, and let* $Y^- \times Y^+$ *be denoted by* Y.

Then the following assertions are true:

(1) *There exists a* C^{k-1} *manifold* γ^0 *that is invariant with respect to the germ* (4.6)*, and the manifold can be represented as the graph of mapping* $\gamma^0 : X \times \mathbb{R}^l \to Y$, $y = \gamma^0(x, \varepsilon)$ *and satisfies* $\gamma^0(0, 0) = 0$ *and* $\partial_x \gamma^0(0, 0) = 0$.

(2) *The germ of the family* (4.6) *is homeomorphic to the product of the multidimensional saddle* $\dot{y}^+ = y^+$, $\dot{y}^- = -y^-$*, and the germ of the family*

$$\begin{cases} \dot{x} = Bx + r_1(x, \varepsilon), \\ \dot{\varepsilon} = 0, \end{cases}$$

where $r_1(x, \varepsilon)$ *is the* x *component of the vector* $r(z, \varepsilon)$, $z = (x, \gamma^0(x, \varepsilon))$*, i.e. the germ of* Eq. 4.6 *is homeomorphic to the germ of the family*

$$\begin{cases} \dot{y}^+ = y^+, \ \dot{y}^- = -y^- \\ \dot{x} = Bx + r_1(x, \varepsilon), \ \dot{\varepsilon} = 0. \end{cases}$$

Let us investigate which are the eigenvalues of the Jacobian of system (4.5).

Since $\bar{\lambda}_{\theta_-}$ and $\bar{\lambda}_{\theta_+}$ belong to the invariant subset $\{\rho = 0\}$, where the components ρ, h_3 and x are fixed, at $\bar{\lambda}_{\theta_\pm}$ the eigenvalues are $\partial_\rho \rho'_{|\bar{\lambda}_{\theta_\pm}} = h_{0\theta\,|\bar{\lambda}_{\theta_\pm}}$ and $\partial_\theta \theta'_{|\bar{\lambda}_{\theta_\pm}} = -h_{0\theta\,|\bar{\lambda}_{\theta_\pm}}$ and four 0.

Moreover, $h_{0\theta\,|\bar{\lambda}_{\theta_-}}$ and $h_{0\theta\,|\bar{\lambda}_{\theta_+}}$ are not null with opposite sign, since

$$\begin{aligned} h_{0\theta\,|\bar{\lambda}_{\theta_\pm}} &= \cos(\theta_\pm) h_{01} + \sin(\theta_\pm) h_{02} \\ &= r\cos(\theta_\pm - \phi) = \pm\sqrt{r^2 - h_{12}^2}, \end{aligned}$$

namely $h_{0\theta\,|\bar{\lambda}_{\theta_-}} = -\sqrt{r^2 - h_{12}^2}$ and $h_{0\theta\,|\bar{\lambda}_{\theta_+}} = \sqrt{r^2 - h_{12}^2}$.

Central manifolds γ^0 of Theorem 4.5 applied to the equilibria $\bar{\lambda}_{\theta_\pm}$ are 4D submanifolds defined by the equations $\rho = 0$, $\theta = \theta_\pm$. The dynamics on the central manifold is trivial: all points are equilibria. Hence, according to Shoshitaishvili theorem, only trajectories from the 1D asymptotically stable (unstable) invariant submanifold tends to the equilibrium point $\bar{\lambda}_{\theta_\pm}$ as $t \to +\infty$ ($t \to \infty$). Moreover, $\rho = 0$ is a 5D invariant submanifold with a very simple dynamics: θ is moving from θ_- to θ_+ (Fig. 2).

Keeping in mind that only a part of the phase portrait where $\rho \geq 0$ is relevant for our study, we obtain that exactly one extremal enters submanifold $\rho = 0$ at $\bar{\lambda}_{\theta_-}$ and exactly one extremal goes out of this submanifold at $\bar{\lambda}_{\theta_+}$. Moreover, the same result in the same neighbourhood is valid for any $\hat{\lambda} \in \Lambda$ sufficiently close to $\bar{\lambda}$ with $\hat{\lambda}$ playing the role of parameter ε in Shoshitaishvili theorem.

Finally, we are going to show that the extremal that we found passes through $\bar{\lambda}$ in finite time. So far, we have proven that there exists $\lambda(t(s))$ which satisfies (4.5) and that it reaches $\bar{\lambda}$ at an equilibrium, so $\lambda(t(s))$ attains and escapes from $\bar{\lambda}$ in infinite time s.

Thus, let us estimate the time Δt that this extremal needs to reach $\bar{\lambda}$.

Due to the facts that $h_{0\theta\,|\bar{\lambda}_{\theta_-}} < 0$ and $h_{0\theta}$ are continuous in $\bar{\lambda}_{\theta_-}$, there exist a neighbourhood $O_{\bar{\lambda}_{\theta_-}}$ of $\bar{\lambda}_{\theta_-}$, in which $h_{0\theta}$ is bounded from above by a negative constant $c_1 < 0$, namely $h_{0\theta\,|O_{\bar{\lambda}_{\theta_-}}} < c_1 < 0$.

Hence, in $O_{\bar{\lambda}_{\theta_-}}$, we have the following estimate of the derivative ρ':

$$\rho' = \rho\, h_{0\theta} < \rho\, c_1,$$

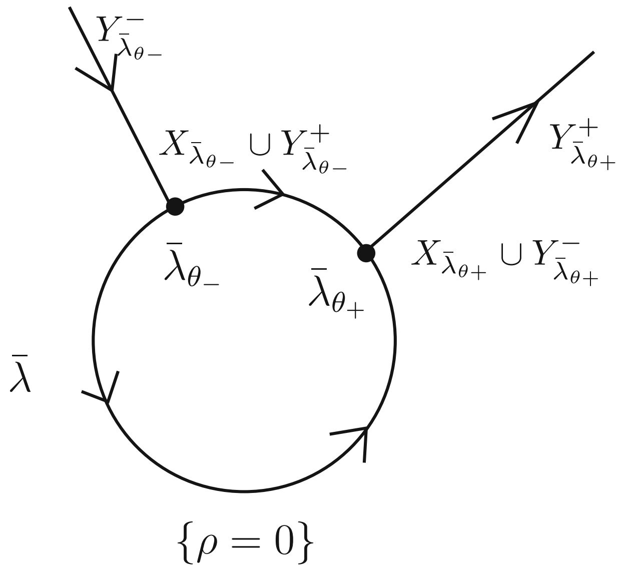

Fig. 2 Description of stable and unstable components of the equilibria

consequently, until $\rho(s) > 0$, it holds

$$\int_{s_0}^{s} \frac{\rho'}{\rho} ds < \int_{s_0}^{s} c_1 ds,$$

then this inequality implies $\log(\rho(s)) < c_1(s - s_0) + \log(\rho(s_0))$, and so

$$\rho(s) < \rho(s_0) e^{c_1(s-s_0)}.$$

Since $\frac{d}{ds} t(s) = \rho(s)$, the amount of time that we want to estimate is given by the following:

$$\Delta t = \lim_{s \to \infty} t(s) - t(s_0) = \int_{s_0}^{\infty} \rho(s) ds;$$

therefore,

$$\Delta t = \int_{s_0}^{\infty} \rho(s) ds < \rho(s_0) \int_{s_0}^{\infty} e^{c_1(s-s_0)} ds = \frac{\rho(s_0)}{-c_1} < \infty.$$

The amount of time in which this extremal goes out from $\bar\lambda$ may be estimated in an analogous way. □

By the previous Proposition and since every extremal out of Λ is smooth, it is proven that there exist a neighbourhood $O_{\bar\lambda} \subset T^*M$ and an interval (α, β), $\alpha < 0 < \beta$, such that for any $z \in O_{\bar\lambda}$ there exists a unique extremal $t \mapsto \lambda(t; z)$ with the initial condition $\lambda(0; z) = z$ defined on the interval $t \in (\alpha + \hat t, \beta + \hat t)$, with $\hat t \in (-\beta, -\alpha)$. Furthermore, every extremal in $O_{\bar\lambda}$ that passes through the singular locus is piecewise smooth with only one switching. The control u correspondent to the extremal that passes through $\bar\lambda$ jumps at the switching time $\bar t$ from $u(\bar t - 0) = (\cos(\theta_-), \sin(\theta_-))$ to $u(\bar t + 0) = (\cos(\theta_+), \sin(\theta_+))$; hence,

$$\begin{aligned} u(\bar t \pm 0) &= (\cos(\phi - (\theta_\pm - \phi)), \sin(\phi - (\theta_\pm - \phi))) \\ &= \left(-\tfrac{h_{12}}{r} \sin(\phi) \pm \frac{\sqrt{r^2 - h_{12}^2}}{r} \cos(\phi), \tfrac{h_{12}}{r} \cos(\phi) \pm \frac{\sqrt{r^2 - h_{12}^2}}{r} \sin(\phi) \right). \end{aligned}$$

Let us conclude the proof with the following Proposition.

Proposition 4.8 *The map* $(t; z) \to \lambda(t; z)$*,* $(t, z) \in (\alpha, \beta) \times O_\lambda$ *is continuous.*

Proof It remains to prove the continuity of the flow $(t; z) \to \lambda(t; z)$ with respect to z; thus, we prove that for each $\varepsilon > 0$ there exists a neighbourhood $O^\varepsilon_{\bar\lambda}$ such that the maximum time interval of the extremals in this neighbourhood $\Delta_{O^\varepsilon_{\bar\lambda}} t$ is less than ε.

As we saw previously, the extremal through $\bar\lambda$ will arrive in and go out of Λ with angles θ_- and θ_+, then we can distinguish three parts of the extremals close to $\bar\lambda$: the parts in $O_{\bar\lambda_{\theta_-}}$ and in $O_{\bar\lambda_{\theta_+}}$, and that part in the middle that is close to $\rho = 0$.

In this last region, since for each extremal ρ is close to 0 and the correspondent time interval with time s is bounded, then Δt is arbitrarily small with respect to $O_{\bar\lambda}$.

Hence, in $O_{\bar\lambda_{\theta_\pm}}$, we are going to show that there exists a sequence of neighbourhoods of $\bar\lambda_{\theta_\pm}$

$$\left(O^R_{\theta_\pm}\right)_R,$$

such that

$$\lim_{R\to 0^+} \Delta_{O^R_{\theta_\pm}} t = 0.$$

For simplicity, we are going to prove this fact in $O_{\bar\lambda_{\theta_-}}$, because the situations in $O_{\bar\lambda_{\theta_-}}$ and $O_{\bar\lambda_{\theta_+}}$ are equivalent.

Let us denote $O^R_{\theta_-}$ a neighbourhood of $\bar\lambda_{\theta_-}$ such that $O^R_{\theta_-} \subseteq O_{\bar\lambda_{\theta_-}}$, for each $((\rho, \theta, h_3), x) \in O^R_{\theta_-}$ $\rho < R$ and $|\theta - \theta_-| < R$. Therefore, we can define

$$M_R = \sup_{\lambda \in O^R_{\theta_-}} h_{0\theta}(\lambda),$$

and assume that it is strictly negative and finite, due to the fact that we can choose $O_{\bar\lambda_{\theta_-}}$ in which $h_{0\theta}(\lambda)$ is strictly negative and finite.

Hence, for every extremal $\lambda(s)$ in $O^R_{\theta_-}$, until its ρ component is different from zero, it holds

$$\frac{\dot\rho(s)}{\rho(s)} < M_R,$$

then

$$\rho(s) < \rho(s_0) e^{M_R(s-s_0)},$$

for every $s > s_0$.

Consequently, $\Delta_{O^R_{\theta_-}} t$ can be estimated in the following way:

$$\Delta_{O^R_{\theta_-}} t < \int_{s_0}^{\infty} \rho(s_0) e^{M_R(s-s_0)} ds = \frac{\rho(s_0)}{-M_R} < \frac{R}{-M_R}.$$

Due to the fact that $\lim_{R\to 0^+} \frac{R}{-M_R} = 0$, we have proved that for each $\varepsilon > 0$ there exists $O^R_{\theta_-}$ such that $\Delta_{O^R_{\theta_-}} t < \varepsilon$. □

4.2 Proof of Proposition 3.4

Let us assume that there exist a time-optimal control $\tilde{u}$ and an interval (τ_1, τ_2) such that $\tilde{u}$ corresponds to an extremal $\lambda(t)$ in $O_{\bar{\lambda}}$ and $\lambda(t) \in \Lambda$, $\forall t \in (\tau_1, \tau_2)$. By construction, for $t \in (\tau_1, \tau_2)$, it holds

$$\begin{cases} \frac{d}{dt} h_1(\lambda(t)) = 0 \\ \frac{d}{dt} h_2(\lambda(t)) = 0. \end{cases} \tag{4.7}$$

Since the maximized Hamiltonian associated with $\tilde{u}$ is

$$\mathcal{H}_{\tilde{u}}(\lambda) = h_0(\lambda) + \tilde{u}_1 h_1(\lambda) + \tilde{u}_2 h_2(\lambda),$$

by Remark 2.17, Eq. 4.7 implies

$$\begin{cases} h_{01}(\lambda(t)) - \tilde{u}_2 h_{12}(\lambda(t)) = 0 \\ h_{02}(\lambda(t)) + \tilde{u}_1 h_{12}(\lambda(t)) = 0. \end{cases} \tag{4.8}$$

Moreover, due to condition (3.3), we can claim that $h_{12}(\lambda(t)) \neq 0$ is along this singular arc; therefore, we have an explicit formulation of $\tilde{u}$ in a singular arc

$$\begin{cases} \tilde{u}_1 = -\frac{h_{02}(\lambda(t))}{h_{12}(\lambda(t))} \\ \tilde{u}_2 = \frac{h_{01}(\lambda(t))}{h_{12}(\lambda(t))}. \end{cases} \tag{4.9}$$

In particular, its norm is the following:

$$||\tilde{u}||^2 = \frac{h_{02}^2(\lambda(t)) + h_{01}^2(\lambda(t))}{h_{12}^2(\lambda(t))}.$$

If $r^2 > h_{12}^2$, we arrive to a contradiction, because in this case, $||\tilde{u}||^2 > 1$ but the norm of admissible controls is less equal than 1. On the other hand, if $r^2 < h_{12}^2$, such extremals might exist, but they are not optimal by the Goh condition, presented at Section 2.2.2.

Hence, we have proved that if $r^2 \neq h_{12}^2$, there are no optimal extremals that lie in Λ for a time interval.

On the other hand, by these observations, if $r^2 = h_{12}^2$, optimal singular arcs could exist.

4.3 Proof of Proposition 3.8

In the limit case $r^2 = h_{12}^2$, by what we have just seen at the proof of Proposition 3.4, we can claim that there could be optimal trajectories, whose extremals lie in Eq. 3.8, and the correspondent controls take values on the boundary of the disk U, with Eq. 4.9.

Now, we are going to show that given a time-optimal trajectory through $\bar{q}$, whose extremal has a point out of the singular locus, it does not attain Λ in finite time.

From Claim 4.3 and Lemma 4.4, we already have an estimate of the behaviour of the extremal out of a small neighbourhood of $\bar{\lambda}_{\bar{\theta}}$, where $\bar{\theta}$ is the unique angle such that $h_{12} + \cos(\bar{\theta})h_{02} - \sin(\bar{\theta})h_{01} = 0$. We are going to extend the estimate to a neighbourhood $O_{\bar{\lambda}_{\bar{\theta}}}$ of $\bar{\lambda}_{\bar{\theta}}$.

Without loss of generality, we assume that $\bar{\theta} = 0$.

Let us omit some boring routine details and focus on the essential part of the estimate. First, we freeze slow coordinates x, h_3 and study the system (4.4) with only two variables ρ, θ. In the worst scenario, we get the following system:

$$\begin{cases} \dot{\rho} = -\sin(\theta) - \rho \\ \dot{\theta} = \frac{1}{\rho}(1 - \cos(\theta)) + 1. \end{cases}$$

Consequently, the behaviour of ρ component with respect to the θ component is described by the following equation:

$$\rho'(\theta) = \frac{-\rho(\sin(\theta) + \rho)}{1 - \cos(\theta) + \rho}. \tag{4.10}$$

With the next Lemma 4.9, we analyze (4.10) and prove that, on the θ axis, there exists an interval I containing 0, on which ρ has a positive increment for any sufficiently small initial condition $\rho(0) = \rho_0 > 0$.

Lemma 4.9 with Lemma 4.4 implies the thesis of Proposition 3.8.

Lemma 4.9 *Given $O_{\bar{\lambda}_{\bar{\theta}}}$, there exist $\eta > 0$ small enough and $\theta_1 > 0$, such that for every initial values $(\rho(0), \theta(0)) = (\rho_0, 0)$ with $\rho_0 \neq 0$, the solution of system (4.10) satisfies the following implication: if $\theta > \theta_1$, then*

$$\rho(-\theta) < \rho(\eta\theta).$$

Proof Given any $\eta > 0$ and any solution of Eq. 4.10 $\rho(\theta)$, we are going to compare the behaviour of $\tilde{\rho}(\theta) = \rho(-\theta)$ and $\hat{\rho}(\theta) = \rho(\eta\theta)$ for $\theta > 0$.

They will be solutions for $\theta > 0$ of the following two systems:

$$\tilde{\rho}'(\theta) = \frac{\tilde{\rho}(\tilde{\rho} - \sin(\theta))}{1 - \cos(\theta) + \tilde{\rho}}$$

and

$$\hat{\rho}'(\theta) = -\eta \frac{\hat{\rho}(\hat{\rho} + \sin(\eta\theta))}{1 - \cos(\eta\theta) + \hat{\rho}}.$$

We can see that $\tilde{\rho}'(0) > \hat{\rho}'(0)$; thus, if θ is very small, it holds $\tilde{\rho}(\theta) > \hat{\rho}(\theta)$.

On the other hand, let us notice that choosing $\eta > 0$ small there exists $\nu > 1$ such that if $\theta > \nu\rho$, then $\hat{\rho}'(\theta) > \tilde{\rho}'(\theta)$. By the classical theory of dynamical system, this implies that in the domain

$$\{(\rho, \theta) \,|\, \theta > \nu\rho\}$$

if $\hat{\rho}(\theta) > \tilde{\rho}(\theta)$ at a certain $\theta > 0$, then the inequality remains true for every bigger value.

In order to compare the behaviour of $\tilde{\rho}(\theta)$ and $\hat{\rho}(\theta)$ when ρ_0 tends to zero, we consider the following rescaling:

$$\begin{cases} \theta = st \\ \tilde{\rho} = s + s^2 x(t) \\ \hat{\rho} = s + s^2 y(t) \end{cases}$$

where s is the initial value ρ_0 and $x(0) = y(0) = 0$.

One can easily notice that if s tends to 0, then

$$\begin{cases} x'(t) = 1 - t + O(s) \\ y'(t) = \eta(-1 - \eta t) + O(s); \end{cases}$$

hence, it holds

$$\begin{cases} x_0(t) = t - \frac{1}{2}t^2 + O(s) \\ y_0(t) = -\eta t - \frac{\eta^2}{2}t^2 + O(s), \end{cases}$$

and

$$x_0(t) - y_0(t) = t\left((1+\eta) - \frac{(1-\eta^2)}{2}t\right) + O(s).$$

Hence, there exist $T > 2\frac{1+\eta}{1-\eta^2} > 2$, such that, denoting ρ_0^{MAX} the maximum among the initial values ρ_0 in $O_{\bar{\lambda}_{\bar{\theta}}}$, and calling $\theta_1 = \rho_0^{\mathrm{MAX}} T$, it holds that if $\theta > \theta_1$, then $\tilde{\rho}(\theta) < \hat{\rho}(\theta)$, namely

$$\rho(-\theta) < \rho(\eta\theta).$$

□

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