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Null Controllability of a Coupled System of Two Korteweg-de Vries Equations from the Left Dirichlet Boundary Conditions

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Abstract In this paper, we consider a coupled system of two Korteweg-de Vries equations on a bounded domain. We establish the null controllability of this system from the left Dirichlet boundary conditions. Combining the analysis of a linearized system and a fixed point argument, this controllability result is reduced to prove the null controllability of a linearized system with two distributed controls.

Keywords Null controllability · Korteweg-de Vries equation · Coupled system · Boundary control

Mathematics Subject Classification (2010) 35Q53 · 93B05

1 Introduction

In this paper, we consider the following coupled system of two Korteweg-de Vries (KdV) equations

$$\begin{cases} u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x = 0, \\ b_1 v_t + rv_x + vv_x + b_2 a_3 u_{xxx} + v_{xxx} + b_2 a_2 uu_x + b_2 a_1 (uv)_x = 0, \end{cases} \tag{1.1}$$

where $0 < x < L$ $(L > 0)$ and $0 < t < T$ $(T > 0)$ with boundary conditions

$$\begin{cases} u(0,t) = h_1(t),\ u(L,t) = u_x(L,t) = 0 \text{ in } (0,T), \\ v(0,t) = h_2(t),\ v(L,t) = v_x(L,t) = 0 \text{ in } (0,T) \end{cases} \tag{1.2}$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in } (0,L). \tag{1.3}$$

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In Eqs. 1.1–1.3, $a_1, a_2, a_3, b_1, b_2, r$ are real constants with b_1, b_2 positive and $a_3^2 b_2 < 1$. $u = u(x,t)$, $v = v(x,t)$ are real valued functions and h_1, h_2 are two control functions.

System (1.1) was derived by Gear and Grimshaw [1] as a model to describe the strong interaction of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms. This system has been studied by many authors from various aspects of physics and mathematics ([2–12]).

The KdV equation

$$u_t + u_x + u_{xxx} + uu_x = 0$$

was first derived by Korteweg-de and Vries [13] in 1895 (or by Boussinesq [14] in 1876) as a model for the propagation of water waves along a channel. The equation furnishes also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems.

The controllability of the KdV equation

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \quad x \in (0, L),\ t > 0, \\ u(0,t) = h_3(t),\ u(L,t) = h_4(t),\ u_x(L,t) = h_5(t), \quad t > 0 \end{cases} \tag{1.4}$$

has been intensively studied (see [15] and the references therein). If only the left control input h_3 is in action, the system (1.4) behaves like a parabolic system and is only null controllable. However, if the system is allowed to control from the right end of the spacial domain, then the system behaves like a hyperbolic system and is exactly controllable.

For system (1.1), to our knowledge, the only known controllability results are due to [7, 10]. In [10], the authors established the exact controllability of Eq. 1.1 with the boundary conditions

$$\begin{cases} u(0,t) = 0,\ u(1,t) = h_6(t),\ u_x(1,t) = h_8(t) \text{ in } (0,T), \\ v(0,t) = 0,\ v(1,t) = h_7(t),\ v_x(1,t) = h_9(t) \text{ in } (0,T). \end{cases} \tag{1.5}$$

More precisely, they proved that for sufficient small $u_0, v_0, u_1, v_1 \in L^2(0,L)$, there exist four control functions $h_6, h_7 \in H^1(0,T)$ and $h_8, h_9 \in L^2(0,T)$, such that the solution $(u,v) \in C([0,T]; (L^2(0,L))^2) \cap L^2(0,T; (H^1(0,L))^2)$ of system (1.1), (1.3), and (1.5) verifies $u(\cdot,T) = u_1$, $v(\cdot,T) = v_1$. Later, Cerpa and Pazoto [7] improved this result by using only two control functions h_8 and h_9 for $L > 0$, $T > 0$ satisfying

$$1 > \frac{\max\{b_1, b_2\}}{\min\left\{b_2(1-\hat{\varepsilon}^2), \left(1 - \frac{a_3^2 b_2}{\hat{\varepsilon}^2}\right)\right\}} \left(\frac{rL^2}{3b_1\pi^2} + \frac{L^3}{3T\pi^2}\right),$$

where

$$\hat{\varepsilon} = \sqrt{\frac{-(1-b_2) + \sqrt{(1-b_2)^2 + 4a_3^2 b_2}}{2b_2}}.$$

Compared with the controllability results of Eq. 1.4 and motivated by [16, 18], it is natural to consider the null controllability of system (1.1)–(1.3).

The main result in this paper reads as follows:

Theorem 1.1 *Let $L > 0$ and $T > 0$. Then, there exists a constant $\delta > 0$ such that for any initial data $(u_0, v_0) \in (L^2(0, L))^2$ verifying $\|(u_0, v_0)\|_{(L^2(0,L))^2} \leq \delta$, there exist two control functions $h_1, h_2 \in H^{1/2-\varepsilon}(0, T)$ for any $\varepsilon > 0$, such that the solution*

$$(u, v) \in C([0, T]; (L^2(0, L))^2) \cap L^2(0, T; (H^1(0, L))^2)$$

of system (1.1)–(1.3) verifies

$$u(\cdot, T) = v(\cdot, T) = 0 \ \text{ in } \ (0, L).$$

Remark 1.1 *In [18], the authors considered the null controllability of* Eq. 1.4 *for $h_4 = h_5 = 0$. To obtain a control which is more regular than $L^2(0, T)$, they extended the spatial domain into $(-L, L)$ and proved an internal controllability result for the linear system. In this paper, we use the same method to prove Theorem 1.1.*

The rest of this paper is organized as follows. In Section 2, we consider the null controllability of a linearized system on $(-L, L)$ with two distributed controls. Section 3 is devoted to proof of Theorem 1.1 by a fixed point argument.

2 Internal Controllability of a Linearized System on $(-L, L)$

In this paper, we use the same extension domain as in [18]. Let us introduce a linear extension operator Π, which maps functions on $[0, L]$ to functions on $[-L, L]$ which support in $[-L/2, L]$, and which is continuous from $L^2(0, L)$ to $L^2(-L, L)$ and from $H^1(0, L)$ to $H^1(-L, L)$. We define

$$\widetilde{u}_0 = \Pi(u_0) \ \text{ and } \ \widetilde{v}_0 = \Pi(v_0).$$

In this section, we consider the following linearized system.

$$\begin{cases} \widetilde{u}_t + \widetilde{u}_{xxx} + a_3\widetilde{v}_{xxx} + (M_1\widetilde{u})_x + (N_1\widetilde{v})_x = \chi_\omega f_1 & \text{in } (-L, L) \times (0, T), \\ b_1\widetilde{v}_t + b_2a_3\widetilde{u}_{xxx} + \widetilde{v}_{xxx} + (M_2\widetilde{u})_x + (N_2\widetilde{v})_x = \chi_\omega f_2 & \text{in } (-L, L) \times (0, T), \\ \widetilde{u}(-L, t) = \widetilde{u}(L, t) = \widetilde{u}_x(L, t) = 0 & \text{in } (0, T), \\ \widetilde{v}(-L, t) = \widetilde{v}(L, t) = \widetilde{v}_x(L, t) = 0 & \text{in } (0, T), \\ \widetilde{u}(x, 0) = \widetilde{u}_0(x), \ \ \widetilde{v}(x, 0) = \widetilde{v}_0(x) & \text{in } (-L, L), \end{cases} \tag{2.1}$$

where $\omega = (l_1, l_2)$ with $-L < l_1 < l_2 < 0$ and $f_1 = f_1(x, t)$, $f_2 = f_2(x, t)$ are two control functions.

Remark 2.1 *In order to prove Theorem 1.1, we study linearized system (2.1) for $\omega = (l_1, l_2) \subset (-L, 0)$. In fact, all results in this section can be extended to the case where ω is any open set of $(-L, L)$ without any difficulty.*

2.1 Well-Posedness

In this subsection, we study the existence and the regularity of the solution to system (2.1). The methods in this subsection are motivated by [18].

First, let us introduce a functional space which will be used in the sequel:

$$X = C([0, T]; L^2(-L, L)) \cap L^2(0, T; H^1(-L, L)).$$

The space X is equipped with its natural norm.

Proposition 2.1 *Let* $(\widetilde{u}_0, \widetilde{v}_0) \in (L^2(-L, L))^2, (f_1, f_2) \in L^2(0, T; (L^2(\omega))^2)$ *and* $M_i, N_i \in X$ $(i = 1, 2)$. *There exists unique solution* $(\widetilde{u}, \widetilde{v})$ *of* Eq. 2.1 *such that*

$$(\widetilde{u}, \widetilde{v}) \in C([0, T]; (L^2(-L, L))^2) \cap L^2(0, T; (H^1(-L, L))^2)$$

and the following estimate holds

$$\begin{aligned} &\|(\widetilde{u}, \widetilde{v})\|_{L^\infty(0,T;(L^2(-L,L))^2)\bigcap L^2(0,T;(H^1(-L,L))^2)} \\ &\leq C(\|(\widetilde{u}_0, \widetilde{v}_0)\|_{(L^2(-L,L))^2} + \|(f_1, f_2)\|_{L^2(0,T;(L^2(\omega))^2)}), \end{aligned} \tag{2.2}$$

where $C = C(\|M_1\|_X, \|M_2\|_X, \|N_1\|_X, \|N_2\|_X)$.

Proof Let $(L^2(-L, L))^2$ endowed with the inner product

$$\langle (u, v), (\varphi, \psi)\rangle = \frac{b_2}{b_1}\int_{-L}^{L} u\varphi dx + \int_{-L}^{L} v\psi dx,$$

and consider the operator

$$A : D(A) \subset (L^2(-L, L))^2 \to (L^2(-L, L))^2,$$

where

$$\begin{aligned} D(A) &= \{(u, v) \in (H^3(-L, L))^2 : u(-L) \\ &= v(-L) = u(L) = v(L) = u_x(L) = v_x(L) = 0\} \end{aligned}$$

and

$$A(u, v) = \begin{pmatrix} -u_{xxx} - a_3 v_{xxx} \\ -\frac{1}{b_1} v_{xxx} - \frac{b_2 a_3}{b_1} u_{xxx} \end{pmatrix}, \quad \forall\, (u, v) \in D(A).$$

The calculations in [10] $(r = 0)$ proved that the operator A and its adjoint A^* are dissipative in $(L^2(-L, L))^2$. Hence, A generates a strong continuous semigroup of contractions on $(L^2(-L, L))^2$ which will be denoted by $\{S(t)\}_{t\geq 0}$.

To simplify notation, we define

$$B_\theta = C([0, \theta]; (L^2(-L, L))^2) \cap L^2(0, \theta; (H^1(-L, L))^2)$$

endowed with its natural norm, where $\theta > 0$ will be determined later.

Write system (2.1) in its integral form

$$\begin{aligned} (\widetilde{u}, \widetilde{v})(t) &= S(t)(\widetilde{u}_0, \widetilde{v}_0) + \int_0^t S(t - s)(-(M_1\widetilde{u})_x \\ &\quad -(N_1\widetilde{v})_x + \chi_\omega f_1, -(M_2\widetilde{u})_x - (N_2\widetilde{v})_x + \chi_\omega f_2)ds. \end{aligned}$$

Define a map on Γ on B_θ by

$$\begin{aligned} \Gamma(\widetilde{u}, \widetilde{v})(t) &= S(t)(\widetilde{u}_0, \widetilde{v}_0) + \int_0^t S(t - s)(-(M_1\widetilde{u})_x \\ &\quad -(N_1\widetilde{v})_x + \chi_\omega f_1, -(M_2\widetilde{u})_x - (N_2\widetilde{v})_x + \chi_\omega f_2)ds \end{aligned}$$

for $(\widetilde{u}, \widetilde{v}) \in B_\theta$.

Simple computation shows that

$$\begin{aligned}\|\Gamma(\widetilde{u},\widetilde{v})\|_{B_\theta} \le\; & C\Big(\|(\widetilde{u}_0,\widetilde{v}_0)\|_{(L^2(-L,L))^2} + \|(f_1,f_2)\|_{L^2(0,\theta;(L^2(\omega))^2)}\Big) \\ & + C_1\sum_{i=1}^{2}\left(\|M_i\|_{L^2(0,\theta;H^1(-L,L))} + \|N_i\|_{L^2(0,\theta;H^1(-L,L))}\right)\|(\widetilde{u},\widetilde{v})\|_{B_\theta},\end{aligned} \tag{2.3}$$

$$\begin{aligned}\|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{B_\theta} \le\; & C_1\sum_{i=1}^{2}\Big(\|M_i\|_{L^2(0,\theta;H^1(-L,L))} \\ & + \|N_i\|_{L^2(0,\theta;H^1(-L,L))}\Big)\|(u_1,v_1)-(u_2,v_2)\|_{B_\theta}\end{aligned}$$

for any (u_1,v_1), (u_2,v_2) and $(\widetilde{u},\widetilde{v}) \in B_\theta$.

We can choose $\theta > 0$ small enough such that

$$C_1\sum_{i=1}^{2}\left(\|M_i\|_{L^2(0,\theta;H^1(-L,L))} + \|N_i\|_{L^2(0,\theta;H^1(-L,L))}\right) \le \frac{1}{2}. \tag{2.4}$$

Then, it follows immediately that

$$\Gamma(\widetilde{u},\widetilde{v}) \in B_\theta \;\text{ and }\; \|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{B_\theta} \le \frac{1}{2}\|(u_1,v_1)-(u_2,v_2)\|_{B_\theta}$$

for any (u_1,v_1), (u_2,v_2), $(\widetilde{u},\widetilde{v}) \in B_\theta$. Thus, Γ is a contraction mapping of B_θ. Its fixed point $(\widetilde{u},\widetilde{v}) = \Gamma(\widetilde{u},\widetilde{v})$ is the unique solution of system (2.1) in B_θ. Moreover, combining Eqs. 2.3 and 2.4, it is shown that

$$\|(\widetilde{u},\widetilde{v})\|_{B_\theta} \le C\left(\|(\widetilde{u}_0,\widetilde{v}_0)\|_{(L^2(-L,L))^2} + \|(f_1,f_2)\|_{L^2(0,\theta;(L^2(\omega))^2)}\right).$$

Note that θ depends only on M_i, N_i $(i=1,2)$, by standard extension argument, one may extend θ to T. Moreover, we have Eq. 2.2.

This proves Proposition 2.1. □

In the next section, we will need a regularity estimate for Eq. 2.1.

Proposition 2.2 *Let* $(\widetilde{u}_0,\widetilde{v}_0) \in (H^1(-L,L))^2, (f_1,f_2) \in L^2(0,T;(L^2(\omega))^2)$ *and* $M_i, N_i \in X$ $(i=1,2)$. *Then, the solution* $(\widetilde{u},\widetilde{v})$ *of* Eq. 2.1 *belongs to*

$$C([0,T];(H^1(-L,L))^2) \cap L^2(0,T;(H^2(-L,L))^2),$$

and moreover, it satisfies the following estimate:

$$\begin{aligned}& \|(\widetilde{u},\widetilde{v})\|_{L^\infty(0,T;(H^1(-L,L))^2)\bigcap L^2(0,T;(H^2(-L,L))^2)} \\ & \le C\left(\|(\widetilde{u}_0,\widetilde{v}_0)\|_{(H^1(-L,L))^2} + \|(f_1,f_2)\|_{L^2(0,T;(L^2(\omega))^2)}\right),\end{aligned} \tag{2.5}$$

where $C = C(\|M_1\|_X, \|M_2\|_X, \|N_1\|_X, \|N_2\|_X)$.

Proof First, we prove an additional regularity result for the following system:

$$\begin{cases} \widetilde{u}_t + \widetilde{u}_{xxx} + a_3\widetilde{v}_{xxx} = g_1 & \text{in } (-L,L)\times(0,T), \\ b_1\widetilde{v}_t + b_2a_3\widetilde{u}_{xxx} + \widetilde{v}_{xxx} = g_2 & \text{in } (-L,L)\times(0,T), \\ \widetilde{u}(-L,t) = \widetilde{u}(L,t) = \widetilde{u}_x(L,t) = 0 & \text{in } (0,T), \\ \widetilde{v}(-L,t) = \widetilde{v}(L,t) = \widetilde{v}_x(L,t) = 0 & \text{in } (0,T), \\ \widetilde{u}(x,0) = \widetilde{u}_0(x), \quad \widetilde{v}(x,0) = \widetilde{v}_0(x) & \text{in } (-L,L). \end{cases} \tag{2.6}$$

Case 1: $(\widetilde{u}_0, \widetilde{v}_0) \in (L^2(-L, L))^2$ and $(g_1, g_2) \in L^2(0, T; (H^{-1}(-L, L))^2)$.

Multiplying the first equation in Eq. 2.6 by $b_2\widetilde{u}$ and the second equation in Eq. 2.6 by $\widetilde{v}$, adding the two obtained equations and integrating by parts in $(-L, L) \times (0, t)$ that

$$\begin{aligned}\frac{1}{2}\int_{-L}^{L}\left(b_2\widetilde{u}^2 + b_1\widetilde{v}^2\right)(x,\tau)|_{\tau=0}^{\tau=t}dx + \frac{1}{2}\int_0^t (b_2\widetilde{u}_x^2 + \widetilde{v}_x^2 + 2b_2a_3\widetilde{u}_x\widetilde{v}_x)(-L,\tau)d\tau \\ = \int_0^t\int_{-L}^{L}(b_2g_1\widetilde{u} + g_2\widetilde{v})dxd\tau \\ \le \delta_1\int_0^t\int_{-L}^{L}(\widetilde{u}_x^2 + \widetilde{v}_x^2)dxd\tau \\ + C(\delta_1)\int_0^t(\|g_1\|^2_{H^{-1}(-L,L)} + \|g_2\|^2_{H^{-1}(-L,L)})d\tau\end{aligned}$$

for any $\delta_1 > 0$.

The rest of the proof follows the proof of Theorem 2.2 in [10]. Same as in [10], choosing $\varepsilon > 0$ such that $\sqrt{a_3^2b_2} < \varepsilon < 1$, we obtain that

$$b_2\widetilde{u}_x^2 + \widetilde{v}_x^2 + 2a_3b_2\widetilde{u}_x\widetilde{v}_x > b_2(1-\varepsilon^2)\widetilde{u}_x^2 + (1 - \frac{a_3^2b_2}{\varepsilon^2})\widetilde{v}_x^2. \tag{2.7}$$

This implies

$$\begin{aligned}\int_{-L}^{L}(\widetilde{u}^2 + \widetilde{v}^2)(x,t)dx \le C\int_{-L}^{L}(\widetilde{u}_0^2 + \widetilde{v}_0^2)dx + \delta\int_0^t\int_{-L}^{L}(\widetilde{u}_x^2 + \widetilde{v}_x^2)dxd\tau \\ + C(\delta)\int_0^t(\|g_1\|^2_{H^{-1}(-L,L)} + \|g_2\|^2_{H^{-1}(-L,L)})d\tau\end{aligned} \tag{2.8}$$

for any $\delta > 0$. Multiplying the first equation in multiplying the first equation in Eq. 2.6 by $b_2(x+L)\widetilde{u}$ and the second equation in Eq. 2.6 by $(x+L)\widetilde{v}$, adding the two obtained equations and integrating by parts in $(-L, L) \times (0, t)$ that

$$\begin{aligned}\tfrac{1}{2}\int_{-L}^{L}(x+L)\left(b_2\widetilde{u}^2 + b_1\widetilde{v}^2)(x,\tau)\right|_{\tau=0}^{\tau=t}dx + \tfrac{3}{2}\int_0^t\int_{-L}^{L}(b_2\widetilde{u}_x^2 + \widetilde{v}_x^2 + 2b_2a_3\widetilde{u}_x\widetilde{v}_x)dxd\tau \\ = \int_0^t\int_{-L}^{L}(b_2(x+L)g_1\widetilde{u} + (x+L)g_2\widetilde{v})dxd\tau \\ \le \delta_2\int_0^t\int_{-L}^{L}(\widetilde{u}_x^2 + \widetilde{v}_x^2)dxd\tau + C(\delta_2)\int_0^t(\|g_1\|^2_{H^{-1}(-L,L)} + \|g_2\|^2_{H^{-1}(-L,L)})d\tau\end{aligned}$$

for any $\delta_2 > 0$. Taking (2.7) into consideration again and choosing δ_2 small enough, we can deduce that

$$\begin{aligned}\int_0^t\int_{-L}^{L}(\widetilde{u}^2 + \widetilde{v}^2)dxd\tau + \int_0^t\int_{-L}^{L}(\widetilde{u}_x^2 + \widetilde{v}_x^2)dxd\tau \\ \le C\Big(\int_{-L}^{L}(\widetilde{u}_0^2 + \widetilde{v}_0^2)dx + \int_0^t(\|g_1\|^2_{H^{-1}(-L,L)} + \|g_2\|^2_{H^{-1}(-L,L)})d\tau\Big).\end{aligned} \tag{2.9}$$

Combining (2.8)–(2.9) and choosing suitable δ, it follows that

$$\begin{aligned}|(\widetilde{u}, \widetilde{v})\|_{L^\infty(0,T;(L^2(-L,L))^2)\bigcap L^2(0,T;(H^1(-L,L))^2)} \\ \le C\left(\|(\widetilde{u}_0, \widetilde{v}_0)\|_{(L^2(-L,L))^2} + \|(g_1, g_2)\|_{L^2(0,T;(H^{-1}(-L,L))^2)}\right).\end{aligned}$$

Case 2: $(\widetilde{u}_0, \widetilde{v}_0) \in D(A)$ and $(g_1, g_2) \in L^2(0, T; (H_0^2(-L, L))^2)$

Let us apply the operator $P = \partial_x^3$ to system (2.6). Considering the conditions on the traces of g_1, g_2 on the boundaries $-L$ and L, $(P\widetilde{u}, P\widetilde{v})$ solves the following system:

$$\begin{cases} (P\widetilde{u})_t + (P\widetilde{u})_{xxx} + a_3(P\widetilde{v})_{xxx} = Pg_1 & \text{in } (-L, L) \times (0, T), \\ b_1(P\widetilde{v})_t + b_2a_3(P\widetilde{u})_{xxx} + (P\widetilde{v})_{xxx} = Pg_2 & \text{in } (-L, L) \times (0, T), \\ (P\widetilde{u})(-L, t) = (P\widetilde{u})(L, t) = (P\widetilde{u})_x(L, t) = 0 & \text{in } (0, T), \\ (P\widetilde{v})(-L, t) = (P\widetilde{v})(L, t) = (P\widetilde{v})_x(L, t) = 0 & \text{in } (0, T), \\ (P\widetilde{u})(x, 0) = (P\widetilde{u}_0)(x), \quad (P\widetilde{v})(x, 0) = (P\widetilde{v}_0)(x) & \text{in } (-L, L). \end{cases}$$

By the similar method as in case 1, it is not difficult to obtain that

$$\begin{aligned} &\|(P\widetilde{u}, P\widetilde{v})\|_{L^\infty(0,T;(L^2(-L,L))^2)\bigcap L^2(0,T;(H^1(-L,L))^2)} \\ &\quad \le C\left(\|(P\widetilde{u}_0, P\widetilde{v}_0)\|_{(L^2(-L,L))^2} + \|(Pg_1, Pg_2)\|_{L^2(0,T;(H^{-1}(-L,L))^2)}\right). \end{aligned}$$

This implies

$$\begin{aligned} &\|(\widetilde{u}, \widetilde{v})\|_{L^\infty(0,T;(H^3(-L,L))^2)\bigcap L^2(0,T;(H^4(-L,L))^2)} \\ &\quad \le C\left(\|(\widetilde{u}_0, \widetilde{v}_0)\|_{(H^3(-L,L))^2} + \|(g_1, g_2)\|_{L^2(0,T;(H^2(-L,L))^2)}\right), \end{aligned}$$

where we use the Poincaré's inequalities: $\forall\, u \in H^4(-L, L)$ such that $u(-L) = u(L) = u'(L) = 0$, one has

$$\|u\|_{H^3(-L,L)} \le C\|Pu\|_{L^2(-L,L)} \quad \text{and} \quad \|u\|_{H^4(-L,L)} \le C\|Pu\|_{H^1(-L,L)}.$$

By interpolation arguments, we can deduce that

$$\begin{aligned} &\|(\widetilde{u}, \widetilde{v})\|_{L^\infty(0,T;(H^1(-L,L))^2)\bigcap L^2(0,T;(H^2(-L,L))^2)} \\ &\quad \le C\left(\|(\widetilde{u}_0, \widetilde{v}_0)\|_{(H^1(-L,L))^2} + \|(g_1, g_2)\|_{L^2(0,T;(L^2(-L,L))^2)}\right). \end{aligned} \tag{2.10}$$

Then, let

$$g_1 = \chi_\omega f_1 - (M_1\widetilde{u})_x - (N_1\widetilde{v})_x, \quad g_2 = \chi_\omega f_2 - (M_2\widetilde{u})_x - (N_2\widetilde{v})_x.$$

It is easy to obtain that

$$\begin{aligned} \|g_1\|_{L^2(0,T;L^2(-L,L))} \le C\Big(&\|f_1\|_{L^2(0,T;L^2(\omega))} + \|M_1\widetilde{u}_x\|_{L^2(0,T;L^2(-L,L))} \\ &+\|M_{1x}\widetilde{u}\|_{L^2(0,T;L^2(-L,L))} \\ &+\|N_1\widetilde{v}_x\|_{L^2(0,T;L^2(-L,L))} + \|N_{1x}\widetilde{v}\|_{L^2(0,T;L^2(-L,L))}\Big). \end{aligned}$$

Direct computation shows that

$$\begin{aligned} \|M_1\widetilde{u}_x\|_{L^2(0,T;L^2(-L,L))} &\le \|M_1\|_{L^\infty(0,T;L^2(-L,L))}\|\widetilde{u}_x\|_{L^2(0,T;L^\infty(-L,L))} \\ &\le \|M_1\|_{L^\infty(0,T;L^2(-L,L))}\|\widetilde{u}\|_{L^2(0,T;H^{7/4}(-L,L))} \\ &\le \varepsilon\|M_1\|_X\|\widetilde{u}\|_{L^2(0,T;H^2(-L,L))} + C(\varepsilon)\|M_1\|_X\|\widetilde{u}\|_{L^2(0,T;H^1(-L,L))} \end{aligned}$$

and

$$\begin{aligned} \|M_{1x}\widetilde{u}\|_{L^2(0,T;L^2(-L,L))} &\le \|M_1\|_{L^2(0,T;H^1(-L,L))}\|\widetilde{u}\|_{L^\infty(0,T;L^\infty(-L,L))} \\ &\le \|M_1\|_{L^2(0,T;H^1(-L,L))}\|\widetilde{u}\|_{L^\infty(0,T;H^{3/4}(-L,L))} \\ &\le \varepsilon\|M_1\|_X\|\widetilde{u}\|_{L^\infty(0,T;H^1(-L,L))} + C(\varepsilon)\|M_1\|_X\|\widetilde{u}\|_{L^\infty(0,T;L^2(-L,L))} \end{aligned}$$

for any $\varepsilon > 0$. Similarly, we can estimate $\|N_1\widetilde{v}_x\|_{L^2(0,T;L^2(-L,L))}$ and $\|N_{1x}\widetilde{v}\|_{L^2(0,T;L^2(-L,L))}$.

Thus, for any $\varepsilon > 0$, we have

$$\begin{aligned} \|g_1\|_{L^2(0,T;L^2(-L,L))} &\le C\|f_1\|_{L^2(0,T;L^2(\omega))} \\ &+ \varepsilon(\|M_1\|_X + \|N_1\|_X)\|(\widetilde{u},\widetilde{v})\|_{L^\infty(0,T;(H^1(-L,L))^2)\bigcap L^2(0,T;(H^2(-L,L))^2)} \\ &+ C(\varepsilon)(\|M_1\|_X + \|N_1\|_X)\|(\widetilde{u},\widetilde{v})\|_{L^\infty(0,T;(L^2(-L,L))^2)\bigcap L^2(0,T;(H^1(-L,L))^2)}. \end{aligned} \tag{2.11}$$

By the same methods, we have

$$\begin{aligned} \|g_2\|_{L^2(0,T;L^2(-L,L))} &\le C\|f_2\|_{L^2(0,T;L^2(\omega))} \\ &+ \varepsilon(\|M_2\|_X + \|N_2\|_X)\|(\widetilde{u},\widetilde{v})\|_{L^\infty(0,T;(H^1(-L,L))^2)\bigcap L^2(0,T;(H^2(-L,L))^2)} \\ &+ C(\varepsilon)(\|M_2\|_X + \|N_2\|_X)\|(\widetilde{u},\widetilde{v})\|_{L^\infty(0,T;(L^2(-L,L))^2)\bigcap L^2(0,T;(H^1(-L,L))^2)}. \end{aligned} \tag{2.12}$$

Gathering together Eqs. 2.2 and 2.10–2.12 and choosing ε sufficient small, we can obtain estimate 2.5. □

2.2 Internal Controllability

In this part, we study the null controllability of system (2.1).

First, we review an estimate for the following system:

$$\begin{cases} y_t + a y_{xxx} = f & \text{in } (-L,L)\times(0,T), \\ y(-L,t) = y(L,t) = y_x(-L,t) = 0 & \text{in } (0,T), \\ y(x,T) = y_T(x) & \text{in } (-L,L), \end{cases} \tag{2.13}$$

where $a > 0$ is a constant.

Pick any function $\psi \in C^3([-L,L])$ such that

$$\begin{cases} \psi > 0 \ \text{ in } \ [-L,L]; \\ |\psi'| > 0, \ \psi'' < 0 \ \text{ and } \ \psi'\psi''' < 0 \ \text{ in } \ [-L,L]\backslash\omega; \\ \psi'(-L) < 0 \ \text{ and } \ \psi'(L) > 0; \\ \min\limits_{x\in[l_1,l_2]} \psi(x) = \psi(l_3) < \max\limits_{x\in[l_1,l_2]} \psi(x) = \psi(l_1) = \psi(l_2), \\ \max\limits_{x\in[-L,L]} \psi(x) = \psi(-L) = \psi(L) \ \text{ and } \ \psi(-L) < \frac{4}{3}\psi(l_3) \ \text{ for some } l_3 \in (l_1,l_2). \end{cases}$$

The existence of such ψ can be found in [16].

To simplify notations, let $Q = (-L,L)\times(0,T)$ and $Q^\omega = \omega\times(0,T)$. Set

$$\varphi(x,t) = \frac{\psi(-x)}{t(T-t)} \ \text{ and } \ \widetilde{y} = e^{-s\varphi}y.$$

Following the methods developed in [16] with minor changes, we have

Proposition 2.3 *There exists some positive constant s_0 such that for all $s \ge s_0$ and all $y_T \in L^2(-L,L)$, the solution y of* Eq. 2.13 *fulfills*

$$\begin{aligned} &\int_Q \left((s\varphi)^5\widetilde{y}^2 + (s\varphi)^3\widetilde{y}_x^2 + s\varphi\widetilde{y}_{xx}^2\right)dxdt \\ &\le C\Big(\int_Q f^2 e^{-2s\varphi}dxdt + \int_{Q^\omega}\left((s\varphi)^5\widetilde{y}^2 + (s\varphi)^3\widetilde{y}_x^2 + s\varphi\widetilde{y}_{xx}^2\right)dxdt\Big). \end{aligned}$$

Next, we are in a position to prove the main result in this subsection.

Theorem 2.1 *Let $(\widetilde{u}_0,\widetilde{v}_0) \in (L^2(-L,L))^2$ and $M_i, N_i \in X$ $(i = 1,2)$. Then, there exists $(f_1, f_2) \in L^2(0,T;(L^2(\omega))^2)$ such that the solution $(\widetilde{u},\widetilde{v})$ of* Eq. 2.1 *satisfies*

$$\widetilde{u}(\cdot,T) = \widetilde{v}(\cdot,T) = 0 \ \text{ in } \ (-L,L).$$

Moreover, there exists constant $C = C(\|M_1\|_X, \|M_2\|_X, \|N_1\|_X, \|N_2\|_X)$ *such that*

$$\|(f_1, f_2)\|_{L^2(0,T;(L^2(\omega)))^2} \leq C\|(\widetilde{u}_0, \widetilde{v}_0)\|_{(L^2(-L,L)^2)}. \tag{2.14}$$

Proof Define

$$\mu = \sqrt{\left(\frac{1}{b_1}\right)^2 + \frac{4b_2a_3^2}{b_1}} \text{ and } \lambda_1^{\pm} = \frac{1}{b_1} - 1 \pm \mu,\ \lambda_2^{\pm} = \frac{1}{2}\left(\frac{1}{b_1} + 1 \pm \mu\right).$$

Our assumption $a_3^2 b_2 < 1$ guarantees that $\lambda_2^{\pm} > 0$. Using the change of variable in [17], i.e.,

$$\widetilde{u} = 2a_3\overline{u} + 2a_3\overline{v} \text{ and } \widetilde{v} = \lambda_1^+\overline{u} + \lambda_1^-\overline{v},$$

we can transform linear system (2.1) into the following system

$$\begin{cases} \overline{u}_t + \lambda_2^+\overline{u}_{xxx} + (\overline{M_1}\overline{u})_x + (\overline{N_1}\overline{v})_x = \chi_\omega \overline{f_1} & \text{in } (-L, L) \times (0, T), \\ \overline{v}_t + \lambda_2^-\overline{v}_{xxx} + (\overline{M_2}\overline{u})_x + (\overline{N_2}\overline{v})_x = \chi_\omega \overline{f_2} & \text{in } (-L, L) \times (0, T), \\ \overline{u}(-L, t) = \overline{u}(L, t) = \overline{u}_x(L, t) = 0 & \text{in } (0, T), \\ \overline{v}(-L, t) = \overline{v}(L, t) = \overline{v}_x(L, t) = 0 & \text{in } (0, T), \\ \overline{u}(x, 0) = \overline{u}_0(x),\ \ \overline{v}(x, 0) = \overline{v}_0(x) & \text{in } (-L, L), \end{cases} \tag{2.15}$$

where

$$\begin{cases} \overline{M_1} = -\frac{\lambda_1^-}{4a_3}(2a_3M_1 + \lambda_1^+N_1) + \frac{1}{2\mu b_1}(2a_3M_2 + \lambda_1^+N_2), \\ \overline{N_1} = -\frac{\lambda_1^-}{4a_3}(2a_3M_1 + \lambda_1^-N_1) + \frac{1}{2\mu b_1}(2a_3M_2 + \lambda_1^-N_2), \\ \overline{M_2} = \frac{\lambda_1^+}{4a_3}(2a_3M_1 + \lambda_1^+N_1) - \frac{1}{2\mu b_1}(2a_3M_2 + \lambda_1^+N_2), \\ \overline{N_2} = \frac{\lambda_1^+}{4a_3}(2a_3M_1 + \lambda_1^-N_1) - \frac{1}{2\mu b_1}(2a_3M_2 + \lambda_1^-N_2), \\ \overline{f_1} = -\frac{\lambda_1^-}{4a_3}f_1 + \frac{1}{2\mu b_1}f_2,\ \ \overline{f_2} = \frac{\lambda_1^+}{4a_3}f_1 - \frac{1}{2\mu b_1}f_2, \\ \overline{u}_0 = -\frac{\lambda_1^-}{4a_3}\widetilde{u}_0 + \frac{1}{2\mu}\widetilde{v}_0,\ \ \ \overline{v}_0 = \frac{\lambda_1^+}{4a_3}\widetilde{u}_0 - \frac{1}{2\mu}\widetilde{v}_0. \end{cases} \tag{2.16}$$

Next, we consider the adjoint system associated to Eq. 2.15:

$$\begin{cases} \eta_t + \lambda_2^+\eta_{xxx} + \overline{M_1}\eta_x + \overline{M_2}w_x = 0 & \text{in } (-L, L) \times (0, T), \\ w_t + \lambda_2^- w_{xxx} + \overline{N_1}\eta_x + \overline{N_2}w_x = 0 & \text{in } (-L, L) \times (0, T), \\ \eta(-L, t) = \eta(L, t) = \eta_x(-L, t) = 0 & \text{in } (0, T), \\ w(-L, t) = w(L, t) = w_x(-L, t) = 0 & \text{in } (0, T), \\ \eta(x, T) = \eta_T(x),\ \ w(x, T) = w_T(x) & \text{in } (-L, L). \end{cases} \tag{2.17}$$

The following part is close to [16, 18]; therefore, we just sketch it.

Claim that for any $(\eta_T, w_T) \in (L^2(-L, L))^2$ and $\overline{M_i}, \overline{N_i} \in X$ $(i = 1, 2)$, the solution (η, w) of Eq. 2.15 satisfies

$$\|(\eta(\cdot, 0), w(\cdot, 0))\|_{(L^2(0,L))^2} \leq C\|(\eta, w)\|_{L^2(0,T;(L^2(\omega))^2)}, \tag{2.18}$$

where $C = C(\|\overline{M_1}\|_X, \|\overline{M_2}\|_X, \|\overline{N_1}\|_X, \|\overline{N_2}\|_X)$.

Indeed, let

$$\widetilde{\eta} = e^{-s\varphi}\eta,\ \ \widetilde{w} = e^{-s\varphi}w.$$

Then, it follows that

$$\eta_x = e^{s\varphi}\widetilde{\eta}_x + s\varphi_x e^{s\varphi}\widetilde{\eta} \text{ and } w_x = e^{s\varphi}\widetilde{w}_x + s\varphi_x e^{s\varphi}\widetilde{w}.$$

Applying Proposition 2.3 to the first two equations in Eq. 2.17 , we have the following estimate

$$\begin{aligned}&\int_Q \Big((s\varphi)^5(\widetilde{\eta}^2+\widetilde{w}^2)+(s\varphi)^3(\widetilde{\eta}_x^2+\widetilde{w}_x^2)+s\varphi(\widetilde{\eta}_{xx}^2+\widetilde{w}_{xx}^2)\Big)dxdt\\ \leq\ & C\int_Q \Big(|\overline{M_1}\eta_x+\overline{M_2}w_x|^2e^{-2s\varphi}+|\overline{N_1}\eta_x+\overline{N_2}w_x\ |^2\ e^{-2s\varphi}\Big)dxdt\\ &+C\int_{Q^\omega}\Big((s\varphi)^5(\widetilde{\eta}^2+\widetilde{w}^2)+(s\varphi)^3(\widetilde{\eta}_x^2+\widetilde{w}_x^2)+s\varphi(\widetilde{\eta}_{xx}^2+\widetilde{w}_{xx}^2)\Big)dxdt.\end{aligned}$$

It is clear that

$$\begin{aligned}&\int_Q \Big(|\overline{M_1}\eta_x+\overline{M_2}w_x|^2e^{-2s\varphi}+|\overline{N_1}\eta_x+\overline{N_2}w_x|^2e^{-2s\varphi}\Big)dxdt\\ \leq\ & C\int_Q \Big(|\overline{M_1}\widetilde{\eta}_x|^2+|\overline{M_2}\widetilde{w}_x|^2+|\overline{N_1}\widetilde{\eta}_x|^2+|\overline{N_2}\widetilde{w}_x|^2\\ &\qquad+s^2\varphi^2(|\overline{M_1}\widetilde{\eta}|^2+|\overline{M_2}\widetilde{w}|^2+|\overline{N_1}\widetilde{\eta}|^2+|\overline{N_2}\widetilde{w}|^2)\Big)dxdt\\ \leq\ & C(\|\overline{M_1}\|_X,\|\overline{M_2}\|_X,\|\overline{N_1}\|_X,\|\overline{N_2}\|_X)\int_Q \Big(|\widetilde{\eta}_{xx}|^2+|\widetilde{w}_{xx}|^2+s^2\varphi^2(|\widetilde{\eta}_x|^2+|\widetilde{\eta}_x|^2)\Big)dxdt.\end{aligned}$$

For $s\geq s_1$ with $s_1=s_1(\|\overline{M_1}\|_X,\|\overline{M_2}\|_X,\|\overline{N_1}\|_X,\|\overline{N_2}\|_X)$ large enough, we infer that

$$\begin{aligned}&\int_Q \Big((s\varphi)^5(\eta^2+w^2)+(s\varphi)^3(\eta_x^2+w_x^2)+s\varphi(\eta_{xx}^2+w_{xx}^2)\Big)e^{-2s\varphi}dxdt\\ \leq\ & C\int_{Q^\omega}\Big((s\varphi)^5(\eta^2+w^2)+(s\varphi)^3(\eta_x^2+w_x^2)+s\varphi(\eta_{xx}^2+w_{xx}^2)\Big)e^{-2s\varphi}dxdt.\end{aligned}\tag{2.19}$$

We introduce the functions

$$\hat{\varphi}=\frac{1}{t(T-t)}\max_{x\in[-L,L]}\psi(x)=\frac{\psi(L)}{t(T-t)}\ \text{ and }\ \check{\varphi}=\frac{1}{t(T-t)}\min_{x\in[-L,L]}\psi(x)=\frac{\psi(l_3)}{t(T-t)}.\tag{2.20}$$

From Eqs. 2.19 and 2.20, we have

$$\begin{aligned}&\int_Q \Big((s\check{\varphi})^5(\eta^2+w^2)+(s\check{\varphi})^3(\eta_x^2+w_x^2)+s\check{\varphi}(\eta_{xx}^2+w_{xx}^2)\Big)e^{-2s\hat{\varphi}}dxdt\\ \leq\ & C\int_{Q^\omega}\Big((s\check{\varphi})^5(\eta^2+w^2)+(s\check{\varphi})^3(\eta_x^2+w_x^2)+s\check{\varphi}(\eta_{xx}^2+w_{xx}^2)\Big)e^{-2s\check{\varphi}}dxdt.\end{aligned}\tag{2.21}$$

Using interpolation in the Sobolev spaces and Young's inequality, it holds that

$$\begin{aligned}&\int_{Q^\omega}\Big((s\check{\varphi})^5(\eta^2+w^2)+(s\check{\varphi})^3(\eta_x^2+w_x^2)+s\check{\varphi}(\eta_{xx}^2+w_{xx}^2)\Big)e^{-2s\hat{\varphi}}dxdt\\ \leq\ & Cs^{10}\int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})}\check{\varphi}^{31}\|(\eta(\cdot,t),w(\cdot,t))\|^2_{(L^2(\omega))^2}dt\\ &\qquad+\varepsilon s^{-2}\int_0^T e^{-2s\hat{\varphi}}\check{\varphi}^{-9}\|(\eta(\cdot,t),w(\cdot,t))\|^2_{(H^{8/3}(\omega))^2}dt\end{aligned}\tag{2.22}$$

for any $\varepsilon>0$. Some complicated estimates yield

$$\begin{aligned}&\int_0^T e^{-2s\hat{\varphi}}\check{\varphi}^{-9}\|(\eta(\cdot,t),w(\cdot,t))\|^2_{(H^{8/3}(\omega))^2}dt\\ &\qquad\leq Cs^2\int_Q \Big((s\check{\varphi})^5(\eta^2+w^2)+(s\check{\varphi})^3(\eta_x^2+w_x^2)+s\check{\varphi}(\eta_{xx}^2+w_{xx}^2)\Big)e^{-2s\hat{\varphi}}dxdt.\end{aligned}\tag{2.23}$$

Taking (2.21)–(2.23) into consideration and choosing suitable ε, we conclude that

$$\begin{aligned}&\int_Q \Big((s\check{\varphi})^5(\eta^2+w^2)+(s\check{\varphi})^3(\eta_x^2+w_x^2)+s\check{\varphi}(\eta_{xx}^2+w_{xx}^2)\Big)e^{-2s\hat{\varphi}}dxdt\\ &\qquad\leq Cs^{10}\int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})}\check{\varphi}^{31}\|(\eta(\cdot,t),w(\cdot,t))\|^2_{(L^2(\omega))^2}dt\end{aligned}\tag{2.24}$$

for $s\geq\max\{s_0,s_1\}$.

Proceeding as in the proof of Proposition 2.3, we can find a constant $C=C(\|\overline{M_1}\|_X,\|\overline{M_2}\|_X,\|\overline{N_1}\|_X,\|\overline{N_2}\|_X)$ such that

$$\|(\eta,w)\|^2_{L^\infty(0,T;(L^2(0,L))^2)\bigcap L^2(0,T;(H^1(0,L))^2)}\leq C\|(\eta_T,w_T)\|^2_{(L^2(0,L))^2}.\tag{2.25}$$

Replacing $(\eta(\cdot,t), w(\cdot,t))$ by $(\eta(\cdot,0), w(\cdot,0))$ and (η_T, w_T) by $(\eta(\cdot,\tau), w(\cdot,\tau))$ for $T/3 < \tau < 2T/3$ in Eq. 2.25, and integrating over $\tau \in (T/3, 2T/3)$, we obtain that

$$\|(\eta(\cdot,0), w(\cdot,0))\|^2_{(L^2(0,L))^2} \leq C \int_{T/3}^{2T/3} \|(\eta(\cdot,\tau), w(\cdot,\tau)))\|^2_{(L^2(-L,L))^2}.$$

Combining (2.24), we derive the observability estimate (2.18).

From Eq. 2.18 and the classical dual arguments, we establish the null controllability of Eq. 2.15. Combining (2.16), the proof of Theorem 2.1 is complete. □

3 Proof of Theorem 1.1

Now, we can prove the main result in this paper.

For $\xi, \zeta \in X$, let

$$M_1 = \frac{1}{2}\xi + a_2\zeta, \quad N_1 = \frac{a_1}{2}\zeta, \quad M_2 = \frac{b_2a_2}{2}\xi + b_2a_1\zeta, \quad N_2 = r + \frac{1}{2}\zeta \tag{3.1}$$

and

$$(u, v) = (\widetilde{u}, \widetilde{v})|_{[0,L]\times(0,T)}, \tag{3.2}$$

where $(\widetilde{u}, \widetilde{v})$ is the solution of Eq. 2.15 with M_1, M_2, N_1, N_2 defined as in Eq. 3.1 and (f_1, f_2) chosen as in Theorem 2.1. According to Theorem 2.1, it is clear that

$$\widetilde{u}(\cdot,T) = \widetilde{v}(\cdot,T) = 0 \ \text{ in } \ (-L, L).$$

Then, (u, v) solves the following system

$$\begin{cases} u_t + u_{xxx} + a_3v_{xxx} + \left((\frac{1}{2}\xi + a_2\zeta)u\right)_x + \left(\frac{a_1}{2}\zeta v\right)_x = 0 & \text{in } (0,L)\times(0,T), \\ b_1v_t + b_2a_3u_{xxx} + v_{xxx} + \left((\frac{b_2a_2}{2}\xi + b_2a_1\zeta)u\right)_x + \left((r + \frac{1}{2}\zeta)v\right)_x = 0 & \text{in } (0,L)\times(0,T), \\ u(0,t) = h_1(t), \ u(L,t) = u_x(L,t) = 0 & \text{in } (0,T), \\ v(0,t) = h_2(t), \ v(L,t) = v_x(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } (0,L) \end{cases}$$

for

$$h_1 = \widetilde{u}|_{x=0} \ \text{ and } \ h_2 = \widetilde{v}|_{x=0}.$$

We clearly have that $u(\cdot,T) = v(\cdot,T) = 0$ and

$$\|(f_1, f_2)\|_{L^2(0,T;(L^2(\omega)))^2} \leq C_*\|(u_0, v_0)\|_{(L^2(-L,L))^2}, \tag{3.3}$$

where $C_* = C_*(\|\xi\|_X, \|\zeta\|_X)$.

In the sequel, we suppose that $(u_0, v_0) \in (H^1(0,L))^2$ and that $\|(u_0, v_0)\|_{(H^1(0,L))^2}$ is sufficiently small.

In fact, for any $\alpha \in (0,T)$, assuming that $h_1 = h_2 = 0$ in $(0,\alpha)$, by the Banach fixed point theorem, system (1.1)–(1.3) admits a unique solution $(u, v) \in C([0,\alpha]; (L^2(0,L))^2) \cap L^2(0,\alpha; (H^1(0,L))^2)$, and if $\|(u_0, v_0)\|_{(L^2(0,L))^2}$ is sufficiently small, we have

$$\|(u, v)\|_{L^\infty(0,\alpha;(L^2(0,L))^2)\bigcap L^2(0,\alpha;(H^1(0,L))^2)} \leq \varepsilon$$

for any $\varepsilon > 0$. In particular, we can find a constant $\alpha_0 \in (0,\alpha)$ such that $(u(\cdot,\alpha_0), v(\cdot,\alpha_0)) \in (H^1(0,L))^2$ and that $\|(u(\cdot,\alpha_0), v(\cdot,\alpha_0))\|_{(H^1(0,L))^2}$ is sufficiently small. Then, we consider system (1.1)–(1.3) in $(0,L)\times(\alpha_0,T)$.

According to Proposition 2.2, (2.14) and interpolation arguments, we get that $h_1, h_2 \in H^{(1/2)-\varepsilon}(0, T)$ for any $\varepsilon > 0$, and

$$\|(h_1, h_2)\|_{(H^{(1/2)-\varepsilon}(0,T))^2} \leq C\|(u_0, v_0)\|_{(H^1(0,L))^2},$$

for some $C = C(\xi, \zeta)$.

Next, we introduce the space

$$E = C([0, T]; (L^2(0, L))^2) \cap L^2(0, T; (H^1(0, L))^2) \cap H^1(0, T; (H^{-2}(0, L))^2)$$

endowed with its natural norm. We consider in $L^2(0, T; (L^2(0, L))^2)$ the following compact subset:

$$B = \{(\xi, \zeta) \in E \mid \|(\xi, \zeta)\|_E \leq 1\}.$$

Let us define

$\Lambda((\xi, \zeta)) := \{(u, v) = (\widetilde{u}, \widetilde{v})|_{[0,L]\times(0,T)} \mid \exists (f_1, f_2) \in L^2(0, T; (L^2(\omega))^2)$ such that (f_1, f_2) satisfies (3.3) and $(\widetilde{u}, \widetilde{v})$ solves (2.1) with M_i, N_i defined in Eq. (3.1) and $\widetilde{u}(\cdot, T) = \widetilde{u}(\cdot, T) = 0\}$.

We shall use the following Banach space version of Kakutani's fixed point theorem (see [19]).

Theorem 3.1 *Let F be a locally convex space, let $B \subset Z$ and let $\Lambda : B \to 2^B$ be a set-valued mapping. Assume that:*

- *B is a nonempty, compact, convex set.*
- *$\Lambda(z)$ is a nonempty, closed, convex set of F for every $z \in B$.*
- *Λ is upper semicontinuous, i.e., for every closed subset A of F, $\Lambda^{-1}(A) = \{z \in B;\ \Lambda(z) \cap A \neq \emptyset\}$ is closed.*

Then, Λ possesses a fixed point in the set B, i.e., there exists $z \in B$ such that $z \in \Lambda(z)$.

Let us check that Theorem 3.1 can be applied to $F = L^2(0, T; (L^2(0, L))^2)$.

It follows from Eqs. 2.5, 2.14, and 3.2 that Λ maps B into 2^B for $\|(u_0, v_0)\|_{(H^1(0,L))^2}$ sufficiently small. The convexity of B and $\Lambda((\xi, \zeta))$ for all $(\xi, \zeta) \in B$ is clear. By Aubin-Lions' lemma, B is compact in F. Applying Theorem 2.1, we see that $\Lambda((\xi, \zeta))$ is nonempty for all $(\xi, \zeta) \in B$. The fact that $\Lambda((\xi, \zeta))$ is closed in F for every $(\xi, \zeta) \in B$ and that Λ is upper semicontinuous can be obtained following the methods developed in [16] with minor changes.

Consequently, Theorem 3.1 applied and this implies that there exists $(u, v) \in \Lambda(u, v)$, that is to say, we have found two controls $h_1, h_2 \in H^{(1/2)-\varepsilon}(0, T)$ for all $\varepsilon > 0$, such that the solution of Eqs. 1.1–1.3 satisfies $u(\cdot, T) = v(\cdot, T) = 0$ in $(0, L)$.

The proof of Theorem 1.1 is finished.

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