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Constructive Solutions to the Riemann–Hilbert Problem and Middle Convolution

Yulia Bibilo[1] · Galina Filipuk[2]

Abstract In this paper, we present a general scheme to generate constructive solutions to the Riemann–Hilbert problem via middle convolution and illustrate this approach for a Fuchsian system with four singular points.

1 Introduction

A Fuchsian system with a given number of poles and a given monodromy representation does not always exist. There have been a number of studies, discussed below in more details, to obtain necessary and sufficient conditions for such system to exist. In this paper, we present a new approach to get explicit solutions of this problem via middle convolution. Middle convolution is related to the Euler transformation of solutions of the Fuchsian systems.

The paper is organized as follows. We shall first review definitions of the Fuchsian system and its monodromy representation, formulate the Riemann–Hilbert problem, and

✉ Yulia Bibilo
y.bibilo@gmail.com

Galina Filipuk
filipuk@mimuw.edu.pl

[1] Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoy Karetny per. 19, Moscow, 127994, Russia

[2] Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097, Warsaw, Poland

present some of the known methods of its constructive solutions (other methods and further references can be found in, for example, a survey paper [14]). Then, we shall briefly describe an algorithm of middle convolution following [7, 8] and, finally, we shall present our new approach (the general scheme) to constructive solutions of the Riemann–Hilbert problem via middle convolution with two illustrative examples.

1.1 Fuchsian Systems

Let us consider a ($p \times p$) linear system of differential equations on the Riemann sphere $\bar{\mathbb{C}}$

$$\frac{dy}{dz} = A(z)y, \tag{1}$$

where $A(z)$ is a rational function.

Definition 1.1 Singular points of system (1) are the poles of the coefficient matrix $A(z)$. A singular point a is called regular if the fundamental matrix solution of system (1) has at most power growth in every sector (with the vertex in a) of an opening less then 2π. Otherwise, the singularity a is called irregular.

Definition 1.2 If the coefficient matrix $A(z)$ of system (1) has a first-order pole at a, then the singularity a is called Fuchsian.

It is known [4] that the Fuchsian singularities are regular. A system with all Fuchsian singularities is called a Fuchsian differential system. For a differential system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{A_i}{z - a_i} \right) y, \quad \sum_{i=1}^{n} A_i = -A_{n+1} \tag{2}$$

the points $a_1, \ldots, a_n$, $a_{n+1} = \infty$ are the Fuchsian singularities.

Definition 1.3 A Fuchsian singularity a_i of system (2) is called non-resonant if the corresponding matrix A_i has no eigenvalues which differ by a natural number. Otherwise, the point a_i is called a resonant singularity.

For system (2), one can define its monodromy representation. Consider a fundamental matrix solution $Y(z)$ of system (2) and its analytic continuation $\tilde{Y}(z)$ along a loop γ. Then, the fundamental solutions are related by $Y(z) = \tilde{Y}(z)G_\gamma$, where the matrix G_γ is non-degenerate, does not dependent on z, and depends only on the homotopy class $[\gamma]$ of the loop γ. The mapping $\gamma \to G_\gamma$ defines a representation of the fundamental group of the Riemann sphere with removed singularities into the group of non-degenerate matrices, i.e.,

$$\chi : \pi_1(\bar{\mathbb{C}} \backslash \{a_1, \ldots, a_n, a_{n+1}\}, z_0) \to \mathbb{GL}(p, \mathbb{C}). \tag{3}$$

The generators of the fundamental group $\gamma_1, \ldots, \gamma_n, \gamma_{n+1}$ (simple loops encircling the corresponding singular points $\{a_1, \ldots, a_n, a_{n+1}\}$) satisfy $\gamma_1 \cdot \ldots \cdot \gamma_n = \gamma_{n+1}^{-1}$. Let G_i be a matrix equal to $\chi([\gamma_i])$. The matrices $G_1, \ldots, G_n, G_{n+1}$ are called monodromy matrices. The set of monodromy matrices generate a matrix group with a single group relation $G_1 \cdot \ldots \cdot G_n = G_{n+1}^{-1}$.

If one takes another fundamental matrix solution $X(z)$ of system (2) different from $Y(z)$, then $X(z) = Y(z)C$, where C is a constant non-degenerate matrix. After an analytic con-

tinuation of $X(z)$ along a loop γ, one gets a new fundamental solution $\tilde{X}(z)$ given by $\tilde{X}(z) = X(z)\tilde{G}_\gamma$. Then the monodromy matrices G_γ and $\tilde{G}_\gamma$ satisfy

$$\tilde{G}_\gamma = C^{-1} G_\gamma C.$$

Hence, the Fuchsian system (2) does not only define a unique monodromy representation χ, but it defines a conjugacy class (by a constant matrix) of the monodromy representations.

The monodromy matrices corresponding to non-resonant singularities can be obtained much simpler than that of the resonant singularities.

Lemma 1.1 [4, Lect. 6] *If the point a_i is a non-resonant Fuchsian singularity of system (2), then $G_i \sim e^{2\pi \mathbf{i} A_i}$.*

1.2 The Riemann–Hilbert Problem

The so-called Riemann–Hilbert problem was first mentioned by Riemann at the end of 1850s. In 1900, Hilbert included it in his famous list of mathematical problems under number 21. It is formulated as follows: show that there always exists a linear differential equation of a Fuchsian type (or a Fuchsian system) with given poles and a given monodromy group (see [1, 4, 5, 13] for the details).

If one compares dimensions of spaces of parameters (of monodromy matrices and coefficients of a scalar equation), then it becomes clear that the problem should be stated only for Fuchsian systems, and it has a negative solution for scalar Fuchsian equations in general. However, in the case of Fuchsian systems, it does not always have a positive solution [1, 4, 5]. Bolibruch constructed counterexample to the Riemann–Hilbert in 1989 [1]. Therefore, results containing necessary or sufficient conditions of the existence of a Fuchsian system with given poles and a given monodromy representation are of particular importance nowadays.

Most of the known results are obtained by non-constructive methods. A review of such results can be found in [14]. However, there also exist constructive approaches. There are several methods to construct a solution to the Riemann–Hilbert problem (i. e., to construct a Fuchsian differential system with given singular points and a given monodromy group). They include the methods using matrix series and investigating their convergence and analytic continuation (Lappo–Danilevsky's method [17–20], Krylov's method [16], Erugin's method [9–12], and others) and the methods coming from the theory of isomonodromic deformations (see, for instance, the papers by Boalch [3], Korotkin [15] and others). We shall overview some of the methods in the next section.

1.3 Constructive Solutions

1.3.1 Lappo–Danilevsky's Method

The Lappo–Danilevsky (Lappo–Danilevskij) method [17–20] allows one to express the matrices $A_1, \ldots, A_n$ of system (2) in the form of certain matrix series depending on either exponents of monodromy $W_1, \ldots, W_n$ defined below or the monodromy matrices $G_1, \ldots, G_n$ with the requirement that the matrices $W_1, \ldots, W_n$ are close to zero matrices or the monodromy matrices $G_1, \ldots, G_n$ are close to the identity matrices (with respect to a matrix norm).

The exponents of monodromy of system (2) are defined as follows. In the neighborhood of each singularity a_i, the fundamental matrix solution of Eq. 2 can be written as

$$Y(z) = \bar{Y}(z)(z - a_i)^{W_i}, \tag{4}$$

where $\bar{Y}(z)$ is a matrix holomorphic at a_i, $\bar{Y}(a_i) \neq 0$ and W_i is a constant matrix. One can find the following fact in [17, II] and [9, XII, §1][1].

Proposition 1.1 *The matrices W_i and A_i corresponding to the Fuchsian singularity a_i of system (2) have the same eigenvalues.*

The eigenvalues of W_i coincide with the so-called exponents of the Fuchsian system. If the singularity a_i is non-resonant, then the matrix $\bar{Y}^{-1}(z)$ is also holomorphic at a_i.

Explicitly, the matrices $A_1, \ldots, A_n$ of system (2) can be expressed in the form

$$A_i = \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,n} R_i(a_{j_1}, \ldots, a_{j_\nu}|b) W_{j_1} \cdot \ldots \cdot W_{j_\nu}, \tag{5}$$

where $R_i(a_{j_1}, \ldots, a_{j_m}|b)$ can be determined recursively. Here, the indices $j_1, \ldots, j_\nu$ take values $1, \ldots, n$ independently in the sum $\sum_{j_1,\ldots,j_\nu}^{1,\ldots,n}$ and b is a complex number used to define the fundamental matrix Y by $Y(b) = I$. The series converges when the norms of the matrices $\|W_i\|$ are close to zero.

The matrices W_i and the monodrony matrices G_i corresponding to both resonant and non-resonant Fuchsian singularity a_i are related by

$$W_i = \frac{1}{2\pi \mathbf{i}} \ln G_i, \quad \mathbf{i} = \sqrt{-1}. \tag{6}$$

It is possible to construct a solution to the Riemann–Hilbert problem in a class of Fuchsian systems with finite non-resonant singularities (one should choose a branch of $\ln G_i$ in such a way that the real parts of the eigenvalues of W_i are in the interval $[0, 1)$).

Note that Lappo–Danilevsky proved that the following formula holds for the coefficients of the system:

$$A_i = \frac{1}{2\pi \mathbf{i}} \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{1,\ldots,n} Q_i(a_{j_1}, \ldots, a_{j_\nu}|b)(G_{j_1} - I) \cdot \ldots \cdot (G_{j_\nu} - I), \tag{7}$$

where the coefficients $Q_i(a_{j_1}, \ldots, a_{j_\nu}|b)$ can be found recursively. The series converges when every $\|G_i - I\|$ is small.

1.3.2 The Gauss System

Lappo–Danilevsky also considered a particular case of a Fuchsian (2×2) system with three singularities

$$\frac{dy}{dz} = \left(\frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} \right) y, \quad A_1 + A_2 = -A_3. \tag{8}$$

System (8) is called the Gauss system. Lappo–Danilevsky fully solved the Riemann–Hilbert problem in terms of the series (7) for arbitrary monodromy matrices (not only

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[1]This fact can also be proved by using the Levelt basis, see [4, Lect. 6].

for matrices lying in the neighborhood of the identity matrix). Later on, Krylov [16] substantially improved this result and described the solutions in terms of the hypergeometric series and studied their multivaluedness. Krylov gave all possible solutions for the Riemann–Hilbert problem. It follows from his results that it is always possible to construct a non-resonant Fuchsian system for irreducible monodromy matrices.

1.3.3 Erugin's Method

Erugin extended Lappo–Danilevsky's results for (2×2) systems. He constructed a solution to the Riemann–Hilbert problem in case of four singular points and (2×2) monodromy matrices [9–12]. More precisely, Erugin solved the Riemann–Hilbert problem formulated as follows. Assume that the (2×2) matrices W_1, W_2, W_3 and the singularities $a_1, a_2, a_3, a_4 = \infty$ are given. Then find a (2×2) linear differential system

$$\frac{dy}{dz} = \left(\frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} + \frac{A_3}{z - a_3} \right) y, \quad A_1 + A_2 + A_3 = -A_4, \qquad (9)$$

such that the fundamental matrix solution of the system $Y(z)$ normalized by $Y(b) = I$ is given by $Y(z) = \bar{Y}_i(z)(z - a_i)^{W_i}$ in the neighborhood of each singularity a_i. Here, $\bar{Y}_i(z)$ is a second-order matrix homomorphic in the neighborhood of a_i. Furthermore, Erugin studied properties of A_i as the function of W_i.

Due to the formulation of the problem in this way and formula (6), it is possible to find a Fuchsian system (9) with non-resonant finite singularities. Indeed, the eigenvalues of the matrix W_i depend on the choice of a branch of the logarithm in formula (6), so we can choose eigenvalues of W_i such that their real parts belong to $[0, 1)$. The eigenvalues of A_i are equal to the eigenvalues of W_i by the Prop. 1.1. Then there is no pair of eigenvalues of A_i with the integer difference.

Remark 1.1 In [9, XII,§3,(3.11)] it is shown that $W_4 = \frac{1}{2\pi i} G_4$ with the main branch of the logarithm, where G_4 is the monodromy matrix at the infinity. Thus, the resulting Fuchsian system (9) is non-resonant at infinity as well.

2 Middle Convolution

Dettweiler and Reiter's algebraic analogue [7, 8] of Katz' middle convolution is a certain transformation of Fuchsian systems which preserves an index of rigidity. Middle convolution is related to the Euler transformation of the Fuchsian system. There have been numerous studies on middle convolution in recent years, including applications to special functions, extensions to irregular systems, and others. In this section, we shall explain main definitions and state some of the results from [7, 8].

The idea of the algebraic construction is as follows. For a given parameter, one defines monodromy or residue matrices of dimension $(np \times np)$ which are partitioned into blocks and have only one row of blocks consisting of initial monodromy or residue matrices and the parameter. By finding invariant subspaces and reducing the size of the matrices (if the invariant subspaces are non-zero), one gets a new set of monodromy matrices (or a new Fuchsian system with the same singularities but with new residue matrices). Note that the size of the resulting matrices (or the system) depends on the choice of the parameter.

2.1 The Multiplicative Case MC_λ

Let us consider a tuple of matrices $G = (G_1, \ldots, G_n)$, $G_i \in \mathbb{GL}(p, \mathbb{C})$, and define $G_{n+1}^{-1} = G_1 \cdot \ldots \cdot G_n$.

Definition 2.1 A convolution of the tuple G is a tuple $C_\lambda(G) = (M_1, \ldots, M_n)$ given by

$$
M_k = \begin{pmatrix}
I & 0 & & & \cdots & & & & 0 \\
 & \ddots & & & & & & & \\
 & & I & & & & & & \\
\lambda(G_1 - I) & \cdots & \lambda(G_{k-1} - I) & \lambda G_k & G_{k+1} - I & \cdots & G_n - I & & \\
 & & & & I & & & & \\
 & & & & & \ddots & & & \\
0 & & & & \cdots & & 0 & I &
\end{pmatrix} \in \mathbb{C}^{np \times np}, \quad (10)
$$

where $k = 1, \ldots, n$ and I is an $(p \times p)$ identity matrix.

There are invariant subspaces under each $M_1, \ldots, M_n$

$$
\mathcal{K}_k = \begin{pmatrix}
0 \\
\vdots \\
\ker(G_k - I) \\
\vdots \\
0
\end{pmatrix} \subset \mathbb{C}^{np}, \quad k = 1, \ldots, n, \quad (11)
$$

$$
\mathcal{L} = \cap_{k=1}^n \ker(M_k - I) = \ker(M_1 \cdot \ldots \cdot M_n - I). \quad (12)
$$

Let

$$
\mathcal{K} = \oplus_{i=1}^n \mathcal{K}_i. \quad (13)
$$

Note that if $\lambda \neq 1$, then

$$
\mathcal{L} = \left\langle \begin{pmatrix}
G_2 \cdot \ldots \cdot G_n v \\
G_3 \cdot \ldots \cdot G_n v \\
\cdots \\
v
\end{pmatrix} \middle| v \in \ker(\lambda G_1 \cdot \ldots \cdot G_n - I) \right\rangle \quad (14)
$$

and $\mathcal{K} + \mathcal{L} = \mathcal{K} \oplus \mathcal{L}$.

Definition 2.2 Fix an isomorphism between $\mathbb{C}^{np}/(\mathcal{K}+\mathcal{L})$ and $\mathbb{C}^m$ for some m. Then a tuple of matrices $MC_\lambda(G) = (\tilde{G}_1, \ldots, \tilde{G}_n)$, $\tilde{G}_i \in \mathbb{GL}(m, \mathbb{C})$, is called a middle convolution of the tuple G, where $\tilde{G}_k$ is induced by the action of M_k on $\mathbb{C}^m \cong \mathbb{C}^{np}/(\mathcal{K} + \mathcal{L})$.

Note that $MC_\lambda(G)$ depends on the choice of an isomorphism between $\mathbb{C}^m$ and $\mathbb{C}^{np}/(\mathcal{K}+\mathcal{L})$. We will denote $\dim(MC_\lambda(G)) = m$.

Let us define the following conditions for any $\tau \in \mathbb{C}\backslash\{0\}$:

$$
(*) \quad \cap_{j \neq i} \ker(G_j - I) \cap \ker(\tau G_i - I) = 0, \quad i = 1, \ldots, n,
$$

$$
(**) \quad \dim\left[\sum_{j \neq i} \mathrm{Im}(G_j - I) + \mathrm{Im}(\tau G_i - I)\right] = p, \quad i = 1, \ldots, n.
$$

Remark 2.1 The conditions $(*), (**)$ are always fulfilled if $p > 1$ and the matrices $G_1, \ldots, G_n$ generate an irreducible subgroup of $\mathbb{GL}(p, \mathbb{C})$; or $p = 1$ and at least two elements of $G_1, \ldots, G_n, n \geq 2$, are not identity matrices [6, Remark 3.1].

Lemma 2.1 [7, 8] *Middle convolution MC_λ satisfies the following properties:*
1. If $\lambda \neq 1$, $\lambda \neq 0$, then

$$\dim(MC_\lambda(G)) = \sum_{k=1}^{n} rk(G_k - I) - p + rk(\lambda G_1 \cdot \ldots \cdot G_n - I). \tag{15}$$

2. If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda = \lambda_1 \cdot \lambda_2$ and conditions $(*), (**)$ hold, then

$$MC_{\lambda_2}(MC_{\lambda_1}(G)) \cong MC_\lambda(G). \tag{16}$$

Moreover, $MC_\lambda(MC_{\lambda^{-1}}(G)) \cong G$.
3. If $\lambda \neq 0$, conditions $(*), (**)$ hold and G is irreducible, then $MC_\lambda(G)$ is irreducible.

2.2 The Additive Case mc_μ

Suppose we have a tuple A of matrices $A_1, \ldots, A_n, A_{n+1}$, where $A_i \in \mathbb{C}^{p \times p}$ and $A_{n+1} = -(A_1 + \ldots + A_n)$.
   Define

$$B_k = \begin{pmatrix} O & \ldots & O & O & O & \ldots & O \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ A_1 & \ldots & A_{k-1} & A_k + \mu I & A_{k+1} & \ldots & A_n \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ O & \ldots & O & O & O & \ldots & O \end{pmatrix} \in \mathbb{C}^{np \times np}, \tag{17}$$

where O denotes a zero $(p \times p)$ matrix. And there are invariant subspaces under each $B_1, \ldots, B_n$

$$l_k = \begin{pmatrix} 0 \\ \vdots \\ \ker(A_k) \\ \vdots \\ 0 \end{pmatrix} \subset \mathbb{C}^{np}, \quad k = 1, \ldots, n, \tag{18}$$

$$l = \cap_{k=1}^{n} \ker(B_k) = \ker(B_1 + \ldots + B_n). \tag{19}$$

   Let

$$\tilde{l} = \oplus_{i=1}^{n} l_k. \tag{20}$$

   Note that if $\mu \neq 0$, then

$$l = \left\langle \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \mid v \in \ker(A_1 + \ldots + A_n + \mu I) \right\rangle$$

and $\tilde{l} + l = \tilde{l} \oplus l$.

Definition 2.3 A tuple of matrices $c_\mu(A) = (B_1, \ldots, B_n)$ is called a convolution of the tuple A.

Definition 2.4 Fix an isomorphism between $\mathbb{C}^{np}/(\tilde{l} + l)$ and $\mathbb{C}^m$ for some m. Then a tuple of matrices

$$mc_\mu(A) = (\tilde{A}_1, \ldots, \tilde{A}_n), \quad \tilde{A}_i \in \mathbb{C}^{m \times m}$$

is called a middle convolution, where $\tilde{A}_i$ is induced by the action of B_i on $\mathbb{C}^m \cong \mathbb{C}^{np}/(\tilde{l}+l)$.

2.3 A Connection Between MC_λ and mc_μ

The following statement is given in [7].

Lemma 2.2 *Let* $G = (G_1, \ldots, G_n)$, $G_i \in \mathbb{GL}(p, \mathbb{C})$ *and conditions* (∗), (∗∗) *hold. Assume that* $MC_\lambda(G) = (\tilde{G}_1, \ldots, \tilde{G}_n)$, $\lambda \neq 1$, $\lambda \neq 0$. *Then*
(1) under the action of MC_λ, *every Jordan block* $J(\alpha, l)$ *appearing in the Jordan decomposition of* G_i *transforms to the Jordan block* $J(\alpha\lambda, l')$ *of the Jordan decomposition of* $\tilde{G}_i$, *where*

$$l' := \begin{cases} l, & \alpha \neq 1, \lambda^{-1}, \\ l - 1, & \alpha = 1, \\ l + 1, & \alpha = \lambda^{-1}. \end{cases}$$

All other Jordan blocks which appear in the Jordan decomposition of $\tilde{G}_i$ *are the blocks of the form* $J(1, 1)$.
(2) Under the action of MC_λ, *every Jordan block* $J(\alpha, l)$ *appearing in the Jordan decomposition of* G_{n+1}^{-1} *transforms to the Jordan block* $J(\alpha\lambda, l')$ *of the Jordan decomposition of* $\tilde{G}_{n+1}^{-1}$, *where*

$$l' := \begin{cases} l, & \alpha \neq 1, \lambda^{-1}, \\ l + 1, & \alpha = 1, \\ l - 1, & \alpha = \lambda^{-1}. \end{cases}$$

All other Jordan blocks which appear in the Jordan decomposition of $\tilde{G}_{n+1}^{-1}$ *are the blocks of the form* $J(\lambda, 1)$.

The following theorem characterizes a connection between the additive and multiplicative cases of middle convolution. We will use the following notations: D_A is a Fuchsian system of differential equations with fixed singularities and with residue matrices given by the tuple A, i.e.,

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{A_i}{z - a_i} \right) y, \quad A = (A_1, \ldots, A_n),$$

and $Mon(D_A)$ is a tuple of monodromy matrices of the system D_A.

Theorem 2.1 [7] *Let* $A = (A_1, \ldots, A_n)$, $A_i \in \mathbb{C}^{p \times p}$, $Mon(D_A) = (G_1, \ldots, G_n)$, $\mu \in \mathbb{C} \backslash \mathbb{Z}$, $\lambda = e^{2\pi \mathbf{i} \mu}$. *Moreover, assume that the following conditions hold:*
1. condition (*)
2. condition (**)
3. $rk(A_i) = rk(G_i - I)$
4. $rk(A_1 + \ldots + A_n + \mu I) = rk(\lambda G_1 \cdot \ldots \cdot G_n - I)$ *(or, equivalently,* $rk(\mu I - A_{n+1}) = rk(\lambda G_{n+1}^{-1} - I))$
Then $Mon(D_{mc_{\mu-1}(A)}) = MC_\lambda(Mon(D_A))$ *(the fundamental matrix solutions defining monodromy matrices should be chosen in a special way).*

Remark 2.2 It follows from Rem. 2.1, that conditions (1) and (2) of the Theorem 2.1 is true when $p > 1$ and $G_1, \ldots, G_n$ generate an irreducible subgroup of $\mathbb{GL}(p, \mathbb{C})$.

Remark 2.3 Theorem 2.1 also states that $\dim mc_{\mu-1}(A) = \dim MC_\lambda(Mon(D_A))$.

The following diagram illustrates Theorem 2.1:

$$
\begin{array}{ccc}
Mon(D_A) = (G_1, \ldots, G_n) & \xrightarrow{MC_\lambda} & (\tilde{G}_1, \ldots, \tilde{G}_n) = Mon(D_{\tilde{A}}) \\
\chi \Big\uparrow \Big\downarrow RH & & \chi \Big\uparrow \Big\downarrow RH \\
A = (A_1, \ldots, A_n) & \xrightarrow{mc_{\mu-1}} & (\tilde{A}_1, \ldots, \tilde{A}_n) = \tilde{A}
\end{array}
$$

3 Main Results

3.1 A General Scheme to Generate Constructive Solutions

If one has some explicit examples of solutions to the Riemann–Hilbert problem, then it is possible to extend the set of these constructive solutions by applying MC to monodromy matrices and mc to the residue matrices (provided that conditions of Theorem 2.1 are fulfilled). In addition, one can also obtain some sufficient results about the constructive solutions to the Riemann–Hilbert problem.

Suppose that we are given a tuple of matrices G and we need to construct a Fuchsian differential system D_A (with fixed singularities $a_1, \ldots, a_n$) such that $Mon(D_A) = G$, i.e., we need to obtain a constructive solution to the Riemann–Hilbert problem. In some cases, it is possible to reduce the problem to the known solutions by using the additive and multiplicative cases of middle convolution. We can use the following idea. We can build a new tuple of matrices $\tilde{G} = MC_\lambda(G)$ for a given set G and for some proper parameter λ such that the Riemann–Hilbert problem has a constructive solution for the new set $\tilde{G}$. In this way, we can find a tuple of matrices $\tilde{A}$ defining the Fuchsian system with given monodromy matrices $\tilde{G}$. Then if conditions of Theorem 2.1 are satisfied for $\tilde{G}$ and $\tilde{A}$, then by choosing $MC_{\lambda^{-1}}$, the Fuchsian system defined by $mc_{-\mu+1}(\tilde{A})$ has monodromy $MC_{\lambda^{-1}}(MC_\lambda(G))$. The tuples of matrices $MC_{\lambda^{-1}}(MC_\lambda(G))$ and G are equal up to a conjugation by a constant non-degenerate matrix by Lemma 2.1. This conjugation corresponds to the choice of the fundamental solution defining monodromy matrices. Therefore, $mc_{-\mu+1}(\tilde{A})$ is the required solution to our problem.

We can illustrate this idea by the following diagram, where the question mark means that we want to find a Fuchsian system for a given tuple of monodromy matrices (and singularities), i.e., want to solve the Riemann–Hilbert problem constructively:

$$Mon(D_A) = (G_1, \ldots, G_n) \xrightarrow[MC_{\lambda^{-1}}]{MC_{\lambda}} (\tilde{G}_1, \ldots, \tilde{G}_n) = Mon(D_{\tilde{A}})$$

$$\chi \uparrow \quad \downarrow RH \qquad\qquad\qquad \chi \uparrow \quad \downarrow RH$$

$$? \xrightarrow[mc_{-\mu+1}]{mc_{\mu-1}} (\tilde{A}_1, \ldots, \tilde{A}_n) = \tilde{A}.$$

In other words,

$$Mon(D_A) = Mon(D_{mc_{-\mu+1}(\tilde{A})}) = MC_{\lambda^{-1}}(Mon(D_{\tilde{A}})) = MC_{\lambda^{-1}}(\tilde{G}) =$$
$$= MC_{\lambda^{-1}}(MC_{\lambda}(G)) = (CG_1C^{-1}, \ldots, CG_nC^{-1}), \quad C \in \mathbb{GL}(p, \mathbb{C}).$$

3.2 Extension of a class of Monodromy Data for which the Riemann–Hilbert Problem Has a Constructive Solution Via Erugin's Method

In this section, we will prove a theorem that allows us to reduce the solution of the Riemann–Hilbert problem to Erugin's solution.

Theorem 3.1 *Let $a_1, a_2, a_3, a_4 = \infty$ be four singular points and assume that the matrices G_1, G_2, G_3, G_4, $G_k \in \mathbb{GL}(p, \mathbb{C})$, $p > 2$, with $G_1 \cdot \ldots \cdot G_4 = I$, satisfy the following conditions:*

1. *The tuple $G = (G_1, G_2, G_3)$ is irreducible.*
2. *There exists $\lambda \in \mathbb{C}\backslash\{0, 1\}$, such that by formula (15)*

$$\dim MC_{\lambda}(G) = 2.$$

Then for a given tuple G of monodromy matrices, there exists a constructive solution to the Riemann–Hilbert problem.

Remark 3.1 To construct solutions to the Riemann–Hilbert problem with our scheme, it is important to use method that allows to construct non-resonant Fuchsian system, such as Erugin's method.

Proof We note that condition $\dim MC_{\lambda}(G) = 2$ means that the matrices should be such that the dimension of the invariant subspaces $\mathcal{K} + \mathcal{L}$ is equal to $4p - 2$.

To prove the theorem, we will use the general scheme of the constructive solution to the Riemann–Hilbert problem discussed above. Let $MC_{\lambda}(G) = \tilde{G} = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_3)$. Then the tuple of matrices $\tilde{G}$ can be realized as the tuple of monodromy matrices of some Fuchsian system with non-resonant singular points (by Erugin's method, see Section 1.3.3)

$$\frac{dy}{dz} = \left(\frac{\tilde{A}_1}{z - a_1} + \frac{\tilde{A}_2}{z - a_2} + \frac{\tilde{A}_3}{z - a_3} \right) y. \tag{21}$$

It remains to prove that the tuple of monodromy matrices $\tilde{G}$ and the corresponding differential system (21) satisfy conditions of Theorem 2.1. Then $Mon(mc_{-\mu+1}(D_{\tilde{A}})) = MC_{\lambda-1}(\tilde{G})$ (for $\mu = \frac{1}{2\pi \mathbf{i}} \log \lambda$) will be true.

We have $\dim \tilde{G} = \dim MC_\lambda(G) = 2 > 1$, so to prove conditions 1 and 2 of Theorem 2.1, we should prove that the group generated by $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3$ is an irreducible subgroup of $\mathbb{GL}(2, \mathbb{C})$ (see Remark 2.2). It is irreducible subgroup due to case 3) in Lemma 2.1.

Let us prove condition 3: $\mathrm{rk}(\tilde{A}_i) = \mathrm{rk}(\tilde{G}_i - I)$, $i = 1, 2, 3$. System (21) is non-resonant in finite singular points, thus Lemma 1.1 can be used. We have chosen the eigenvalues of the matrix W_i (for Erugin's method) such that their real parts belong to the interval $[0, 1)$. Then the eigenvalues of the matrix $\tilde{A}_i$ are also such that their real parts belong to the interval $[0, 1)$ by Prop. 1.1. Then Jordan forms of $\tilde{A}_i$ and $\tilde{G}_i$ are agreed (they have the same number of Jordan blocks of equal sizes) because of formula $\tilde{G}_i \sim e^{2\pi \mathbf{i} \tilde{A}_i}$ in Lemma 1.1. Also, the Jordan form of $\tilde{A}_i$ has a Jordan block $J(0, l)$ if and only if the Jordan form of $\tilde{G}_i$ has a block $J(1, l)$. Thus, condition 3 also holds true.

To check condition 4, i.e., $\mathrm{rk}(\mu I + \tilde{A}_4) = \mathrm{rk}(\lambda^{-1}\tilde{G}_4^{-1} - I)$, we will use that ∞ is a non-resonant singularity of system (21) (see Remark 1.1). Then again Jordan forms of $\tilde{A}_4$ and $\tilde{G}_4$ are agreed and the Jordan form of $\tilde{A}_4$ has a Jordan block $J(0, l)$ if and only if the Jordan form of $\tilde{G}_4^{-1}$ has a block $J(1, l)$ because of the formula $\tilde{G}_i \sim e^{2\pi \mathbf{i} \tilde{A}_i}$ in Lemma 1.1. $\square$

Remark 3.2 We can use Krylov's method instead of Erugin's method to construct system with three Fuchsian singular points. So Theorem 3.1 is true for three singular points and three monodromy matrices.

3.3 The First Illustrative Example

At first, we illustrate Theorem 3.1 with the following simple example. Assume that we are given three matrices

$$G_1 = \begin{pmatrix} \mathbf{i} & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2\mathbf{i} & 2\mathbf{i} & 0 \\ 2\mathbf{i} & -\mathbf{i} & -2\mathbf{i} \end{pmatrix}, \quad G_3 = (G_1 G_2)^{-1},$$

and we need to construct a Fuchsian system with singular points $a_1, a_2, a_3 = \infty$ realizing the given monodromy matrices G_1, G_2, G_3.

The tuple (G_1, G_2) is irreducible. We apply $MC_{\mathbf{i}}$ and get

$$\tilde{G}_1 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix}.$$

We could get residue matrices according to Erugin's or Krylov's method but we will use here the well-known facts from analytic differential equations theory.

The next lemma comes from linear algebra.

Lemma 3.1 [16] *Every pair of (2×2)-matrices $\tilde{A}_i$, $\tilde{A}_h$ can be transformed (conjugating by a constant non-degenerate matrix C) to one of the four types*

$$C\tilde{A}_i C^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 1 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_h C^{-1} = \begin{pmatrix} \chi_h^{(1)} & \nabla \\ 0 & \chi_h^{(2)} \end{pmatrix}; \tag{I}$$

$$C\tilde{A}_i C^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 1 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_h C^{-1} = \begin{pmatrix} \chi_h^{(1)} & 0 \\ 0 & \chi_h^{(2)} \end{pmatrix}; \tag{II}$$

$$C\tilde{A}_i C^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 0 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_h C^{-1} = \begin{pmatrix} \chi_h^{(1)} & 0 \\ 0 & \chi_h^{(2)} \end{pmatrix}; \tag{III}$$

$$C\tilde{A}_i C^{-1} = \begin{pmatrix} \chi_i^{(1)} & 0 \\ 1 & \chi_i^{(2)} \end{pmatrix}, \quad C\tilde{A}_h C^{-1} = \begin{pmatrix} \chi_h^{(1)} & 0 \\ \nabla & \chi_h^{(2)} \end{pmatrix}; \tag{IV}$$

where $i \neq h, i, h \in \{1, 2\}$, $\nabla \in \mathbb{C}$, $\nabla \neq 0$, $\nabla + (\chi_1^{(1)} - \chi_1^{(2)})(\chi_2^{(1)} - \chi_2^{(2)}) \neq 0$.

Monodromy matrices $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3$ generate an irreducible subgroup. Then the corresponding Fuchsian system has to be irreducible as well. So we are looking for residue matrices of the type (I).

The formula

$$\chi_i^{(k)} = \frac{1}{2\pi \mathbf{i}} \text{Log}(\eta_i^{(k)}), \quad \mathbf{i} = \sqrt{-1}, \tag{22}$$

follows from Proposition 1.1, where $\eta_i^{(1)}$, $\eta_i^{(2)}$ are the eigenvalues of the monodromy matrix $\tilde{G}_i$, $\chi_i^{(1)}$, $\chi_i^{(2)}$ are the eigenvalues of the residue matrix $\tilde{A}_i$. Brunches of logarithm in Eq. 22 should be chosen in such a way that eigenvalues satisfy the following relation

$$\chi_1^{(1)} + \chi_1^{(2)} + \chi_2^{(1)} + \chi_2^{(2)} + \chi_3^{(1)} + \chi_3^{(2)} = 0,$$

since $\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 = 0$.

The value ∇ can be expressed in terms of the eigenvalues $\{\chi_i^{(k)}\}$:

$$\nabla = (\chi_1^{(1)} + \chi_2^{(1)})(\chi_1^{(2)} + \chi_2^{(2)}) - \chi_3^{(1)} \chi_3^{(2)}.$$

Indeed

$$C\tilde{A}_3 C^{-1} = -C(\tilde{A}_1 + \tilde{A}_2)C^{-1} = -\begin{pmatrix} \chi_1^{(1)} + \chi_2^{(1)} & \nabla \\ 1 & \chi_1^{(2)} + \chi_2^{(2)} \end{pmatrix},$$

$$\det \tilde{A}_3 = \chi_3^{(1)} \chi_3^{(2)} = (\chi_1^{(1)} + \chi_2^{(1)})(\chi_1^{(2)} + \chi_2^{(2)}) - \nabla.$$

So we have (here we take $C = I$)

$$\tilde{A}_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\pi \mathbf{i}} \text{Log}(2) & \frac{1}{9} \\ 0 & \frac{1}{2\pi \mathbf{i}} \text{Log}(2) \end{pmatrix},$$

$$\tilde{A}_3 = -\tilde{A}_1 - \tilde{A}_2 = \begin{pmatrix} -\frac{1}{2\pi \mathbf{i}} \text{Log}(2) & -\frac{1}{9} \\ -1 & -\frac{1}{2\pi \mathbf{i}} \text{Log}(2) \end{pmatrix}.$$

The system

$$\frac{dy}{dz} = \frac{\tilde{A}_1}{z - a_1} + \frac{\tilde{A}_2}{z - a_2}, \quad a_1 \neq a_2,$$

has only non-resonant singular points and it is irreducible. Thus, it has monodromy $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3$ which follows from the analogous reasoning.

It is evident that this solution of the Riemann–Hilbert problem is not unique.

Next, we apply $mc_{\frac{3}{4}}$ according to the main idea. The resulting system for G_1, G_2, G_3 is

$$\frac{dy}{dz} = \frac{A_1}{z - a_1} + \frac{A_2}{z - a_2},$$

$$A_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} + \frac{\log(2)}{2\pi\mathbf{i}} & \frac{1}{9} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{5}{4} + \frac{\log(2)}{2\pi\mathbf{i}} & \frac{1}{9} \\ 1 & 0 & \frac{3}{4} + \frac{\log(2)}{2\pi\mathbf{i}} \end{pmatrix}.$$

3.4 The Second Illustrative Example

In the second example of the constructive solution to the Riemann–Hilbert problem, we use middle convolution and Erugin's method [9, 11] (see also [10, 12, 13]). In this example, we use some of the results from [2] in which Erugin's method for four (2×2) matrices in a special form is studied.

Let us fix four singular points $a_1, a_2, a_3, a_4 = \infty$. Define $G = (G_1, G_2, G_3)$, where the matrices $G_1, \ldots, G_4$, $G_4 = (G_1 G_2 G_3)^{-1}$ are given as follows:

$$G_1 = \begin{pmatrix} a & -2\pi\mathbf{i}\tau & 2\pi\mathbf{i}\tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2\pi\mathbf{i}\kappa & a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\pi\mathbf{i}\kappa & 0 & a \end{pmatrix}, \qquad \tau, \kappa \in (0, 1), \ a \in \mathbb{C}\backslash\{0, 1\}. \tag{23}$$

We cannot apply the Lappo–Danilevsky method for the matrices $G_1, \ldots, G_4$ since the norms $\|G_i - I\|$ are not small when $|a|$ is big. Notice that the matrices $G_1, \ldots, G_4$ satisfy conditions of Theorem 3.1. Indeed, the matrices G_1, G_2, G_3 generate an irreducible subgroup[2] and, as shown below,

$$\dim MC_{a^{-1}}G = 2.$$

Hence, by the Erugin method (see [9, 11] and Section 1.3.3), one can find a constructive solution to the Riemann–Hilbert problem for these matrices.

Let us apply the multiplicative version of middle convolution MC_λ with the parameter $\lambda = a^{-1}$ to the tuple G. At first we find the matrices M_1, M_2, M_3 according to formula (10). For each matrix G_1, G_2, G_3, one has that $\dim \ker(G_i - I) = 2$ and, moreover, $\dim \ker(M_1 M_2 M_3 - I) = 1$. Thus, in the result of the application of the algorithm of middle convolution, we get a tuple of (2×2) matrices

$$\tilde{G}_1 = \begin{pmatrix} 1 & 0 \\ -2\pi\mathbf{i}\kappa & 1 \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} 1 & -2\pi\mathbf{i}\tau \\ 0 & 1 \end{pmatrix}, \quad \tilde{G}_3 = \begin{pmatrix} 1 & 2\pi\mathbf{i}\tau \\ 0 & 1 \end{pmatrix}. \tag{24}$$

Next, we use Erugin's method for tuple (24) following [2]. We first define the matrices W_i using formula (6) choosing a branch of the logarithm to avoid the resonances. We have

$$W_1 = \begin{pmatrix} 0 & 0 \\ -\kappa & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & -\tau \\ 0 & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix}. \tag{25}$$

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[2] One can verify it by direct calculations, i.e., there is no matrix C, such that all matrices CG_iC^{-1} have the same (2×1) or (1×2) zero block under the leading diagonal when $a \notin \{0, 1\}$, $\kappa \neq 0$, $\tau \neq 0$.

Furthermore, from $\tilde{G}_4 = (\tilde{G}_1 \tilde{G}_2 \tilde{G}_3)^{-1}$, we get

$$\tilde{G}_4 = \begin{pmatrix} 1 & 0 \\ 2\pi i\kappa & 1 \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0 & 0 \\ \kappa & 0 \end{pmatrix}. \tag{26}$$

Let us apply Erugin's construction to the set W_1, W_2, W_3, W_4. We remark that in [2], the authors study a more general case. In the result, we get the system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{3} \frac{\tilde{A}_i}{z - a_i} \right) y, \quad \tilde{A}_i = \begin{pmatrix} -\sqrt{\frac{-\sigma_{i1}\sigma_{i2}}{\tau\kappa}} & \frac{\sigma_{i2}}{-\tau} \\ \frac{\sigma_{i1}}{-\kappa} & \sqrt{\frac{-\sigma_{i1}\sigma_{i2}}{\tau\kappa}} \end{pmatrix}, \tag{27}$$

where the values σ_{i1}, σ_{i2} are the sums of the power series in $t = \frac{a_3 - a_1}{a_3 - a_2}$, $|t| < 1$ (see [2] for details and calculations), in particular,

$$\sigma_{ik} = \text{tr}(\tilde{A}_i W_k), \tag{28}$$

where

$$\tilde{W}_k = \tilde{A}_k + \text{tr}(\tilde{A}_i \tilde{A}_k) + \sum_{v=2}^{\infty} \sum_{j_1,\ldots,j_v}^{1,2,3} P_k^*(a_{j_1}, \ldots, a_{j_v}|b) \text{tr}(\tilde{A}_{j_1} \cdot \ldots \cdot \tilde{A}_{j_v})$$

and

$$P_k^*(a_{j_1}, \ldots, a_{j_v}|b) = \frac{1}{2\pi i} P_k(a_{j_1}, \ldots, a_{j_v}|b),$$

$$P_k(a_{j_1}|b) = \begin{cases} 2\pi i, & k = j_1, \\ 0, & k \neq j_1, \end{cases}$$

$$P_k(a_{j_1}, \ldots, a_{j_v}|b) = \frac{(2\pi i)^v}{v!}, \quad j_1 = \ldots = j_v = k,$$

$$P_k(a_{j_1}, \ldots, a_{j_v}|b) = \int_{a_k}^{b} \left(\frac{P_k(a_{j_1}, \ldots, a_{j_{v-1}}|b)}{b - a_{j_v}} - \frac{P_k(a_{j_2}, \ldots, a_{j_v}|b)}{b - a_{j_1}} \right) db.$$

We also have

$$\tilde{A}_j^2 = 0, \quad \tilde{A}_j \tilde{A}_k = \rho_{jk} I - \tilde{A}_k \tilde{A}_j, \quad \tilde{A}_j \tilde{A}_k \tilde{A}_j = \tilde{A}_j \rho_{jk},$$

where $\rho_{jk} = \text{tr}(\tilde{A}_j \tilde{A}_k)$. Note that ρ_{jk} can be expressed as certain power series in t [2].

To get a Fuchsian system with the monodromy matrices $G_1, \ldots, G_4$, we use $mc_{-\mu+1}$ with the parameter $\mu = \frac{1}{2\pi i} \ln a^{-1}$ for system (27). In the result, we get a (3×3) system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{3} \frac{A_i}{z - a_i} \right) y, \tag{29}$$

$$A_1 = \begin{pmatrix} -\mu + 1 & \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}} \frac{\sigma_{21}}{\sigma_{11}} - \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}} & \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}} \frac{\sigma_{31}}{\sigma_{11}} - \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}} \frac{\sigma_{11}}{\sigma_{21}} - \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}} & -\mu + 1 & \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}} \frac{\sigma_{31}}{\sigma_{21}} - \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}} \\ 0 & 0 & 0 \end{pmatrix}, \tag{30}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}} \frac{\sigma_{11}}{\sigma_{31}} - \sqrt{\frac{-\sigma_{11}\sigma_{12}}{\tau\kappa}} & \sqrt{\frac{-\sigma_{31}\sigma_{32}}{\tau\kappa}} \frac{\sigma_{21}}{\sigma_{31}} - \sqrt{\frac{-\sigma_{21}\sigma_{22}}{\tau\kappa}} & -\mu + 1 \end{pmatrix}.$$

System (29) solves the Riemann–Hilbert problem for monodromy matrices (23).

It is easy to see that the constructed system (29) has non-resonant finite singularities. Note that the general case in [2] with matrices

$$\tilde{A}_i = \begin{pmatrix} -\eta_i\theta_i & \eta_i^2 \\ -\theta_i^2 & \eta_i\theta_i \end{pmatrix}, \quad (\eta_1\theta_2 - \eta_2\theta_1)^2 + (\eta_1\theta_3 - \eta_3\theta_1)^2 + (\eta_2\theta_3 - \eta_3\theta_2)^2 = 0, \quad \eta_i, \theta_i \in \mathbb{C}$$

can be treated in a similar way.

4 Discussion

In this paper, we showed that it is possible to extend conditions of constructive solutions of the Riemann–Hilbert problem via middle convolution. We illustrated this idea on Erugin's method and the results in [2] for the Fuchsian system of order 2 with four singularities.

Essentially, theorems similar to Theorem 3.1 can be obtained for other methods of constructive solutions to the Riemann–Hilbert problem (see [14] for the overview and further references) by using the general scheme proposed in Section 3.1 (for any number of singularities of the Fuchsian system). This considerably extends the class of monodromy data for which for which the Riemann–Hilbert problem has a (constructive) solution.

There are certain analogues of middle convolution for linear systems with irregular singularities. Therefore, the method used in this paper can be extended to such systems and generalized Riemann–Hilbert problem (if the statement similar to Theorem 2.1 can be proved for linear systems with irregular singularities and their monodromy data).

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