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On Sub-Riemannian and Riemannian Structures on the Heisenberg Groups

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Abstract We consider the left-invariant sub-Riemannian and Riemannian structures on the Heisenberg groups. A classification of these structures was found previously. In the present paper, we find (for each normalized structure) the isometry group, the exponential map, the totally geodesic subgroups, and the conjugate locus. Finally, we determine the minimizing geodesics from identity to any given endpoint. (Several of these points have been covered, to varying degrees, by other authors.)

Keywords Sub-Riemannian geometry · Riemannian geometry · Heisenberg group · Isometries · Geodesics · Totally geodesic subgroups · Conjugate locus

Mathematics Subject Classification (2010) 53C17 · 22E25 · 53C22 · 49J15

1 Introduction

Over the last two decades, invariant sub-Riemannian structures on Lie groups have received quite some attention, both from the point of view of classifying such structures (see, e.g., [1, 8]; also [27, 56]) as well as the investigation of geodesics, conjugate loci, and cut loci

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(see, e.g., [10, 11, 18, 19, 44, 46–48, 54, 55]). Invariant Riemannian structures, on the other hand, have been studied for several decades. In particular, such structures (especially on nilpotent Lie groups) have proved to be a rich source of examples and counterexamples for a number of questions and conjectures in Riemannian geometry (see, e.g., [23–25, 29, 30, 34, 35, 37–41, 45, 61–63]).

Sub-Riemannian (and Riemannian) structures on nilpotent Lie groups (and particularly Carnot groups) are arguably the simplest and serve as prototypes; amongst the most basic of these structures are the ones on the Heisenberg groups. Sub-Riemannian structures on the Heisenberg group H_{2n+1} have been studied by quite a number of authors; we give only a partial overview here. Vershik and Gershkovich [58, 59] studied the three-dimensional case in the 1980s; in particular, they described the sub-Riemannian geodesics and wave front. (The three-dimensional case was also treated as a fundamental example in [13].) Around the year 2000, Beals, Gaveau, and Greiner [12] (see also the monograph [21] and the references therein) worked out in detail the subelliptic geometry of the Heisenberg groups and related it to complex Hamiltonian mechanics. In particular, they describe the minimizing geodesics for a class of sub-Riemannian structures on H_{2n+1} (which it turns out covers all sub-Riemannian structures on H_{2n+1}, up to isometry). Around the same time, Monroy-Pérez and Anzaldo-Meneses [47] considered a class of optimal control problems on H_{2n+1} (which again it turns out covers all sub-Riemannian structures on H_{2n+1}, up to isometry). In particular, they computed the sub-Riemannian exponential map and determined the conjugate locus. Monti [50] studied some properties of Carnot-Carathéodory balls in the Heisenberg group and in particular described the geodesics, but it turns out only for the maximally symmetric case (i.e., when the isometry group is maximal). A couple of years later, Ambrosio and Rigot [9] considered optimal mass transportation in the Heisenberg group; toward that end, they described the minimizing geodesics, but it turns out only for the maximally symmetric case. Meanwhile, Tan and Yang [57] explicitly described the minimizing sub-Riemannian geodesics for Heisenberg type groups, as well as describing the isometry groups; however, this again covers only the case of maximal symmetry on the Heisenberg group. More recently, Agrachev, Barilari and Boscain [3] investigated some properties of the Hausdorff volume in sub-Riemannian geometry. In particular, they investigated geodesics of a class of contact sub-Riemannian structures on nilpotent Lie groups (it turns out that these structures are, up to isometry, exactly the sub-Riemannian structures on H_{2n+1}); it is shown that geodesics are optimal up to the first conjugate point (and so the cut and conjugate loci coincide).

Riemannian structures on the Heisenberg group have received comparatively less attention. Nonetheless, more general classes of Riemannian structures on nilpotent Lie groups have been considered by quite a few authors; we mention a few relevant papers (mostly from the 1990s). Eberlein [24, 25] (see also [26]) investigated the geometry of invariant Riemannian structures on two-step nilpotent Lie groups; in particular, geodesics are explicitly described and the totally geodesic submanifolds are characterized. Walschap [61] showed that the cut and conjugate loci coincide for a large class of invariant Riemannian structures on two-step nilpotent Lie groups; this class includes the invariant Riemannian structures on H_{2n+1}. For the maximally symmetric Riemannian structure on H_{2n+1}, a derivation of the geodesics, using the Levi-Civita connection, can be found in [42]. We note that for Riemannian structures on the Heisenberg groups, only the maximally symmetric case is a Heisenberg type group. (Heisenberg type groups, introduced by Kaplan [34], are a special subclass of two-step nilpotent Lie groups endowed with an invariant Riemannian metric.) The geometry of Heisenberg type groups is fairly well understood (see, e.g., the monograph [14] and references therein).

In this paper, we revisit the invariant sub-Riemannian and Riemannian structures on H_{2n+1}. With regard to geodesics, we follow the approach given by geometric control theory [2, 6, 33]. That is, the sub-Riemannian geodesic problem is regarded as an invariant optimal control problem and the Pontryagin Maximum Principle is used to obtain first-order necessary conditions for minimizing geodesics. Furthermore, we shall regard a Riemannian structure simply as a special case of a sub-Riemannian structure; in the words of Agrachev and Gamkrelidze [5] "Even in the classical case of Riemannian geometry, the maximum principle approach to finding geodesics leads to a final result much simpler and shorter than the traditional method of using the Levi-Civita connection."

Section 2 contains the preliminaries (with regard to invariant sub-Riemannian structures on Lie groups and the Pontryagin Maximum Principle).

In a previous paper [16], we classified the sub-Riemannian and Riemannian structures on H_{2n+1}; we restate this classification in Section 3.1. (Recently, there has also appeared a classification of Riemannian and Lorentzian metrics on H_{2n+1} in [60].) In Section 3.2, the Riemannian and sub-Riemannian structures on the H_{2n+1} are shown to be (compatible) central extensions of the Euclidean space $\mathbb{E}^{2n}$ (in the sense explored in [15]); in particular, this implies that the sub-Riemannian geodesics are (in a sense) simply projections of the Riemannian geodesics. For each normalized sub-Riemannian and Riemannian structure, we determine the isometry group (Section 3.3). As each isometry preserving the identity is an automorphism of H_{2n+1}, this problem is essentially one of determining the subgroup of automorphisms of the Lie algebra preserving the metric at identity. In the maximally symmetric sub-Riemannian (resp. Riemannian) case, the isometry group was previously found in [57] (resp. [34, 42, 57]); our result is a simple extension of those results. (We note, however, that there is a small mistake in the description of the isometry group given in [42].) It was noticed in [57] that the isometry groups for the (maximally symmetric) sub-Riemannian and Riemannian structures coincide; this result turns out to hold generally on H_{2n+1} (when the Riemannian structure tames the sub-Riemannian one). In Section 3.4, the (sub-Riemannian) exponential map is calculated.

In Section 4, we investigate the totally geodesic subgroups of the normalized sub-Riemannian and Riemannian structures exhibited in Section 3.1. (We say that a closed subgroup N is totally geodesic if any geodesic tangent to N remains on N.) A simple characterization for a subgroup to be totally geodesic is given in terms of invariant subspaces of the associated Hamilton–Poisson system on the corresponding Lie–Poisson space. Not many authors appear to have considered totally geodesic submanifolds of sub-Riemannian structures; we note, however, that in [7], some sufficient conditions for a submanifold to be totally geodesic were given (although their definition of a totally geodesic submanifold differs from ours). In Section 4.2, we specialize Eberlein's [25] characterization of totally geodesic subgroups of Riemannian structures on two-step nilpotent Lie groups to the Heisenberg group. It is shown that the sub-Riemannian case is closely related to the Riemannian case. The totally geodesic subgroups are enumerated (up to isometry) in both the Riemannian and sub-Riemannian cases. We say that a totally geodesic subgroup is representative, if all geodesics of the ambient structure can be recovered via isometries from the geodesics of the restricted structure (along with some other regularity conditions). In Section 4.3, we identify a representative totally geodesic subgroup of minimal dimension for each normalized structure. In particular, the maximally symmetric structures on H_{2n+1} turn out to have three-dimensional representative totally geodesic subgroups.

In Section 5, we investigate the minimizing geodesics (of the normalized structures exhibited in Section 3.1). First, in Section 5.1, we calculate the first conjugate time and consequently the conjugate locus (of identity). In [47], it was concluded that the conjugate locus (for the sub-Riemannian structures) is simply the center of the group. We show, however, that is only true in the case of maximal symmetry. Nonetheless, our result concerning the first conjugate time agrees with that in [47]. Even though our calculations agree with and are very similar to those presented in [47] (see also [3]), details are given for the sake of clarity and completeness. In Section 5.2, we proceed to describe the minimizing geodesics from identity to any given endpoint. We make use of the fact that no geodesic is minimizing beyond the first conjugate time; this yields a subset of geodesics to consider when searching for the minimizer. We show that there exists either only one such geodesic from identity to the endpoint or a family of such geodesics, all of the same length. Consequently, we find that the conjugate and cut loci coincide both in the Riemannian case ([61]) and in the sub-Riemannian case (cf. [3]). Furthermore, we give a simple expression for Carnot–Carathéodory distance. The results presented in this paper concerning the sub-Riemannian minimizing geodesics (and Carnot–Carathéodory distance) are consistent with those presented in [12]. However, we believe that both our presentation and our proof are clearer and simpler. (In particular, the necessary condition for minimality involving the first conjugate time is not utilized in [12].) To our knowledge, an explicit description of the Riemannian minimizing geodesics (from identity to any given endpoint) has not appeared before.

A remarkable feature of our presentation is the extent to which the Riemannian and sub-Riemannian cases could be treated simultaneously.

2 Preliminaries

2.1 Left-Invariant Sub-Riemannian Structures

A *left-invariant sub-Riemannian structure* on a (real, finite-dimensional, connected) Lie group G with identity $\mathbf{1}$ is a triplet $(\mathsf{G}, \mathscr{D}, \mathbf{g})$. Here, $\mathscr{D}$ is a smooth nonintegrable left-invariant distribution on G and $\mathbf{g}$ is a left-invariant Riemannian metric on $\mathscr{D}$. More precisely, $\mathscr{D}(\mathbf{1})$ is a linear subspace of the Lie algebra $\mathfrak{g}$ of G and $\mathscr{D}(g) = g\mathscr{D}(\mathbf{1})$; the metric $\mathbf{g}_\mathbf{1}$ is a positive definite symmetric bilinear form on $\mathscr{D}(\mathbf{1})$ and $\mathbf{g}_g(gA, gB) = \mathbf{g}_\mathbf{1}(A, B)$ for $A, B \in \mathfrak{g}$, $g \in \mathsf{G}$. The product gA is given by $T_\mathbf{1}L_g \cdot A$, where $L_g : h \mapsto gh$ is the left translation by g and $T_\mathbf{1}L_g$ is the tangent map of L_g at identity. ($T\mathsf{G}$ has the left trivialization $T\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}$, $gA \leftrightarrow (g, A)$.) We shall denote by ι the inclusion map $\iota : \mathscr{D}(\mathbf{1}) \to \mathfrak{g} = T_\mathbf{1}\mathsf{G}$ and by ι^* its dual $\iota^* : \mathfrak{g}^* \to \mathscr{D}(\mathbf{1})^*$. Furthermore, we have the musical isomorphisms $\flat : \mathscr{D}(\mathbf{1}) \to \mathscr{D}(\mathbf{1})^*$, $A \mapsto \mathbf{g}_\mathbf{1}(A, \cdot)$ and $\sharp = \flat^{-1} : \mathscr{D}(\mathbf{1})^* \to \mathscr{D}(\mathbf{1})$. When $\mathscr{D} = T\mathsf{G}$ (i.e., $\mathscr{D}(\mathbf{1}) = \mathfrak{g}$ or equivalently $\iota = \mathrm{id}_\mathfrak{g}$), we recover a left-invariant Riemannian structure $(\mathsf{G}, \mathbf{g})$. A Riemannian structure $(\mathsf{G}, \tilde{\mathbf{g}})$ is said to *tame* a sub-Riemannian structure $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ if $\tilde{\mathbf{g}}|_\mathscr{D} = \mathbf{g}$. Note that any left-invariant structure $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ on G is uniquely determined by the subspace $\mathscr{D}(\mathbf{1}) \subseteq \mathfrak{g}$ and the scalar product $\mathbf{g}_\mathbf{1}$ on $\mathscr{D}(\mathbf{1})$. A list of k smooth vector fields $(X_1, \ldots, X_k)$ is said to be an orthonormal frame for $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ if $\mathscr{D}(g) = \mathrm{span}(X_1(g), \ldots, X_k(g))$ and $\mathbf{g}(X_i, X_j) = \delta_{ij}$; we note that any left-invariant sub-Riemannian structure on a Lie group admits a global orthonormal frame of left-invariant vector fields.

An absolutely continuous curve $g(\cdot) : [0, t_1] \to \mathsf{G}$ is called a $\mathscr{D}$-curve if $\dot{g}(t) \in \mathscr{D}(g(t))$ for almost every $t \in [0, t_1]$. We shall assume that $\mathscr{D}$ satisfies the bracket generating condition, i.e., $\mathscr{D}(\mathbf{1})$ generates $\mathfrak{g}$; by the Chow–Rashevskii theorem, this condition is necessary and sufficient for any two points in G to be connected by a $\mathscr{D}$-curve (see, e.g., [20, 49]).

The *length* of a $\mathscr{D}$-curve $g(\cdot)$ is given by

$$\ell(g(\cdot)) = \int_0^{t_1} \sqrt{\mathbf{g}(\dot{g}(t), \dot{g}(t))}\, dt.$$

A sub-Riemannian structure $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ is endowed with a natural metric space structure, namely the *Carnot–Carathéodory distance*:

$$d(g_1, g_2) = \inf\{\ell(g(\cdot)) \ : \ g(\cdot) \text{ is a } \mathscr{D}\text{-curve curve joining } g_1 \text{ and } g_2\}.$$

When $\mathscr{D} = T\mathsf{G}$, we recover the Riemannian distance. By left invariance $d(g_1, g_2) = d(\mathbf{1}, g_1^{-1} g_2)$. A $\mathscr{D}$-curve curve $g(\cdot)$ that realizes the Carnot–Carathéodory distance between two points is called a *minimizing geodesic*.

Proposition 1 (cf. [13]) *Let* $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ *be a left-invariant sub-Riemannian structure on a (connected) Lie group* G*. Then:*

1. *The Carnot–Carathéodory metric d is complete.*
2. *Any two points in* G *can be joined by a minimizing geodesic.*

Proof Let $(\mathsf{G}, \tilde{\mathbf{g}})$ be a left-invariant Riemannian structure taming $(\mathsf{G}, \mathscr{D}, \mathbf{g})$. Furthermore, let d^R be the distance associated to $(\mathsf{G}, \tilde{\mathbf{g}})$ and d^{SR} be the distance associated to $(\mathsf{G}, \mathscr{D}, \mathbf{g})$. As $(\mathsf{G}, \tilde{\mathbf{g}})$ is homogeneous, d^R is complete. Hence, as $(\mathsf{G}, \tilde{\mathbf{g}})$ tames $(\mathsf{G}, \mathscr{D}, \mathbf{g})$, it follows that d^{SR} is complete (see, e.g., [56]). Item 1 implies item 2 (see, e.g., [2, 49]). $\square$

An *isometry* between two left-invariant sub-Riemannian (or Riemannian) structures $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ and $(\mathsf{G}', \mathscr{D}', \mathbf{g}')$ is a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$\phi_* \mathscr{D} = \mathscr{D}' \qquad \text{and} \qquad \mathbf{g} = \phi^* \mathbf{g}'$$

i.e., $T_g\phi \cdot \mathscr{D}(g) = \mathscr{D}'(\phi(g))$ and $\mathbf{g}_g(gA, gB) = \mathbf{g}'_{\phi(g)}(T_g\phi \cdot gA, T_g\phi \cdot gB)$. We shall denote the group of isometries of a structure $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ by $\mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$. The isotropy subgroup of $g \in \mathsf{G}$ (i.e., the subgroup of isometries fixing g) will be denoted by $\mathsf{Iso}_g(\mathsf{G}, \mathscr{D}, \mathbf{g})$.

Remark 1 Every isometry is distance preserving (i.e., for any isometry ϕ we have that $d(g, h) = d(\phi(g), \phi(h))$). Conversely, every distance-preserving diffeomorphisms ϕ is an isometry (see, e.g., [56]). Moreover, if all geodesics are normal, then any distance-preserving homeomorphism is smooth ([31], see also [22]).

Every left translation is an isometry by definition. Hence, the group of isometries $\mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ is generated by the left translations $L_g : h \mapsto gh$, $g \in \mathsf{G}$ and the isotropy subgroup of identity. Indeed, any isometry $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ can be written as $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, where $\phi' \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}.\mathscr{D}, \mathbf{g})$.

2.1.1 Sub-Riemannian and Riemannian Structures on Nilpotent Groups

A k-step *Carnot* group G is a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ has stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ with $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{1+j}$, $j = 1, \ldots, k-1$, $[\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}$, and $\mathfrak{g}_k \neq \{0\}$. Let $\mathscr{D}$ be the left-invariant distribution on G specified by $\mathscr{D}(\mathbf{1}) = \mathfrak{g}_1$. Once we fix a left-invariant metric $\mathbf{g}$ on $\mathscr{D}$, we have a left-invariant sub-Riemannian structure on G (note that $\mathscr{D}$ is bracket generating). We shall refer to such a structure as a *sub-Riemannian Carnot group*.

For any Carnot group G, there exists a (one-parameter) family of automorphisms $\delta_r \in \mathsf{Aut}(\mathsf{G})$, $r \neq 0$ given by the corresponding Lie algebra automorphisms (cf. [31, 36])

$$T_{\mathbf{1}}\delta_r : A_1 + A_2 + \cdots + A_k \longmapsto rA_1 + r^2 A_2 + \cdots + r^k A_k, \qquad A_i \in \mathfrak{g}_i.$$

Consequently, for a sub-Riemannian Carnot group $(\mathsf{G}, \mathscr{D}, \mathbf{g})$, we have that $(\delta_r)_* \mathscr{D} = \mathscr{D}$ and $\delta_r^* \mathbf{g} = r^2 \mathbf{g}$. Accordingly, if d is the Carnot–Carathéodry distance associated to $(\mathsf{G}, \mathscr{D}, \mathbf{g})$, then $d(\delta_r(g_1), \delta_r(g_2)) = |r|\, d(g_1, g_2)$ for $g_1, g_2 \in \mathsf{G}$. Also, δ_r maps minimizing geodesics (of length ℓ) to minimizing geodesics (of length $|r|\ell$).

We have the following characterization of isometries between sub-Riemannian Carnot groups.

Proposition 2 (cf. [31, 36]; see also [22]) *Let* $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathscr{D}', \mathbf{g}')$ *be two sub-Riemannian Carnot groups. A diffeomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *is an isometry between* $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathscr{D}', \mathbf{g}')$ *if and only if* ϕ *is the composition* $\phi = L_{\phi(\mathbf{1})} \circ \phi'$ *of a left translation* $L_{\phi(\mathbf{1})}$ *on* G' *and a Lie group isomorphism* $\phi' = L_{\phi(\mathbf{1})^{-1}} \circ \phi : \mathsf{G} \to \mathsf{G}'$ *such that* $T_{\mathbf{1}}\phi' \cdot \mathscr{D}(\mathbf{1}) = \mathscr{D}'(\mathbf{1}')$ *and* $\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi' \cdot A, T_{\mathbf{1}}\phi' \cdot B)$.

Likewise, in the Riemannian case, we have the following result.

Proposition 3 (cf. [62]; see also [40]) *Let* $(\mathsf{G}, \mathbf{g})$ *and* $(\mathsf{G}', \mathbf{g}')$ *be two left-invariant Riemannian structures on simply connected nilpotent Lie groups* G *and* G'*, respectively. A diffeomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *is an isometry between* $(\mathsf{G}, \mathbf{g})$ *and* $(\mathsf{G}', \mathbf{g}')$ *if and only if* ϕ *is the composition* $\phi = L_{\phi(\mathbf{1})} \circ \phi'$ *of a left translation* $L_{\phi(\mathbf{1})}$ *on* G' *and a Lie group isomorphism* $\phi' = L_{\phi(\mathbf{1})^{-1}} \circ \phi : \mathsf{G} \to \mathsf{G}'$ *such that* $\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi' \cdot A, T_{\mathbf{1}}\phi' \cdot B)$.

For a sub-Riemannian Carnot group $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ or a Riemannian structure $(\mathsf{G}, \mathscr{D} = T\mathsf{G}, \mathbf{g})$ on a simply connected nilpotent Lie group, we have that the isotropy subgroup is given by

$$\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathscr{D}, \mathbf{g}) = \left\{\phi \in \mathsf{Aut}(\mathsf{G}) : T_{\mathbf{1}}\phi \cdot \mathscr{D}(\mathbf{1}) = \mathscr{D}(\mathbf{1}),\ \mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}_{\mathbf{1}}(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot B)\right\}.$$

Accordingly, the isometry group $\mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ decomposes as a semidirect product of the normal subgroup $L_{\mathsf{G}} = \{L_g : g \in \mathsf{G}\}$ of left translations and the isotropy subgroup $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ of the identity. Consequently, any isometry $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ is uniquely expressible as $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, where $\phi' \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}.\mathscr{D}, \mathbf{g})$. We shall denote by $d\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ the group $\{T_{\mathbf{1}}\phi : \phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathscr{D}, \mathbf{g})\} \cong \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ of *linearized isotropies.*

2.2 Pontryagin Maximum Principle

A drift-free left-invariant control affine system on a (real, finite-dimensional, connected) Lie group G consists of a family of left-invariant vector fields Ξ_u on G, linearly parametrized by controls $u \in \mathbb{R}^k$. Such a system is written as

$$\dot{g} = \Xi_u(g) = g(u_1 B_1 + \cdots + u_k B_k), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^k.$$

Here, $B_1, \ldots, B_k$ are linearly independent elements of the Lie algebra $\mathfrak{g}$. Admissible controls are bounded and measurable maps $u(\cdot) : [0, t_1] \to \mathbb{R}^k$. A trajectory for an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0, t_1] \to \mathsf{G}$ such that $\dot{g}(t) = \Xi_{u(t)}(g(t))$.

A standard argument shows that the length minimization problem

$$\dot{g}(t) \in \mathscr{D}(g(t)), \qquad g(0) = g_0, \quad g(t_1) = g_1,$$
$$\int_0^{t_1} \sqrt{\mathbf{g}(\dot{g}(t), \dot{g}(t))} \to \min$$

is equivalent to the energy minimization problem, or invariant optimal control problem:

$$\begin{gathered} \dot{g} = \Xi_u(g), \ \ g \in \mathsf{G}, \ u \in \mathbb{R}^k, \qquad g(0) = g_0, \ g(t_1) = g_1 \\ \int_0^{t_1} L(u(t))\, dt \to \min . \end{gathered} \tag{1}$$

Here, $\Xi_u(g) = g(u_1 B_1 + \cdots + u_k B_k)$ with $B_1, \ldots, B_k$ being some linearly independent elements of $\mathfrak{g}$ such that $\operatorname{span}(B_1, \ldots, B_k) = \mathscr{D}(\mathbf{1})$; $L(u(t)) = \mathbf{g_1}(\Xi_{u(t)}(\mathbf{1}), \Xi_{u(t)}(\mathbf{1}))$. (Energy minimizers are exactly those length minimizers which have constant speed.)

The Pontryagin Maximum Principle provides necessary conditions for optimality which are naturally expressed in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (see [6, 33]). The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$; here $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$. More precisely, each element $(g, p) \in \mathsf{G} \times \mathfrak{g}^*$ is identified with $(T_g L_{g^{-1}})^* \cdot p \in T_g^*\mathsf{G}$. To an optimal control problem (1), we associate, for each real number λ and each control parameter $u \in \mathbb{R}^k$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda L(u) + \xi(\Xi_u(g)) \\ &= \lambda L(u) + p(\Xi_u(\mathbf{1})), \qquad \xi = (g, p) \in \mathsf{G} \times \mathfrak{g}^*. \end{aligned}$$

We denote by $\vec{H}_u^\lambda$ the corresponding Hamiltonian vector field (with respect to the symplectic structure on $T^*\mathsf{G}$). In terms of the above Hamiltonians, the Maximum Principle can be stated as follows.

Maximum Principle *Suppose that the controlled trajectory* $(\bar{g}(\cdot), \bar{u}(\cdot))$*, defined over the interval* $[0, t_1]$*, is a solution for the optimal control problem* (1)*. Then, there exist a curve* $\xi(\cdot) : [0, t_1] \to T^*\mathsf{G}$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, t_1]$*, and a real number* $\lambda \leq 0$*, such that the following conditions hold for almost every* $t \in [0, t_1]$*:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{2}$$
$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{3}$$
$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H_u^\lambda(\xi(t)) = \textit{constant}. \tag{4}$$

Any optimal trajectory, $\bar{g}(\cdot) : [0, t_1] \to \mathsf{G}$, is the projection of some integral curve $\xi(\cdot)$ of the Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$. A pair $(\xi(\cdot), u(\cdot))$ is said to be an extremal pair if it satisfies the conditions (2), (3), and (4). The component $\xi(\cdot)$ of an extremal pair is called an extremal. An extremal curve is called normal if $\lambda < 0$ and abnormal if $\lambda = 0$. The *normal* (resp. *abnormal*) *geodesics* of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ are the projection to G of the normal (resp. abnormal) extremal curves $\xi(\cdot)$.

For the class of optimal control problems under consideration, the maximum condition (4) eliminates the parameter u from the family of Hamiltonians (H_u); as a result, we obtain a smooth G-invariant function H (without parameters) on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$; for any orthonormal frame $(X_1, X_2, \ldots, X_k)$ of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$, we have that $H(\xi) = \sum_{i=1}^k \xi(X_i(g))^2 =$

$\sum_{i=1}^k p(X_{\mathbf{1}}(\mathbf{1}))^2$, $\xi = (g, p) \in T_g^*\mathsf{G}$. The dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ admits a natural Poisson structure, the (minus) Lie-Poisson structure (see, e.g., [43]), given by

$$\{F, G\}(p) = -p([dF(p), dG(p)]).$$

Here, $F, G \in C^\infty(\mathfrak{g}^*)$ and $dF(p), dG(p)$ are elements of the double dual $\mathfrak{g}^{**}$ which is canonically identified with the Lie algebra $\mathfrak{g}$. The Hamiltonian vector field associated to a function $F \in C^\infty(\mathfrak{g}^*)$ is specified by $\vec{F}[G] = \{G, F\}$. The Hamilton–Poisson system H on $T^*\mathsf{G}$ can be reduced to a Hamilton–Poisson system on the Lie–Poisson space $\mathfrak{g}^*$. The normal geodesics of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ can then be described as follows.

Proposition 4 (cf. [6, 33]; see also [17]) *The normal geodesics $g(\cdot)$ of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ are given by*

$$\begin{cases} \dot{g} = T_{\mathbf{1}}L_g \cdot (\iota^* p)^\sharp \\ \dot{p} = \vec{H}(p) \end{cases} \tag{5}$$

where $H(p) = \frac{1}{2}(\iota^* p) \cdot (\iota^* p)^\sharp$, $g \in \mathsf{G}$, $p \in \mathfrak{g}^*$.

For a normal geodesic $g(\cdot)$, we have

$$\mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t)) = \mathbf{g}_{\mathbf{1}}((\iota^* p(t))^\sharp, (\iota^* p(t))^\sharp) = 2H(p(t)).$$

Therefore, the unit-speed geodesics correspond to $H(p(0)) = \frac{1}{2}$.

The *sub-Riemannian exponential* map $\mathrm{Exp}_g : T_g^*\mathsf{G} \to \mathsf{G}$ from $g \in \mathsf{G}$ is defined as $\mathrm{Exp}_g(\xi) = \pi \circ e^{\vec{H}}(\xi)$, where $\pi : T^*\mathsf{G} \to \mathsf{G}$ is the canonical projection and $\vec{H}$ is the Hamiltonian vector field on $T^*\mathsf{G}$. Note that, as $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ is complete (Proposition 1), the exponential map Exp_g is defined for all $\xi \in T_g^*\mathsf{G}$ ([2]). Isometries are compatible with the exponential map in the following sense.

Proposition 5 *If $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$, then $\phi \circ Exp_g = Exp_{\phi(g)} \circ (T_{\phi(g)}\phi^{-1})^*$.*

Proof Let $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ and let

$$T^*\phi : T^*\mathsf{G} \to T^*\mathsf{G}, \quad T_{\phi(g)}^*\mathsf{G} \ni \xi \longmapsto (T_g\phi)^* \cdot \xi \in T_g^*\mathsf{G}$$

be the cotangent lift of ϕ. $T^*\phi$ is a symplectic transformation of the symplectic structure on $T^*\mathsf{G}$ (see, e.g., [43]). Note that $T^*\phi^{-1}$ covers ϕ (i.e., $\phi \circ \pi = \pi \circ T^*\phi^{-1}$). Let $X_1, \dots X_k$ be an orthonormal frame for $(\mathsf{G}, \mathscr{D}, \mathbf{g})$. The Hamiltonian H on $T^*\mathsf{G}$ is given by $H(\xi) = \frac{1}{2}\sum_{i=1}^k \xi(X_i(g))^2$ for $\xi \in T_g\mathsf{G}$. We have that $(H \circ T^*\phi^{-1})(\xi) = \frac{1}{2}\sum_{i=1}^k \xi(\phi_*^{-1}X_i(g))^2$ for $\xi \in T_g\mathsf{G}$. As ϕ^{-1} is an isometry, we have that $\phi_*^{-1}X_1, \dots, \phi_*^{-1}X_k$ is also an orthonormal frame. Hence $H \circ T^*\phi^{-1} = H$. It follows that $T^*\phi^{-1}$ maps integral curves of $\vec{H}$ to integral curves of $\vec{H}$. Let $\xi(\cdot)$ be an integral curve of $\vec{H}$. Then $\pi(\xi(t)) = \mathrm{Exp}_{\pi(\xi(0))}(t\xi(0))$. Hence $(\phi \circ \pi)(\xi(t)) = \pi(T^*\phi^{-1} \cdot \xi(t)) = \mathrm{Exp}_{(\phi\circ\pi)(\xi(0))}(tT^*\phi^{-1} \cdot \xi(0))$. □

The exponential map is left invariant, i.e., for $\xi = (g, p)$ we have that $\mathrm{Exp}_g(g, p) = g\,\mathrm{Exp}_{\mathbf{1}}(\mathbf{1}, p)$. Hence, we shall consider only the exponential map $\mathrm{Exp}_{\mathbf{1}}$ from identity, which we simply write as Exp. Moreover, we have that $\mathrm{Exp} : \mathfrak{g}^* \to \mathsf{G}$ is given by $\mathrm{Exp}(t\,p(0)) = g(t)$ where $g(\cdot)$ and $p(\cdot)$ are solutions to Eq. 5 with $g(0) = \mathbf{1}$. Note that, for any element p contained in the annihilator $\mathscr{D}(\mathbf{1})^\circ = \{p \in \mathfrak{g}^* \,:\, p(A) = 0 \text{ for all } A \in \mathscr{D}(\mathbf{1})\} = \ker \iota^*$, we have $\mathrm{Exp}(p) = \mathbf{1}$.

Remark 2 In the Riemannian case, the usual (Riemannian) exponential map $\widetilde{\text{Exp}} : T_{\mathbf{1}}\mathsf{G} = \mathfrak{g} \to \mathsf{G}$ is simply given by $\widetilde{\text{Exp}}(A) = \text{Exp}(A^\flat)$. Also, for any isometry ϕ, we have that $\phi(\text{Exp}(A^\flat)) = \text{Exp}((T_{\mathbf{1}}\phi \cdot A)^\flat)$ as expected.

3 Structures, Isometries, and Geodesics

3.1 Sub-Riemannian and Riemannian Structures on the Heisenberg Groups

The $(2n+1)$-dimensional Heisenberg group is, up to isomorphism, the only $(2n+1)$-dimensional simply connected two-step nilpotent Lie group with a one-dimensional center. The Heisenberg group is often represented as $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ endowed with group product

$$(z, x, y) \cdot (z', x', y') = (z + z' + \tfrac{1}{2} \sum_{i=1}^{n} x_i y_i' - x_i' y_i,\ x + x',\ y + y').$$

We shall, however, prefer to represent the Heisenberg group as the matrix Lie group

$$\mathsf{H}_{2n+1} = \left\{ \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n & z + \frac{1}{2}\sum_{i=1}^n x_i y_i \\ 0 & 1 & 0 & \cdots & 0 & y_1 \\ 0 & 0 & 1 & \cdots & 0 & y_2 \\ & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & y_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = m(z, x_1, y_1, \ldots, x_n, y_n) : x_i, y_i, z \in \mathbb{R} \right\}.$$

Nonetheless, with this choice of parametrization $m : \mathbb{R}^{2n+1} \to \mathsf{H}_{2n+1}$, we have that

$$\begin{aligned} &m(z', x_1', y_1', \ldots, x_n', y_n')\, m(z', x_1', y_1', \ldots, x_n', y_n') \\ &\quad = m(z + z' + \tfrac{1}{2} \sum_{i=1}^{n} x_i y_i' - x_i' y_i,\ x_1 + x_1',\ y_1 + y_1',\ \ldots,\ x_n + x_n',\ y_n + y_n'). \end{aligned}$$

The Lie algebra of H_{2n+1} is given by

$$\mathfrak{h}_{2n+1} = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 0 & 0 & \cdots & 0 & y_1 \\ 0 & 0 & 0 & \cdots & 0 & y_2 \\ & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & y_n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = zZ + \sum_{i=1}^{n} x_i X_i + y_i Y_i \ :\ x_i, y_i, z \in \mathbb{R} \right\}$$

and has non-zero commutators $[X_i, Y_i] = Z$. If $\mathfrak{g}_1 = \text{span}(X_1, Y_1, \ldots, X_n, Y_n)$ and $\mathfrak{g}_2 = \text{span}(Z)$, then $\mathfrak{h}_{2n+1}$ decomposes as a direct sum of subspaces $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$. Therefore H_{2n+1} is a two-step Carnot group. Whenever convenient, the elements $Z, X_i, Y_i \in \mathfrak{h}_{2n+1}$ will be identified with their corresponding left-invariant vector fields.

Throughout, we use the ordered basis $(Z, X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$ for $\mathfrak{h}_{2n+1}$. Typically, we shall write an element $A \in \mathfrak{h}_{2n+1}$ as $A = a_z Z + \sum_{i=1}^n a_{x_i} X_i + a_{y_i} Y_i$. Let $(Z^*, X_1^*, Y_1^*, \ldots, X_n^*, Y_n^*)$ be the basis for $\mathfrak{h}_{2n+1}^*$ dual to $(Z, X_1, Y_1, \ldots, X_n, Y_n)$; we likewise write an element $p \in \mathfrak{h}_{2n+1}^*$ as $p = p_z Z^* + \sum_{i=1}^n p_{x_i} X_i^* + p_{y_i} Y_i^*$ or simply as $p = (p_z, p_{x_1}, p_{y_1}, \ldots, p_{x_n}, p_{y_n})$.

The automorphisms of $\mathfrak{h}_{2n+1}$ are exactly those linear isomorphisms that preserve the center $\mathfrak{z} = \mathrm{span}(Z)$ of $\mathfrak{h}_{2n+1}$ and for which the induced map on $\mathfrak{h}_{2n+1}/\mathfrak{z}$ preserves an appropriate symplectic structure (cf. [28]). More precisely, let ω be the skew-symmetric bilinear form on $\mathfrak{h}_{2n+1}$ specified by $[A, B] = \omega(A, B)Z$ for $A, B \in \mathfrak{h}_{2n+1}$. A linear isomorphism $\psi : \mathfrak{h}_{2n+1} \to \mathfrak{h}_{2n+1}$ is a Lie algebra automorphism if and only if $\psi \cdot Z = cZ$ and $\omega(\psi \cdot A, \psi \cdot B) = c\,\omega(A, B)$ for some $c \neq 0$.

We give a matrix representation for the group of automorphisms. The bilinear form ω takes the form

$$\omega = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{bmatrix}.$$

We note that the linear map

$$\sigma = \begin{bmatrix} -1 & 0 & & & & 0 \\ 0 & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ 0 & & & & 1 & 0 \end{bmatrix} \tag{6}$$

is an automorphism.

Proposition 6 (cf. [16, 53]) *The group of automorphisms* $\mathsf{Aut}(\mathfrak{h}_{2n+1})$ *is given by*

$$\left\{ \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}, \ \sigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} : r > 0, \ v \in \mathbb{R}^{1\times 2n}, \ g \in \mathsf{Sp}(n, \mathbb{R}) \right\}$$

where

$$\mathsf{Sp}(n, \mathbb{R}) = \left\{ g \in \mathbb{R}^{2n\times 2n} \ : \ g^\top J g = J \right\}$$

is the $n(2n+1)$*-dimensional symplectic group over* $\mathbb{R}$.

The Lie group exponential $\exp : \mathfrak{h}_{2n+1} \to \mathsf{H}_{2n+1}$ is simply given by

$$\exp\left(zZ + \textstyle\sum_{i=1}^n x_i X_i + y_i Y_i\right) = m(z, x_1, y_1, \ldots, x_n, y_n).$$

Accordingly, as $\phi(\exp(A)) = \exp(T_\mathbf{1}\phi \cdot A)$ for any automorphism $\phi \in \mathsf{Aut}(\mathsf{H}_{2n+1})$, the action of ϕ on an element $m(z, x_1, x_2, \ldots, x_n, y_n)$ is identical to the action of the linear map $T_\mathbf{1}\phi$ on the (exponential) coordinates $(z, x_1, y_1, \ldots, x_n, y_n)$. Also, as H_{2n+1} is simply connected, for every $\psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})$, there exists a $\phi \in \mathsf{Aut}(\mathsf{H}_{2n+1})$ such that $T_\mathbf{1}\phi = \psi$.

In a previous paper [16], we classified the left-invariant sub-Riemannian and Riemannian structures on H_{2n+1} up to isometric Lie group automorphisms (i.e., isometries which are also Lie group automorphisms). In light of Propositions 2 and 3, we restate these results as a classification of the left-invariant sub-Riemannian and Riemannian structures up to isometry. (A classification of Riemannian and Lorentzian metrics on the Heisenberg groups was also recently obtained in [60].)

Theorem 1 *Any left-invariant sub-Riemannian structure on* H_{2n+1} *is isometric to exactly one of the structures* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *specified by*

$$\begin{cases} \mathscr{H}(\mathbf{1}) &= \operatorname{span}(X_1, Y_1, \ldots, X_n, Y_n) \\ \mathbf{g}^\lambda_{\mathbf{1}} &= \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n) \end{cases}$$

i.e., with orthonormal frame

$$\left(\tfrac{1}{\sqrt{\lambda_1}} X_1, \tfrac{1}{\sqrt{\lambda_1}} Y_1, \tfrac{1}{\sqrt{\lambda_2}} X_2, \tfrac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n, \tfrac{1}{\sqrt{\lambda_n}} Y_n\right).$$

Here, $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ *parametrize a family of (non-isometric) class representatives.*

Note 1 As the distribution $\mathscr{H}$ is strongly bracket-generating all geodesics of the sub-Riemannian structure $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ are normal (see, e.g., [20, 49]).

Note 2 $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ is a sub-Riemannian Carnot group.

Theorem 2 *Any left-invariant Riemannian structure on* H_{2n+1} *is isometric to exactly one of the structures* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *specified by*

$$\tilde{\mathbf{g}}^\lambda_{\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}, \qquad \Lambda = \mathit{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)$$

i.e., with orthonormal frame

$$\left(Z, \tfrac{1}{\sqrt{\lambda_1}} X_1, \tfrac{1}{\sqrt{\lambda_1}} Y_1, \tfrac{1}{\sqrt{\lambda_2}} X_2, \tfrac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n, \tfrac{1}{\sqrt{\lambda_n}} Y_n\right).$$

Here $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ *parametrize a family of (non-isometric) class representatives.*

Note 3 Although λ_1 can be normalized to $\lambda_1 = 1$ in the sub-Riemannian case, we shall find it more convenient not to impose this normalization explicitly in the ensuing discussions. (This allows for a more unified presentation of the sub-Riemannian and Riemannian cases.)

3.2 Extensions of Sub-Riemannian Structures

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and center $\mathsf{Z}(\mathsf{G}) = \mathsf{Z}$; let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and let $q : \mathsf{G} \to \mathsf{G}/\mathsf{Z}$ be the canonical quotient map. We say that a sub-Riemannian structure $(\mathsf{G}, \tilde{\mathscr{D}}, \tilde{\mathbf{g}})$ is an *central extension* of a sub-Riemannian structure $(\mathsf{G}/\mathsf{Z}, \mathscr{D}, \mathbf{g})$ if

1. $T_{\mathbf{1}} q \cdot \tilde{\mathscr{D}}(\mathbf{1}) = \mathscr{D}(\mathbf{1})$;
2. $\tilde{\mathbf{g}}_{\mathbf{1}}(A, B) = \mathbf{g}_{\mathbf{1}}(T_{\mathbf{1}} q \cdot A, T_{\mathbf{1}} q \cdot B)$ for $A, B \in \hat{\mathscr{D}}(\mathbf{1})$, where
$$\hat{\mathscr{D}}(\mathbf{1}) = (\mathfrak{z} \cap \tilde{\mathscr{D}}(\mathbf{1}))^\perp = \{A \in \tilde{\mathscr{D}}(\mathbf{1}) \,:\, \tilde{\mathbf{g}}_{\mathbf{1}}(A, B) = 0 \text{ for } B \in \mathfrak{z} \cap \tilde{\mathscr{D}}(\mathbf{1})\}.$$

We call the sub-Riemannian structure $(\mathsf{G}, \hat{\mathscr{D}}, \hat{\mathbf{g}})$ with $\hat{\mathbf{g}} = \tilde{\mathbf{g}}\big|_{\hat{\mathscr{D}}}$ the *corresponding shrunk extension*.

For any $\mathscr{D}$-curve $g(\cdot)$ on G/Z and any $\hat{g}_0 \in \mathsf{G}$ such that $q(\hat{g}_0) = g(0)$, there exists a unique $\hat{\mathscr{D}}$-curve $\hat{g}(\cdot)$ on G such that $\hat{g}(0) = \hat{g}_0$ and $q \circ \hat{g}(\cdot) = g(\cdot)$. We say that $\hat{g}(\cdot)$ is the $\hat{\mathscr{D}}$*-lift* through $\hat{g}_0$ of $g(\cdot)$. On the other hand, for any $\tilde{\mathscr{D}}$-curve $\tilde{g}(\cdot)$, there exists a unique $\hat{\mathscr{D}}$-curve $\hat{g}(\cdot)$ such that $\hat{g}(0) = \tilde{g}(0)$ and $q \circ \hat{g}(\cdot) = q \circ \tilde{g}(\cdot)$. We call $\hat{g}(\cdot)$ the $\hat{\mathscr{D}}$*-projection* of $\tilde{g}(\cdot)$. Note that the $\hat{\mathscr{D}}$-projection of $\tilde{g}(\cdot)$ is exactly the $\hat{\mathscr{D}}$-lift through $\tilde{g}(0)$ of $q \circ \tilde{g}(\cdot)$. $\hat{\mathscr{D}}$-lifts and $\hat{\mathscr{D}}$-projections are compatible with (normal) geodesics in the following sense.

Proposition 7 ([15]) *The $\hat{\mathscr{D}}$-lift of any (minimizing, normal, or abnormal) geodesic of $(\mathsf{G}/\mathsf{Z}, \mathscr{D}, \mathbf{g})$ is a (minimizing, normal, or abnormal, respectively) geodesic of both $(\mathsf{G}, \hat{\mathscr{D}}, \hat{\mathbf{g}})$ and $(\mathsf{G}, \tilde{\mathscr{D}}, \tilde{\mathbf{g}})$. Furthermore, the normal geodesics of $(\mathsf{G}, \hat{\mathscr{D}}, \hat{\mathbf{g}})$ are exactly the $\hat{\mathscr{D}}$-projections of the normal geodesics of $(\mathsf{G}, \tilde{\mathscr{D}}, \tilde{\mathbf{g}})$.*

The Riemannian structure $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ is a central extension of the Riemannian structure on the Abelian group $\mathsf{H}_{2n+1}/\mathsf{Z}(\mathsf{H}_{2n+1}) \cong \mathbb{R}^{2n}$ with orthonormal frame

$$\left(\tfrac{1}{\sqrt{\lambda_1}} T_{\mathbf{1}}q \cdot X_1, \tfrac{1}{\sqrt{\lambda_1}} T_{\mathbf{1}}q \cdot Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} T_{\mathbf{1}}q \cdot X_n, \tfrac{1}{\sqrt{\lambda_n}} T_{\mathbf{1}}q \cdot Y_n\right). \tag{7}$$

Here, $q : \mathsf{H}_{2n+1} \to \mathsf{H}_{2n+1}/\mathsf{Z}(\mathsf{H}_{2n+1}) \cong \mathbb{R}^{2n}$. Moreover, the sub-Riemannian structure $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ tamed by $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ is the corresponding shrunk extension. Accordingly, we have the following result.

Proposition 8 *Suppose $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ tames $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$. Then:*

1. *The normal geodesics of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ are exactly the $\mathscr{H}$-projections of the normal geodesics of $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$.*
2. *The curve $g(t) = m(0, \frac{\bar{x}_1}{t_1}t, \frac{\bar{y}_1}{t_1}t, \ldots, \frac{\bar{x}_n}{t_1}t, \frac{\bar{y}_n}{t_1}t)$ with $t_1 = \sqrt{\sum_{i=1}^n \lambda_i(\bar{x}_i^2 + \bar{y}_i^2)}$ is a unit-speed minimizing geodesic from identity to $g(t_1) = m(0, \bar{x}_1, \bar{y}_1, \ldots, \bar{x}_n, \bar{y}_n)$ for both $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$.*

Proof The first item is simply a consequence of Proposition 7. The Euclidean space with orthonormal frame (7) has minimizing geodesics $g'(t) = (\frac{\bar{x}_1}{t_1}t, \frac{\bar{y}_1}{t_1}t, \ldots, \frac{\bar{x}_n}{t_1}t, \frac{\bar{y}_n}{t_1}t)$ for $(\bar{x}_1, \bar{y}_1, \ldots, \bar{x}_n, \bar{y}_n) \in \mathbb{R}^{2n}$. The $\mathscr{H}$-lift of $g'(t)$ through $g(0) = \mathbf{1}$ is $g(t)$. Hence by Proposition 7, these are minimizing geodesics. □

Remark 3 The first item in the above proposition explains (in part) the similarity between the Riemannian and sub-Riemannian cases in the ensuing sections.

3.3 Isometry Groups

We calculate the group of isometries for $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$. Let $\mu_1 > \mu_2 > \cdots > \mu_k > 0$ denote the distinct values in the list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and let $\nu_1, \ldots, \nu_k$ denote the corresponding multiplicities. We refer to the pair (μ, ν) as the *metric data* for the structure. It turns out that if $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ and $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ have the same metric data, then their respective isometry groups are identical.

Theorem 3 *Suppose $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ have metric data (μ, ν). The groups $d\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $d\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ of linearized isotropies are identical and given by*

$$\left\{ \begin{bmatrix} 1 & & & 0 \\ & g_1 & & \\ & & \ddots & \\ 0 & & & g_k \end{bmatrix}, \ \sigma \begin{bmatrix} 1 & & & 0 \\ & g_1 & & \\ & & \ddots & \\ 0 & & & g_k \end{bmatrix} \ : \ g_i \in \mathsf{U}(\nu_i) \right\}.$$

Here the unitary group $\mathsf{U}(\nu_i) = \mathsf{Sp}(\nu_i, \mathbb{R}) \cap \mathsf{O}(2\nu_i)$, the orthogonal group $\mathsf{O}(m) = \{g \in \mathbb{R}^{m\times m} : g^\top g = I_m\}$ and σ is the involutive automorphism (6).

Proof Suppose that $\psi \in d\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$, i.e., $\psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})$, $\psi \cdot \mathscr{H}(\mathbf{1}) = \mathscr{H}(\mathbf{1})$, and $\mathbf{g}^\lambda_{\mathbf{1}}(A, B) = \mathbf{g}^\lambda_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$. As ψ is a Lie algebra automorphism, we have

$$\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \quad \text{or} \quad \psi = \sigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$$

for some $r > 0$, $v \in \mathbb{R}^{1\times 2n}$ and $g \in \mathsf{Sp}(n, \mathbb{R})$. We need only consider the former case, as a simple computation shows that $\sigma \in d\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$.

For the sub-Riemannian case, we have $v = 0$ as $\psi \cdot \mathscr{H}(\mathbf{1}) = \mathscr{H}(\mathbf{1})$. Moreover, as $\mathbf{g}^\lambda_{\mathbf{1}}(A, B) = \mathbf{g}^\lambda_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathscr{H}(\mathbf{1})$, we get that $\Lambda = r^2 g^\top \Lambda g$. Since $Jg^\top = g^{-1}J$, the eigenvalues of $J\Lambda$ and $J\, g^\top \Lambda g$ coincide. Hence, as $\Lambda = r^2 g^\top \Lambda g$, the eigenvalues of $J\Lambda$ and $r^2 J\Lambda$ must also coincide. The eigenvalues of $J\Lambda$ are purely imaginary and appear in conjugate pairs $\pm i\lambda_j$, $j = 1, \dots, n$; likewise, the eigenvalues of $r^2 J\Lambda$ are $\pm i r^2 \lambda_j$, $j = 1, \dots, n$. Therefore, it follows that $r = 1$. On the other hand, for the Riemannian case, the condition $\mathbf{g}^\lambda_{\mathbf{1}}(A, B) = \mathbf{g}^\lambda_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathfrak{h}_{2n+1}$ implies that $r = 1$, $v = 0$, and $\Lambda = g^\top \Lambda g$.

In either case, we have

$$\psi = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}, \quad g^\top J g = J, \quad \text{and} \quad g^\top \Lambda g = \Lambda.$$

It is a simple matter to show that $J\Lambda g = gJ\Lambda$. Thus, as $\Lambda^2 = -J\Lambda J\Lambda$, we have $\Lambda^2 g = g\Lambda^2$. Hence $\Lambda^2 g w = \mu_i^2 g w$ whenever w is an eigenvector of Λ^2 associated to the eigenvalue μ_i^2. It follows that g (resp. $g^\top$) preserves each eigenspace of Λ^2. That is, $g\, \mathfrak{e}_i = \mathfrak{e}_i$ and $g^\top\, \mathfrak{e}_i = \mathfrak{e}_i$, $i = 1, \dots, k$ where $\mathfrak{e}_i$ denotes the eigenspace of Λ^2 corresponding to μ_i^2. Therefore, g takes block diagonal form

$$g = \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_k \end{bmatrix}, \qquad g_i \in \mathsf{GL}(2\nu_i, \mathbb{R}).$$

Moreover, as $g^\top J g = J$ and $g^\top \Lambda g = \Lambda$, we have $g_i \in \mathsf{Sp}(\nu_i, \mathbb{R}) \cap \mathsf{O}(2\nu_i) = \mathsf{U}(\nu_i)$. Finally, it is easy to verify that each mapping $\psi = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$ of this form satisfies $\psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})$, $\psi \cdot \mathscr{H}(\mathbf{1}) = \mathscr{H}(\mathbf{1})$, and $\mathbf{g}^\lambda_{\mathbf{1}}(A, B) = \mathbf{g}^\lambda_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$ and so is a linearized isotropy. □

Remark 4 We have that $\phi \in \mathsf{Aut}(\mathsf{H}_{2n+1})$,

$$T_{\mathbf{1}}\phi = \begin{bmatrix} 1 & & & 0 \\ & g_1 & & \\ & & \ddots & \\ 0 & & & g_n \end{bmatrix}, \qquad g_i \in \mathsf{SO}(2)$$

is always an isotropy of identity for $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ irrespective of the metric data. Hence, the isotropy subgroup is at least n-dimensional.

Corollary 1 *For the structures* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *and* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *with metric data* (μ, ν) *we have that*

$$\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda) = \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda) \cong (\mathsf{U}(\nu_1) \times \cdots \times \mathsf{U}(\nu_k)) \rtimes \{1, \sigma\}$$

and

$$n \leq \dim \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda}) = \dim \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^{\lambda}) = \sum_{i=1}^{k} \nu_i^2 \leq n^2.$$

The minimal dimension is attained when the values $\lambda_1, \ldots, \lambda_n$ are all distinct; the maximal dimension is attained when the values $\lambda_1, \ldots, \lambda_n$ are all identical.

Remark 5 The Riemannian structures $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^{\lambda})$ are not symmetric, i.e., there exists no involutive isotropy ϕ such that $T_{\mathbf{1}}\phi = -I$. However, the sub-Riemannian structures $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$ are sub-symmetric, i.e., there exists an involutive isotropy ϕ such that $T_{\mathbf{1}}\phi|_{\mathscr{H}(\mathbf{1})} = -I$ (cf. [56]), namely the automorphism δ_{-1} given by $T_{\mathbf{1}}\delta_{-1} : a_z Z + \sum_{i=1}^{n} a_{x_i} X_i + a_{y_i} Y_i \mapsto a_z Z - \sum_{i=1}^{n} a_{x_i} X_i + a_{y_i} Y_i$.

Remark 6 As noted in Section 2.1.1, the sub-Riemannian Carnot group $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$ admits the homotheties

$$\delta_r : m(z, x_1, y_1, \ldots, x_n, y_m) \mapsto m(r^2 z, r\, x_1, r\, y_1, \ldots, r\, x_n, r\, y_n), \qquad r > 0.$$

That is, $(\delta_r)_* \mathscr{H} = \mathscr{H}$ and $\delta_r^* \mathbf{g}^{\lambda} = r^2\, \mathbf{g}^{\lambda}$ and so $d^{SR}(\delta_r(g_1), \delta_r(g_2)) = r\, d^{SR}(g_1, g_2)$ for $g_1, g_2 \in \mathsf{H}_{2n+1}$. (Here, d^{SR} is the Carnot–Carathéodory distance associated with $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$.) On the other hand, $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^{\lambda})$ admits no homotheties which are not isometries (i.e., if $\phi^* \tilde{\mathbf{g}}^{\lambda} = r \tilde{\mathbf{g}}^{\lambda}$ for some $r > 0$, then $r = 1$ and ϕ is an isometry).

3.4 Exponential Map

The sub-Riemannian structure $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$ has (reduced) Hamiltonian

$$H^{SR} : \mathfrak{h}_{2n+1}^* \to \mathbb{R}, \qquad H^{SR}(p) = \sum_{i=1}^{n} \frac{p_{x_i}^2 + p_{y_i}^2}{\lambda_i}.$$

Likewise, the Riemannian structure $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^{\lambda})$ has Hamiltonian

$$H^{R} : \mathfrak{h}_{2n+1}^* \to \mathbb{R}, \qquad H^{R}(p) = p_z^2 + \sum_{i=1}^{n} \frac{p_{x_i}^2 + p_{y_i}^2}{\lambda_i}.$$

Here, $p = p_z Z^* + \sum_{i=1}^{n} p_{x_i} X_i^* + p_{y_i} Y_i^*$. The Hamiltonian vector fields $\vec{H}^{SR}$ and $\vec{H}^{R}$ coincide (as $C(p) = p_z^2$ is a Casimir function for $\mathfrak{h}_{2n+1}^*$) and are given by

$$\begin{cases} \dot{p}_z = 0 \\ \begin{bmatrix} \dot{p}_{x_i} \\ \dot{p}_{y_i} \end{bmatrix} = \begin{bmatrix} 0 & -p_z \\ p_z & 0 \end{bmatrix} \begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix}, \qquad i = 1, \ldots, n. \end{cases}$$

The normal geodesic $\dot{g} = m(z, x_1, y_1, \ldots, x_n, y_n)$ corresponding to an integral curve $p(\cdot)$ of $\vec{H}$ is then given by (see Proposition 4)

$$\begin{cases} \dot{z} = \frac{1}{2} \sum_{i=1}^{n} \frac{x_i\, p_{y_i}(t) - y_i\, p_{x_i}(t)}{\lambda_i} \\ \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \frac{1}{\lambda_i} \begin{bmatrix} p_{x_i}(t) \\ p_{y_i}(t) \end{bmatrix}, \qquad i = 1, \ldots, n \end{cases}$$

in the sub-Riemannian case and by

$$\begin{cases} \dot{z}(t) = p_z + \frac{1}{2} \sum_{i=1}^{n} \frac{x_i p_{y_i}(t) - y_i p_{x_i}(t)}{\lambda_i} \\ \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \frac{1}{\lambda_i} \begin{bmatrix} p_{x_i}(t) \\ p_{y_i}(t) \end{bmatrix}, \qquad i = 1, \ldots, n \end{cases}$$

in the Riemannian case. These equations are easily integrated (with $g(0) = \mathbf{1}$) to yield the exponential $\mathrm{Exp}(p(0)) = g(1)$.

Proposition 9 *The exponential map Exp* : $\mathfrak{h}^*_{2n+1} \to \mathsf{H}_{2n+1}$ *for* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *is given by* $Exp(p_z, p_{x_1}, p_{y_1}, \ldots, p_{x_n}, p_{y_n}) = m(z, x_1, y_1, \ldots, x_n, y_n)$*, where*

$$\begin{cases} z = \frac{1}{2p_z^2} \sum_{i=1}^{n} \left(p_{x_i}^2 + p_{y_i}^2\right)\left(\frac{p_z}{\lambda_i} - \sin\frac{p_z}{\lambda_i}\right) \\ \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \frac{1}{p_z} \begin{bmatrix} \sin\frac{p_z}{\lambda_i} & -\left(1 - \cos\frac{p_z}{\lambda_i}\right) \\ 1 - \cos\frac{p_z}{\lambda_i} & \sin\frac{p_z}{\lambda_i} \end{bmatrix} \begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix}, \qquad i = 1, \ldots, n \end{cases}$$

when $p_z \neq 0$ *and* $(z, x_1, y_1, \ldots, x_n, y_n) = (0, \frac{p_{x_1}}{\lambda_1}, \frac{p_{y_1}}{\lambda_1}, \ldots, \frac{p_{x_n}}{\lambda_n}, \frac{p_{y_n}}{\lambda_n})$ *when* $p_z = 0$.

Note 4 The annihilator of $\mathscr{H}(\mathbf{1})$ is given by $\mathscr{H}(\mathbf{1})^\circ = \mathrm{span}\{Z^*\}$.

Proposition 10 *The exponential map Exp* : $\mathfrak{h}^*_{2n+1} \to \mathsf{H}_{2n+1}$ *for* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *is given by* $Exp(p_z, p_{x_1}, p_{y_1}, \ldots, p_{x_n}, p_{y_n}) = m(z, x_1, y_1, \ldots, x_n, y_n)$*, where*

$$\begin{cases} z = p_z + \frac{1}{2p_z^2} \sum_{i=1}^{n} \left(p_{x_i}^2 + p_{y_i}^2\right)\left(\frac{p_z}{\lambda_i} - \sin\frac{p_z}{\lambda_i}\right) \\ \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \frac{1}{p_z} \begin{bmatrix} \sin\frac{p_z}{\lambda_i} & -\left(1 - \cos\frac{p_z}{\lambda_i}\right) \\ 1 - \cos\frac{p_z}{\lambda_i} & \sin\frac{p_z}{\lambda_i} \end{bmatrix} \begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix}, \qquad i = 1, \ldots, n \end{cases}$$

when $p_z \neq 0$ *and* $(z, x_1, y_1, \ldots, x_n, y_n) = (0, \frac{p_{x_1}}{\lambda_1}, \frac{p_{y_1}}{\lambda_1}, \ldots, \frac{p_{x_n}}{\lambda_n}, \frac{p_{y_n}}{\lambda_n})$ *when* $p_z = 0$.

Remark 7 The x_i and y_i coordinates for the sub-Riemannian and Riemannian exponential maps coincide; this is in accordance with the sub-Riemannian geodesics being $\mathscr{H}$-projections of the Riemannian geodesics (see Section 3.2).

Remark 8 The projection of a geodesic of $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ to $\mathsf{H}_{2n+1}/\mathsf{Z}(\mathsf{H}_{2n+1})$ (endowed with the Riemannian structure admitting the orthonormal frame (7)) has constant curvature (cf. [32]). Moreover, in the maximally symmetric case, this projection is a point, a line, or a circle. The same two comments hold true for $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$.

4 Totally Geodesic Subgroups

By analogy to the Riemannian case, there are two prospective definitions for a (connected) closed subgroup N to be a (normal) totally geodesic subgroup of a left-invariant sub-Riemannian structure $(\mathsf{G}, \mathscr{D}, \mathbf{g})$:

$\mathfrak{D}1.$ Whenever a normal geodesic $g(\cdot)$ of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ is tangent to N at some point $g(t_1) \in \mathsf{N}$, then the entire trace of $g(\cdot)$ is contained in N.

$\mathfrak{D}2.$ Any normal geodesic of the restricted structure $(\mathsf{N}, \mathscr{E}, \mathbf{g}|_{\mathscr{E}})$, $\mathscr{E}(g) = \mathscr{D}(g) \cap T_g\mathsf{N}$ is a normal geodesic of the ambient structure $(\mathsf{G}, \mathscr{D}, \mathbf{g})$.

Remark 9 As the sub-Riemannian structures on the Heisenberg group admit no abnormal geodesics (Note 1), we find it convenient here to restrict our definition to normal geodesics.

In the Riemannian case, $\mathfrak{D}1$ and $\mathfrak{D}2$ are equivalent. However, for sub-Riemannian structures, they do not coincide. Indeed, for the structure $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ on the Heisenberg group, any Abelian subgroup N with Lie algebra $\mathfrak{n} \subset \mathscr{H}(\mathbf{1})$ satisfies $\mathfrak{D}2$ but not $\mathfrak{D}1$ (see Proposition 9). On the other hand, it turns out that $\mathfrak{D}1$ implies $\mathfrak{D}2$ (at least on the Heisenberg group). Accordingly, we shall say that a closed subgroup N is a *totally geodesic subgroup* of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ if it satisfies $\mathfrak{D}1$. (We note, however, that in [7] totally geodesic submanifolds are defined in accordance to $\mathfrak{D}2$.)

Proposition 11 *Let N be a closed subgroup with Lie algebra $\mathfrak{n}$. The following statements are equivalent:*

1. *N is a totally geodesic subgroup of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$.*
2. *$Exp(p) \in \mathsf{N}$ for every $p \in (\sharp \circ \iota^*)^{-1}(\mathfrak{n}) \subseteq \mathfrak{g}^*$.*
3. *$(\sharp \circ \iota^*)^{-1}(\mathfrak{n}) \subseteq \mathfrak{g}^*$ is an invariant subspace of the Hamiltonian vector field $\vec{H}$.*

Remark 10 $(\sharp \circ \iota^*)^{-1}(\mathfrak{n}) = (\sharp \circ \iota^*)^{-1}(\mathfrak{n} \cap \mathscr{D}(\mathbf{1})) = (\iota^*)^{-1}((\mathfrak{n} \cap \mathscr{D}(\mathbf{1}))^\flat)$. So in the Riemannian case, items 2 and 3 can be restated as: (2) $\mathrm{Exp}(A^\flat) \in \mathsf{N}$ for $A \in \mathfrak{n}$, (3) $\mathfrak{n}^\flat \subseteq \mathfrak{g}^*$ is an invariant subspace of the Hamiltonian vector field $\vec{H}$.

Proof (1$\Rightarrow$3) Suppose N is a totally geodesic subgroup of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$. Let $p \in \mathfrak{g}^*$, $(\iota^* p)^\sharp \in \mathfrak{n}$. Then $g(t) = \mathrm{Exp}(t\,p)$ is a geodesic such that $\dot{g}(0) = (\iota^* p)^\sharp \in \mathfrak{n}$ (see Proposition 4). Therefore, $\mathrm{Exp}(t\,p) \in \mathsf{N}$ for all t including $t = 1$.

(2$\Rightarrow$3) Suppose $\mathrm{Exp}(p) \in \mathsf{N}$ for every $p \in (\sharp \circ \iota^*)^{-1}(\mathfrak{n})$. Let $p \in (\sharp \circ \iota^*)^{-1}(\mathfrak{n})$ and $p(\cdot)$ be the integral curve of $\vec{H}$ through $p(0) = p$. Let $g(t) = \mathrm{Exp}(t\,p(0))$. As $t\,p(0) \in (\sharp \circ \iota^*)^{-1}(\mathfrak{n})$ we have that $g(t) \in \mathsf{N}$ for all t. Hence, as $\dot{g}(t) = T_{\mathbf{1}} L_{g(t)} \cdot (\iota^* p(t))^\sharp$, we have that $(\iota^* p(t))^\sharp = T_{g(t)} L_{g(t)^{-1}} \cdot \dot{g}(t) \in \mathfrak{n}$. Thus $p(t) \in (\sharp \circ \iota^*)^{-1}(\mathfrak{n})$ for all t.

(3$\Rightarrow$1) Suppose $(\sharp \circ \iota^*)^{-1}(\mathfrak{n}) \subseteq \mathfrak{g}^*$ is an invariant subspace of the Hamiltonian vector field $\vec{H}$. Let $g(\cdot)$ be a normal geodesic tangent to N at $g(0)$. We have that $\dot{g}(t) = T_{\mathbf{1}} L_{g(t)} \cdot (\iota^* p(t))^\sharp$ for some $p(\cdot)$, $\dot{p}(t) = \vec{H}(p(t))$. Furthermore, $g(0)^{-1}\,\dot{g}(0) = (\iota^* p(0))^\sharp \in \mathfrak{n}$. Therefore, $p(0) \in (\sharp \circ \iota^*)^{-1}(\mathfrak{n})$ and so $p(t) \in (\sharp \circ \iota^*)^{-1}(\mathfrak{n})$ for all t. Consequently $g(t)^{-1}\,\dot{g}(t) \in \mathfrak{n}$ for all t and so (as $g(0) \in \mathsf{N}$) it follows that $g(t) \in \mathsf{N}$ for all t. $\square$

4.1 Notation

We fix some notation to be used throughout Sections 4.2 and 4.3. Let (μ, ν) be the metric data (see Section 3.3) of the sub-Riemannian structure $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ or Riemannian structure $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ under consideration (i.e., $\mu_1 > \mu_2 > \cdots > \mu_k > 0$ denote the distinct values in the list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\nu_1, \ldots, \nu_k$ denote the corresponding multiplicities). Furthermore, let $\chi_i = \nu_1 + \nu_2 + \cdots + \nu_i$ and $\chi_0 = 0$.

In the sub-Riemannian case, the Lie algebra of $\mathfrak{h}_{2n+1}$ decomposes as a direct sum $\mathfrak{h}_{2n+1} = \mathfrak{z} \oplus \mathscr{H}(\mathbf{1})$. Likewise, in the Riemannian case, $\mathfrak{h}_{2n+1} = \mathfrak{z} \oplus \mathfrak{z}^\perp$. Furthermore, the subspace $\mathscr{H}(\mathbf{1}) = \mathfrak{z}^\perp \subset \mathfrak{h}_{2n+1}$ decomposes as a direct sum $\mathscr{H}(\mathbf{1}) = \mathfrak{e}_1 \oplus \mathfrak{e}_2 \oplus \cdots \oplus \mathfrak{e}_k$ of eigenspaces of Λ corresponding to values $\mu_1, \ldots, \mu_k$ and with dimension $\nu_1, \ldots, \nu_k$, respectively. Hence, we have the following direct sum decomposition $\mathfrak{h}_{2n+1} = \mathfrak{z} \oplus \mathfrak{e}_1 \oplus \cdots \oplus \mathfrak{e}_k$.

We shall denote by $\rho_j(\theta)$ the isotropy specified by (see Remark 4)

$$T_{\mathbf{1}}\rho_j(\theta)\,\big|_{(X_j,Y_j)} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$T_{\mathbf{1}}\rho_j(\theta)\big|_{(Z,X_1,Y_1,\ldots,X_{j-1},Y_{j-1},X_{j+1},Y_{j+1},\ldots,X_n,Y_n)} = I_{2n-1}.$$

(We note that $\rho_j(\theta)$ is simply a rotation in the x_i, y_i plane.) Likewise, when $\lambda_i = \lambda_j$, we shall denote by $\rho_{i,j}(\theta)$ the isotropy specified by (see Theorem 3)

$$T_{\mathbf{1}}\rho_{j,k}(\theta)\,\big|_{(X_j,Y_j,X_k,Y_k)} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{bmatrix} \in \mathsf{U}(2) = \mathsf{Sp}(2,\mathbb{R}) \cap \mathsf{O}(4)$$
$$T_{\mathbf{1}}\rho_{j,k}(\theta)\,\big|_{(Z,X_1,Y_1,\ldots,X_{j-1},Y_{j-1},X_{j+1},Y_{j+1},\ldots,X_{k-1},Y_{k-1},X_{k+1},Y_{k+1},\ldots,X_n,Y_n)} = I_{2n-3}.$$

Furthermore, let ς denote the isotropy

$$\varsigma = \rho_1(\tfrac{\pi}{2}) \circ \rho_2(\tfrac{\pi}{2}) \circ \cdots \circ \rho_n(\tfrac{\pi}{2}) \tag{8}$$

($T_{\mathbf{1}}\varsigma$ is given by $Z \mapsto Z$, $X_i \mapsto Y_i$, and $Y_i \mapsto -X_i$).

4.2 Characterization and Classification

In the Riemannian case, the totally geodesic subalgebras of nonsingular two-step nilpotent metric Lie algebras were classified by Eberlein in [25]. Let $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{z}^\perp)$ be the linear map given by

$$\tilde{\mathbf{g}}^\lambda_{\mathbf{1}}([A,B],C) = \tilde{\mathbf{g}}^\lambda_{\mathbf{1}}(j(C)\cdot A, B), \qquad A, B \in \mathfrak{z}^\perp, \quad C \in \mathfrak{z}.$$

We have (with respect to $(X_1, Y_1, \ldots, X_n, Y_n)$)

$$j(Z) = \begin{bmatrix} 0 & -\frac{1}{\lambda_1} & & & 0 \\ \frac{1}{\lambda_1} & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\frac{1}{\lambda_n} \\ 0 & & & \frac{1}{\lambda_n} & 0 \end{bmatrix}.$$

Remark 11 A two-step nilpotent metric Lie algebra is said to be of Heisenberg type if $j(Z)^2 = -\|Z\|^2\mathrm{id}_{\mathfrak{z}^\perp}$. We note that $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ is of Heisenberg type exactly when $\lambda_1 = \cdots = \lambda_n = 1$.

By Eberlein's result, a subgroup N with Lie algebra $\mathfrak{n}$ is a totally geodesic subgroup of $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ if and only if exactly one of the following occurs:

1. $\mathfrak{n}$ is an Abelian subalgebra contained in $\mathfrak{z}^\perp$
2. $\mathfrak{n}$ is a subspace of $\mathfrak{z}$
3. $j(Z)(\mathfrak{n} \cap \mathfrak{z}^\perp) = \mathfrak{n} \cap \mathfrak{z}^\perp$ and $\mathfrak{n}$ is a direct sum of nonzero subspaces $\mathfrak{n} \cap \mathfrak{z}$ and $\mathfrak{n} \cap \mathfrak{z}^\perp$.

For the Riemannian structure $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ on the Heisenberg group, the above characterization can be specialized as follows.

Theorem 4 *A subgroup* N *with Lie algebra* $\mathfrak{n}$ *is a totally geodesic subgroup of* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *if and only if* $\mathfrak{n}$ *is an Abelian subalgebra contained in* $\mathfrak{z}^\perp$ *or*

$$T_{\mathbf{1}}\varsigma(\mathfrak{n}) = \mathfrak{n} \qquad \text{and} \qquad \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k, \quad \mathfrak{n}_i = \mathfrak{n} \cap \mathfrak{e}_i.$$

We now proceed to describe the totally geodesic subgroups of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$. It turns out that the totally geodesic subgroups of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ are closely related to those of $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$.

Proposition 12 *Let* N *be a subgroup with Lie algebra* $\mathfrak{n}$ *and let* $\widetilde{\mathsf{N}}$ *be the subgroup with Lie algebra* $\mathfrak{n} + \mathfrak{z}$. *Furthermore, let* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *be the Riemannian structure taming* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$. N *is a totally geodesic subgroup of* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *if and only if* $\widetilde{\mathsf{N}}$ *is a totally geodesic subgroup of* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$.

Proof Let $H^{SR} \in C^\infty(\mathfrak{h}_{2n+1}^*)$ be the Hamiltonian corresponding to $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and let $H^R \in C^\infty(\mathfrak{h}_{2n+1}^*)$ be the Hamiltonian corresponding to $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$. Likewise, let $\sharp_{SR} : \mathscr{H}(\mathbf{1})^* \to \mathscr{H}(\mathbf{1})$ and $\iota_{SR}^* : \mathfrak{h}_{2n+1}^* \to \mathscr{H}(\mathbf{1})^*$ be the maps associated to $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and let $\flat_R : \mathfrak{h}_{2n+1} \to \mathfrak{h}_{2n+1}^*$ be the map associated with $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$. We have that $\vec{H}^{SR} = \vec{H}^R$ (see Section 3.4). N is a totally geodesic subgroup of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ if and only if $(\sharp_{SR} \circ \iota_{SR}^*)^{-1}(\mathfrak{n}) \subseteq \mathfrak{g}^*$ is an invariant subspace of the Hamiltonian vector field $\vec{H}^{SR}$ (Proposition 11). It is a simple matter to show that $(\sharp_{SR} \circ \iota_{SR}^*)^{-1}(\mathfrak{n}) = (\mathfrak{n} + \mathfrak{z})^{\flat_R}$. The result then follows as $\widetilde{\mathsf{N}}$ is a totally geodesic subgroup of $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ if and only if $(\mathfrak{n}+\mathfrak{z})^{\flat_R}$ is an invariant subspace of $\vec{H}^R = \vec{H}^{SR}$. □

Accordingly, we get the following characterization of the totally geodesic subgroups of $(\mathsf{H}_{2n+1}, \mathscr{H}, \tilde{\mathbf{g}}^\lambda)$.

Theorem 5 *A subgroup* N *with Lie algebra* $\mathfrak{n}$ *is a totally geodesic subgroup of* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *if and only if*

$$T_{\mathbf{1}}\varsigma(\mathfrak{n}) = \mathfrak{n} \qquad \text{and} \qquad \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k, \quad \mathfrak{n}_i = \mathfrak{n} \cap \mathfrak{e}_i.$$

Remark 12 Suppose $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ tames $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$. The only totally geodesic subgroups of $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ which are not totally geodesic subgroups of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ are the Abelian subgroups with Lie algebra contained in $\mathfrak{z}^\perp$. Excluding the center (which is a totally geodesic subgroup of the sub-Riemannian structure to which no geodesic is tangent), these are exactly the totally geodesic subgroups of the Riemannian structure that are flat (cf. [51]).

Next, we enumerate the totally geodesic subgroups up to isometry.

Corollary 2 *Any subgroup of* H_{2n+1} *with Lie algebra spanned by*

$$(Z, X_{i_1}, Y_{i_1}, \ldots, X_{i_k}, Y_{i_k}), \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad 0 \leq k \leq n \tag{9}$$

is a totally geodesic subgroup of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$*; any other nontrivial totally geodesic subgroup is the image under an isotropy* $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *of one of these subgroups.*

Corollary 3 *Any subgroup of* H_{2n+1} *with Lie algebra spanned by Eq. 9 is a totally geodesic subgroup of* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$. *Moreover, any other non-Abelian totally geodesic subgroup is the image under an isotropy* $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *of one of these subgroups.*

Corollaries 2 and 3 By Theorems 4 and 5, the given subgroups are totally geodesic. It remains to be shown that any other totally geodesic subgroup is the image under an isotropy of identity of one of the subgroups given. We consider only the sub-Riemannian case (the Riemannian case is completely analogous).

Let $\varpi_i : \mathfrak{h}_{2n+1} \to \mathfrak{e}_i$ denote the projection corresponding to the direct sum decomposition $\mathfrak{h}_{2n+1} = \mathfrak{z} \oplus \mathfrak{e}_1 \oplus \cdots \oplus \mathfrak{e}_k$. Let N be a totally geodesic subgroup with Lie algebra $\mathfrak{n}$. Suppose $\varpi_r(\mathfrak{n}) \neq \{0\}$ for some $r \in \{1, 2, \ldots, k\}$. Note that, by Theorem 5, we have that $\varpi_i(\mathfrak{n}) = \mathfrak{n} \cap \mathfrak{e}_i$. Let $A_1, \ldots, A_s$ be a basis for $\varpi_r(\mathfrak{n})$ and let $A_j = \sum_{i=\chi_{r-1}+1}^{\chi_r} a^j_{x_i} X_i + a^j_{y_i} Y_i$. There exists t such that $\chi_{r-1} + 1 \leq t \leq \chi_r$,

$$a^j_{x_i} = a^j_{y_i} = 0, \qquad i = \chi_{r-1} + 1, \ldots, t - 1, \quad j = 1, \ldots, s \tag{10}$$

and $(a^u_{x_t})^2 + (a^u_{y_t})^2 \neq 0$ for some u; we may assume $u = 1$. (Note that when $t = \chi_{r-1} + 1$, the condition (10) falls away.) Let $a^1_{x_t} = \varepsilon \cos\theta$ and $a^1_{y_t} = \varepsilon \sin\theta$; we may assume $\varepsilon = 1$. We have that $\rho_r(-\theta) \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$. Hence $B_1 = T_{\mathbf{1}}\rho_r(-\theta) \cdot A_1 = X_t + \sum_{i=t+1}^{\chi_r} a^1_{x_i} X_i + a^1_{y_i} Y_i$. Suppose $(a^1_{x_{\bar{t}}})^2 + (a^1_{y_{\bar{t}}})^2 \neq 0$ for some $\bar{t} \geq t + 1$; we may assume $a^1_{x_{\bar{t}}} \neq 0$ and $a^1_{y_{\bar{t}}} = 0$ by application of an appropriate isotropy $\rho_{\bar{t}}(\theta)$. Let $a^1_{x_t} = \varepsilon \cos\theta$ and $a^1_{x_{\bar{t}}} = \varepsilon \sin\theta$. Then $\rho_{t,\bar{t}}(-\theta) \cdot B_1 = \varepsilon X_t + \sum_{i=t+1,\, i\neq\bar{t}}^{\chi_r} a^1_{x_i} X_i + a^1_{y_i} Y_i$; by rescaling, we may assume $\varepsilon = 1$. Accordingly, (repeating this process several times if necessary) there exists an isotropy ϕ such that $T_{\mathbf{1}}\phi \cdot B_1 = X_t$.

As $T_{\mathbf{1}}\varsigma(\mathfrak{n}) = \mathfrak{n}$ and $T_{\mathbf{1}}\varsigma \circ \varpi_i = \varpi_i \circ T_{\mathbf{1}}\varsigma$, we have $T_{\mathbf{1}}\varsigma(\varpi_r(\mathfrak{n})) = \varpi_r(\mathfrak{n})$. Hence, we get that $T_{\mathbf{1}}\varsigma(X_t) = Y_t \in T_{\mathbf{1}}\phi(\varpi_r(\mathfrak{n}))$. Accordingly, there exists an isotropy ϕ such that $X_t, Y_t \in T_{\mathbf{1}}\phi(\varpi_r(\mathfrak{n}))$. Therefore, $T_{\mathbf{1}}\phi(\varpi_r(\mathfrak{n}))$ has basis $X_t, Y_t, C_3, \ldots, C_s$ where $C_3, \ldots, C_s$ are some elements of $\mathfrak{e}_i$ with $c^i_{x_j} = c^i_{y_j} = 0$ for $j = \chi_{r-1}+1, \ldots, t$. Repeating the above procedure (possibly several times), it follows that there exists an isotropy ϕ such that $T_{\mathbf{1}}\phi(\varpi_r(\mathfrak{n}))$ has basis $\{X_{t_1}, Y_{t_1}, \ldots, X_{t_s}, Y_{t_s}\}$ for some $\chi_{r-1} + 1 \leq t_1 < t_2 < \ldots < t_s \leq \chi_r$ and some $0 \leq s \leq \nu_r$. As $\varpi_r(\mathfrak{n}) = \mathfrak{n} \cap \mathfrak{e}_r$, it follows that $X_{t_1}, Y_{t_1}, \ldots, X_{t_s}, Y_{t_s} \in T_{\mathbf{1}}\phi(\mathfrak{n})$. Moreover, we have that the isotropy ϕ constructed fixes all elements in $\mathfrak{z}$ and $\varpi_i(\mathfrak{n})$, $i \neq r$. Therefore, the procedure can be carried out for each subspace $\varpi_i(\mathfrak{n})$. Lastly, as $X_i, Y_i \in T_{\mathbf{1}}\phi(\mathfrak{n})$ for some $i \in \{1, 2, \ldots, n\}$, it follows that $Z = [X_i, Y_i] \in T_{\mathbf{1}}\phi(\mathfrak{n})$. □

4.3 Representative Subgroups

We say that a totally geodesic subgroup N of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ is *representative* if the following four conditions hold:

1. The distribution $\mathscr{E}$ on N given by $\mathscr{E}(g) = \mathscr{D}(g) \cap T_g\mathsf{N}$ is bracket generating.
2. For every geodesic $g(\cdot)$ of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$, there exists an isometry $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ such that the trace of $\phi \circ g(\cdot)$ is contained in N.
3. Every minimizing geodesic of $(\mathsf{N}, \mathscr{E}, \mathbf{g}|_{\mathscr{E}})$ is a minimizing geodesic of the ambient structure $(\mathsf{G}, \mathscr{D}, \mathbf{g})$.
4. For every isometry $\phi \in \mathsf{Iso}(\mathsf{N}, \mathscr{E}, \mathbf{g}|_{\mathscr{E}})$, there exists an isometry $\tilde{\phi} \in (\mathsf{G}, \mathscr{D}, \mathbf{g})$ such that $\tilde{\phi}(\mathsf{N}) = \mathsf{N}$ and $\tilde{\phi}|_{\mathsf{N}} = \phi$.

For example, the Euclidean space $\mathbb{E}^n$ (viewed as a left-invariant Riemannian structure on the Abelian group $\mathbb{R}^n$) has representative totally geodesic subgroup $\mathbb{E}^1$. If N is a representative totally geodesic subgroup of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$, then the problem of finding the minimizing geodesics of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ can effectively be reduced to the problem of finding the minimizing geodesics of $(\mathsf{N}, \mathscr{E}, \mathbf{g}|_{\mathscr{E}})$ in the following way.

Proposition 13 *Let* N *be a representative totally geodesic subgroup of* $(\mathsf{G}, \mathscr{D}, \mathbf{g})$*. If* $g(\cdot)$ *is a minimizing geodesic of* $(\mathsf{N}, \mathscr{E}, \mathbf{g}\,|_{\mathscr{E}})$*, then* $\phi \circ g(\cdot)$ *is a minimizing geodesic of* $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ *for every* $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$*. Moreover, for every minimizing geodesic* $\bar{g}(\cdot)$ *of* $(\mathsf{G}, \mathscr{D}, \mathbf{g})$*, there exists an isometry* $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ *and a minimizing geodesic* $g(\cdot)$ *of* $(\mathsf{N}, \mathscr{E}, \mathbf{g}\,|_{\mathscr{E}})$ *such that* $\bar{g}(\cdot) = \phi \circ g(\cdot)$*.*

Proof Let $g(\cdot)$ be a minimizing geodesic of $(\mathsf{N}, \mathscr{E}, \mathbf{g}\,|_{\mathscr{E}})$. Then $g(\cdot)$ is a minimizing geodesic of the ambient structure $(\mathsf{G}, \mathsf{N}, \mathbf{g})$. Hence, as isometries map minimizing geodesics to minimizing geodesics, it follows that $\phi \circ g(\cdot)$ is a minimizing geodesic of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$ for every $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$. On the other hand, let $\bar{g}(\cdot)$ be a minimizing geodesic of $(\mathsf{G}, \mathscr{D}, \mathbf{g})$. Then there exists an isometry $\phi \in \mathsf{Iso}(\mathsf{G}, \mathscr{D}, \mathbf{g})$ such that the trace of $\phi \circ \bar{g}(\cdot)$ is contained in N. Consequently, $g(\cdot) = \phi \circ \bar{g}(\cdot)$ is a minimizing geodesic of $(\mathsf{N}, \mathscr{E}, \mathbf{g}\,|_{\mathscr{E}})$ such that $\bar{g}(\cdot) = \phi^{-1} \circ g(\cdot)$. □

We now proceed to find representative totally geodesic subgroups of minimal dimension for $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^{\lambda})$.

Theorem 6 *For both* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$ *and* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^{\lambda})$*, the* $(2k+1)$*-dimensional subgroup* N *with Lie algebra spanned by*

$$(Z, X_1, Y_1, X_{\chi_1+1}, Y_{\chi_1+1}, X_{\chi_2+1}, Y_{\chi_2+1}, \ldots, X_{\chi_{k-1}+1}, Y_{\chi_{k-1}+1})$$

is a representative totally geodesic subgroup of minimal dimension.

Proof We provide a proof for the sub-Riemannian case; the Riemannian case can be proved similarly. Suppose $\lambda_i = \lambda_{i+1}$. Let N be the totally geodesic subgroup with Lie algebra spanned by $(Z, X_1, Y_1, \ldots, X_i, Y_i, X_{i+2}, Y_{i+2}, \ldots, X_n, Y_n)$. We claim that N is a representative totally geodesic subgroup. Clearly, $\mathscr{E}$ is bracket generating. By Theorem 3, it follows that, for every isometry $\phi \in \mathsf{Iso}(\mathsf{N}, \mathscr{E}, \mathbf{g}^{\lambda}\,|_{\mathscr{E}})$, there exists an isometry $\tilde{\phi} \in (\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$ such that $\tilde{\phi}(\mathsf{N}) = \mathsf{N}$ and $\tilde{\phi}\,|_{\mathsf{N}} = \phi$.

Let $g(t) = \mathrm{Exp}(t\,p)$ be a (normal) geodesic on H_{2n+1} and let $p_{x_i} = r_i \cos\theta_i$, $p_{y_i} = r_i \sin\theta_i$. Then $\rho_i(-\theta_i)(\mathrm{Exp}(t\,p)) = \mathrm{Exp}(t\,(T_{\mathbf{1}}\rho_i(\theta_i))^* \cdot p)$, where the X_i^*, Y_i^* components of $p' = (T_{\mathbf{1}}\rho_i(\theta_i))^* \cdot p$ are $p'_{x_i} = r_i$ and $p'_{y_i} = 0$ (see Proposition 5). Likewise, there then exists an isotropy $\rho_{i+1}(-\theta_{i+1})$ such that $p'' = (T_{\mathbf{1}}\rho_{i+1}(\theta_{i+1}))^* \cdot p'$ takes the form

$$p'' = (p_z, p_{x_1}, p_{y_1}, \ldots, p_{x_{i-1}}, p_{y_{i-1}}, r_i, 0, r_{i+1}, 0, p_{x_{i+1}}, p_{y_{i+1}}, \ldots, p_{x_n}, p_{y_n}).$$

Let $r_i = r\cos\theta$ and $r_{i+1} = r\sin\theta$. We have that $\rho_{i,i+1}(-\theta) \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$ and $p''' = (T_{\mathbf{1}}\rho_{i,i+1}(\theta))^* \cdot p''$ takes the form

$$p''' = (p_z, p_{x_1}, p_{y_1}, \ldots, p_{x_{i-1}}, p_{y_{i-1}}, r, 0, 0, 0, p_{x_{i+1}}, p_{y_{i+1}}, \ldots, p_{x_n}, p_{y_n}).$$

If $m(\bar{z}(t), \bar{x}_1(t), \bar{y}_1(t), \ldots, \bar{x}_n(t), \bar{y}_n(t)) = \mathrm{Exp}(t\,p''')$, then $\bar{x}_{i+1}(t) = \bar{y}_{i+1}(t) = 0$. Thus, for every geodesic $g(\cdot)$ of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$, there exists an isotropy ϕ such that the trace of $\phi \circ g(\cdot)$ is contained in N.

It remains to be shown that every minimizing geodesic of $(\mathsf{N}, \mathscr{E}, \mathbf{g}^{\lambda}\,|_{\mathscr{E}})$ is a minimizing geodesic of the ambient structure $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$. Let $g(\cdot)$ be a minimizing geodesic of $(\mathsf{N}, \mathscr{E}, \mathbf{g}^{\lambda}\,|_{\mathscr{E}})$. We may assume $g(0) = \mathbf{1}$ (by left invariance). Suppose that $g(\cdot)$ is not a minimizing geodesic of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})$, i.e., there exists a minimizing geodesic $\bar{g}(\cdot)$, $\bar{g}(0) = \mathbf{1}$, $\bar{g}(t_1) = g(t_1)$ such that

$$\ell(\bar{g}(\cdot)) = d^{(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^{\lambda})}(\mathbf{1}, g(t_1)) < d^{(\mathsf{N}, \mathscr{E}, \mathbf{g}^{\lambda}|_{\mathscr{E}})}(\mathbf{1}, g(t_1)) = \ell(g(\cdot)).$$

We have $\bar{g}(t) = \mathrm{Exp}(t\,p)$ for some $p \in \mathfrak{h}^*_{2n+1}$. Therefore

$$\begin{bmatrix} \bar{x}_j(t) \\ \bar{y}_j(t) \end{bmatrix} = \frac{1}{p_z} \begin{bmatrix} \sin \frac{p_z t}{\lambda_j} & -\left(1 - \cos \frac{p_z t}{\lambda_j}\right) \\ 1 - \cos \frac{p_z t}{\lambda_j} & \sin \frac{p_z}{\lambda_j t} \end{bmatrix} \begin{bmatrix} p_{x_j} \\ p_{y_j} \end{bmatrix}, \qquad j = i, i+1$$

However, as $\bar{g}(t_1) = g(t_1) \in \mathsf{N}$, we have that $\bar{x}_{i+1}(t_1) = \bar{y}_{i+1}(t_1) = 0$. Thus either $p_{x_{i+1}} = p_{y_{i+1}} = 0$ or $p_z t_1 \in 2\pi \lambda_{i+1}\mathbb{Z}$. Suppose $p_{x_{i+1}} = p_{y_{i+1}} = 0$. Then $\bar{g}(\cdot)$ is a curve on N and so is a minimizing geodesic of $(\mathsf{N}, \mathscr{E}, \mathbf{g}^\lambda\big|_{\mathscr{E}})$. Thus

$$\ell(g(\cdot)) = d^{(\mathsf{N}, \mathscr{E}, \mathbf{g}^\lambda|_{\mathscr{E}})}(\mathbf{1}, g(t_1)) = \ell(\bar{g}(\cdot))$$

which contradicts $\ell(\bar{g}(\cdot)) < \ell(g(\cdot))$. On the other hand, suppose $p_z t_1 \in 2\pi \lambda_{i+1}\mathbb{Z}$, $p_z \neq 0$. Then $\bar{x}_i(t_1) = \bar{y}_i(t_1) = 0$ (as $\lambda_i = \lambda_{i+1}$). Consequently, the isotropy ϕ constructed above fixes the point $g(t_1)$. Moreover, $\phi \circ \bar{g}(\cdot)$ is a minimizing geodesic of $(\mathsf{N}, \mathscr{E}, \mathbf{g}^\lambda\big|_{\mathscr{E}})$. Thus

$$\ell(g(\cdot)) = d^{(\mathsf{N}, \mathscr{E}, \mathbf{g}^\lambda|_{\mathscr{E}})}(\mathbf{1}, g(t_1)) = \ell(\phi \circ \bar{g}(\cdot)) = \ell(\bar{g}(\cdot))$$

which contradicts $\ell(\bar{g}(\cdot)) < \ell(g(\cdot))$. Lastly, suppose $p_z = 0$. In this case $g(t_1) = (0, x_1(t_1), y_1(t_1), \ldots, x_i(t_1), y_i(t_1), 0, 0, x_{i+2}(t_1), y_{i+2}(t_1), \ldots, x_n(t_1), y_n(t_1))$. Therefore, by Proposition 8, it follows that $d^{(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)}(\mathbf{1}, g(t_1)) = d^{(\mathsf{N}, \mathscr{E}, \mathbf{g}^\lambda|_{\mathscr{E}})}(\mathbf{1}, g(t_1))$, which again contradicts $\ell(\bar{g}(\cdot)) < \ell(g(\cdot))$.

It is easy to show that the property of being a representative totally geodesic subgroup is transitive. Consequently, we can apply the above process until we end up with a representative totally geodesic subgroup of the given form (i.e., one that is isometric to a structure for which $\lambda_1 > \cdots > \lambda_k > 0$). Every (linearized) isotropy preserves $\mathfrak{z}$ and each subspace $\mathfrak{e}_i$; moreover, $\dim \mathfrak{e}_i \geq 2$, $i = 1, \ldots, k$. Therefore, no representative totally geodesic subgroup can have dimension less than $2k + 1$. $\square$

Remark 13 The maximally symmetric structures (i.e., ones for which $\lambda_1 = \lambda_2 = \cdots = \lambda_n$) have minimal representative totally geodesic subgroup (isomorphic to) H_3. The minimally symmetric structures (i.e., ones for which $\lambda_1 > \lambda_2 > \cdots > \lambda_n$) have no representative totally geodesic subgroups (except trivially the group H_{2n+1} itself).

Note 5 Henceforth, we will consider only the minimally symmetric structures (i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_n$), as any structure $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ or $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ has a representative totally geodesic subgroup isometric to one of these structures.

5 Minimizing Geodesics

5.1 Conjugate Locus

A point $\bar{g} \in \mathsf{H}_{2n+1}$ is said to be *conjugate* to $\mathbf{1}$ if it is a critical value of $\mathrm{Exp} : \mathfrak{g}^* \to \mathsf{G}$. Likewise, $\bar{g}$ is conjugate to $\mathbf{1}$ along the geodesic $g(t) = \mathrm{Exp}(t\,p)$ if $\bar{g} = g(t_1)$ and $t_1\,p_1$ is a critical point of the exponential map. A point $\bar{g} \in \mathsf{H}_{2n+1}$ is said to be a *first conjugate point* if there exists a geodesic $g(t) = \mathrm{Exp}(t\,p)$ with $g(t_1) = \bar{g}$ conjugate to $\mathbf{1}$ and $g(t)$ not conjugate to $\mathbf{1}$ for $0 < t < t_1$. The collection of first conjugate points is referred as the *first conjugate locus*. It is known that a geodesic $g(t) = \mathrm{Exp}(t\,p)$ is not minimizing after passing through a conjugate point (see, e.g., [2, 6, 33]).

We proceed to determine the first conjugate locus for both $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$. Throughout, we assume $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ (see Note 5). In both cases, the Jacobian $\mathscr{J}$ of Exp takes the form (cf. [47])

$$\mathscr{J} = \begin{bmatrix} \frac{\partial z}{\partial p_z} & z_{x_1} & z_{x_2} & \cdots & z_{x_i} & z_{y_i} & \cdots & z_{x_n} & z_{y_n} \\ b_{x_1} & a^1_{11} & a^1_{12} & & & & & & \\ b_{y_1} & a^1_{21} & a^1_{22} & & & & & & \\ \vdots & & & \ddots & & & & & \\ b_{x_i} & & & & a^i_{11} & a^i_{12} & & & \\ b_{y_i} & & & & a^i_{21} & a^i_{22} & & & \\ \vdots & & & & & & \ddots & & \\ b_{x_n} & & & & & & & a^n_{11} & a^n_{12} \\ b_{y_n} & & & & & & & a^n_{21} & a^n_{22} \end{bmatrix}$$

where

$$\begin{aligned} z_{x_i} &= \frac{\partial z}{\partial p_{x_i}} = \frac{p_{x_i}}{p_z^2}\left(\frac{p_z}{\lambda_i} - \sin\frac{p_z}{\lambda_i}\right) \\ z_{y_i} &= \frac{\partial z}{\partial p_{y_i}} = \frac{p_{y_i}}{p_z^2}\left(\frac{p_z}{\lambda_i} - \sin\frac{p_z}{\lambda_i}\right) \\ b_{x_i} &= \frac{\partial x_i}{\partial p_z} = \frac{1}{p_z^2}\left(p_{y_i} - \frac{p_z p_{y_i} + p_{x_i}\lambda_i}{\lambda_i}\sin\frac{p_z}{\lambda_i} + \frac{p_z p_{x_i} - p_{y_i}\lambda_i}{\lambda_i}\cos\frac{p_z}{\lambda_i}\right) \\ b_{y_i} &= \frac{\partial y_i}{\partial p_z} = \frac{1}{p_z^2}\left(-p_{x_i} + \frac{p_z p_{y_i} + p_{x_i}\lambda_i}{\lambda_i}\cos\frac{p_z}{\lambda_i} + \frac{p_z p_{x_i} - p_{y_i}\lambda_i}{\lambda_i}\sin\frac{p_z}{\lambda_i}\right) \end{aligned}$$

and

$$A_i = \begin{bmatrix} a^i_{11} & a^i_{12} \\ a^i_{21} & a^i_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_i}{\partial p_{x_i}} & \frac{\partial x_i}{\partial p_{y_i}} \\ \frac{\partial y_i}{\partial p_{x_i}} & \frac{\partial y_i}{\partial p_{y_i}} \end{bmatrix} = \frac{1}{p_z}\begin{bmatrix} \sin\frac{p_z}{\lambda_i} & -(1-\cos\frac{p_z}{\lambda_i}) \\ 1-\cos\frac{p_z}{\lambda_i} & \sin\frac{p_z}{\lambda_i} \end{bmatrix}.$$

In the sub-Riemannian case

$$\frac{\partial z}{\partial p_z} = \frac{1}{2p_z^2}\sum_{i=1}^n (p_{x_i}^2 + p_{y_i}^2)\left(\frac{2}{p_z}\sin\frac{p_z}{\lambda_i} - \frac{1}{\lambda_i}\left(1+\cos\frac{p_z}{\lambda_i}\right)\right)$$

and so

$$\begin{aligned} \det\mathscr{J} &= \frac{1}{2p_z^2}\sum_{i=1}^n (p_{x_i}^2 + p_{y_i}^2)\left(\frac{2}{p_z}\sin\frac{p_z}{\lambda_i} - \frac{1}{\lambda_i}\left(1+\cos\frac{p_z}{\lambda_i}\right)\right)\prod_{i=1}^n |A_i| \\ &\quad + \sum_{i=1}^n \left(-z_{x_i}(b_{x_i}a^i_{22} - b_{y_i}a^i_{12}) + z_{y_i}(b_{x_i}a^i_{21} - b_{y_i}a^i_{11})\right)\prod_{j\neq i}|A_j| \\ &= \sum_{i=1}^n \Bigg[\frac{1}{2p_z^2}\left(p_{x_i}^2 + p_{y_i}^2\right)\left(\frac{2}{p_z}\sin\frac{p_z}{\lambda_i} - \frac{1}{\lambda_i}\left(1+\cos\frac{p_z}{\lambda_i}\right)\right)\left(\frac{2}{p_z^2}\left(1-\cos\frac{p_z}{\lambda_i}\right)\right) \\ &\qquad - z_{x_i}\left(b_{x_i}a^i_{22} - b_{y_i}a^i_{12}\right) + z_{y_i}\left(b_{x_i}a^i_{21} - b_{y_i}a^i_{11}\right)\Bigg]\prod_{j\neq i}\frac{2}{p_z^2}\left(1-\cos\frac{p_z}{\lambda_i}\right) \\ &= \frac{2^n}{p_z^{2(n+1)}}\sum_{i=1}^n \frac{1}{\lambda_i}\left(p_{x_i}^2 + p_{y_i}^2\right)\left(1-\cos\frac{p_z}{\lambda_i} - \frac{p_z}{2\lambda_i}\sin\frac{p_z}{\lambda_i}\right)\prod_{j\neq i}\left(1-\cos\frac{p_z}{\lambda_j}\right). \end{aligned}$$

Likewise, in the Riemannian case

$$\frac{\partial z}{\partial p_z} = 1 + \frac{1}{2p_z^2}\sum_{i=1}^{n}(p_{x_i}^2 + p_{y_i}^2)\left(\frac{2}{p_z}\sin\frac{p_z}{\lambda_i} - \frac{1}{\lambda_i}\left(1+\cos\frac{p_z}{\lambda_i}\right)\right)$$

and so

$$\begin{aligned}\det \mathscr{J} &= \frac{2^n}{p_z^{2n}}\prod_{i=1}^{n}\left(1-\cos\frac{p_z}{\lambda_i}\right)\\ &\quad+\frac{2^n}{p_z^{2(n+1)}}\sum_{i=1}^{n}\frac{1}{\lambda_i}(p_{x_i}^2+p_{y_i}^2)\left(1-\cos\frac{p_z}{\lambda_i}-\frac{p_z}{2\lambda_i}\sin\frac{p_z}{\lambda_i}\right)\prod_{j\neq i}\left(1-\cos\frac{p_z}{\lambda_j}\right).\end{aligned}$$

The term $\eta_i(p_z) = 1 - \cos\frac{p_z}{\lambda_i} - \frac{p_z}{2\lambda_i}\sin\frac{p_z}{\lambda_i}$ is positive for

$$p_z \in (-2\pi\lambda_i, 0) \cup (0, 2\pi\lambda_i) \supseteq (-2\pi\lambda_n, 0) \cup (0, 2\pi\lambda_n)$$

and zero for $p_z = 0$ and $p_z = \pm 2\pi\lambda_i$. Indeed, $\eta_i(0) = 0$, $\frac{d\eta_i}{p_z}\big|_{p_z=0} = 0$ and $\frac{d^2\eta_i}{dp_z^2} = \frac{p_z}{2\lambda_i^3}\sin\frac{p_z}{\lambda_i} > 0$ for $p_z \in (0, \pi\lambda_i)$; hence $\eta_i(p_z) > 0$ for $p_z \in (0, \pi\lambda_i)$. On the other hand, $1 - \cos\frac{p_z}{\lambda_i} > 0$ and $-\frac{p_z}{2\lambda_i}\sin\frac{p_z}{\lambda_i} \geq 0$ for $p_z \in [\pi\lambda_i, 2\pi\lambda_i)$. Finally, note that $\eta_i(-p_z) = \eta_i(p_z)$.

Accordingly, for $p = (p_z, p_{x_1}, p_{y_1}, \ldots, p_{x_n}, p_{y_n}) \in \mathfrak{h}_{2n+1}^*$, $p_z \in [-2\pi\lambda_n, 0) \cup (0, 2\pi\lambda_n]$, we have that p is a critical point for the sub-Riemannian (resp. Riemannian) exponential map exactly when $p_z = \pm 2\pi\lambda_n$.

Theorem 7 *For both* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *and* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$*:*

1. *If $p_z = 0$, then there are no conjugate points along the geodesic $t \mapsto Exp(t\,p)$.*
2. *If $p_z \neq 0$ and $p \notin \mathscr{H}(\mathbf{1})^\circ$, then the first conjugate point along the geodesic $t \mapsto Exp(t\,p)$ is attained at $t = \frac{2\pi\lambda_n}{|p_z|}$.*

Next, we determine the first conjugate locus. In Fig. 1, we graph the conjugate loci for the three- and five-dimensional cases. (In the five-dimensional case, we graph the projection of the loci under the mapping $(z, x_1, y_1, x_2, y_2) \mapsto (z, x_1, y_1)$.)

Theorem 8 *For* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$*, the first conjugate locus of identity is*

$$\mathscr{C}^{SR} = \left\{ m(z, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, 0, 0) \in \mathsf{H}_{2n+1}\backslash\{\mathbf{1}\} \,:\, |z| \geq \frac{1}{8}\sum_{i=1}^{n-1}\varepsilon_i(x_i^2 + y_i^2) \right\}.$$

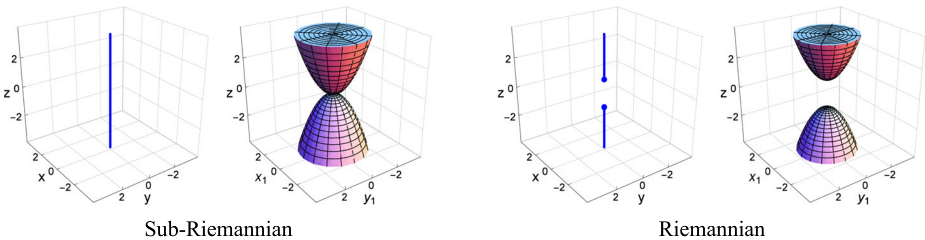


Fig. 1 First conjugate loci for sub-Riemannian and Riemannian structures in three and five dimensions

For $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$*, the first conjugate locus of identity is*

$$\mathscr{C}^R = \left\{ m(z, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, 0, 0) \in \mathsf{H}_{2n+1} \;:\; |z| \geq 2\lambda_n \pi + \frac{1}{8} \sum_{i=1}^{n-1} \varepsilon_i (x_i^2 + y_i^2) \right\}.$$

Here, $\varepsilon_i = \frac{2\pi\lambda_n - \lambda_i \sin \frac{2\pi\lambda_n}{\lambda_i}}{\lambda_i \sin^2 \frac{\pi\lambda_n}{\lambda_i}} > 0$*,* $i = 1, 2, \ldots, n-1$*.*

Proof By Theorem 7, the first conjugate locus $\mathscr{C}$ is given by $\mathscr{C} = \{\mathrm{Exp}(p) \;:\; p \in \mathfrak{h}^*_{2n+1} \backslash \mathscr{H}(\mathbf{1})^\circ,\ |p_z| = 2\pi\lambda_n\}$. Let $m(z, x_1, y_1, \ldots, x_n, y_n) = \mathrm{Exp}(p)$ with $|p_z| = 2\pi\lambda_n$. Then

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \frac{1}{p_z} \begin{bmatrix} \sin \frac{p_z}{\lambda_i} & -\left(1 - \cos \frac{p_z}{\lambda_i}\right) \\ 1 - \cos \frac{p_z}{\lambda_i} & \sin \frac{p_z}{\lambda_i} \end{bmatrix} \begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix}, \qquad i = 1, \ldots, n-1$$

and $x_n = y_n = 0$. Therefore,

$$\frac{1}{p_z^2}(p_{x_i}^2 + p_{y_i}^2) = \frac{x_i^2 + y_i^2}{4 \sin^2 \frac{\pi\lambda_n}{\lambda_i}}, \qquad i = 1, \ldots, n-1.$$

Consequently, (as $\mathrm{sgn}(p_z) = \mathrm{sgn}(z)$)

$$|z| = \frac{1}{8} \sum_{i=1}^{n-1} \frac{2\pi\lambda_n - \lambda_i \sin \frac{2\pi\lambda_n}{\lambda_i}}{\lambda_i \sin^2 \frac{\pi\lambda_n}{\lambda_i}} \left(x_i^2 + y_i^2\right) + \frac{1}{4\pi\lambda_n^2} \left(p_{x_n}^2 + p_{y_n}^2\right)$$

in the sub-Riemannian case and

$$|z| = 2\pi\lambda_n + \frac{1}{8} \sum_{i=1}^{n-1} \frac{2\pi\lambda_n - \lambda_i \sin \frac{2\pi\lambda_n}{\lambda_i}}{\lambda_i \sin^2 \frac{\pi\lambda_n}{\lambda_i}} \left(x_i^2 + y_i^2\right) + \frac{1}{4\pi\lambda_n^2} \left(p_{x_n}^2 + p_{y_n}^2\right)$$

in the Riemannian case. By ranging through $p_{x_n}, p_{y_n} \in \mathbb{R}$, we arrive at the given expression. □

Remark 14 In the general case (i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$), the conjugate locus is identical to the orbit, under the action of isotropy group $\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$, of the conjugate locus of a minimal representative totally geodesic subgroup. For example, in the case of maximal symmetry (i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_n$), the conjugate locus of the minimal representative totally geodesic subgroup H_3 (with Lie algebra spanned by Z, X_1, X_2) is the center of the group (or a subset of the center in the Riemannian case). Consequently, as any isotropy fixes each element in the center, the conjugate locus in this case coincides with the conjugate locus of the minimal representative totally geodesic subgroup H_3.

We shall find it convenient (for both $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$) to partition H_{2n+1} into the following four subsets.

- $\mathscr{I}$: the collection of points $\mathrm{Exp}(t\,p)$ for which there are no conjugate points along $t \mapsto \mathrm{Exp}(t\,p)$. ($\mathscr{I}$ coincides with collection of endpoints of the minimizing geodesics described in Proposition 8.)
- $\mathsf{H}_{2n+1} \backslash (\mathscr{C} \cup \mathscr{I})$: the collection of points which are not first conjugate points and do not belong to $\mathscr{I}$.
- $\partial\mathscr{C}$: the boundary of $\mathscr{C}$ (excluding identity, with respect to the subgroup H_{2n-1} of codimension two containing $\mathscr{C}$).

– int$\mathscr{C}$: the interior of the first conjugate locus (again with respect to the subgroup H_{2n-1} of codimension two containing $\mathscr{C}$).

These subsets can be characterized as follows:

$$\begin{aligned}
g \in \mathscr{I} \quad &\iff \quad z = 0\\
g \notin \mathscr{C} \cup \mathscr{I} \quad &\iff \quad z \neq 0 \text{ and } x_n^2 + y_n^2 \neq 0, \text{ or } 0 < |z| < \zeta(g)\\
g \in \partial\mathscr{C} \quad &\iff \quad x_n^2 + y_n^2 = 0 \text{ and } |z| = \zeta(g) > 0\\
g \in \operatorname{int}\mathscr{C} \quad &\iff \quad x_n^2 + y_n^2 = 0 \text{ and } |z| > \zeta(g).
\end{aligned}$$

Here, $g = m(z, x_1, y_1, \ldots, x_n, y_n)$, $\varepsilon_i = \frac{2\pi\lambda_n - \lambda_i \sin\frac{2\pi\lambda_n}{\lambda_i}}{\lambda_i \sin^2\frac{\pi\lambda_n}{\lambda_i}}$, and

$$\zeta^{SR}(g) = \frac{1}{8}\sum_{i=1}^{n-1}\varepsilon_i(x_i^2 + y_i^2), \qquad \zeta^R(g) = 2\pi\lambda_n + \frac{1}{8}\sum_{i=1}^{n-1}\varepsilon_i(x_i^2 + y_i^2)$$

in the sub-Riemannian and Riemannian cases, respectively. We note that in the three-dimensional case $\partial\mathscr{C}^{SR}$ is empty and $\partial\mathscr{C}^R = \{\pm 2\pi\lambda_1\}$.

5.2 Optimal Synthesis

For every $\bar{g} \in \mathsf{H}_{2n+1}$, we describe the minimizing geodesics from identity to $\bar{g}$. (All other minimizing geodesics can be recovered by left translations.) Again, it is assumed that $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ (see Note 5).

Theorem 9 *Let $\bar{g} = m(\bar{z}, \bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \ldots, \bar{x}_n, \bar{y}_n) \in \mathsf{H}_{2n+1}$. In the sub-Riemannian case $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$, let*

$$\tau_n(s_1, s_2) = \frac{1}{8}\sum_{i=1}^{n}\frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s_1}{2\lambda_i}}\left(s_2 - \lambda_i \sin\frac{s_2}{\lambda_i}\right), \qquad \kappa_n(s) = \frac{1}{4}\sum_{i=1}^{n}\frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s}{2\lambda_i}}$$

and in the Riemannian case $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$, let

$$\tau_n(s_1, s_2) = s_2 + \frac{1}{8}\sum_{i=1}^{n}\frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s_1}{2\lambda_i}}\left(s_2 - \lambda_i \sin\frac{s_2}{\lambda_i}\right), \quad \kappa_n(s) = 1 + \frac{1}{4}\sum_{i=1}^{n}\frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s}{2\lambda_i}}.$$

Furthermore, let

$$\zeta = \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n), \qquad R_i(t, t_1, \alpha) = \frac{\sin\frac{\alpha t}{2t_1\lambda_i}}{\sin\frac{\alpha}{2\lambda_i}}\begin{bmatrix}\cos\frac{\alpha(t-t_1)}{2t_1\lambda_i} & -\sin\frac{\alpha(t-t_1)}{2t_1\lambda_i}\\ \sin\frac{\alpha(t-t_1)}{2t_1\lambda_i} & \cos\frac{\alpha(t-t_1)}{2t_1\lambda_i}\end{bmatrix}.$$

For both $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$, the unit-speed minimizing geodesics

$$g(t) = m(z(t), x_1(t), y_1(t), \ldots, x_n(t), y_n(t))$$

from $g(0) = \mathbf{1}$ to $g(t_1) = \bar{g}$ (having length t_1) are given below.

1. *If $\bar{g} \in \mathscr{I}$, then there exists a unique unit-speed minimizing geodesic from $\mathbf{1}$ to $\bar{g}$ given by*

$$z(t) = 0, \quad x_i(t) = \frac{\bar{x}_i}{t_1}t, \quad y_i(t) = \frac{\bar{y}_i}{t_1}t$$

where $t_1 = \sqrt{\sum_{i=1}^{n}\lambda_i(\bar{x}_i^2 + \bar{y}_i^2)}$.

2. *If $\bar{g} \notin \mathscr{I} \cup \mathscr{C}$, then there exists a unique unit-speed minimizing geodesic from* $\mathbf{1}$ *to* $\bar{g}$ *given by*
$$\begin{cases} z(t) = \tau_n\left(\alpha, \frac{\alpha t}{t_1}\right) \\ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = R_i(t, t_1, \alpha) \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix}, \qquad i = 1, \ldots, n \end{cases}$$
where $\operatorname{sgn}(\bar{z})\alpha$ *is the unique solution to* $\tau_n(s, s) = |\bar{z}|$ *on the interval* $(0, 2\pi\lambda_n)$ *and* $t_1 = |\alpha|\sqrt{\kappa_n(\alpha)}$.
3. *If $\bar{g} \in \partial\mathscr{C}$, then there exists a unique unit-speed minimizing geodesic from* $\mathbf{1}$ *to* $\bar{g}$ *given by*
$$\begin{cases} z(t) = \tau_{n-1}\left(\alpha, \frac{\alpha t}{t_1}\right) \\ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = R_i(t, t_1, \alpha) \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix}, \qquad i = 1, \ldots, n-1 \\ x_n(t) = y_n(t) = 0 \end{cases}$$
where $\alpha = 2\operatorname{sgn}(\bar{z})\pi\lambda_n$ *and* $t_1 = 2\pi\lambda_n\sqrt{\kappa_{n-1}(2\pi\lambda_n)}$.
4. *If $\bar{g} \in \operatorname{int}\mathscr{C}$, then there exists a family of unit-speed minimizing geodesics*
$$\begin{cases} z(t) = \tau_{n-1}\left(\alpha, \frac{\alpha t}{t_1}\right) + \frac{|\bar{z}|-\zeta}{2\pi}\left(\frac{\alpha t}{\lambda_n t_1} - \sin\frac{\alpha t}{\lambda_n t_1}\right) \\ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = R_i(t, t_1, \alpha) \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix}, \qquad i = 1, \ldots, n-1 \\ \begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix} = \frac{\operatorname{sgn}(\bar{z})\sqrt{|\bar{z}|-\zeta}}{\sqrt{\pi}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\frac{\alpha t}{\lambda_n t_1} \\ 1 - \cos\frac{\alpha t}{\lambda_n t_1} \end{bmatrix} \end{cases}$$
from $\mathbf{1}$ *to* $\bar{g}$*, parametrized by* $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathsf{SO}(2)$. *Here* $\alpha = 2\operatorname{sgn}(\bar{z})\pi\lambda_n$ *and* $t_1 = 2\sqrt{\pi\lambda_n}\sqrt{|\bar{z}| - \zeta + \pi\lambda_n\kappa_{n-1}(2\pi\lambda_n)}$. *Moreover, the subgroup*
$$\left\{ \phi \in \mathsf{Aut}(\mathsf{H}_{2n+1}) \;:\; T_{\mathbf{1}}\phi = \begin{bmatrix} 1 & 0 & & & 0 \\ 0 & I_2 & & & 0 \\ & & \ddots & & \\ & & & I_2 & \\ 0 & 0 & & & g_n \end{bmatrix}, \; g_n \in \mathsf{SO}(2) \right\}$$
of the isotropy subgroup $\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda) = \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *acts transitively on this family of minimal geodesics.*

Proof We note that the existence of minimizing geodesics is assured by Proposition 1. (Also, on the Heisenberg group, every minimizing geodesic is a normal geodesic, i.e., takes the form $t \mapsto \operatorname{Exp}(t\,p)$.) We prove only the sub-Riemannian case (the Riemannian case follows similarly). Throughout, we make use of the characterizations of $\mathscr{I}$, $\mathsf{H}_{2n+1}\backslash(\mathscr{I} \cup \mathscr{C})$, $\partial\mathscr{C}$, and $\operatorname{int}\mathscr{C}$ given in Section 5.1.

If $\bar{z} = 0$, then there is exactly one unit-speed geodesic from identity to $\bar{g}$, namely the one described in Proposition 8. Henceforth, we assume $\bar{z} \neq 0$. We seek $p \in \mathfrak{h}_{2n+1}^*$, $H(p) = \frac{1}{2}$ and minimal $t_1 > 0$ such that $\operatorname{Exp}(t_1\,p) = \bar{g}$, i.e.,
$$\bar{z} = \frac{1}{2p_z^2}\sum_{i=1}^{n}\left(p_{x_i}^2 + p_{y_i}^2\right)\left(\frac{p_z t_1}{\lambda_i} - \sin\frac{p_z t_1}{\lambda_i}\right)$$
$$\begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix} = \frac{1}{p_z}\begin{bmatrix} \sin\frac{p_z t_1}{\lambda_i} & -\left(1 - \cos\frac{p_z t_1}{\lambda_i}\right) \\ 1 - \cos\frac{p_z t_1}{\lambda_i} & \sin\frac{p_z t_1}{\lambda_i} \end{bmatrix}\begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix}, \qquad i = 1, \ldots, n.$$

Note that $\mathrm{sgn}(p_z) = \mathrm{sgn}(\bar{z})$. The condition $H(p) = \frac{1}{2}$ corresponds to the restriction to unit-speed geodesics. By Theorem 7, we have that $|p_z|t_1 \in (0, 2\pi\lambda_n]$.

Suppose $x_n^2 + y_n^2 \neq 0$; then $|p_z|t_1 \neq 2\pi\lambda_n$ and so $|p_z|t_1 \in (0, 2\pi\lambda_n)$. We have

$$\begin{vmatrix} \sin\frac{p_z t_1}{\lambda_i} & -\left(1-\cos\frac{p_z t_1}{\lambda_i}\right) \\ 1-\cos\frac{p_z t_1}{\lambda_i} & \sin\frac{p_z t_1}{\lambda_i} \end{vmatrix} = 2 - 2\cos\frac{p_z t_1}{\lambda_i} \neq 0 \quad \text{for} \quad |p_z|t \in (0, 2\pi\lambda_n).$$

Hence,

$$\begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix} = p_z \begin{bmatrix} \sin\frac{p_z t_1}{\lambda_i} & -\left(1-\cos\frac{p_z t_1}{\lambda_i}\right) \\ 1-\cos\frac{p_z t_1}{\lambda_i} & \sin\frac{p_z t_1}{\lambda_i} \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix}, \quad i = 1, \ldots, n \tag{11}$$

and so $\bar{z} = \tau_n(p_z t_1, p_z t_1)$. We have that $\lim_{s\to 0}\tau_n(s,s) = 0$, $\lim_{s\to 2\pi\lambda_n}\tau_n(s,s) = \infty$, and $\frac{d\tau_n(s,s)}{ds} = \frac{(\bar{x}_i^2+\bar{y}_i^2)\left(2\lambda_i - s\cot\frac{s}{2\lambda_i}\right)}{\lambda_i^2\sin^2\frac{s}{2\lambda_i}} > 0$ for $s \in (0, 2\pi\lambda_n)$. Hence, there exists a unique solution s' to $\tau_n(s,s) = |\bar{z}|$ on $(0, 2\pi\lambda_n)$. As $\tau_n(-s,-s) = -\tau_n(s,s)$, it follows that $\tau_n(\mathrm{sgn}(\bar{z})s', \mathrm{sgn}(\bar{z})s') = \bar{z}$. Let $\alpha = \mathrm{sgn}(\bar{z})s'$; then we have that $p_z t_1 = \alpha$. As $H(p) = \frac{1}{2}$, we have that

$$\sum_{i=1}^{n}\frac{p_{x_i}^2 + p_{y_i}^2}{\lambda_i} = p_z^2\sum_{i=1}^{n}\frac{\bar{x}_i^2+\bar{y}_i^2}{4\lambda_i\sin^2\frac{\alpha}{2\lambda_i}} = 1.$$

Therefore, $t_1 = \frac{\alpha}{p_z} = \mathrm{sgn}(\bar{z})\alpha\sqrt{\kappa_n(\alpha)}$. Thus, we have obtained a unique p and t_1 such that $\mathrm{Exp}(t_1 p) = g_1$ with $t_1 \in (0, \frac{2\pi\lambda_n}{|p_z|})$. Substituting the value for p (with $p_z = \frac{\alpha}{t_1}$) back into expression for $\mathrm{Exp}(t\,p)$ yields the expression given.

Henceforth, suppose $\bar{x}_n^2 + \bar{y}_n^2 = 0$. As $|p_z|t_1 \in (0, 2\pi\lambda_n]$ it follows that Eq. 11 holds, but only for $i = 1, 2, \ldots, n-1$. Accordingly

$$|\bar{z}| = \tau_{n-1}(|p_z|t_1, |p_z|t_1) + \frac{1}{2p_z^2}\left(p_{x_n}^2 + p_{y_n}^2\right)\left(\frac{|p_z|t_1}{\lambda_n} - \sin\frac{|p_z|t_1}{\lambda_n}\right) \geq \tau_{n-1}(|p_z|t_1, |p_z|t_1).$$

Suppose $|\bar{z}| < \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n)$. If $|p_z|t_1 = 2\pi\lambda_n$, then $|\bar{z}| \geq \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n)$, which is a contradiction. Hence, $|p_z|t_1 \in (0, 2\pi\lambda_n)$ and so the above argument holds for this case.

Suppose $|\bar{z}| = \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n)$. As $\bar{x}_n^2 + \bar{y}_n^2 = 0$, it follows that $|p_z|t_1 = 2\pi\lambda_n$ or $p_{x_i}^2 + p_{y_i}^2 = 0$. If $p_{x_i}^2 + p_{y_i}^2 = 0$, then $\tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n) = \tau_{n-1}(|p_z|t_1, |p_z|t_1)$ and so (as $\tau_{n-1}(s,s)$ is increasing on $(0, 2\pi\lambda_n]$) we have that $|p_z|t_1 = 2\pi\lambda_n$. If $|p_z|t_1 = 2\pi\lambda_n$, then

$$\tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n) = |z| = \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n) + \frac{\pi}{p_z^2}\left(p_{x_n}^2 + p_{y_n}^2\right)$$

and so $p_{x_n}^2 + p_{y_n}^2 = 0$. Therefore, in either case, we have that $|p_z|t_1 = 2\pi\lambda_n$ and $p_{x_n}^2 + p_{y_n}^2 = 0$. As $H(p) = \frac{1}{2}$, we have that

$$\sum_{i=1}^{n}\frac{p_{x_i}^2 + p_{y_i}^2}{\lambda_i} = p_z^2\sum_{i=1}^{n-1}\frac{\bar{x}_i^2+\bar{y}_i^2}{4\lambda_i\sin^2\frac{\pi\lambda_n}{\lambda_i}} = 1.$$

Therefore, $t_1 = \frac{2\pi\lambda_n}{|p_z|} = 2\pi\lambda_n\sqrt{\kappa_{n-1}(2\pi\lambda_n)}$. Again, substituting the unique value for p back into expression for $\mathrm{Exp}(t\,p)$ yields the expression given.

Lastly, suppose $\bar{z} > \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n)$. If $p_{x_n}^2 + p_{y_n}^2 = 0$, then we have that $|\bar{z}| = \tau_{n-1}(|p_z|t_1, |p_z|t_1) \leq \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n)$, which is a contradiction. Hence, as $x_n^2 + y_n^2 = 0$, it follows that $|p_z|t_1 = 2\pi\lambda_n$. Consequently, as $H(p) = \frac{1}{2}$, we have that

$$\sum_{i=1}^{n}\frac{p_{x_i}^2 + p_{y_i}^2}{\lambda_i} = \frac{p_{x_n}^2 + p_{y_n}^2}{\lambda_n} + p_z^2\sum_{i=1}^{n-1}\frac{\bar{x}_i^2+\bar{y}_i^2}{4\lambda_i\sin^2\frac{\pi\lambda_n}{\lambda_i}} = 1.$$

Therefore, $p_{x_n}^2 + p_{y_n}^2 = \lambda_n - \lambda_n p_z^2 \kappa_{n-1}(2\pi\lambda_n)$ and so

$$|\bar{z}| - \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n) = \pi\lambda_n \left(\frac{1}{p_z^2} - \kappa_{n-1}(2\pi\lambda_n)\right).$$

Thus

$$t_1 = \frac{2\pi\lambda_n}{|p_z|} = 2\sqrt{\pi\lambda_n}\sqrt{|\bar{z}| - \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n) + \pi\lambda_n\kappa_{n-1}(2\pi\lambda_n)}.$$

Let $p_{x_n} = r\cos\theta$ and $p_{y_n} = r\sin\theta$. As $p_{x_n}^2 + p_{y_n}^2 = \lambda_n - \lambda_n p_z^2 \kappa_{n-1}(2\pi\lambda_n)$ and $|p_z| = \frac{2\pi\lambda_n}{t_1}$, we have that

$$r^2 = \frac{\lambda_n(|z| - \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n))}{|z| - \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n) + \pi\lambda_n\kappa_{n-1}(2\pi\lambda_n)} = \frac{4\pi\lambda_n^2}{t_1^2}\left(|z| - \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n)\right).$$

We have determined p up to a rotation in the p_{x_n}, p_{y_n} plane. Nonetheless, all these geodesics have the same unique length t_1 and so are minimal. Hence, we have a family of minimizing geodesics parametrized by $\mathsf{SO}(2)$. Again, substituting the value for p (with $p_{x_n} = r\cos\theta$, $p_{y_n} = r\sin\theta$, and r as given above) back into expression for $\operatorname{Exp}(t\,p)$ yields the expression given. □

Corollary 4 *The associated (Carnot–Carathéodory) distance for* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *and* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *is given by*

$$d(\mathbf{1}, g) = \begin{cases} \sqrt{\sum_{i=1}^n \lambda_i(x_i^2 + y_i^2)} & \text{if } g \in \mathscr{I} \\ \alpha\sqrt{\kappa_n(\alpha)} & \text{if } g \notin \mathscr{C} \cup \mathscr{I} \\ 2\sqrt{\pi\lambda_n}\sqrt{|z| - \tau_{n-1}(2\pi\lambda_n) + \pi\lambda_n\kappa_{n-1}(2n\pi\lambda_n)} & \text{if } g \in \mathscr{C}. \end{cases}$$

Here, $g = m(z, x_1, y_1, \ldots, x_n, y_n)$*,* α *is the unique solution to* $\tau_n(s) = |z|$ *on* $(0, 2\pi\lambda_n)$*,*

$$\tau_n(s) = \frac{1}{8}\sum_{i=1}^n \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s}{2\lambda_i}}\left(s - \lambda_i \sin\frac{s}{\lambda_i}\right), \qquad \kappa_n(s) = \frac{1}{4}\sum_{i=1}^n \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s}{2\lambda_i}}$$

in the sub-Riemannian case and

$$\tau_n(s) = s + \frac{1}{8}\sum_{i=1}^n \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s}{2\lambda_i}}\left(s - \lambda_i \sin\frac{s}{\lambda_i}\right), \qquad \kappa_n(s) = 1 + \frac{1}{4}\sum_{i=1}^n \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2\frac{s}{2\lambda_i}}$$

in the Riemannian case.

Remark 15 We note that, in the above corollary,

$$-\tau_{n-1}(2\pi\lambda_n) + \pi\lambda_n\kappa_{n-1}(2\pi\lambda_n) = \frac{1}{4}\sum_{i=1}^{n-1}(x_i^2 + y_i^2)\cot\frac{\pi\lambda_n}{\lambda_i}$$

in the sub-Riemannian case, and

$$-\tau_{n-1}(2\pi\lambda_n) + \pi\lambda_n\kappa_{n-1}(2\pi\lambda_n) = -\pi\lambda_n + \frac{1}{4}\sum_{i=1}^{n-1}(x_i^2 + y_i^2)\cot\frac{\pi\lambda_n}{\lambda_i}$$

in the Riemannian case.

Remark 16 If a Riemannian structure tames a sub-Riemannian one, then $d^{SR}(\mathbf{1}, g) \geq d^R(\mathbf{1}, g)$. On the Heisenberg group, we have that $d^{SR}(\mathbf{1}, g) = d^R(\mathbf{1}, g)$ if and only if $g \in \mathscr{I}$.

For the sub-Riemannian and Riemannian structures on the Heisenberg group, we have that $\mathscr{H}$-projections are compatible with minimizing geodesics (see Section 3.2).

Corollary 5 *Suppose that* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *is tamed by* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$. *An absolutely continuous curve on* H_{2n+1} *is a minimizing geodesic of* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *if and only if its* $\mathscr{H}$*-projection is a minimizing geodesic of* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$.

Furthermore, it turns out that the cut and conjugate loci coincide. (We say $\bar{g} \in \mathsf{H}_{2n+1}$ is a cut point if there exists a geodesic $g(\cdot)$ with $g(0) = \mathbf{1}$ and $g(t_1) = \bar{g}$ such that $d(\mathbf{1}, g(t)) = \ell(g(t))$ for $0 < t \leq t_1$, but $d(\mathbf{1}, g(t)) < \ell(g(t))$ for $t > t_1$. The cut locus is the collection of cut points.)

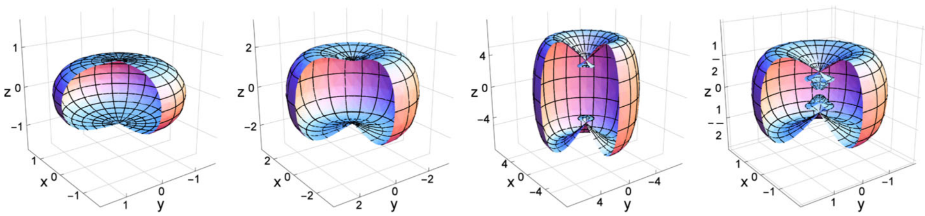
Corollary 6 (cf. [3, 61]) *For both* $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ *and* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$*, the first conjugate locus and the cut locus coincide.*

Remark 17 (cf. [61]) If a (strictly normal) geodesic $g(\cdot)$ from identity is minimizing up to (but not beyond) a cut point $\bar{g} = g(t_1)$, then $\bar{g}$ is a conjugate point or there exists another geodesic of the same length from $\mathbf{1}$ to $\bar{g}$. (This is well known in the Riemannian case, see, e.g., [52]; for the sub-Riemannian case, see [2, 4].) For both $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$, the points $\bar{g} \in \partial\mathscr{C}$ are examples of cut points which are conjugate points but for which there exists only one geodesic from $\mathbf{1}$ to $\bar{g}$ of length $d(\mathbf{1}, \bar{g})$. (However, note that this does not occur in the maximally symmetric sub-Riemannian case, as $\partial\mathscr{C}^{SR}$ is empty.)

In Fig. 2, we plot the wavefronts $\mathscr{W}(s) = \{\mathrm{Exp}(s\,p) \;:\; p \in \mathfrak{g}^* \backslash \mathscr{H}(\mathbf{1})^\circ,\; H(p) = \frac{1}{2}\}$ for $s = \pi\lambda_n, 2\pi\lambda_n, 4\pi\lambda_n$ in the Riemannian case and for $s = 1$ in the sub-Riemannian case. (In the sub-Riemannian case $\mathscr{W}^{SR}(s) = \delta_s(\mathscr{W}^{SR}(1))$, see Remark 6.) In Fig. 3, we graph some typical minimizing geodesics.

Beals, Gaveau, and Greiner [12] previously described the minimizing geodesics (and Carnot–Carathéodory distance) of the sub-Riemannian structures on the Heisenberg groups, though from a different point of view. To facilitate the comparison of results, we give an isometry between the normalized sub-Riemannian structure considered in [12] and $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$. In [12], the Heisenberg group is represented as $\mathbb{R}^{2n} \times \mathbb{R}$ with group law

$$(x, t) \cdot (x', t') = \left(x + x', t + t' + 2 \sum_{j=1}^{n} a_j \left(x_{2j} x'_{2j-1} - x_{2j-1} x'_{2j} \right) \right)$$



$\mathscr{W}^R(\pi\lambda_1)$ $\mathscr{W}^R(2\pi\lambda_1)$ $\mathscr{W}^R(4\pi\lambda_1)$ $\mathscr{W}^{SR}(1)$

Fig. 2 Wave fronts for sub-Riemannian and Riemannian structures in three dimensions. (The sub-Riemannian wavefront is plotted only for $|p_z| \leq 12\pi\lambda_n$)

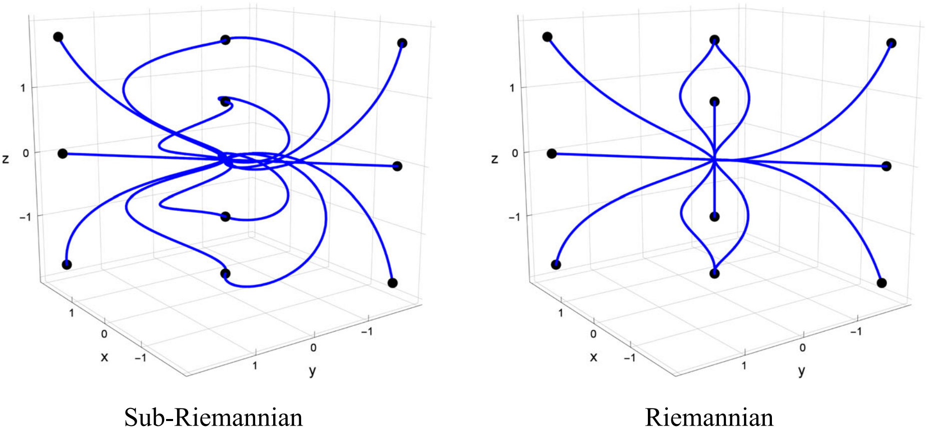


Fig. 3 Some minimizing geodesics from identity for a sub-Riemannian structure and a Riemannian structure (taming the sub-Riemannian one) in three dimensions, corresponding to the same set of endpoints (on the $x = y$ plane)

where $a_1, a_2, \ldots, a_n$ are constants such that $0 < a_1 \leq a_2 \leq \cdots \leq a_p < a_{p+1} = \cdots = a_n$. The vector fields

$$V_{2j-1} = \frac{\partial}{\partial x_{2j-1}} + 2a_j x_{2j} \frac{\partial}{\partial t}, \qquad V_{2j} = \frac{\partial}{\partial x_{2j}} - 2a_j x_{2j-1} \frac{\partial}{\partial t}$$

are left invariant and generate the Lie algebra; we have $[V_{2j-1}, V_{2j}] = -4a_j \frac{\partial}{\partial t}$. The sub-Riemannian structure considered in [12] is exactly the structure $(\mathbb{R}^{2n} \times \mathbb{R}, \mathscr{E}, \mathbf{h})$ on $\mathbb{R}^{2n} \times \mathbb{R}$ admitting the orthonormal frame $(V_1, V_2, \ldots, V_{2n})$. It is a simple matter to show that the mapping $\phi : \mathbb{R}^{2n} \times \mathbb{R} \to \mathsf{H}_{2n+1}$ given by

$$(x, t) \longmapsto m\left(t, \tfrac{1}{\sqrt{\lambda_1}} x_2, \tfrac{1}{\sqrt{\lambda_1}} x_1, \tfrac{1}{\sqrt{\lambda_2}} x_4, \tfrac{1}{\sqrt{\lambda_2}} x_3, \ldots, \tfrac{1}{\sqrt{\lambda_n}} x_n, \tfrac{1}{\sqrt{\lambda_n}} x_{n-1}\right)$$

with $\lambda_j = \frac{1}{4a_j}$ is a Lie group isomorphism. Moreover, $\phi_* V_{2j-1} = \frac{1}{\sqrt{\lambda_j}} Y_j$ and $\phi_* V_{2j} = \frac{1}{\sqrt{\lambda_j}} X_j$. Accordingly, ϕ is an isometry between $(\mathbb{R}^{2n} \times \mathbb{R}, \mathscr{E}, \mathbf{h})$ and $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$. Hence, ϕ maps minimizing geodesics of $(\mathbb{R}^{2n} \times \mathbb{R}, \mathscr{E}, \mathbf{h})$ to minimizing geodesics of $(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)$ and, of course, $d^{(\mathbb{R}^{2n} \times \mathbb{R}, \mathscr{E}, \mathbf{h})}((0,0), (x,t)) = d^{(\mathsf{H}_{2n+1}, \mathscr{H}, \mathbf{g}^\lambda)}(\mathbf{1}, \phi(x,t))$.

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