J Dyn Control Syst (2017) 23:249–268
DOI 10.1007/s10883-016-9312-0

Approximate and Near Weak Invariance for Nonautonomous Differential Inclusions

Omar Benniche[1,2] · Ovidiu Cârjă[3,4]

Received: 11 June 2015 / Revised: 1 November 2015 / Published online: 26 February 2016

Abstract Let X be a real Banach space and I a nonempty interval. Let $K : I \rightsquigarrow X$ be a multi-function with the graph $\mathcal{K}$. We give here a characterization for $\mathcal{K}$ to be approximate/near weakly invariant with respect to the differential inclusion $x'(t) \in F(t, x(t))$ by means of an appropriate tangency concept and Lipschitz conditions on F. The tangency concept introduced in this paper extends in a natural way the quasi-tangency concept introduced by Cârjă et al. (Trans Amer Math Soc. 2009;361:343–90) (see also Cârjă et al. (2007)). Viability, invariance and applications. Amsterdam: Elsevier Science B V) in the case when F is independent of t. As an application, we give some results concerning the set of solutions for the differential inclusion $x'(t) \in F(t, x(t))$.

Keywords Approximate/near weak invariance · Tangency conditions · Differential inclusions

Mathematics Subject Classification (2010) 34G25 · 47H04

✉ Ovidiu Cârjă
ocarja@uaic.ro

Omar Benniche
obenniche@gmail.com; o.benniche@univ-dbkm.dz

[1] Department of Mathematics, Laboratory Energie et les Systémes Intelligents (LESI), Djilali Bounaama University, Khemis Miliana, 442500, Algeria

[2] Laboratory Théorie de Point Fixe et Applications (TPFA), ENS, BP 92, 16050, Algiers, Algeria

[3] Department of Mathematics, Al. I. Cuza University, Iaşi 700506, Romania

[4] Octav Mayer Institute of Mathematics (Romanian Academy), Iaşi 700505, Romania

1 Introduction

Let X be a real Banach space and $I = [a, b)$ where $a < b \leq +\infty$. Let $F : I \times X \rightsquigarrow X$ be a given multi-function. Let us recall that $W^{1,1}(\tau, T; X)$ denotes the space of all functions $x : [\tau, T] \to X$, which are a.e. differentiable on $[\tau, T]$ with $x' \in L^1(\tau, T; X)$ and for each $t \in [\tau, T]$, we have

$$x(t) = x(\tau) + \int_\tau^t x'(s)ds.$$

As usual, an (exact) solution of

$$x'(t) \in F(t, x(t)) \tag{1}$$

on $[\tau, T] \subset I$ is a function $x \in W^{1,1}(\tau, T; X)$ which satisfies Eq. (1) a.e. on $[\tau, T]$. Let $K : I \rightsquigarrow X$ be a multi-function. We denote by $\mathcal{K}$ the graph of K, i.e.,

$$\mathcal{K} = \{(t, x); t \in I \text{ and } x \in K(t)\}.$$

Roughly speaking, the classical concept of (exact) weak invariance (or viability, in other terminology) for $\mathcal{K}$ with respect to Eq. (1) requires that for each initial data (τ, ξ) in $\mathcal{K}$, there exists a solution of Eq. (1), starting from ξ at $t = \tau$ and such that its graph remains in $\mathcal{K}$ at least for a short time. We point out that in [11], necessary and sufficient conditions for $\mathcal{K}$ to be weakly invariant with respect to Eq. (1), expressed in terms of the tangency concept given in Eq. (6) below and when F is β−compact with convex values, were given (see [11, Theorem 4.1]).

A more general concept, called approximate weak invariance, was introduced in [6] in Hilbert spaces and the autonomous case, i.e., $x'(t) \in F(x(t))$ and K is independent of t. It involves ε-solutions, which are in fact solutions of $x'(t) \in F(x(t) + \varepsilon\mathbb{B})$, where $\mathbb{B}$ is the closed unit ball. Approximate weak invariance requires the existence of ε-solution for which

$$\text{dist}(x(t); K) \leq \varepsilon, \tag{2}$$

for t in some interval. We mention that in this paper, we adopt the terminology from [6]. It is interesting to note that the approximate weak invariance is equivalent to a tangency condition under very general assumptions on F. Adding natural convexity and compactness assumptions on F, passing to limit when $\varepsilon \to 0$, we get exact weak invariance. Again the autonomous case, but in general Banach spaces, was considered in [2]. We point out that in [2], the authors used the quasi-tangency concept introduced in [3] and [4]. In the case when F is Lipshitz, it was proved in [2] that the tangency condition is equivalent to the existence of exact solution of $x'(t) \in F(x(t))$ which satisfies Eq. (2). This means that K is near weakly invariant with respect to $x'(t) \in F(x(t))$. We point out that near weak invariance of K is equivalent to the tangency condition under the Lipshitz property of F but with no compactness assumptions. Further, it is not required that F has convex values, which is an essential condition for exact weak invariance. To illustrate this, we consider the following example taken from [3]. Let the multi-function $F : \mathbb{R}^2 \rightsquigarrow \mathbb{R}^2$ defined by $F(x) = \{(1, 0), (-1, 0)\}$ for every $x \in \mathbb{R}^2$ and we let $K = \mathbb{B}$. It is easy to see that K is not weakly invariant with respect to $x'(t) \in F(x(t))$. However, by using Theorem 8 in this paper, we check that K is near weakly invariant with respect to $x'(t) \in F(x(t))$.

In this paper, we extend the results of [2] to the nonautonomous differential inclusion in Eq. (1) and the multi-function $K : I \rightsquigarrow X$. We introduce a new tangency concept which extends in a natural way the quasi-tangency concept used in [2] and is appropriate for our setting.

Following [6], for a given $\varepsilon > 0$, we introduce the notion of ε-solution that we are dealing with in our paper.

Definition 1 A function $x : [\tau, T] \to X$ is called an ε-solution of Eq. (1), if it is a solution of the differential inclusion

$$x'(t) \in F(t, x(t) + \varepsilon\mathbb{B}) \tag{3}$$

on $[\tau, T]$.

We introduce the concept of approximate weak invariance for $\mathcal{K}$ with respect to Eq. (1), which generalizes that one given in [6, Definition 3.1] when F is independent of t.

Definition 2 We say that $\mathcal{K}$ is approximate weakly invariant with respect to Eq. (1), if for any $(\tau, \xi) \in \mathcal{K}$, there exists $T > \tau$ such that $[\tau, T] \subset I$ and for any $\varepsilon > 0$, there exist a function $\sigma : [\tau, T] \to [\tau, T]$, satisfying $t - \varepsilon \le \sigma(t) \le t$ for each $t \in [\tau, T]$ and an ε-solution $x : [\tau, T] \to X$ of Eq. (1) with $x(\tau) = \xi$ and $\operatorname{dist}(x(t); K(\sigma(t))) \le \varepsilon$, for all $t \in [\tau, T]$.

Remark 1 Notice that in the case when $K : I \rightsquigarrow X$ is Lipschitz, i.e.,

$$K(t_1) \subset K(t_2) + M|t_1 - t_2|\mathbb{B},$$

for some $M > 0$ and each $t_1, t_2 \in I$, a simple calculation shows that one can rephrase Definition 2 by taking $\sigma(t) = t$, for all $t \in [\tau, T]$.

Definition 3 We say that $\mathcal{K}$ is near weakly invariant with respect to Eq. (1), if for any $(\tau, \xi) \in \mathcal{K}$, there exists $T > \tau$ such that $[\tau, T] \subset I$ and for any $\varepsilon > 0$, there exist a function $\sigma : [\tau, T] \to [\tau, T]$, satisfying $t - \varepsilon \le \sigma(t) \le t$ for each $t \in [\tau, T]$ and a solution $x : [\tau, T] \to X$ of Eq. (1) with $x(\tau) = \xi$ and $\operatorname{dist}(x(t); K(\sigma(t))) \le \varepsilon$, for each $t \in [\tau, T]$.

This paper is organized as follows. Necessary prerequisites for the proofs of main results are summarized in Section 2; in Section 3, we define some tangency conditions which have a crucial role of our results. In Sections 4 and 5, we establish some sufficient and/or necessary conditions for approximate and near weak invariance. Finally, in Section 6, we give applications to the study of the qualitative theory of the solutions of Eq. (1).

2 Preliminary Results

In this section, we gather several basic concepts and results concerning measurable multi-functions we shall refer to in the sequel. We recall that the usual distance between two subsets A and B in a Banach space is defined by

$$\operatorname{dist}(A; B) = \inf\{\|a - b\|, a \in A,\ b \in B\},$$

in particular, for $x \in X$, $\operatorname{dist}(x; B)$ stands for $\operatorname{dist}(\{x\}; B)$. For $\xi \in X$ and $\rho > 0$, we denote by $B(\xi, \rho)$ the closed ball with center ξ and radius ρ. As we have mentioned in Section 1, $\mathbb{B} = B(0, 1)$. We begin by introducing some definitions for measurable multi-functions; for more details, we refer the reader to [1, Chapter 8].

Definition 4 Let X be a Banach space and $I \subset \mathbb{R}$. A multi-function $F : I \rightsquigarrow X$ is called measurable, if for every open set $\mathcal{O} \subset X$, the inverse image of $\mathcal{O}$, i.e., $F^{-}(\mathcal{O}) = \{t \in I, F(t) \cap \mathcal{O} \neq \emptyset\}$ is measurable.

The existence of measurable selections is given below (see, e.g., [1, Theorem 8.1.3]).

Theorem 1 *Let X be a separable Banach space. Assume that $F : I \rightsquigarrow X$ is a measurable multi-function with nonempty and closed values. Then, F admits at least one measurable selection.*

Remark 2 We note here that, if F is integrably bounded, i.e.,

$$F(t) \subset l(t)\mathbb{B}, \text{ a.e. for } t \in I,$$

for some $l \in L^1(I; \mathbb{R}_+)$, then every measurable selection of F is integrable, thanks to Lebesgue's theorem.

Let $F : I \rightsquigarrow X$ be a multi-function. We define for each nonempty sub-interval J of I the set

$$S_J F(\cdot) = \{f \in L^1(J; X); f(t) \in F(t), \text{ a.e. for } t \in J\}.$$

As we have mentioned above, if X is a separable Banach space and F is measurable and integrably bounded with nonempty and closed values then $S_J F(\cdot) \neq \emptyset$.

Definition 5 Let X be a Banach space and $F : I \rightsquigarrow X$ a multi-function. The Aumann integral of F on $J \subset I$ is the set of integrals of integrable selections of F on J, i.e.,

$$\int_J F(s)d\lambda := \left\{ \int_J f(s)d\lambda, f \in S_J F(\cdot) \right\}.$$

Here, λ is the Lebesgue measure on I and the vector-valued integrals are taken in the sense of Bochner. Next, we recall a well-known result concerning measurable multi-functions which will be an important tool in the proofs of the main results.

Lemma 1 *Let X be a separable Banach space, $U : [\tau, T] \rightsquigarrow X$ a measurable multi-function with nonempty closed values and $g : [\tau, T] \to X$, $k : [\tau, T] \to \mathbb{R}_+$ measurable functions. Assume that*

$$W(t) := U(t) \cap (g(t) + k(t)\mathbb{B}) \neq \emptyset,$$

a.e. for $t \in [\tau, T]$. Then, there exists a measurable function $w : [\tau, T] \to X$ such that $w(t) \in W(t)$ a.e. for $t \in [\tau, T]$.

The proof of the above Lemma follows from [7, pp. 87-88]. We continue by stating a Filippov type result (see, e.g., [13, Theorem 3.1]).

Theorem 2 *Let X be a Banach space and $\Omega \subset X$ be an open bounded set. Let $F : [\tau, T] \times \Omega \to X$ be a multi-function with nonempty and closed values. Assume that $F(\cdot, x)$ is measurable for each $x \in \Omega$ and there exists an integrable function $k : [\tau, T] \to \mathbb{R}_+$ satisfying*

$$F(t, x) \subset F(t, y) + k(t)\|x - y\|\mathbb{B}, \tag{4}$$

for any $x, y \in \Omega$ and a.e. for $t \in [\tau, T]$. Let $\varepsilon > 0$ and $u \in W^{1,1}(\tau, T; X)$ be such that

$$\int_{\tau}^{T} \operatorname{dist}(u'(t); F(t, u(t)))dt < \varepsilon. \tag{5}$$

Then, there exists $\delta > 0$ such that for any $x_\tau \in u(\tau) + \delta\mathbb{B}$, there exists a solution $x : [\tau, T] \to X$ of the differential inclusion in Eq. (1) with $x(\tau) = x_\tau$ and such that

$$\|x(t) - u(t)\| < l\varepsilon,$$

for each $t \in [\tau, T]$, where $l = \exp \int_{\tau}^{T} k(t)dt$.

We end this section by recalling some useful definitions.

Definition 6 A multi-function $F : I \times X \rightsquigarrow X$ is said to be integrably bounded, if for each $(\tau, \xi) \in I \times X$, there exist $\rho_1 > 0$ and $l \in L^1(I, \mathbb{R}^+)$ such that $F(t, x) \subset l(t)\mathbb{B}$, a.e. for $t \in I$ and for all $x \in B(\xi, \rho_1)$.

Definition 7

(i) The multi-function $F : I \rightsquigarrow X$ is said to be $\varepsilon - \delta$ lower semicontinuous ($\varepsilon - \delta$ l.s.c) at $\tau \in I$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $F(\tau) \subseteq F(s) + \varepsilon\mathbb{B}$, for any $s \in [\tau - \delta, \tau + \delta]$.

(ii) The multi-function $F : I \rightsquigarrow X$ is said to be $\varepsilon - \delta$ upper semicontinuous ($\varepsilon - \delta$ u.s.c) at $\tau \in I$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $F(s) \subseteq F(\tau) + \varepsilon\mathbb{B}$, for any $s \in [\tau - \delta, \tau + \delta]$.

3 Tangency Conditions

There is a rich literature on the invariance problem which uses tangency conditions expressed in terms of the tangency concept introduced recently by Cârjă et al. (see [4] and [3]) in order to establish several sufficient and/or necessary and sufficient conditions for exact/approximate weak invariance for a cylindrical domain $\mathcal{K} = I \times K$ and $K \subset X$. We recall that for $(\tau, \xi) \in \mathcal{K}$, the bounded subset $F(\tau, \xi)$ is said to be quasi-tangent to $\mathcal{K}$ at (τ, ξ) if

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}\left(\xi + \int_{\tau}^{\tau+h} F(\tau, \xi)ds; K\right) = 0.$$

Notice that in [11] and in the case when $\mathcal{K}$ is not necessarily a cylindrical domain, the above tangency concept was suitably adapted, in order to express (exact) weak invariance results for $\mathcal{K}$ with respect to Eq. (1), by saying that for $(\tau, \xi) \in \mathcal{K}$ the bounded subset $F(\tau, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ) if

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}\left(\xi + \int_{\tau}^{\tau+h} F(\tau, \xi)ds; K(\tau + h)\right) = 0. \tag{6}$$

The t-dependence of the multi-function F leads us to extend, in a natural way, the quasi-tangency concept defined by Eq. (6) for an integrably bounded multi-function F by saying that for $(\tau, \xi) \in \mathcal{K}$ the multi-function $F(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ) if

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}\left(\xi + \int_{\tau}^{\tau+h} F(s, \xi)ds; K(\tau + h)\right) = 0. \tag{7}$$

In the next proposition, we give some characterizations for the quasi-tangency concept defined by Eq. (7) that will be used in the sequel.

Proposition 1 *Let F be an integrably bounded multi-function. Let $\mathcal{K}$ be a graph and let $(\tau, \xi) \in \mathcal{K}$. The following conditions are equivalent:*

(i) *The multi-function $F(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ);*

(ii) *for each $\varepsilon > 0$, there exist $h \in (0, \varepsilon)$, $f \in S_{[\tau,\tau+h]}F(\cdot, \xi)$ and $p \in X$ with $\|p\| \leq \varepsilon$, such that*

$$\xi + \int_{\tau}^{\tau+h} f(s)ds + hp \in K(\tau + h);$$

(iii) *there exist two sequences, $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(f_n)_n$ such that $f_n \in S_{[\tau,\tau+h_n]}F(\cdot, \xi)$ for each $n \in \mathbb{N}^\star$, satisfying*

$$\lim_{n\to+\infty} \frac{1}{h_n}\text{dist}\left(\xi + \int_{\tau}^{\tau+h_n} f_n(s)ds;\, K(\tau + h_n)\right) = 0;$$

(iv) *there exist three sequences, $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(f_n)_n$ such that $f_n \in S_{[\tau,\tau+h_n]}F(\cdot, \xi)$ for each $n \in \mathbb{N}^\star$, and $(p_n)_n$ in X with $\lim_{n\to+\infty} p_n = 0$, satisfying*

$$\xi + \int_{\tau}^{\tau+h_n} f_n(s)ds + h_n p_n \in K(\tau + h_n)$$

for, $n = 1, 2, ...$

In addition, if X is a separable Banach space and $F(\cdot, \xi)$ is a measurable multi-function with nonempty and closed values, then (i) ∼(iv) are equivalent to

(v) *the multi-function $\overline{co}F\,(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ).*

Proof The proof of the equivalences (i)∼(iv) follows the very same arguments as those used in Problem 2.3.2 in [3, p. 292]. For the reader's convenience, we sketch the proof of the equivalence between (i) and (ii). Indeed, by Eq. (7), $F(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ), if and only if, we have

$$\sup_{\delta>0}\ \inf_{h\in(0,\delta)} \frac{1}{h}\text{dist}\left(\xi + \int_{\tau}^{\tau+h} F(s, \xi)ds;\, K(\tau + h)\right) = 0,$$

which in its turn is equivalent to the following: for each $\varepsilon > 0$ and each $\delta > 0$, there exists $h \in (0, \delta)$ such that

$$\text{dist}\left(\xi + \int_{\tau}^{\tau+h} F(s, \xi)ds;\, K(\tau + h)\right) < h\varepsilon.$$

For the proof of (v) one can use [1, Theorem 8.6.4], which says that, whenever X is a separable Banach space and $F\,(\cdot, \xi)$ is a measurable and integrably bounded multi-function with nonempty and closed values, we have

$$\overline{\int_a^b F(s, \xi)ds} = \int_a^b \overline{co}F(s, \xi)ds.$$

□

Remark 3 A stronger variant of Eq. (7) can be obtained by using the excess distance $H^*(\cdot; \cdot)$ instead of $\text{dist}(\cdot; \cdot)$. Namely, let F be an integrably bounded multi-function and let $\mathcal{K}$ be a graph. For $(\tau, \xi) \in \mathcal{K}$, we say that the multi-function $F(\cdot, \xi)$ is strongly tangent to $\mathcal{K}$ at (τ, ξ) if

$$\liminf_{h \to 0^+} \frac{1}{h} H^* \left(\xi + \int_\tau^{\tau+h} F(s, \xi) ds; K(\tau + h) \right) = 0. \tag{8}$$

We recall that the excess distance of A over B is defined by $H^*(A; B) = \sup_{a \in A} \operatorname{dist}(a; B)$. We point out that by using a tangency condition expressed in terms of Eq. (8), a local weak invariance result for a cylindrical domain $\mathcal{K}$ with respect to Eq. (1) was given in [12], as we shall mention in the end of Section 5.

The relationship between the quasi-tangency concepts in Eqs. (6) and (7) is clarified in the next proposition.

Proposition 2 *Let X be a separable Banach space. Let $(\tau, \xi) \in \mathcal{K}$. Assume that $F(\cdot, \xi) : I \rightsquigarrow X$ is an integrably bounded and measurable multi-function with nonempty and closed values.*

(i) If $F(\cdot, \xi)$ is $\varepsilon - \delta$ l.s.c at τ and $F(\tau, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ), then Eq. (7) holds true.

(ii) If $F(\cdot, \xi)$ is $\varepsilon - \delta$ u.s.c at τ and Eq. (7) holds true, then $F(\tau, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ).

Proof We begin by proving (i). Let $(\tau, \xi) \in \mathcal{K}$. Assume that $F(\tau, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ). In view of Proposition 1 applied with the multi-function $s \mapsto F(\tau, \xi)$, we infer that there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(f_n)_n$ such that $f_n \in S_{[\tau, \tau+h_n]} F(\tau, \xi)$ for each $n \in \mathbb{N}^\star$, and $(p_n)_n$ in X with $\lim_{n \to +\infty} p_n = 0$, such that

$$\xi + \int_\tau^{\tau+h_n} f_n(s) ds + h_n p_n \in K(\tau + h_n),$$

for all $n = 1, 2, \ldots$. Since $F(\cdot, \xi)$ is $\varepsilon - \delta$ l.s.c at τ, for each $n = 1, 2\ldots$ there exists $\delta_n > 0$ such that

$$F(\tau, \xi) \subset F(s, \xi) + \frac{1}{n} \mathbb{B},$$

for all $s \in [\tau, \tau + \delta_n]$. Let $k_n \in \mathbb{N}^*$ be such that $h_{k_n} < \delta_n$. Then,

$$f_{k_n}(s) \in F(s, \xi) + \frac{1}{n} \mathbb{B},$$

for all $s \in [\tau, \tau + \delta_n]$. Therefore,

$$F(s, \xi) \cap \left(f_{k_n}(s) + \frac{1}{n} \mathbb{B} \right) \neq \emptyset,$$

for all $s \in [\tau, \tau + h_{k_n}]$. By Lemma 1, for all $n = 1, 2, \ldots$ there exist measurable functions g_{k_n} and b_{k_n} such that, $g_{k_n}(s) \in F(s, \xi)$ and $b_{k_n}(s) \in \mathbb{B}$ and

$$g_{k_n}(s) = f_{k_n}(s) + \frac{1}{n} b_{k_n}(s),$$

a.e. for $s \in [\tau, \tau + h_{k_n}]$. Let us set

$$q_{k_n} = \frac{1}{h_{k_n}} \int_\tau^{\tau+h_{k_n}} \left(g_{k_n}(s) - f_{k_n}(s) \right) ds$$

and $r_{k_n} = p_{k_n} - q_{k_n}$. It is clear that, $\lim_n q_{k_n} = 0$ and, so, $\lim_n r_{k_n} = 0$. On the other hand, one has

$$\begin{aligned}\xi + \int_\tau^{\tau+h_{k_n}} g_{k_n}(s)ds + h_{k_n} r_{k_n} &= \xi + \int_\tau^{\tau+h_{k_n}} g_{k_n}(s)ds + h_{k_n}(p_{k_n} - q_{k_n}) \\ &= \xi + \int_\tau^{\tau+h_{k_n}} f_{k_n}(s)ds + h_{k_n} p_{k_n} \in K(\tau + h_{k_n}).\end{aligned}$$

In view of (iv) in Proposition 1, we deduce that $F(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ).

For the proof of (ii), let us point out that if Eq. (7) holds true, then in view of Proposition 1, there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(f_n)_n$ such that $f_n \in S_{[\tau,\tau+h_n]}F(\cdot, \xi)$ for each $n \in \mathbb{N}^\star$, and $(p_n)_n$ in X with $\lim_{n\to+\infty} p_n = 0$, such that

$$\xi + \int_\tau^{\tau+h_n} f_n(s)ds + h_n p_n \in K(\tau + h_n),$$

for all $n = 1, 2, ...$ Since $F(\cdot, \xi)$ is $\varepsilon - \delta$ u.s.c at τ, for each $n = 1, 2...$ there exists $\delta_n > 0$ such that

$$F(s, \xi) \subset F(\tau, \xi) + \frac{1}{n}\mathbb{B},$$

for all $s \in [\tau, \tau + \delta_n]$. Therefore,

$$F(\tau, \xi) \cap \left(f_{k_n}(s) + \frac{1}{n}\mathbb{B}\right) \neq \emptyset,$$

for all $s \in [\tau, \tau + h_{k_n}]$. Using now Lemma 1 and following the same steps as in the proof of (i), we get the conclusion. □

Remark 4 We end this section by a simple remark, which will be used later, on the relationship between the quasi-tangent concept in Eq. (6) and the Bouligand weak tangent cone $T^w_{\mathcal{K}}(\tau, \xi)$ defined for $(\tau, \xi) \in \mathcal{K}$ by

$$T^w_{\mathcal{K}}(\tau, \xi) = \{\eta \in X;\ \exists h_n \downarrow 0,\ \eta_n \overset{\text{weakly}}{\to} \eta,\ \text{and}\ \xi + h_n\eta_n \in K(\tau + h_n),\ \text{for}\ n = 1.2.....\}.$$

We shall show, as in Problem 2.4.2 in [3, p. 294], that in reflexive Banach spaces setting in the case when $F(\tau, \xi)$ is nonempty, closed, and convex if $F(\tau, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ), then there exists at least one vector η in $F(\tau, \xi)$ and belonging to $T^w_{\mathcal{K}}(\tau, \xi)$. Indeed, by virtue of Proposition 1, there exist three sequences, $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(f_n)_n$ such that $f_n \in S_{[\tau,\tau+h_n]}F(\tau, \xi)$ for each $n \in \mathbb{N}^\star$ and $(p_n)_n$ in X with $\lim_{n\to+\infty} p_n = 0$, satisfying

$$\xi + h_n\eta_n + h_n p_n \in K(\tau + h_n)$$

for $n = 1, 2, ...$ where

$$\eta_n = \frac{1}{h_n}\int_\tau^{\tau+h_n} f_n(s)ds$$

for $n = 1, 2, ...$ Taking into account that $F(\tau, \xi)$ is closed, convex, and bounded (from the definition of the quasi-tangent concept), we deduce that it is weakly compact. Therefore, on a subsequence, at least, $(\eta_n)_n$ converges weakly to some $\eta \in F(\tau, \xi)$. Accordingly,

$$T^w_{\mathcal{K}}(\tau, \xi) \cap F(\tau, \xi) \neq \emptyset.$$

4 Approximate Weak Invariance

This section is devoted to establish some results concerning approximate weak invariance for $\mathcal{K}$ with respect to the differential inclusion in Eq. (1). First, let us recall the definition of a graph $\mathcal{K}$ that is locally closed from the left.

Definition 8 The graph $\mathcal{K}$ is said to be locally closed from the left if for each $(\tau, \xi) \in \mathcal{K}$ there exist $T > \tau$ and $\rho_2 > 0$ such that for each $(\tau_n, \xi_n) \in ([\tau, T] \times B(\xi, \rho_2)) \cap \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \overline{\tau}$ and $\lim_n \xi_n = \overline{\xi}$, we have $(\overline{\tau}, \overline{\xi}) \in \mathcal{K}$.

Now, we state a basic tangential hypothesis we shall refer to in the sequel. Let $(\tau, \xi) \in \mathcal{K}$ and consider the following hypothesis

(H1) $F(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ).

The next result gives sufficient conditions for a locally closed from the left graph $\mathcal{K}$ to be approximate weakly invariant with respect to Eq. (1).

Theorem 3 *Let $\mathcal{K}$ be locally closed from the left and F an integrably bounded multi-function. If* (H1) *is satisfied for each $(\tau, \xi) \in \mathcal{K}$, then $\mathcal{K}$ is approximate weakly invariant with respect to Eq.* (1).

The proof of Theorem 3 is based on a result regarding the existence of "approximate solutions" given below (see, e.g., [3, Lemma 6.3.1]). We point out that in [3, Lemma 6.3.1] the authors consider the multi-function $F : K \rightsquigarrow X$ and use the assumption that $F(\xi)$ is quasi-tangent to K at ξ for each $\xi \in K$. Here, we assume that (H1) is satisfied for each $(\tau, \xi) \in \mathcal{K}$ and the proof is similar to [3, Lemma 6.3.1]. Let $(\tau, \xi) \in \mathcal{K}$. Let $\rho_2 > 0$ and $T > \tau$ be as in Definition 8. Let $\rho_1 > 0$ and l be as in Definition 6. We take $\rho = \min\{\rho_1, \rho_2\}$. Diminishing $T > \tau$, if necessary, we may assume that

$$T - \tau + \int_{\tau}^{T} l(s)ds < \rho. \tag{9}$$

Lemma 2 *Let $\mathcal{K}$ be locally closed from the left and let F be an integrably bounded multi-function. Let $(\tau, \xi) \in \mathcal{K}$, $T > \tau$, l and $\rho > 0$ be fixed as above. If* (H1) *is satisfied for each $(t, x) \in \mathcal{K}$ then for each $\varepsilon \in (0, 1)$, there exist $\sigma : [\tau, T] \to [\tau, T]$ nondecreasing, $f, g \in L^1(\tau, T; X)$ and $u : [\tau, T] \to X$ continuous such that:*

(i) $t - \epsilon \leq \sigma(t) \leq t$, *for all* $t \in [\tau, T]$ *and* $\sigma(T) = T$*;*
(ii) $u(\sigma(t)) \in K(\sigma(t)) \cap B(\xi, \rho)$, *for all* $t \in [\tau, T]$*;*
(iii) $f(t) \in F(t, u(\sigma(t)))$, *a.e. for* $t \in [\tau, T]$ *and* $\|f(t)\| \leq l(t)$ *a.e. for* $t \in [\tau, T]$*;*
(iv) $\|g(t)\| \leq \varepsilon$, *a.e. for* $t \in [\tau, T]$*;*
(v) $u(t) = \xi + \int_\tau^t f(s)ds + \int_\tau^t g(s)ds$, *for all* $t \in [\tau, T]$*;*
(vi) $\|u(t) - u(\sigma(t))\| \leq \varepsilon$, *for all* $t \in [\tau, T]$.

In what follows and before proceeding to the proof of Theorem 3, we mention that we can give a version of Lemma 2 where the time T is independent of the initial state (τ, ξ). More exactly, if we assume that $\mathcal{K}$ is X-closed and F has a sublinear growth condition, i.e., there exists $c \in L^1(I, \mathbb{R}_+)$ such that

$$F(t, x) \subset c(t)(1 + \|x\|)\mathbb{B}, \tag{10}$$

for each $x \in X$ and a.e. for $t \in I$, then the following result holds true.

Lemma 3 *Let $\mathcal{K}$ be X-closed and F a multi-function satisfying the sublinear growth condition of Eq.* (10)*. Assume that* (H1) *is satisfied for each $(\tau, \xi) \in \mathcal{K}$. Then, for each $(\tau, \xi) \in \mathcal{K}$, $T > \tau$ with $[\tau, T] \subset I$ and $\varepsilon \in (0, 1)$, there exist $\sigma : [\tau, T] \to [\tau, T]$ nondecreasing, $f, g \in L^1(\tau, T; X)$ and $u : [\tau, T] \to X$ continuous, such that:*

(a) $t - \epsilon \le \sigma(t) \le t$, *for all* $t \in [\tau, T]$ *and* $\sigma(T) = T$*;*
(b) $u(\sigma(t)) \in K(\sigma(t))$, *for all* $t \in [\tau, T]$*;*
(c) $f(t) \in F(t, u(\sigma(t)))$, *a.e. for* $t \in [\tau, T]$*;*
(d) $\|g(t)\| \le \varepsilon$, *a.e. for* $t \in [\tau, T]$*;*
(e) $u(t) = \xi + \int_\tau^t f(s)ds + \int_\tau^t g(s)ds$, *for all* $t \in [\tau, T]$*;*
(f) $\|u(t) - u(\sigma(t))\| \le \varepsilon$, *for all* $t \in [\tau, T]$.

We recall that the graph $\mathcal{K}$ is said to be X-closed, if for each $(\tau_n, \xi_n) \in \mathcal{K}$ with $\lim_n \tau_n = \overline{\tau} \in I$ and $\lim_n \xi_n = \overline{\xi}$, we have $(\overline{\tau}, \overline{\xi}) \in \mathcal{K}$. We continue with the proof of Theorem 3.

Proof of Theorem 3 Let $(\tau, \xi) \in \mathcal{K}$. Let l and T be as in Lemma 2. Let $\varepsilon > 0$ and $\eta(\varepsilon) > 0$ be such that

$$\lambda(E) \le \eta(\varepsilon) \Rightarrow \int_E l(t)dt \le \frac{\varepsilon}{2}. \tag{11}$$

We take $\varepsilon' \in (0, 1)$ such that

$$0 < \varepsilon' < \min\{\eta(\varepsilon); \frac{\varepsilon}{2(T - \tau)}\}.$$

Applying now Lemma 2 for ε', there exist functions $\sigma : [\tau, T] \to [\tau, T]$, $f, g \in L^1(\tau, T; X)$ and $u : [\tau, T] \to X$ continuous such that (i)∼(vi) hold. Let us define $x : [\tau, T] \to X$ by

$$x(t) = u(t) - \int_\tau^t g(s)ds = \xi + \int_\tau^t f(s)ds,$$

for all $t \in [\tau, T]$. Then, $x'(t) = f(t)$ a.e. for $t \in [\tau, T]$. Taking into account that $f \in L^1(\tau, T; X)$, we deduce that $x \in W^{1,1}(\tau, T; X)$. By (iii), we get that

$$x'(t) \in F(t, u(\sigma(t))),$$

a.e. for $t \in [\tau, T]$. Using (v), (i), Eq. (11) and the choice of ε', we obtain

$$\begin{aligned}
\|u(\sigma(t)) - x(t)\| &= \|\int_t^{\sigma(t)} f(s)ds + \int_\tau^{\sigma(t)} g(s)ds\| \\
&\le \int_{\sigma(t)}^t \|f(s)\|ds + \int_\tau^{\sigma(t)} \|g(s)\|ds \\
&\le \int_{\sigma(t)}^t l(s)ds + \varepsilon'(T - \tau) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

for every $t \in [\tau, T]$. Accordingly,

$$x'(t) \in F(t, x(t) + \varepsilon\mathbb{B}),$$

a.e. for $t \in [\tau, T]$. Therefore, x is an ε-solution of Eq. (1) on $[\tau, T]$ with $x(\tau) = \xi$. Finally, taking into account (i), (ii) and the choice of ε' ($\varepsilon' < \varepsilon$), one has

$$t - \varepsilon \le t - \varepsilon' \le \sigma(t) \le t,$$

for all $t \in [\tau, T]$ and

$$\operatorname{dist}(x(t); K(\sigma(t))) \leq \|x(t) - u(\sigma(t))\| \leq \varepsilon$$

for all $t \in [\tau, T]$. The proof is complete.

The next theorem is a variant of Theorem 3 when the graph $\mathcal{K}$ is X-closed (instead of locally closed from the left) and F satisfies the sublinear growth condition of Eq. (10) (instead of integrably bounded).

Theorem 4 *Let $\mathcal{K}$ be X-closed and F a multi-function satisfying the sublinear growth condition of Eq.* (10)*. Assume that* (H1) *is satisfied for each $(\tau, \xi) \in \mathcal{K}$. Then, for each $(\tau, \xi) \in \mathcal{K}$, $T > \tau$ with $[\tau, T] \subset I$ and any $\varepsilon > 0$, there exist a nondecreasing function $\sigma : [\tau, T] \to [\tau, T]$, such that $t - \varepsilon \leq \sigma(t) \leq t$ for each $t \in [\tau, T]$, and an ε-solution $x : [\tau, T] \to X$ of Eq.* (1) *with $x(\tau) = \xi$, satisfying*

$$\operatorname{dist}(x(t); K(\sigma(t))) \leq \varepsilon$$

for all $t \in [\tau, T]$.

Proof The proof remains the same as in Theorem 3, using this time Lemma 3 instead of Lemma 2. □

The next result follows immediately from (i) in Proposition 2 and Theorem 3.

Theorem 5 *Let X be a separable Banach space. Let $\mathcal{K}$ be locally closed from the left and F an integrably bounded multi-function with nonempty and closed values. Assume that for each $(\tau, \xi) \in \mathcal{K}$, $F(\cdot, \xi)$ is $\varepsilon - \delta$ l.s.c at τ. If $F(\tau, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ) for each $(\tau, \xi) \in \mathcal{K}$, then $\mathcal{K}$ is approximate weakly invariant with respect to Eq. (1).*

In what follows and for certain class of Carathéodory multi-functions, we shall show that a necessary condition for approximate weak invariance for a graph $\mathcal{K}$ with respect to Eq. (1), is that (H1) be satisfied a.e for $\tau \in I$ and for all $\xi \in K(\tau)$. To this end, we consider the following standing assumptions.

- (H2) For each $x \in X$, the multi-function $F(\cdot, x)$ is measurable on I.
- (H3) There exists $l \in L^1(I, \mathbb{R}_+)$ such that for each $(\tau, \xi) \in \mathcal{K}$, there exist a nondecreasing function $W : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at 0 with $W(0) = 0$ and a bounded open set $\Omega \subset X$ containing ξ, such that

$$F(t, x) \subset F(t, \xi) + l(t)W(\|x - \xi\|)\mathbb{B},$$

  for each $x \in \Omega$ and a.e. for $t \in I$.
- (H4) The multi-function $K : I \rightsquigarrow X$ is Lipschitz.

Theorem 6 *Let X be a separable Banach space and F be a multi-function with nonempty and closed values satisfying* (H2)*,* (H3) *and the sublinear growth condition of Eq.* (10)*. Let K satisfying* (H4)*. If $\mathcal{K}$ is approximate weakly invariant with respect to Eq.* (1)*, then* (H1) *is satisfied a.e. for $\tau \in I$ and for each $\xi \in K(\tau)$.*

Proof First, let us denote by I_λ the set of all Lebesgue points of the function l. It is well known that $\lambda(I \backslash I_\lambda) = 0$ and for each $\tau \in I_\lambda$ we have

$$\lim_{\lambda(J) \to 0} \frac{1}{\lambda(J)} \int_J |l(s) - l(\tau)| ds = 0, \tag{12}$$

where J denotes an arbitrary interval of positive length containing τ. Let $\tau \in I_\lambda$ and $\xi \in K(\tau)$. Let $T > \tau$ be as in Definition 2 and let $(\varepsilon_n) \subset (0, 1)$ be such that $\varepsilon_n \downarrow 0$ and $\sqrt{\epsilon_n} < T - \tau$ for $n = 1, 2......$ Since $\mathcal{K}$ is approximate weakly invariant with respect to Eq. (1) and taking into account Remark 1, there exists an ε_n-solution $x_n : [\tau, T] \to X$ of Eq. (1), i.e.,

$$x_n'(t) \in F(t, x_n(t) + \varepsilon_n \mathbb{B}), \tag{13}$$

a.e. for $t \in [\tau, T]$, with $x_n(\tau) = \xi$, and

$$\operatorname{dist}(x_n(t); K(t)) \le \varepsilon_n, \tag{14}$$

for each $t \in [\tau, T]$. By Eqs. (13) and (10), we get

$$\|x_n'(t)\| \le c(t)(2 + \|x_n(t)\|), \tag{15}$$

a.e. for $t \in [\tau, T]$. Moreover, since

$$x_n(t) = x_n(\tau) + \int_\tau^t x_n'(s) ds = \xi + \int_\tau^t x_n'(s) ds, \tag{16}$$

for all $t \in [\tau, T]$, we have

$$\|x_n(t)\| \le \|\xi\| + \int_\tau^t \|x_n'(s)\| ds \le \|\xi\| + \int_\tau^t c(s)(2 + \|x_n(s)\|) ds,$$

for all $t \in [\tau, T]$. Applying Gronwall's inequality, we get that

$$\|x_n(t)\| \le k, \tag{17}$$

for all $t \in [\tau, T]$, where $k = e^C(2C + \|\xi\|)$ and $C = \int_\tau^T c(s) ds$. Taking into account Eqs. (15) and (17), we obtain

$$\|x_n'(t)\| < c(t)(2 + k), \tag{18}$$

a.e. for $t \in [\tau, T]$. From Eqs. (16) and (18), it follows

$$\|x_n(t) - \xi\| \le \int_\tau^t \|x_n'(s)\| ds \le \int_\tau^t c(s)(2 + k) ds,$$

for all $t \in [\tau, T]$. Let $t_n = \sqrt{\varepsilon_n} < T - \tau$. Then,

$$\|x_n(t) - \xi\| \le \delta_n,$$

for all $t \in [\tau, \tau + t_n]$, where $\delta_n = \int_\tau^{\tau + t_n} c(s)(2 + k) ds$. Clearly, $\lim_n \delta_n = 0$. Furthermore,

$$x_n(t) \in \xi + \delta_n \mathbb{B},$$

for all $t \in [\tau, \tau + t_n]$. Using now Eq. (13) to get that

$$x_n'(t) \in F(t, \xi + (\delta_n + \varepsilon_n)\mathbb{B}),$$

a.e. for $t \in [\tau, \tau + t_n]$. From hypothesis (H3) and for n sufficiently large, one has

$$F(t, \xi + (\delta_n + \varepsilon_n)\mathbb{B}) \subset F(t, \xi) + l(t) w_n \mathbb{B},$$

where $w_n = W(\delta_n + \varepsilon_n)$. Consequently,

$$x_n'(t) \in F(t, \xi) + l(t) w_n \mathbb{B},$$

a.e. for $t \in [\tau, \tau + t_n]$. Therefore,

$$F(t, \xi) \cap (x_n'(t) + l(t)w_n\mathbb{B}) \neq \emptyset,$$

a.e for $t \in [\tau, \tau + t_n]$. Hence, by Lemma 1 and for n sufficiently large, there exist measurable functions f_n and b_n with $f_n(t) \in F(t, \xi)$ and $b_n(t) \in \mathbb{B}$ a.e. for $t \in [\tau, \tau + t_n]$ and

$$f_n(t) = x_n'(t) + l(t)w_n b_n(t).$$

Then,

$$\begin{aligned} \xi + \int_\tau^{\tau+t_n} f_n(s)ds &= \xi + \int_\tau^{\tau+t_n} [x_n'(s) + l(s)w_n b_n(s)]ds \\ &= x_n(\tau + t_n) + w_n \int_\tau^{\tau+t_n} l(s)b_n(s)ds. \end{aligned}$$

Moreover,

$$\begin{aligned} &\text{dist}\left(\xi + \int_\tau^{\tau+t_n} f_n(s)ds; K(\tau + t_n)\right) \\ &\leq \text{dist}\left(x_n\left(\tau + t_n\right); K\left(\tau + t_n\right)\right) + \left\| w_n \int_\tau^{\tau+t_n} l\left(s\right) b_n\left(s\right) ds \right\| \leq \varepsilon_n + w_n \int_\tau^{\tau+t_n} l\left(s\right) ds. \end{aligned}$$

Finally, using the choice of t_n, one has

$$\begin{aligned} \frac{1}{t_n}\text{dist}\left(\xi + \int_\tau^{\tau+t_n} f_n\left(s\right) ds; K\left(\tau + t_n\right)\right) &\leq \frac{1}{t_n}\left(\varepsilon_n + w_n \int_\tau^{\tau+t_n} l\left(s\right) ds\right) \\ &\leq \sqrt{\varepsilon_n} + w_n \frac{1}{t_n} \int_\tau^{\tau+t_n} \left|l\left(s\right) - l\left(\tau\right)\right| ds + l\left(\tau\right) w_n. \end{aligned}$$

From Eq. (12) and taking into account that $\lim_n w_n = 0$, it follows that

$$\lim_{n\to+\infty} \frac{1}{t_n}\text{dist}\left(\xi + \int_\tau^{\tau+t_n} f_n\left(s\right) ds; K\left(\tau + t_n\right)\right) = 0.$$

By (iii) in Proposition 1, we deduce that (H1) holds true a.e. for $\tau \in I$ and for each $\xi \in K(\tau)$. □

Remark 5 Clearly, if the function l is continuous, then Eq. (12) holds for all $\tau \in I$ and every $\xi \in K(\tau)$. Accordingly, (H1) is satisfied for each $(\tau, \xi) \in \mathcal{K}$.

From the above remark, Theorem 3 and Theorem 6, we obtain the following assertion.

Corollary 1 *Let X be a separable Banach space. Let $\mathcal{K}$ be a locally closed from the left graph and F a multi-function with nonempty and closed values satisfying* (H1), (H3) *(l is supposed continuous) and the sublinear growth condition of Eq.* (10). *Let K satisfying* (H4). *A necessary and sufficient condition in order that $\mathcal{K}$ be approximate weakly invariant with respect to Eq.* (1) *is that* (H1) *be satisfied for each $(\tau, \xi) \in \mathcal{K}$.*

We continue with a remark on the relation between our results and [5, Theorem 2.2].

Remark 6 First, we recall that in [5, Theorem 2.2], the authors gave a criterion of approximate weak invariance for a cylindrical domain, i.e., $\mathcal{K} = I \times K$ where $K \subset X$, with respect to Eq. (1) by using the classical tangent cone $T_{\mathcal{K}}^w(\tau, \xi)$ already defined in Remark 4. More precisely, if we assume that F is $\varepsilon - \delta$ upper semicontinuous in both variables with nonempty convex and closed values satisfying the sublinear growth condition of Eq. (10) and such that

$$F(\tau,\xi)\cap T^{w}_{\mathcal{K}}(\tau,\xi)\neq\emptyset, \tag{19}$$

for each $(\tau,\xi)\in\mathcal{K}$, then $\mathcal{K}$ is approximate weakly invariant with respect to Eq. (1). Let us point out that, under the same hypotheses on F and taking into account (ii) in Proposition 2 and Remark 4, if (H1) is satisfied for each $(\tau,\xi)\in\mathcal{K}$, then Eq. (19) holds true, which means that $\mathcal{K}$ is approximate weakly invariant with respect to Eq. (1). However, we proved in Theorem 3 that it suffices that (H1) be satisfied for each $(\tau,\xi)\in\mathcal{K}$ and F be integrably bounded and without any conditions on the values of F to get an approximate weak invariance property for a not necessarily cylindrical domain $\mathcal{K}$ with respect to Eq. (1).

We conclude this section with a remark on our approach in this paper.

Remark 7 A general method to pass from autonomous to nonautonomous case is to consider the space $\mathcal{X}=\mathbb{R}\times X$, to set $z(s)=(t(s),x(s))$ and the autonomous multi-function $\overline{F}(z)=(1,F(z))$. This trick does not work here because it leads to an ε-variation in the first variable of F in Eq. (3).

5 Near Weak Invariance

In this section, using a Filippov type result, we establish some sufficient conditions for $\mathcal{K}$ to be near weakly invariant with respect to the differential inclusion in Eq. (1). Let us first list some assumptions we shall refer to in what follows:

(H5) For each $(\tau,\xi)\in\mathcal{K}$, there exist $\overline{T}>\tau$, an open set $\Omega\subset X$ containing ξ and $k\in L^1\left(\tau,\overline{T};\mathbb{R}_+\right)$ such that

$$F(t,x)\subset F(t,y)+k(t)\|x-y\|\mathbb{B},$$

for each $x,y\in\Omega$ and a.e. for $t\in\left[\tau,\overline{T}\right]$;

(H6) There exists $k\in L^1(I,\mathbb{R}_+)$ such that

$$F(t,x)\subset F(t,y)+k(t)\|x-y\|\mathbb{B},$$

for each $x,y\in X$ and a.e. for $t\in I$.

Notice that (H6) is nothing but a particular case getting from (H5).

Theorem 7 *Let $\mathcal{K}$ be locally closed from the left and F an integrably bounded multi-function with nonempty and closed values satisfying* (H2) *and* (H5). *If* (H1) *is satisfied for each $(\tau,\xi)\in\mathcal{K}$, then $\mathcal{K}$ is near weakly invariant with respect to Eq.* (1).

Proof Let $(\tau,\xi)\in\mathcal{K}$. Let Ω be as in (H5) and let $\widetilde{\rho}$ be such that $B(\xi,\widetilde{\rho})\subset\Omega$. We take $T>\tau$ and $\rho>0$ as in Eq. (9). Without loss of generality, we may assume that $0<\rho\leq\widehat{\rho}$. In fact, this is possible if we take $\rho=\min\{\rho_1,\rho_2,\widehat{\rho}\}$ where ρ_1 and ρ_2 are, respectively, from Definition 6 and Definition 8.

Let $\varepsilon>0$. We take $\varepsilon'\in(0,1)$ such that

$$0<\varepsilon'\leq\frac{\varepsilon}{e^{l}(l+\overline{T}-\tau)+1},$$

where $l = \int_\tau^{\overline{T}} k(t)dt$. Applying Lemma 2 for ε', there exist σ, f, g and u satisfying (i)∼(vi). Since u is continuous at τ, there exists $T_1 \in (\tau, T)$ such that for each $t \in [\tau, T_1]$, we have $u(t) \in B(\xi, \rho)$. Notice that $u(\sigma(t)) \in B(\xi, \rho)$ for each $t \in [\tau, T]$ follows from (ii) in Lemma 2. Let us set $\widetilde{T} = \min\{T, \overline{T}, T_1\}$. From (v), we get $u'(t) = f(t) + g(t)$ a.e. for $t \in [\tau, \widetilde{T}]$. Since $f, g \in L^1(\tau, \widetilde{T}; X)$, we deduce that $u \in W^{1,1}(\tau, \widetilde{T}; X)$. By using (iii) and hypothesis (H5), we obtain

$$f(t) \in F(t, u(\sigma(t))) \subset F(t, u(t)) + k(t)\|u(\sigma(t)) - u(t)\|\mathbb{B},$$

a.e. for $t \in [\tau, \widetilde{T}]$. Taking into account (vi), (iv) and the above inclusion, we obtain

$$u'(t) \in F(t, u(t)) + k(t)\varepsilon'\mathbb{B} + \varepsilon'\mathbb{B},$$

a.e. for $t \in [\tau, \widetilde{T}]$. Therefore,

$$\int_\tau^{\widetilde{T}} \mathrm{dist}(u'(t); F(t, u(t))dt \le (l + \overline{T} - \tau)\varepsilon',$$

which means that u satisfies Eq. (5) in Theorem 2 with $\varepsilon := (l + \overline{T} - \tau)\varepsilon'$. Applying now Theorem 2, there exists a solution $x : [\tau, \widetilde{T}] \to X$ of Eq. (1) with $x(\tau) = \xi$ satisfying

$$\|x(t) - u(t)\| \le e^l(l + \overline{T} - \tau)\varepsilon'$$

for all $t \in [\tau, \widetilde{T}]$. Furthermore, from (i), (ii), (vi), the above inequality and the choice of ε', we get

$$t - \varepsilon \le t - \varepsilon' \le \sigma(t) \le t,$$

for all $t \in [\tau, \widetilde{T}]$ and

$$\begin{aligned}\mathrm{dist}(x(t); K(\sigma(t))) \le \|x(t) - u(\sigma(t))\| &\le \|x(t) - u(t)\| + \|u(t) - u(\sigma(t)\| \\ &\le e^l(l + \overline{T} - \tau)\varepsilon' + \varepsilon' \le \varepsilon,\end{aligned}$$

for all $t \in [\tau, \widetilde{T}]$. The proof is therefore complete. □

In what follows, we state that under hypotheses (H2), (H6) and the sublinear growth condition of Eq. (10), if (H1) is satisfied for each $(\tau, \xi) \in \mathcal{K}$, then $\mathcal{K}$ is near weakly invariant with respect to Eq. (1), independently of the time T, more precisely.

Theorem 8 *Let $\mathcal{K}$ be X-closed and F a nonempty and closed valued multi-function satisfying* (H2), (H6) *and the sublinear growth condition of Eq.* (10). *Assume that* (H1) *is satisfied for each $(\tau, \xi) \in \mathcal{K}$. Then for each $(\tau, \xi) \in \mathcal{K}$, $T > \tau$ with $[\tau, T] \subset I$ and any $\varepsilon > 0$, there exist a nondecreasing function $\sigma : [\tau, T] \to [\tau, T]$ such that $t - \varepsilon \le \sigma(t) \le t$ for each $t \in [\tau, T]$ and a solution $x : [\tau, T] \to X$ of Eq.* (1) *with $x(\tau) = \xi$, satisfying*

$$\mathrm{dist}(x(t); K(\sigma(t))) \le \varepsilon$$

for all $t \in [\tau, T]$.

Proof The proof follows, except minor modifications, the very same arguments as those of the proof of Theorem 7, using this time Lemma 3 instead of Lemma 2. □

Let us make a comparison between the result obtained in [12, Theorem 2.1] and Theorem 8 of our paper. Let X be a separable Banach space and $\mathcal{K}$ be a cylindrical domain. Assume that F is an integrably bounded multi-function satisfying (H2) and (H6). The author proved in [12, Theorem 2.1] that if $F(\cdot,\xi)$ is strongly tangent to $\mathcal{K}$ at (τ,ξ) for each $(\tau,\xi)\in\mathcal{K}$ in the sense of Eq. (8), then $\mathcal{K}$ is weakly invariant with respect to Eq. (1). It is clear that the tangency condition used in [12] is more restrictive than (H1). However, our result says that under the same hypotheses on F, if (H1) is satisfied for each $(\tau,\xi)\in\mathcal{K}$, then $\mathcal{K}$ is near weakly invariant with respect to Eq. (1).

Finally, let us mentioned that the method of adding a new variable $t(s)$ in order to pass from the autonomous case to the nonautonomous could work here if F is Lipshitz in both variables.

6 Applications

In this section, we give two applications in the study of the regularity of the solutions set of the differential inclusion in Eq. (1). To this end, we assume, in this section, that the sublinear growth condition of Eq. (10) holds with $c(t)=C$, for some $C>0$ and a.e. for $t\in I$. In this case, one proves that each solution of Eq. (1) is Lipschitz. Namely, if $x:[\tau,T]\to X$ is a solution of Eq. (1) and F satisfies Eq. (10) with the above condition, using the same reasoning as in Eq. (13)~(18) (we take exact solution instead of ε-solution), we get that x' is bounded on $[\tau,T]$. Since x is absolutely continuous, we deduce immediately that it is Lipschitz, i.e.,

$$\|x(t_1)-x(t_2)\|\leq m|t_1-t_2|, \tag{20}$$

for some $m>0$ and for all $t_1,t_2\in[\tau,T]$.

6.1 On the Lipschitz Dependence with Respect to the Initial States

It is well known that in the finite-dimensional case, the set of solutions of the differential inclusion in Eq. (1) where F is Lipschitz, depends in a Lipschitz manner upon the initial states. In Banach spaces, a result of this type is proved in [2] . Here, we extend the result of [2] to the nonautonomous case in Eq. (1).

Theorem 9 *Let X be a separable Banach space and let F be a multi-function with nonempty and closed values satisfying* (H2), (H6) *and the sublinear growth condition of Eq.* (10). *Let $x_0,\ y_0\in X$ and $\varepsilon>0$. Then for any solution $x:[t_0,T]\to X$ of Eq.* (1) *with $x(t_0)=x_0$, there exists a solution $y:[t_0,T]\to X$ of Eq.* (1) *with $y(t_0)=y_0$, such that*

$$\|x(t)-y(t)\|\leq e^{\int_{t_0}^{t}k(s)ds}\left(\|x_0-y_0\|+\varepsilon\right) \tag{21}$$

for any $t\in[t_0,T]$.

Proof Let $x:[t_0,T]\to X$ be a solution of Eq. (1) with $x(t_0)=x_0$. We consider the extended space $\widehat{X}=X\times\mathbb{R}$, with the norm

$$\|(x,z)\|=\|x\|+|z|.$$

We define the multi-functions $\widehat{K}:[t_0,T)\rightsquigarrow\widehat{X}$ by

$$\widehat{K}(t)=\{(x,z)\in X\times\mathbb{R}^+,\ \|x-x(t)\|\leq z\}$$

and $\widehat{F}:[t_0,T)\times\widehat{X}\rightsquigarrow\widehat{X}$ by

$$\widehat{F}(s,\widehat{x})=F(s,x)\times\{k(s)\|x-x(s)\|\}$$

where $\widehat{x} = (x, z) \in \widehat{X}$. Let $\widehat{\mathcal{K}} := \operatorname{graph} \widehat{K}$. Obviously $\widehat{\mathcal{K}}$ is X-closed and $\widehat{F}$ is a multifunction with nonempty and closed values satisfying (H2). Taking into account that F satisfies (H6), by a simple calculation, we show that

$$\widehat{F}(s, \widehat{x}_1) \subset \widehat{F}(s, \widehat{x}_2) + \widehat{k}(s)\|\widehat{x}_1 - \widehat{x}_2\|\widehat{\mathbb{B}}$$

for every $\widehat{x}_1, \widehat{x}_2 \in \widehat{X}$ and a.e. for $s \in [t_0, T)$ where $\widehat{k}(s) = 2k(s)$ for every $s \in [t_0, T)$. This means that $\widehat{F}$ satisfies (H6). Further, $\widehat{F}$ satisfies the sublinear growth condition, that is,

$$\widehat{F}(s, \widehat{x}) \subset \widehat{c}(s)(1 + \|\widehat{x}\|)\widehat{\mathbb{B}},$$

for every $\widehat{x} \in \widehat{X}$, a.e. for $s \in [t_0, T)$, and

$$\widehat{c}(s) = 2 \max\{C, k(s), k(s)\|x(s)\|\}$$

for every $s \in [t_0, T)$.
Now, let us consider the differential inclusion

$$\widehat{x}'(s) \in \widehat{F}(s, \widehat{x}(s)). \tag{22}$$

In order to get a near approximate weak invariance result for the graph $\widehat{\mathcal{K}}$ with respect to Eq. (22), we need to prove that $\widehat{F}(\cdot, \widehat{\xi})$ is quasi-tangent to $\widehat{\mathcal{K}}$ at $(\tau, \widehat{\xi})$ for each $(\tau, \widehat{\xi}) \in \widehat{\mathcal{K}}$. Let us fix $(\tau, \widehat{\xi}) \in \widehat{\mathcal{K}}$ with $\widehat{\xi} = (x, z) \in \widehat{K}(\tau)$ and

$$\|x - x(\tau)\| \leq z. \tag{23}$$

By Proposition 1, it suffices to find sequences $(h_n) \subset \mathbb{R}^+$, $h_n \downarrow 0^+$, $(f_n)_n$ such that $f_n \in S_{[\tau,\tau+h_n]}F(\cdot, x)$ for each $n \in \mathbb{N}^*$, $((p_n), (q_n)) \subset X \times \mathbb{R}^+$ with $p_n \to 0$ in X and $q_n \to 0$ in $\mathbb{R}^+$, such that

$$\left\| x + \int_\tau^{\tau+h_n} f_n(s)\, ds + h_n p_n - x(\tau + h_n) \right\| \leq z + \int_\tau^{\tau+h_n} k(s)\, \|x - x(s)\| ds + h_n q_n.$$

Let us choose $(h_n) \subset \mathbb{R}^+$, $h_n \downarrow 0^+$ with $\tau + h_n \in [t_0, T)$ for all $n = 1, 2, \ldots.$ and take $p_n = 0$ in X and $q_n = 0$ in $\mathbb{R}$, for all $n = 1, 2, \ldots.$ Since x is a solution of Eq. (1) on $[t_0, T]$ we have $x'(s) \in F(s, x(s))$ a.e. for $s \in [\tau, T]$. From hypothesis (H6), we get

$$F(s, x(s)) \subset F(s, x) + k(s)\, \|x(s) - x\|\mathbb{B}$$

a.e. for $s \in [t_0, T]$. Therefore,

$$x'(s) \in F(s, x) + k(s)\, \|x(s) - x\|\mathbb{B}$$

a.e. for $s \in [\tau, \tau + h_n]$, for all $n = 1, 2, \ldots..$ Then,

$$F(s, x) \cap \left(x'(s) + k(s)\, \|x(s) - x\|\mathbb{B} \right) \neq \emptyset$$

a.e. for $s \in [\tau, \tau + h_n]$. Hence by Lemma 1, for all $n = 1, 2, \ldots.,$ there exist measurable functions f_n and b_n with $f_n(s) \in F(s, x)$ and $b_n(s) \in \mathbb{B}$ a.e. for $s \in [\tau, \tau + h_n]$ and such that

$$f_n(s) = x'(s) + k(s)\, \|x(s) - x\| b_n(s).$$

Therefore,

$$\begin{aligned}\left\| x + \int_\tau^{\tau+h_n} f_n(s)\, ds - x(\tau + h_n) \right\| &= \left\| x - x(\tau) + \int_\tau^{\tau+h_n} k(s)\, \|x(s) - x\| b_n(s)\, ds \right\| \\ &\leq \|x - x(\tau)\| + \int_\tau^{\tau+h_n} k(s)\, \|x(s) - x\| ds.\end{aligned}$$

Finally, taking into account Eq. (23), we deduce that $\widehat{F}(\cdot, \widehat{\xi})$ is quasi-tangent to $\widehat{\mathcal{K}}$ at $(\tau, \widehat{\xi})$. Let $\varepsilon > 0$ and take $\varepsilon' > 0$ such that $\varepsilon' \leq \frac{\varepsilon}{m+2}$. Applying now Theorem 8, for all $\overline{T} \in [\tau, T)$, there exist a nondecreasing function $\sigma : [t_0, \overline{T}] \to [t_0, \overline{T}]$ satisfying

$$s - \varepsilon' \leq \sigma(s) \leq s \tag{24}$$

for each $s \in [t_0, \overline{T}]$ and a solution $s \mapsto \widehat{x}(s) = (y(s), z(s))$ of the differential inclusion in Eq. (22) on $[t_0, \overline{T}]$ with $\widehat{x}(t_0) = (y_0, \|x_0 - y_0\|)$ and such that

$$\operatorname{dist}(\widehat{x}(s), \widehat{K}(\sigma(s))) \leq \varepsilon' \tag{25}$$

for each $s \in [t_0, \overline{T}]$. It is clear that y is a solution of Eq. (1), with $y(t_0) = y_o$ and z is given by

$$z(s) = \|x_0 - y_0\| + \int_{t_0}^{s} k(t)\|y(t) - x(t)\|dt,$$

for each $s \in [t_0, \overline{T}]$. From Eq. (25), it follows that for each $s \in [t_0, \overline{T}]$, there exists $(\widetilde{y(s)}, \widetilde{z(s)}) \in \widehat{K}(\sigma(s))$ such that

$$\|(y(s), z(s)) - (\widetilde{y(s)}, \widetilde{z(s)})\| = \|y(s) - \widetilde{y(s)}\| + |z(s) - \widetilde{z(s)}| < 2\varepsilon'. \tag{26}$$

Therefore, using Eqs. (20), (24), (26) and the choice of ε', we obtain

$$\begin{aligned}
\|x(s) - y(s)\| &\leq \|x(s) - x(\sigma(s))\| + \|x(\sigma(s)) - \widetilde{y(s)}\| + \|\widetilde{y(s)} - y(s)\| \\
&\leq m|\sigma(s) - s| + \widetilde{z(s)} + \|\widetilde{y(s)} - y(s)\| \\
&\leq m\varepsilon' + \|\widetilde{y(s)} - y(s)\| + |\widetilde{z(s)} - z(s)| + z(s) \\
&\leq m\varepsilon' + 2\varepsilon' + \|x_0 - y_0\| + \int_{t_0}^{s} k(t)\|y(t) - x(t)\|dt \\
&\leq \varepsilon + \|x_0 - y_0\| + \int_{t_0}^{s} k(t)\|y(t) - x(t)\|dt,
\end{aligned}$$

for all $s \in [t_0, \overline{T}]$. By the Gronwall inequality, we get that for each $\overline{T} \in [t_0, T)$ there exists a solution y defined on $[t_0, \overline{T}]$ and such that Eq. (21) is satisfied for each $t \in [t_0, \overline{T}]$. By a standard argument using Zorn Lemma, we get a solution y defined on $[t_0, T)$ of Eq. (1) such that Eq. (21) is verified on $[t_0, T)$. Finally, y can be extended to $[t_0, T]$ verifying the relation of Eq. (21) on $[t_0, T]$. □

6.2 On the Relaxation Theorem

The relaxation problem plays a fundamental role in the qualitative theory of differential equations. It concerns the relation between the set of solutions of the the relaxed (convexified) differential inclusion

$$x'(t) \in \overline{co}F(t, x(t)), \tag{27}$$

and the set of solutions of Eq. (1).

We point out that the relaxation problem has been studied by many authors by using various frames and techniques, see for example, [9, 10, 14] and [13].

Here, we aim to establish, for certain Carathéodory multi-functions, a relation between ε-solutions of the differential inclusion in Eq. (1) and (exact) solutions of the relaxed differential inclusion in Eq. (27). We recall that this part of research originates from [6, Proposition 3.2] in the case of autonomous differential inclusion in Hilbert spaces setting. There the authors established, under very general assumptions on F, a relation between ε-solutions of both differential inclusions $x'(t) \in \overline{co}F(x(t))$ and $x'(t) \in F(x(t))$. In the same context but in finite-dimensional spaces, we should make reference to the result [8, Theorem 1], which is interesting by itself, in which the author established a relation between quasi-trajectory of the differential inclusion $x'(t) \in F(x(t))$ (notion related to the notion of "ε–solution") and solution to the differential inclusion $x'(t) \in G(x(t))$, where G is the regularization of F, i.e.,

$$G(x) = \bigcap_{\varepsilon>0} \overline{co}\{u : u \in F(y), \|y - x\| < \varepsilon\}.$$

The main result of this section is given below.

Theorem 10 *Let X be a separable Banach space and F a multi-function satisfying the sublinear growth condition of Eq.* (10)*. Let $x : [\tau, T] \to X$ be a solution of the differential inclusion $x'(t) \in \overline{co}F(t, x(t))$, with $x(\tau) = \xi$. Assume that $\overline{co}F$ satisfies* (H3) *where $\mathcal{K} = \{(t, x(t)); t \in [\tau, T)\}$ and l is a continuous function. Then, for any $\varepsilon > 0$, there exists an ε-solution $\overline{x} : [\tau, T] \to X$ of Eq.* (1) *with $\overline{x}(\tau) = \xi$, satisfying*

$$\|x(t) - \overline{x}(t)\| \leq \varepsilon,$$

for each $t \in [\tau, T]$.

Proof Let $x : [\tau, T] \to X$ be a solution of the differential inclusion in Eq. (27) with $x(\tau) = \xi$ and let $K : [\tau, T) \rightsquigarrow X$ be defined by

$$K(t) = \{x(t)\}$$

for each $t \in [\tau, T)$. One checks easily that K satisfies (H4) and $\mathcal{K}$ is approximate weakly invariant with respect to Eq. (27) (in fact it is exact weakly invariant). Applying Theorem 6 and Remark 5, we deduce that for each $(\tau, \xi) \in \mathcal{K}$, $\overline{co}F(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ). Therefore, in view of (v) in Proposition 1, we conclude that $F(\cdot, \xi)$ is quasi-tangent to $\mathcal{K}$ at (τ, ξ) for each $(\tau, \xi) \in \mathcal{K}$. Apply now Theorem 4 and Remark 1. For $\varepsilon > 0$ and $\tau < \overline{T} < T$, there exists an ε-solution $\overline{x} : [\tau, \overline{T}] \to X$ for Eq. (1) with $\overline{x}(\tau) = \xi$ and such that

$$\mathrm{dist}(\overline{x}(t), K(t)) = \|\overline{x}(t) - x(t)\| \leq \varepsilon,$$

for all $t \in [\tau, \overline{T}]$. Using the same arguments as in the end of Theorem 9, we conclude the proof. □

Acknowledgments The first author was supported by a grant of the Ministry of Higher Education and Scientific Research Algerian, project number 265/PNE/Roumanie/2014-2015. The second author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0154.

References

1. Aubin JP, Frankowska H. Set-valued analysis Boston: Birkhäuser Inc MA. 1990.
2. Cârjă O, Lazu A. Approximate weak invariance for differential inclusions in Banach spaces. J Dyn Control Syst. 2012;18:215–27.
3. Cârjă O, Necula M, Vrabie II. Viability, invariance and applications. Amsterdam: Elsevier Science B V. 2007.
4. Cârjă O, Necula M, Vrabie II. Necessary and sufficient conditions for viability for semilinear differential inclusions. Trans Amer Math Soc. 2009;361:343–90.
5. Cârjă O, Monteiro Marques MDP. Weak tangency, weak invariance and Caratheodory mappings. J Dyn Control Syst. 2002;8:445–61.

6. Clarke FH, Ledyaev Yu S, Radulescu ML. Approximate invariance and differential inclusions in Hilbert spaces. J Dyn Control Syst. 1997;3:493–518.
7. Castaing C, Valadier M. Convex analysis and measurable multifunctions. Lecture notes in mathematics. Berlin: Springer-Verlag; 1977.
8. Colombo G. Approximate and relaxed solutions of differential inclusions. Rend Sem Mat Univ Padova. 1989;81:229–38.
9. Filippov AF. Classical solutions of differential equations with multi-valued right-hand side. SIAM J Control. 1967;5:609–21.
10. Papageorgiou NS. Relaxation theorem for differential inclusions in Banach spaces. Tohoku Math J. 1987;39:505–517.
11. Necula M, Popescu M, Vrabie II. Viability for differential inclusions on graphs. Set-Valued Var Anal. 2008;16:961–81.
12. Xuan Duc Ha T. Existence of viable solutions for nonconvex-values differential inclusions in Banach spaces. J Portugaliae Mathematica. 1995;52:242–50.
13. Zhu QJ. On the solution set of differential inclusions in Banach space. J Differ Equ. 1991;93:213–37.
14. Wazewski T. Sur une généralisation de la notion des solutions dúne équation au contingent. Bull Acad Polon Sci Sér Sci Math Astronom Phys. 1962;10:11–15.