J Dyn Control Syst (2016) 22:465–483
DOI 10.1007/s10883-015-9295-2

Geodesics in the Engel Group with a Sub-Lorentzian Metric

Qihui Cai[1] · Tiren Huang[1,2] · Yu. L. Sachkov[3] · Xiaoping Yang[1]

Received: 6 February 2015 / Revised: 9 June 2015 / Published online: 3 September 2015

Abstract Let E be the Engel group and D be a rank 2 bracket generating left invariant distribution with a Lorentzian metric, which is a nondegenerate metric of index 1. In this paper, we first study some properties of horizontal curves on E. Second, we prove that time-like normal geodesics are locally maximizers in the Engel group and calculate the explicit expression of non-space-like geodesics.

Keywords Geodesics · Engel group · Sub-Lorentzian metric

Mathematics Subject Classification (2010) 58E10 · 53C50

Supported by NSFC (No.11071119, No.11401531); NSFC-RFBR (No. 11311120055).

✉ Tiren Huang
htiren@zstu.edu.cn

Qihui Cai
caigada@gmail.com

Yu. L. Sachkov
sachkov@sys.botik.ru

Xiaoping Yang
yangxp@njust.edu.cn

[1] Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, China

[2] Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China

[3] Program Systems Institute, Pereslavl-Zalessky 152140, Yaroslav, Russia

1 Introduction

A sub-Riemannian structure on a manifold M is given by a smoothly varying distribution D on M and a smoothly varying positively definite metric g on the distribution. The triple (M, D, g) is called a *sub-Riemannian manifold*, which has been applied in control theory, quantum physics, C-R geometry, and the other areas [1–4, 9, 15, 19]. Some efforts have been made to generalize sub-Riemannian manifold. One of them leads to the following question: what kind of geometrical features the mentioned triple will have if we change the positively definite metric to an indefinite nondegenerate metric? It is natural to start with the Lorentzian metric of index 1. In this case, the triple: manifold, distribution, and Lorentzian metric on the distribution is called a *sub-Lorentzian manifold* by analogy with a Lorentzian manifold. For the details concerning the *sub-Lorentzian geometry*, the reader is referred to [11]. To our knowledge, there are only a few works devoted to this subject (see [10–14, 16, 18]). In [10], Chang, Markina, and Vasiliev have systematically studied the geodesics in an anti-de Sitter space with a sub-Lorentzian metric and a sub-Riemannian metric, respectively. In [13], Grochowski computed reachable sets starting from a point in the Heisenberg sub-Lorentzian manifold on $\mathbb{R}^3$. It was shown in [18] that the Heisenberg group $\mathbb{H}$ with a Lorentzian metric on $\mathbb{R}^3$ possesses the uniqueness of Hamiltonian geodesics of time-like or space-like type.

The Engel group was first named by Cartan [8] in 1901. It is a prolongation of a three-dimensional contact manifold and is a Goursat manifold. In [5–7], A. Ardentov and Yu. L. Sachkov computed minimizers on the sub-Riemannian Engel group. In the present article, we study the Engel group furnished with a sub-Lorentzian metric. This is an interesting example of sub-Lorentzian manifolds because the Engel group is the simplest manifold with nontrivial abnormal extremal trajectories, and the vector distribution of the Engel group is not $2-$ generating, its growth vector is $(2, 3, 4)$. We first study some properties of horizontal curves in the Engel group. Second, we use the Hamiltonian formalism and Pontryagin maximum principle to write the equations for geodesics. Furthermore, we give a complete description of the Hamiltonian geodesics in the Engel group.

Apart from the introduction, this paper contains three sections. Section 2 contains some preliminaries as well as definitions of sub-Lorentzian manifolds, the Engel group. In Section 3, we study some properties of horizontal curves in the Engel group. In Section 4, we prove that the time-like normal geodesics are locally maximal in the Engel group, and explicitly calculate the non-space-like Hamiltonian geodesics.

2 Preliminaries

A sub-Lorentzian manifold is a triple (M, D, g), where M is a smooth n-dimensional manifold, D is a smooth distribution on M, and g is a smoothly varying Lorentzian metric on D. For each point $p \in M$, a vector $v \in D_p$ is said to be horizontal. An absolutely continuous curve $\gamma(t)$ is said to be horizontal if its derivative $\gamma'(t)$ exists almost everywhere and lies in $D_{\gamma(t)}$.

A vector $v \in D_p$ is said to be time-like if $g(v, v) < 0$; space-like if $g(v, v) > 0$ or $v = 0$; null (light-like) if $g(v, v) = 0$ and $v \neq 0$; and non-space-like if $g(v, v) \leq 0$. A curve $\gamma(t)$ is said to be time-like if its tangent vector $\dot{\gamma}(t)$ is time-like a.e.; space-like if $\dot{\gamma}(t)$ is space-like a.e.; null if $\dot{\gamma}(t)$ is null a.e.; non-space-like if $\dot{\gamma}(t)$ is non-space-like a.e.

By a time orientation of (M, D, g), we mean a continuous time-like vector field on M. From now on, we assume that (M, D, g) is time-oriented. If X is a time orientation on $(M, D,$

g), then a non-space-like vector $v \in D_p$ is said to be future directed if $g(v, X(p)) < 0$, and past directed if $g(v, X(p)) > 0$. Throughout this paper, "f.d." stands for "future directed," "t." for "time-like," and "nspc." for "non-space-like."

Let $v, w \in D$ be two non-space-like vectors, we have the following reverse Schwartz inequality (see page 144 in [17]):

$$|g(v, w)| \geq \|v\| \cdot \|w\|,$$

where $\|v\| = \sqrt{|g(v, v)|}$. The equality holds if and only if v and w are linearly dependent.

We introduce the space $H_{\gamma(t)}$ of horizontal nspc. curves:

$$\begin{aligned} H_{\gamma(t)} = \ & \{\gamma : [0, 1] \to M |\ \gamma(t) \text{ is absolutely continuous },\ \ g(\dot{\gamma}(t), \dot{\gamma}(t)) \leq 0, \\ & \dot{\gamma}(t) \in D_{\gamma(t)} \text{ for almost all } t \in [0, 1]\}. \end{aligned} \tag{2.1}$$

The sub-Lorentzian length of a horizontal nspc. curve $\gamma(t)$ is defined as follows:

$$l(\gamma) = \int_0^1 \|\gamma'(t)\| dt,$$

where $\|\gamma'(t)\| = \sqrt{|g(\gamma'(t), \gamma'(t))|}$. We use the length to define the sub-Lorentzian distance $d_U(q_1, q_2)$ with respect to a set $U \subset M$ between two points $q_1, q_2 \in U$:

$$d_U(q_1, q_2) = \begin{cases} \sup\{l(\gamma), \gamma \in H_U(q_1, q_2)\} & \text{if } H_U(q_1, q_2) \neq \emptyset \\ 0 & \text{if } H_U(q_1, q_2) = \emptyset, \end{cases}$$

where $H_U(q_1, q_2)$ is the set of all nspc.f.d curves contained in U and joining q_1 and q_2.

A nspc. curve is said to be a maximizer if it realizes the distance between its endpoints. We also use the name U-geodesic for a curve in U whose each suitably short sub-arc is a U-maximizer.

A distribution $D \subset TM$ is called bracket generating if any local frame $\{X_i\}_{1 \leq i \leq r}$ for D, together with all of its iterated Lie brackets $[X_i, X_j], [X_i, [X_j, X_k]], \cdots$ span the tangent bundle TM. Bracket generating distributions are sometimes also called completely nonholonomic distributions, or distributions satisfying Hörmander's condition.

Theorem 2.1 (Chow) *Fix a point $q \in M$. If the distribution $D \subset TM$ is bracket generating, then the set of points that can be connected to q by a horizontal curve is the component of M containing q.*

By Chow's Theorem, we know that if D is bracket generating and M is connected, then any two points of M can be joined by a horizontal curve.

Now, we describe the Engel group E. We consider the Engel group E with coordinates $q = (x_1, x_2, y, z) \in \mathbb{R}^4$. The group law is denoted by $\odot$ and defined as follows:

$$\begin{aligned} & (x_1, x_2, y, z) \odot (x_1', x_2', y', z') \\ & = \left(x_1 + x_1', x_2 + x_2', y + y' + \frac{x_1 x_2' - x_1' x_2}{2}, z + z' + \frac{x_2 x_2'}{2}(x_2 + x_2') + x_1 y' + \frac{x_1 x_2'}{2}(x_1 + x_1') \right). \end{aligned}$$

A vector field X is said to be left invariant if it satisfies $dL_q X(e) = X(q)$, where L_q denotes the left translation $p \to L_q(p) = q \odot p$ and e is the identity of E. This definition implies that any left invariant vector field on E is a linear combination of the following vector fields:

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial y}; \quad X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial y} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial z};$$
$$X_3 = \frac{\partial}{\partial y} + x_1\frac{\partial}{\partial z}; \quad X_4 = \frac{\partial}{\partial z}. \tag{2.2}$$

The distribution $D = \text{span}\{X_1, X_2\}$ of E satisfies the bracket generating condition, since $X_3 = [X_1, X_2], X_4 = [X_1, X_3]$. The Engel group is a nilpotent Lie group, since $[X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0$. We define a smooth Lorentzian metric $\tilde{g}$ on E such that $\tilde{g}(X_i, X_j) = (-1)^{\delta_{1i}}\delta_{ij}$, $i, j = 1, \cdots, 4$, where δ_{ij} is the Kronecker symbol. It is not difficult to compute the coefficients of $\tilde{g}$ under the local coordinates $(x_1, x_2, y, z) \in \mathbb{R}^4$. The coefficients can be expressed as

$$(\tilde{g}_{ij}) = \begin{pmatrix} -1 + \frac{x_2^2}{4} + \frac{x_1^2x_2^2}{4} & -\frac{x_1x_2}{4} + \frac{x_1x_2^3}{4} & \frac{x_2}{2} + \frac{x_2x_1^2}{2} & -\frac{x_1x_2}{2} \\ -\frac{x_1x_2}{4} + \frac{x_1x_2^3}{4} & 1 + \frac{x_1^2}{4} + \frac{x_2^4}{4} & -\frac{x_1}{2} + \frac{x_1x_2^2}{2} & -\frac{x_2^2}{2} \\ \frac{x_2}{2} + \frac{x_2x_1^2}{2} & -\frac{x_1}{2} + \frac{x_1x_2^2}{2} & 1 + x_1^2 & -x_1 \\ -\frac{x_1x_2}{2} & -\frac{x_2^2}{2} & -x_1 & 1 \end{pmatrix} \tag{2.3}$$

When we restrict $\tilde{g}$ to D, we can get a smooth sub-Lorentzian metric $g = \tilde{g}_D$, which satisfies

$$g(X_1, X_1) = -1, \quad g(X_2, X_2) = 1, \quad g(X_1, X_2) = 0. \tag{2.4}$$

On the other hand, any sub-Lorentzian metric on D can be extended to a (usually not unique) Lorentzian metric on E. In this paper, we assume that X_1 is the time orientation.

3 Horizontal Curves

Chow's theorem states that any two points can be connected by a horizontal curve, but we have no information about the character of horizontal curves. In this section, we will investigate some properties of horizontal curves.

An absolutely continuous curve $\gamma(s) : [0, 1] \to E$ is said to be horizontal if the tangent vector $\dot{\gamma}(s)$ can be expressed linearly by the horizontal directions X_1, X_2; hence, we have the following lemma.

Lemma 3.1 *A curve $\gamma(s) = (x_1(s), x_2(s), y(s), z(s))$ is horizontal with respect to the distribution D, if and only if*

$$\frac{x_2\dot{x_1}}{2} - \frac{x_1\dot{x_2}}{2} + \dot{y} = 0,$$
$$-\frac{x_1^2 + x_2^2}{2}\dot{x}_2 + \dot{z} = 0. \tag{3.1}$$

Proof The distribution D is the annihilator of the one-forms:

$$\omega_1 = \frac{x_2}{2}dx_1 - \frac{x_1}{2}dx_2 + dy, \quad \omega_2 = -\frac{x_1^2 + x_2^2}{2}dx_2 + dz$$

so $\gamma(s)$ is horizontal if and only if (3.1) holds. □

By the same method, we can easily calculate the left invariant coordinates $u_1(s)$ and $u_2(s)$ of the horizontal curve $\gamma(s)$:

$$u_1 = \dot{x_1}, \quad u_2 = \dot{x_2}. \tag{3.2}$$

The square of the velocity vector for the horizontal curve is as follows:

$$g(\dot{\gamma}, \dot{\gamma}) = -u_1^2 + u_2^2 = -\dot{x}_1^2 + \dot{x}_2^2. \tag{3.3}$$

So whether a horizontal curve is time-like (or nspc.) is determined by the sign of $-\dot{x}_1^2 + \dot{x}_2^2$.

Next, we present a left invariant property of horizontal curves in a Lie group with sub-Lorentzian metric. That is to say, the causal character (time-like, space-like, light-like, or non-space-like) of horizontal curves will not change under left translations. Hence, it is also true for the Engel group.

Let us consider a left invariant sub-Lorentzian structure on a Lie group G: $\mathscr{D} = \text{span}(X_1, X_2, \cdots, X_k) \subset TG$, $g(X_i, X_j) = (-1)^{\delta_{1i}}\delta_{ij}$, with a time orientation X_1. The vector fields X_i are assumed to be left invariant, i.e.,

$$L_{x*}X_i(q) = X_i(x \cdot q), \quad x, q \in G, \quad i = 1, \cdots, k.$$

Proposition 3.2 *Left translations preserve the causal character of horizontal curves of a left invariant sub-Lorentzian structure on a Lie group G, and the property of future-directness is also preserved.*

Proof Let $c(t)$ be a causal horizontal curve, and

$$\dot{c}(t) = \sum_{i=1}^{k} u_i(t) X_i(c(t)).$$

Then, the left translation $\gamma(t) = x \odot c(t)$ has the same causal character, since

$$\begin{aligned}\dot{\gamma}(t) &= L_{x*}\dot{c}(t) = L_{x*}\left(\sum_{i=1}^{k} u_i(t) X_i(c(t))\right) = \sum_{i=1}^{k} u_i(t) L_{x*}(X_i(c(t))) \\ &= \sum_{i=1}^{k} u_i(t) X_i(x \odot c(t)) = \sum_{i=1}^{k} u_i(t) X_i(\gamma(t)).\end{aligned}$$

Therefore,

$$\begin{aligned}&g(\dot{c}(t), \dot{c}(t)) = \sum_{i=1}^{k} (-1)^{\delta_{i1}} u_i^2 = g(\dot{\gamma}(t), \dot{\gamma}(t)), \\ &g(\dot{c}(t), X_1) = -u_1 = g(\dot{\gamma}(t), X_1).\end{aligned}$$

□

By Chow's Theorem, we know that any two points on the Engel group can be connected by a horizontal curve. But we do not know its causal character. Next, we will present some particular examples to show its complexity.

Example 1 Let $\dot{x}_2 = 0$. Then, $x_2 = x_2^0$ is constant. The horizontal condition (3.1) becomes

$$\frac{x_2}{2}\dot{x_1} + \dot{y} = 0, \tag{3.4}$$

$$\dot{z} = 0. \tag{3.5}$$

Then, therefore, the square of the velocity vector

$$-u_1^2+u_2^2=-\dot{x}_1^2\le 0. \tag{3.6}$$

It follows that, the curves satisfying (3.4) and (3.5) are all non-space-like curves. Furthermore, we obtain,

$$y(s)=-\frac{1}{2}x_2^0x_1(s)+\frac{1}{2}x_2^0x_1^0+y^0,\quad z(s)=z^0. \tag{3.7}$$

Therefore, all nonconstant horizontal curves $c(s)=\left(x_1(s),x_2^0,-\frac{x_1(s)x_2^0}{2}+\frac{x_1^0x_2^0}{2}+y^0,z^0\right)$ are time-like. These curves are straight lines. If $\dot{x}_1=0$, $c(s)$ degenerate into some points, so there are no null curves in this family.

Example 2 Let $\dot{x}_2\neq 0$. We choose x_2 as a parameter, then the horizontal condition (3.1) becomes

$$\frac{x_2}{2}\dot{x}_1-\frac{x_1}{2}+\dot{y}=0, \tag{3.8}$$

$$-\frac{x_1^2+x_2^2}{2}+\dot{z}=0. \tag{3.9}$$

And the square of the velocity vector

$$-u_1^2+u_2^2=-\dot{x}_1^2+1. \tag{3.10}$$

We consider three different cases.

(a) If $\dot{x}_1=0$, then $x_1=x_1^0$ is constant, (3.8) and (3.9) become

$$-\frac{x_1^0}{2}+\dot{y}=0, \tag{3.11}$$

$$-\frac{\left(x_1^0\right)^2+x_2^2}{2}+\dot{z}=0. \tag{3.12}$$

In this case, $|\dot{c}(s)|^2=1$, so the curves satisfying (3.11) and (3.12) are all space-like. Furthermore, we obtain,

$$y(s)=\frac{x_1^0}{2}x_2+y^0,\quad z(s)=\frac{1}{6}x_2^3+\frac{\left(x_1^0\right)^2}{2}x_2+z_0. \tag{3.13}$$

Therefore, all nonconstant horizontal curves $c(s)=\left(x_1^0,x_2,\frac{x_1^0}{2}x_2+y^0,\frac{1}{6}x_2^3+\frac{\left(x_1^0\right)^2}{2}x_2+z_0\right)$ are space-like. There are no null or time-like horizontal curves in this family.

(b) If $\dot{y}=0$, Eqs. 3.8 and 3.9 become

$$x_2\dot{x}_1-x_1=0, \tag{3.14}$$

$$-\frac{x_1^2+x_2^2}{2}+\dot{z}=0. \tag{3.15}$$

From Eq. 3.14, we get

$$\frac{1}{x_2}=\frac{\dot{x}_1}{x_1},$$

integrating with respect to x_2, we calculate $x_1 = \iota x_2$, where $\iota = \frac{x_1^0}{x_2^0}$, i.e., $x_1 = \frac{x_1^0}{x_2^0} x_2$, substituting x_1 in Eq. 3.15, we obtain

$$z = \frac{1}{6}\left(1 + \iota^2\right) x_2^3 + z^0. \tag{3.16}$$

Therefore, all nonconstant horizontal curves

$$c(s) = \left(\iota x_2, x_2, y^0, \frac{1}{6}\left(1 + \iota^2\right) x_2^3 + z^0\right) \tag{3.17}$$

are time-like when $|\iota| > 1$. If $|\iota| < 1$ $(= 1)$, they are space-like (null).

(c) If $\dot{z} = 0$, the horizontal condition becomes:

$$\frac{x_2}{2}\dot{x}_1 - \frac{x_1}{2} + \dot{y} = 0, \tag{3.18}$$

$$-\frac{x_1^2 + x_2^2}{2} = 0. \tag{3.19}$$

So $x_1 = x_2 = 0$, $y = y_0$. The curves degenerate into some points. There are no causal (time-like, space-like, null) horizontal curves in this family.

Thus, any two points $P_1\left(x_1^0, x_2^0, y^0, z^0\right)$, $Q_1\left(x_1, x_2^0, y^1, z^0\right)$ can be connected by a time-like horizontal curve if $y^1 = -\frac{x_1 x_2^0}{2} + \frac{x_1^0 x_2^0}{2} + y^0$. Especially, any two points $\left(x_1^0, 0, y^0, z^0\right)$, $\left(x_1, 0, y^0, z^0\right)$ can be connected by a time-like horizontal straight line.

Any two points $P_1\left(x_1^0, x_2^0, y^0, z^0\right)$, $Q_2\left(x_1^0, x_2, y^1, z^1\right)$ can be connected by a space-like horizontal curve if $y^1 = \frac{x_1^0}{2} x_2 + y^0$, $z^1 = \frac{1}{6} x_2^3 + \frac{\left(x_1^0\right)^2}{2} x_2 + z_0$.

Any two points $P_1\left(x_1^0, x_2^0, y^0, z^0\right)$, $Q_3\left(x_1, x_2, y^0, z^1\right)$ can be connected by a time-like (space-like, null) horizontal curve if $x_1 = \iota x_2$, $z^1 = \frac{1}{6}\left(1 + \iota^2\right) x_2^3 + z^0$, and $|\ \iota\ | = \left|\frac{x_1^0}{x_2^0}\right| > 1 (< 1,\ = 1)$.

4 Sub-Lorentzian Geodesics

In the Lorentzian geometry, there are no curves of minimal length because two arbitrary points can be connected by a piecewise light-like curve whose length is always 0. For example, let $\mathbb{R}^2$ be the two-dimensional Minkowski space, $\hat{p} = (\hat{x}, \hat{y})$ is any one point in this space. We want to find a light-like curve going from the origin to $\hat{p}$. First, we choose a curve $\gamma_1(t) : (x(t), y(t)) = (t, t)$ which connects the origin and the point $\left(\frac{\hat{x}+\hat{y}}{2}, \frac{\hat{x}+\hat{y}}{2}\right)$; then, we choose the second curve $\gamma_2(t) : (x(t), y(t)) = (t, -t + \hat{x} + \hat{y})$ which joins $\left(\frac{\hat{x}+\hat{y}}{2}, \frac{\hat{x}+\hat{y}}{2}\right)$ and $\hat{p}$. It is easy to check that the curve $\gamma(t)$ consisting of γ_1 and γ_2 is a light-like curve. It goes from the origin to the point $\hat{p}$, and the length is 0. However, there do exist time-like curves with maximal length which are time-like geodesics [17]. For this reason, we will study the optimality of time-like geodesics and compute the longest curve among all horizontal time-like ones on the sub-Lorentzian Engel group. The computation will be given by extremizing the action integral $S = \frac{1}{2}\int\left(-u_1^2 + u_2^2\right) dt$ under constraint (3.1). By Proposition 3.2, horizontal time-like curves are left invariant, so we can assume that the initial point is origin, i.e., $x_1(0) = x_2(0) = y(0) = z(0) = 0$, and time-like initial velocity is $-u_1^2(0) + u_2^2(0) = -1$.

Let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ be the vector of costate variables, so the Hamiltonian function of Pontryagin's maximum principle is

$$H(\xi_0, \xi, q, u) = \xi_0 \frac{-u_1^2 + u_2^2}{2} + \xi_1 u_1 + \xi_2 u_2 + \xi_3 \frac{x_1 u_2 - x_2 u_1}{2} + \xi_4 \frac{x_1^2 + x_2^2}{2} u_2. \quad (4.1)$$

where ξ_0 is a constant which equals to 0 or -1. Also, we get the Hamiltonian system:

$$\dot{x}_1 = H_{\xi_1} = u_1, \quad \dot{x}_2 = H_{\xi_2} = u_2, \quad \dot{y} = H_{\xi_3} = \frac{x_1 u_2 - x_2 u_1}{2}, \quad \dot{z} = H_{\xi_4} = \frac{x_1^2 + x_2^2}{2} u_2,$$
$$\dot{\xi}_1 = -H_{x_1} = -\frac{\xi_3 u_2}{2} - \xi_4 x_1 u_2, \quad \dot{\xi}_2 = -H_{x_2} = \frac{\xi_3 u_1}{2} - \xi_4 x_2 u_2, \quad \dot{\xi}_3 = \dot{\xi}_4 = 0, \quad (4.2)$$

and the maximum condition:

$$H(\xi_0, \xi(t), q(t), u(t)) = \max_{\tilde{u} \in \mathbb{R}^2} H(\xi_0, \xi(t), \tilde{q}(t), \tilde{u}), \quad \xi_0 \leq 0, \quad (4.3)$$

where $u(t)$ is the optimal control, and $(\xi_0, \xi(t)) \neq 0$.

4.1 Abnormal Extremal Trajectories

We shall investigate the abnormal case $\xi_0 = 0$. From the maximum condition (4.3), we obtain

$$H_{u_1} = \xi_1 - \frac{\xi_3 x_2}{2} = 0, \quad (4.4)$$
$$H_{u_2} = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4 (x_1^2 + x_2^2)}{2} = 0. \quad (4.5)$$

Differentiating Eqs. 4.4 and 4.5, we obtain

$$0 = \dot{\xi}_1 - \frac{\xi_3 \dot{x}_2}{2} = \dot{\xi}_1 - \frac{\xi_3 u_2}{2} = -u_2(\xi_3 + \xi_4 x_1), \quad (4.6)$$
$$0 = \dot{\xi}_2 + \frac{\xi_3 \dot{x}_1}{2} + \xi_4 (x_1 \dot{x}_1 + x_2 \dot{x}_2) = u_1(\xi_3 + \xi_4 x_1). \quad (4.7)$$

For a time-like curve, we assume that $-u_1^2 + u_2^2 = -1$, so $\xi_3 + \xi_4 x_1 = 0$. If $\xi_4 = 0$, then, $\xi_3 = 0$, and therefore, $\xi = 0$. It is a contradiction with the nontriviality of the costate variables; hence, $\xi_4 \neq 0$. In this case, $x_1 = \frac{-\xi_3}{\xi_4}$ is a constant, and $u_1 = 0$, $u_2 = \pm i$, so there is no time-like abnormal extremal in the Engel group E.

For a space-like curve, we assume that $-u_1^2 + u_2^2 = 1$, by using the same method, we get that $u_1 = 0$, $u_2 = \pm 1$, so the space-like abnormal extremal trajectories are given by the following expression:

$$\gamma(s) = \left(0, \pm s, 0, \pm \frac{s^3}{6}\right). \quad (4.8)$$

For a null curve, suppose that $-u_1^2 + u_2^2 = 0$, we can easily get that $u_1 = 0$, $u_2 = 0$, so the null abnormal extremal trajectories are trivial curves.

4.2 Normal Geodesics

4.2.1 Normal Hamiltonian System

Now, we look at the normal case $\xi_0 = -1$. It follows from the maximum condition (4.3) that $H_{u_1} = H_{u_2} = 0$. Hence,

$$u_1 = -\left(\xi_1 - \frac{x_2\xi_3}{2}\right), \quad u_2 = \xi_2 + \frac{\xi_3 x_1}{2} + \frac{\xi_4(x_1^2 + x_2^2)}{2}. \tag{4.9}$$

Let $\zeta_i = (\xi, X_i)$, $i = 1, 2$, be the Hamiltonian corresponding to the basis vector fields X_1, X_2 in the cotangent space T_q^*E. They are linear on the fibers of the cotangent space T^*E, and

$$\zeta_1 = \xi_1 - \frac{x_2}{2}\xi_3, \quad \zeta_2 = \xi_2 + \frac{x_1}{2}\xi_3 + \frac{x_1^2 + x_2^2}{2}\xi_4. \tag{4.10}$$

So $u_1 = -\zeta_1$ and $u_2 = \zeta_2$.

The Hamiltonian system in the normal case becomes:

$$\begin{cases} \dot{x}_1 = \frac{\partial H}{\partial \xi_1} = -(\xi_1 - \frac{x_2}{2}\xi_3) = -\zeta_1, \\ \dot{x}_2 = \frac{\partial H}{\partial \xi_2} = \left(\xi_2 + \frac{x_1}{2}\xi_3 + \frac{x_1^2+x_2^2}{2}\xi_4\right) = \zeta_2, \\ \dot{y} = \frac{\partial H}{\partial \xi_3} = \zeta_1\frac{x_2}{2} + \zeta_2\frac{x_1}{2} = \frac{1}{2}(x_1\zeta_2 + x_2\zeta_1), \\ \dot{z} = \frac{\partial H}{\partial \xi_4} = \frac{x_1^2+x_2^2}{2}\zeta_2, \\ \dot{\xi}_1 = -\frac{\partial H}{\partial x_1} = -\zeta_2\left(\frac{\xi_3}{2} + x_1\xi_4\right), \\ \dot{\xi}_2 = -\frac{\partial H}{\partial x_2} = -\frac{1}{2}\xi_3\zeta_1 - x_2\xi_4\zeta_2, \\ \dot{\xi}_3 = -\frac{\partial H}{\partial y} = 0, \\ \dot{\xi}_4 = -\frac{\partial H}{\partial z} = 0. \end{cases} \tag{4.11}$$

Definition 4.1 A normal geodesic in the sub-Lorentzian manifold (E, D, g) is a curve $\gamma : [a, b] \to E$ that admits a lift $\Gamma : [a, b] \to T^*M$, which is a solution of the Hamiltonian equations (4.11). In this case, we say that Γ is a normal lift of γ.

Associate with the expression of H, a sub-Lorentzian geodesic is time-like if $H < 0$; space-like if $H > 0$; light-like if $H = 0$.

Remark 4.1 In fact, abnormal extremal trajectories (4.8) are also normal geodesics, since we can choose the costate variables as $\widetilde{\xi} = (0, \pm 1, 0, 0)$; it is easy to check that $\Gamma(t) = (\gamma, \widetilde{\xi})$ satisfies Hamiltonian equation (4.11). This example also confirms that normal geodesics and abnormal trajectories are sometimes not mutually exclusive.

Lemma 4.2 *The causal character of normal sub-Lorentzian geodesics does not depend on time.*

Proof The Hamiltonian H is an integral of the Hamiltonian system, i.e., $\dot{H}(s) = 0$; this implies that the causality character does not change for all $t \in [0, \infty)$. □

Remark 4.3 If $\gamma(t)$ is a nspc. normal geodesic on the Engel group, then the orientation will not change along the curve. In fact, if $\gamma(t)$ is time-like, and it is future directed at $t = 0$, then, we have $-u_1^2(t)+u_2^2(t) = -1$, $u_1(0) > 0$. We only need to show that $u_1(t)$ will not be equal to 0 along the curve $\gamma(t)$. Actually, if there is a $t_1 > 0$, such that $u_1(t_1) = 0$, then we have $u_2^2(t_1) = -1$; it is impossible. So, $u_1(t)$ will not change the sign (since $u_1(t) = -\zeta_1(t)$ is continuous), and $\gamma(t)$ is future directed along the curve. It is also true for the other cases.

Differentiating ζ_i,

$$\dot{\zeta}_1 = \dot{\xi}_1 - \frac{\xi_3}{2}\dot{x}_2 = -\zeta_2(\xi_3 + x_1\xi_4), \tag{4.12}$$

$$\dot{\zeta}_2 = \dot{\xi}_2 + \frac{1}{2}\dot{x}_1\xi_3 + (x_1\dot{x}_1 + x_2\dot{x}_2)\xi_4 = -\zeta_1(\xi_3 + x_1\xi_4). \tag{4.13}$$

Let

$$\beta(s) = -(\xi_3 + x_1\xi_4), \ \ \dot{\beta} = \xi_4\zeta_1, \tag{4.14}$$

then, we have

$$\dot{\zeta}_1 = \beta\zeta_2, \ \ \dot{\zeta}_2 = \beta\zeta_1, \ \ \dot{\beta} = \xi_4\zeta_1. \tag{4.15}$$

4.2.2 Maximality of Short Arcs of Geodesics

Definition 4.2 Let φ be a smooth function on M, and U an open subset in M. The horizontal gradient $\nabla_H\varphi$ of φ is a smooth horizontal vector field on U such that for each $p \in U$ and $v \in H$, $\partial_v\varphi(p) = g(\nabla_H\varphi(p), v)$.

Locally, we can write

$$\nabla_H\varphi = -(\partial_{X_1}\varphi)X_1 + \sum_{i=2}^{r}(\partial_{X_i}\varphi)X_i.$$

Now, we give a proof that the time-like normal geodesics are locally maximizing curves on the Engel group.

Proposition 4.4 *If γ is a t.f.d. (t.p.d.) normal geodesic on the Engel group, then every sufficiently short sub-arc of γ is a maximizer.*

Proof Assume that $\gamma : (a, b) \to E$ is parameterized by arc-length, $\dot{\gamma}(t) = u_1^0(t)X_1(\gamma(t)) + u_2^0(t)X_2(\gamma(t))$, X_1 is the time orientation, and $\tilde{\Gamma}(t) = (\gamma(t), \lambda(t))$ is the normal lift of γ. So, we have $H(\gamma(t), \lambda(t)) = -\frac{1}{2}$, $t \in (a, b)$. For any $c \in (a, b), \epsilon > 0$, let $J_c = (c - \epsilon, c + \epsilon) \subset (a, b)$ be a neighborhood of c. We will prove that $\gamma|_{J_c}$ is maximal for any $c \in (a, b)$ and small $\epsilon > 0$. Since the sub-Lorentzian metric is left invariant, so we can assume that $\gamma(c) = 0$, $\lambda(c) = \lambda_0$. Consider an three-dimensional hypersurface S passing through the origin 0, and satisfying $\lambda_0(T_0(S)) = 0$. Let $\bar{\lambda}$ be a smooth one-form on an open neighborhood Ω of 0, such that $\bar{\lambda}(0) = \lambda_0$, and $\forall p \in S \cap \Omega$, $\bar{\lambda}(p)(T_pS) = 0$, $H(p, \bar{\lambda}(p)) = -\frac{1}{2}$. Let $\Gamma_p = (\gamma_p, \lambda_p)$ be the solution of $\dot{\Gamma}(t) = \vec{H}(\Gamma(t))$, $\Gamma(c) = (p, \bar{\lambda}(p))$. Then, clearly $\Gamma_0 = \tilde{\Gamma}$. Since $\dot{\gamma}(0) \notin T_0S$, by the Implicit Function Theorem, there exits a diffeomorphism:

$$\nu : (c - \epsilon, c + \epsilon) \times W \to U \subset E,$$

$$(t, p) \to \gamma_p(t),$$

where W is a neighborhood of 0 in S, $U \subset \Omega$ is a neighborhood of 0 in E. Define a smooth function $V : U \to \mathbb{R}$ as follows:

$$V(x) = t, \quad if\ x = \gamma_p(t),$$

we will show that $\|\nabla_H V\| = 1$. For this purpose, let Y_1 be the vector field on U given by

$$Y_1(x) = \dot{\gamma}_p(t) = u_1(p,t)X_1(\gamma_p(t)) + u_2(p,t)X_2(\gamma_p(t)), \quad if\ x = \gamma_p(t),$$

where $u_1(p,t), u_2(p,t)$ are smooth functions on $W \times (c-\epsilon, c+\epsilon)$, and $u_1(0,t) = u_1^0(t), u_2(0,t) = u_2^0(t)$. Since $H(p, \bar{\lambda}(p)) = -\frac{1}{2}$, by the construction of $\Gamma_p(t)$, we have $H(\gamma_p(t), \lambda_p(t)) = -\frac{1}{2}$, and $-u_1^2 + u_2^2 = -1$. It is easy to check that $Y_1 = u_1X_1 + u_2X_2, Y_2 = u_2X_1 + u_1X_2$ is also an orthonormal basis of D, so $\partial_{Y_1}V = 1,\ \partial_{Y_2}V = 0$. Therefore, $\nabla_H V = -Y_1,\ \|\nabla_H V\| = \sqrt{|g(-Y_1,-Y_1)|} = \sqrt{|-u_1^2+u_2^2|} = 1$. Choose t_1, t_2 in the domain of γ. If $\gamma(t)$ is a t.f.d. geodesic, then $|u_1^0| > |u_2^0|$, and $u_1^0 > 0$. Since $u_1(0,t) = u_1^0$, and $u_1(p,t)$ is a smooth function, so there exists a neighborhood $W_1 \times (c-\epsilon_1, c+\epsilon_1) \subset W \times (c-\epsilon, c+\epsilon)$ such that $u_1(p,t) > 0$. Thus, $\nabla_H V = -Y_1$ is past directed. On the other hand, since $-u_1^2 + u_2^2 = -1$, we have $|u_1| > |u_2|$. Let $\eta : [0,\alpha] \to U$ be a t.f.d. curve with $\eta(0) = \gamma(t_1), \eta(\alpha) = \gamma(t_2)$, and $\dot{\eta} = v_1X_1 + v_2X_2$, then $|v_1| > |v_2|, v_1 > 0$, so $g(\dot{\eta}, \nabla_H V) = u_1v_1 - u_2v_2 > 0$, and

$$\begin{aligned} L(\gamma|_{[t_1,t_2]}) &= t_2 - t_1 = V(\gamma(t_2)) - V(\gamma(t_1)) = \int_0^\alpha \frac{dV(\eta(s))}{ds}ds \\ &= \int_0^\alpha g(\dot{\eta}, \nabla_H V)ds \geq \int_0^\alpha \|\dot{\eta}(s)\|ds = L(\eta|_{[0,\alpha]}). \end{aligned}$$

By the reverse Schwartz inequality, $L(\gamma) = L(\eta)$ holds if and only if η can be reparameterized as a trajectory of $-\nabla_H V$. If $\gamma(t)$ is a t.p.d. geodesic, then $|u_1^0| > |u_2^0|$, and $u_1^0 < 0$. By the same method, we choose a neighborhood $W_2 \times (c-\epsilon_2, c+\epsilon_2) \subset W \times (c-\epsilon, c+\epsilon)$ such that $u_1(p,t) < 0$. Thus, $\nabla_H V = -Y_1$ is future directed. Let $\rho : [0,\alpha] \to U$ be a t.p.d. curve with $\rho(0) = \gamma(t_1), \rho(\alpha) = \gamma(t_2)$, and $\dot{\rho} = \mu_1X_1 + \mu_2X_2$, then $|\mu_1| > |\mu_2|, \mu_1 < 0$, so $g(\dot{\rho}, \nabla_H V) = u_1\mu_1 - u_2\mu_2 > 0$, and

$$\begin{aligned} L(\gamma|_{[t_1,t_2]}) &= t_2 - t_1 = V(\gamma(t_2)) - V(\gamma(t_1)) = \int_0^\alpha \frac{dV(\eta(s))}{ds}ds \\ &= \int_0^\alpha g(\dot{\rho}, \nabla_H V)ds \geq \int_0^\alpha \|\dot{\rho}(s)\|ds = L(\rho|_{[0,\alpha]}). \end{aligned}$$

By the reverse Schwartz inequality, $L(\gamma) = L(\rho)$ holds if and only if ρ can be reparameterized as a trajectory of $-\nabla_H V$. In conclusion, the t.f.d (t.p.d.) normal geodesics are locally maximizers. This ends the proof. □

4.2.3 Light-Like Geodesics

Next, we compute the expressions of light-like geodesics and time-like geodesics on the Engel group. Firstly, we study the case of light-like sub-Lorentzian geodesics.

By the definition, we have $H = \frac{1}{2}\left(-\zeta_1^2 + \zeta_2^2\right) = 0$, thus $\zeta_2 = \pm\zeta_1$. If $\zeta_2 = \zeta_1$, then light-like trajectories satisfy the ODE:

$$\dot{\gamma} = -\zeta_1(X_1 - X_2),$$

i.e., they are reparameterizations of the one-parametric subgroup of the field $X_1 - X_2$. We assume $\dot{\gamma} = X_1 - X_2$, so

$$\dot{x}_1 = 1,\ \dot{x}_2 = -1,\ \dot{y} = -\frac{1}{2}(x_1 + x_2),\ \dot{z} = -\frac{1}{2}\left(x_1^2 + x_2^2\right),$$

thus,

$$x_1 = t,\ x_2 = -t,\ y = 0,\ z = -\frac{1}{3}t^3.$$

If $h_2 = -h_1$, similarly, we obtain

$$x_1 = t,\ x_2 = t,\ y = 0,\ z = \frac{1}{3}t^3.$$

In conclusion, we get the following theorem:

Theorem 4.5 *Light-like horizontal geodesics starting from the origin are reparameterizations of the curves:*

$$x_1 = t,\ x_2 = \pm t,\ y = 0,\ z = \pm\frac{1}{3}t^3,$$

i.e., they are reparameterizations of the one-parameter subgroups corresponding to the vector fields $X_1 \pm X_2$.

4.2.4 Time-Like Geodesics

Secondly, we study time-like sub-Lorentzian geodesics on the Engel group.

We consider the case of $\xi_4 = 0$ at first. This case is also of interest since it reproduces the earlier known results for the Heisenberg group [18]. In this case, $\beta = -(\xi_3 + x_1\xi_4) = -\xi_3$ is a constant. Equations 4.15 become

$$\dot{\zeta}_1 = -\xi_3\zeta_2,\ \ \dot{\zeta}_2 = -\xi_3\zeta_1, \tag{4.16}$$

where ξ_3 is a constant. There are two separate cases:

Case 1 If $\xi_3 = 0$, we have ζ_1 and ζ_2 are constants, i.e., $\zeta_1(s) = \zeta_1(0) = \xi_1(0)$ and $\zeta_2(s) = \zeta_2(0) = \xi_2(0)$. According to Eq. 4.11, ξ_1 and ξ_2 are constants. On the other hand, by integrating $\dot{x}_1 = -\zeta_1$ and $\dot{x}_2 = \zeta_2$, we get

$$x_1(s) = -\xi_1 s \quad \text{and} \quad x_2(s) = \xi_2 s. \tag{4.17}$$

Since $\dot{y} = \frac{1}{2}(x_1\zeta_2 + x_2\zeta_1) = 0$, then $y(s) = 0$. Also

$$\dot{z} = \frac{x_1^2 + x_2^2}{2}\zeta_2 = \frac{\xi_1^2 + \xi_2^2}{2}\xi_2 s^2,$$

so

$$z(s) = \frac{\xi_1^2 + \xi_2^2}{6}\xi_2 s^3 = \frac{x_1^2(s)x_2(s) + x_2^3(s)}{6}.$$

Theorem 4.6 *In the case of $\xi_3 = \xi_4 = 0$, there is a unique time-like horizontal geodesic joining the origin to a point (x_1, x_2, y, z), if and only if $y = 0$, z is the following function of x_1, x_2:*

$$z = \frac{x_1^2 x_2 + x_2^3}{6}. \tag{4.18}$$

The expression of the geodesic is

$$x_1(s) = -\xi_1 s,\ \ x_2(s) = \xi_2 s,\ \ y(s) = 0,\ \ z(s) = \frac{\xi_1^2 + \xi_2^2}{6}\xi_2 s^3, \tag{4.19}$$

where ξ_1, ξ_2 are constants. The arc-length is given by

$$l = \sqrt{x_1^2 - x_2^2}. \tag{4.20}$$

Its projection to the (x_1, x_2) *plane is a straight line.*

Case 2 If $\xi_3 \neq 0$, from Eq. 4.16, we have

$$\zeta_1(s) = \xi_1^0 \cosh(\xi_3 s) - \xi_2^0 \sinh(\xi_3 s), \tag{4.21}$$

$$\zeta_2(s) = -\xi_1^0 \sinh(\xi_3 s) + \xi_2^0 \cosh(\xi_3 s), \tag{4.22}$$

where $\xi_1^0 = \xi_1(0)$, $\xi_2^0 = \xi_2(0)$. So,

$$x_1 = -\int_0^s \zeta_1(t)dt = -\frac{\xi_1^0}{\xi_3}\sinh(\xi_3 s) + \frac{\xi_2^0}{\xi_3}\left(\cosh(\xi_3 s) - 1\right), \tag{4.23}$$

$$x_2 = \int_0^t \zeta_2(t)dt = -\frac{\xi_1^0}{\xi_3}\left(\cosh(\xi_3 s) - 1\right) + \frac{\xi_2^0}{\xi_3}\sinh(\xi_3 s). \tag{4.24}$$

Substituting them into the expression of $\dot{y}, \dot{z}$ in Eq. 4.11, and integrating, we get

Theorem 4.7 *In the case of* $\xi_3 \neq 0$, $\xi_4 = 0$, *the time-like horizontal geodesics starting from the origin are given by the following:*

$$x_1(s) = -A_1 \sinh(\xi_3 s) + A_2\left(\cosh(\xi_3 s) - 1\right), \tag{4.25}$$

$$x_2(s) = -A_1\left(\cosh(\xi_3 s) - 1\right) + A_2 \sinh(\xi_3 s), \tag{4.26}$$

$$y(s) = \frac{1}{2}\left(A_2^2 - A_1^2\right)\left(\xi_3 s - \sinh(\xi_3 s)\right), \tag{4.27}$$

$$\begin{aligned} z(s) &= A_2\left(A_1^2 + A_2^2\right)\cosh^2(\xi_3 s)\sinh(\xi_3 s) - \frac{2}{3}A_2^3\sinh^3(\xi_3 s) - \frac{1}{3}A_1\left(A_1^2 + 3A_2^2\right)\cosh^3(\xi_3 s) \\ &+ \frac{1}{2}A_1\left(A_1^2 + 3A_2^2\right)\cosh^2(\xi_3 s) - \frac{1}{2}A_2\left(3A_1^2 + A_2^2\right)\sinh(\xi_3 s)\cosh(\xi_3 s) - \frac{1}{2}A_2\left(3A_1^2 + A_2^2\right)s \\ &- \frac{1}{6}A_1\left(A_1^2 + 3A_2^2\right). \end{aligned} \tag{4.28}$$

where $\xi_1^0 = \xi_1(0)$, $\xi_2^0 = \xi_2(0)$ *is the initial value,* ξ_3, $A_1 = \frac{\xi_1^0}{\xi_3}$, $A_2 = \frac{\xi_2^0}{\xi_3}$ *are constants.*

Projections of geodesics to the plane (x_1, x_2) are hyperbolas, for $\xi(0) = \left(\sqrt{2}, 1, 1, 0\right)$, $\xi(0) = \left(\frac{\sqrt{5}}{2}, \frac{1}{2}, 1, 0\right)$ and $\xi(0) = \left(\frac{\sqrt{5}}{2}, \frac{1}{2}, -1, 0\right)$, they are shown in Fig. 1.

From this theorem, we obtain a description of the reachable set by geodesics $\xi_3 \neq 0$, $\xi_4 = 0$ starting from the origin.

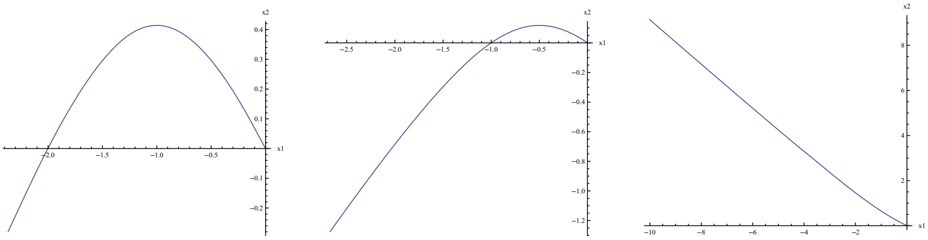

Fig. 1 Projections of geodesics to the plane (x_1, x_2) when $\xi_3 \neq 0$, $\xi_4 = 0$

Corollary 4.8 *In the case of* $\xi_3 \neq 0, \xi_4 = 0$*, let* (x_1, x_2, y, z) *be a point on a time-like geodesic, then we have*

$$-1 < \frac{4y}{-x_1^2 + x_2^2} < 1.$$

Proof By Eqs. 4.25 and 4.26, we get

$$-x_1^2 + x_2^2 = 4\left(A_2^2 - A_1^2\right)\sinh^2\left(\frac{\xi_3}{2}\right), \tag{4.29}$$

substituting Eq. 4.29 into Eq. 4.27, we obtain the following equation:

$$y = \frac{\left(-x_1^2 + x_2^2\right)(\xi_3 - \sinh(\xi_3))}{8\sinh^2\left(\frac{\xi_3}{2}\right)}, \tag{4.30}$$

if we set $\tau = \frac{\xi_3}{2}$, then

$$y = \frac{\left(-x_1^2 + x_2^2\right)}{4}\left(\frac{\tau}{\sinh^2(\tau)} - \coth(\tau)\right), \tag{4.31}$$

or

$$\frac{4y}{\left(-x_1^2 + x_2^2\right)} = \frac{\tau}{\sinh^2(\tau)} - \coth(\tau). \tag{4.32}$$

It is easy to check that the right-hand side of Eq. 4.32 is a decreasing function in $(-\infty, +\infty)$, and its range is $(-1, 1)$. That is to say, the points on the time-like geodesics should satisfy

$$-1 < \frac{4y}{-x_1^2 + x_2^2} < 1.$$

This ends the proof. □

Next, we consider the case $\xi_4 \neq 0$. Recall that

$$\dot{\zeta}_1 = \beta\zeta_2, \quad \dot{\zeta}_2 = \beta\zeta_1, \text{ where } \beta(s) = -(\xi_3 + x_1\xi_4), \; \dot{\beta} = \xi_4\zeta_1. \tag{4.33}$$

Combining the expressions for $\dot{\beta}$ and $\dot{\zeta}_2$ to get

$$\xi_4\dot{\zeta}_2 = \beta\xi_4\zeta_1 = \beta\dot{\beta}. \tag{4.34}$$

Integrating both sides, we have

$$\xi_4\zeta_2 = \frac{\beta^2}{2} + C_1, \text{ where } C_1 = \xi_4\zeta_2(0) - \frac{\beta^2(0)}{2} = \xi_4\xi_2^0 - \frac{\xi_3^2}{2}. \tag{4.35}$$

This yields

$$x_1(s) = -\frac{\beta(s) + \xi_3}{\xi_4}, \tag{4.36}$$

and

$$\zeta_2(s) = \frac{1}{\xi_4}\left(\frac{\beta^2(s)}{2} + C_1\right). \tag{4.37}$$

Since $\dot{x}_2 = \zeta_2$, we deduce

$$x_2(s) = \int_0^s \zeta_2(t)dt = \frac{1}{\xi_4}\int_0^s \left(\frac{\beta^2(t)}{2} + C_1\right)dt. \tag{4.38}$$

To compute $y(s)$ in term of $\beta(s)$, we note that

$$\dot{y} = \frac{1}{2}(x_1\zeta_2 + x_2\zeta_1) = \frac{1}{2}(x_1\dot{x}_2 - x_2\dot{x}_1), \tag{4.39}$$

then integration by parts yields

$$\begin{aligned} y(s) &= \frac{1}{2}\int_0^s (x_1\dot{x}_2 - x_2\dot{x}_1)dt = \int_0^s x_1\zeta_2 dt - \frac{1}{2}x_1x_2 \\ &= -\frac{1}{\xi_4^2}\int_0^s (\beta(t)+\xi_3)\left(\frac{\beta^2(t)}{2} + C_1\right)dt - \frac{1}{2}x_1x_2. \end{aligned} \tag{4.40}$$

Finally, since $\dot{z} = \frac{x_1^2+x_2^2}{2}\zeta_2$,

$$\begin{aligned} z(s) &= \int_0^s \frac{x_1^2+x_2^2}{2}\zeta_2 dt = \frac{1}{2}\int_0^s x_1^2\zeta_2 dt + \frac{1}{6}x_2^3 \\ &= \frac{1}{2\xi_4^3}\int_0^s (\beta(t)+\xi_3)^2\left(\frac{\beta^2(t)}{2} + C_1\right)dt + \frac{1}{6}x_2^3. \end{aligned} \tag{4.41}$$

Once we find β, we can find the geodesic $(x_1(s), x_2(s), y(s), z(s))$ explicitly.

Since $\dot{\beta}(s) = \xi_4\zeta_1$, $\dot{\beta}(0) = \xi_4\zeta_1(0) = \xi_4\xi_1^0$, we have

$$\ddot{\beta}(s) = \xi_4\dot{\zeta}_1 = \xi_4\beta(s)\zeta_2 = \beta(s)(\xi_4\zeta_2) = \beta(s)\left(\frac{\beta^2(s)}{2} + C_1\right). \tag{4.42}$$

Multiplying both sides by $2\dot{\beta}(s)$ and integrating, we have

$$\dot{\beta}^2(s) = \frac{\beta^4(s)}{4} + C_1\beta^2(s) + C_2 = \left(\frac{\beta^2(s)}{2} + C_1\right)^2 + C_2 - C_1^2, \tag{4.43}$$

where C_2 is a constant, and

$$C_2 = \dot{\beta}^2(0) - \frac{\beta^4(0)}{4} - C_1\beta^2(0) = (\xi_1^0)^2\xi_4^2 + \frac{\xi_3^4}{4} - \xi_2^0\xi_3^2\xi_4. \tag{4.44}$$

Then,

$$C_2 - C_1^2 = (\xi_1^0)^2\xi_4^2 + \frac{\xi_3^4}{4} - \xi_2^0\xi_3^2\xi_4 - \left(\xi_2^0\xi_4 - \frac{\xi_3^2}{2}\right)^2 = \xi_4^2((\xi_1^0)^2 - (\xi_2^0)^2) = \xi_4^2, \tag{4.45}$$

since $(\xi_1^0)^2 - (\xi_2^0)^2 = 1$.

Assume $\dot{\beta}(s) > 0$, we have

$$\frac{d\beta(s)}{ds} = \sqrt{\left(\frac{\beta^2(s)}{2} + C_1\right)^2 + \xi_4^2}. \tag{4.46}$$

Hence,

$$ds = \frac{d\beta}{\sqrt{\left(\frac{\beta^2(s)}{2} + C_1\right)^2 + \xi_4^2}}. \tag{4.47}$$

Let $\rho^2 = C_1 + \xi_4 i$, $\bar{\rho}^2 = C_1 - \xi_4 i$ and $u = \frac{\beta}{\sqrt{2}}$, integrating (4.47) from 0 to s, we obtain

$$s = \int_{\frac{\beta(0)}{\sqrt{2}}}^{\frac{\beta(s)}{\sqrt{2}}} \frac{\sqrt{2}du}{\sqrt{(u^2+\rho^2)(u^2+\bar{\rho}^2)}}. \tag{4.48}$$

Set

$$k^2 = -\frac{(\rho-\bar{\rho})^2}{4\rho\bar{\rho}} = \frac{\sqrt{C_1^2+\xi_4^2}-C_1}{2\sqrt{C_1^2+\xi_4^2}},$$

$$\mathfrak{g} = \frac{1}{2\sqrt{\rho\bar{\rho}}} = \frac{1}{2\left(C_1^2+\xi_4^2\right)^{\frac{1}{4}}}.$$

Since

$$\int_y^{\infty} \frac{dt}{\sqrt{(t^2+\rho^2)(t^2+\bar{\rho})}} = \mathfrak{g}\cdot cn^{-1}\,(cos\,\varphi, k) = \mathfrak{g}F(\varphi, k), \tag{4.49}$$

where $cn^{-1}\,(y, k)$ is a Jacobi's Inverse Elliptic Functions, and

$$\varphi = \cos^{-1}\left(\frac{y^2-\rho\bar{\rho}}{y^2+\rho\bar{\rho}}\right), \qquad F(\varphi, k) = \int_0^{\varphi}\frac{dt}{\sqrt{1-k^2 sin^2\,t}}.$$

Hence,

$$\int_{\frac{\beta(0)}{\sqrt{2}}}^{\frac{\beta(s)}{\sqrt{2}}} \frac{\sqrt{2}du}{\sqrt{(u^2+\rho^2)(u^2+\bar{\rho}^2)}} = \int_{\frac{\beta(0)}{\sqrt{2}}}^{\infty} \frac{\sqrt{2}du}{\sqrt{(u^2+\rho^2)(u^2+\bar{\rho}^2)}} - \int_{\frac{\beta(s)}{\sqrt{2}}}^{\infty} \frac{\sqrt{2}du}{\sqrt{(u^2+\rho^2)(u^2+\bar{\rho}^2)}}.$$

According to Eq. 4.49, we have

$$\int_{\frac{\beta(0)}{\sqrt{2}}}^{\infty} \frac{\sqrt{2}du}{\sqrt{(u^2+\rho^2)(u^2+\bar{\rho}^2)}} = \sqrt{2}\mathfrak{g}F(\varphi_1, k) = constant, \tag{4.50}$$

where

$$\varphi_1 = cos^{-1}\left(\frac{\xi_3^2-2\sqrt{C_1+\xi_4^2}}{\xi_3^2+2\sqrt{C_1+\xi_4^2}}\right).$$

Since

$$\int_{\frac{\beta(s)}{\sqrt{2}}}^{\infty} \frac{\sqrt{2}du}{\sqrt{(u^2+\rho^2)(u^2+\bar{\rho}^2)}} = \sqrt{2}\mathfrak{g}\cdot cn^{-1}\left(\frac{\beta^2(s)-2\rho\bar{\rho}}{\beta^2(s)+2\rho\bar{\rho}}\right). \tag{4.51}$$

Hence,

$$cn^{-1}\left(\frac{\beta^2(s)-2\rho\bar{\rho}}{\beta^2(s)+2\rho\bar{\rho}}\right) = F(\varphi_1, k) - \frac{s}{\sqrt{2}\mathfrak{g}}, \tag{4.52}$$

let $F = F(\varphi_1, k)$, we obtain

$$\beta^2(s) = \frac{2\rho\bar{\rho}\left(1+cn\,\left(F-\frac{s}{\sqrt{2}\mathfrak{g}}, k\right)\right)}{\left(1-cn\,\left(F-\frac{s}{\sqrt{2}\mathfrak{g}}, k\right)\right)} = \frac{2\rho\bar{\rho}\,(1+cn\,\,(2\tilde{s}, k))}{(1-cn\,\,(2\tilde{s}, k))}, \tag{4.53}$$

where $2\tilde{s} = F - \frac{s}{\sqrt{2}\mathfrak{g}}$.

Since

$$\frac{1-cn\,(2s)}{1+cn\,(2s)} = tn^2\,(s)dn^2\,(s), \tag{4.54}$$

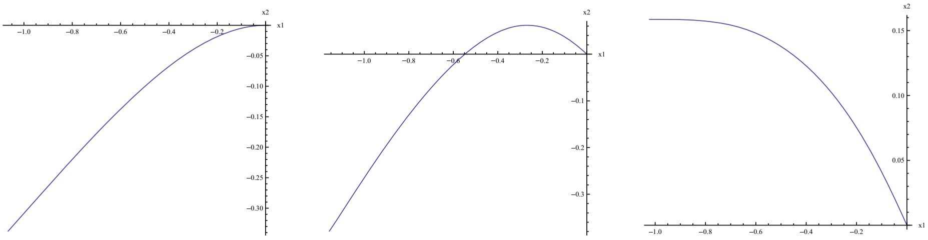

Fig. 2 Projections of geodesics to the plane (x_1, x_2) when $\xi_3 \neq 0,\ \xi_4 \neq 0$

hence,

$$\beta(s) = \frac{\sqrt{2\rho\bar{\rho}}}{tn\ (\tilde{s}, k)dn\ (\tilde{s}, k)} = \sqrt{2\rho\bar{\rho}}cs\ (\tilde{s}, k)nd\ (\tilde{s}, k). \tag{4.55}$$

For the case of $\dot{\beta}(s) < 0$, we can calculate by the same method, and get the same result. But the expression of the parameter $\tilde{s}$ in Eqs. 4.53 and 4.55 should be changed to

$$\frac{1}{2}\left(F + \frac{s}{\sqrt{2\mathfrak{g}}}\right).$$

Thus, the sign of $\dot{\beta}(s)$ will not affect the expression of the geodesics.

Therefore, integrating Eqs. 4.36, 4.38, 4.40, and 4.41, we get a complete description of the Hamiltonian time-like geodesics in the Engel group.

Theorem 4.9 *In the case of $\xi_4 \neq 0$, time-like geodesics starting from the origin are given by the following:*

$$x_1(s) = -\frac{1}{\xi_4}(\beta(s) + \xi_3), \tag{4.56}$$

$$x_2(s) = \frac{1}{2\xi_4}(B_2(s) + 2C_1), \tag{4.57}$$

$$y(s) = -\frac{1}{2\xi_4^2}(B_3(s) + 2C_1B_1(s) + \xi_3B_2(s) + 2C_1\xi_3 s) - \frac{1}{2}x_1(s)x_2(s), \tag{4.58}$$

$$z(s) = \frac{1}{4\xi_4^3}(B_4(s) + 2C_1B_2(s) + 2\xi_3B_3(s) + 4C_1\xi_3B_1(s) + \xi_3^2B_2(s) + 2C_1\xi_3^2) + \frac{1}{6}x_2^3(s), \tag{4.59}$$

where $C_1 = \xi_4\xi_2^0 - \frac{\xi_3^2}{2}$, $B_i(s) = \int_0^s \beta^i(t)dt,\ i = 1, \ldots, 4$, and the expressions of $B_i(s)$ are presented in Appendix.

Projections of geodesics to the plane (x_1, x_2) with $\xi(0) = (1, 0, 1, 1)$, $\xi(0) = \left(\frac{\sqrt{5}}{2}, 1/2, 2, 1\right)$ and $\xi(0) = \left(\frac{\sqrt{5}}{2}, 1/2, 1, 1\right)$ are shown in Fig. 2.

Acknowledgments This work was supported by NSFC(No.11071119, No.11401531) and NSFC-RFBR (No. 11311120055). The authors cordially thank the referees for their careful reading and helpful comments.

Appendix

Denoting

$$\psi_1(s) = ln\ (k'^2 + cs^2\ (s,k)), \qquad \psi_2(s) = \frac{dn\ (2s,k) sn\ (2s,k)}{(1-cn\ (2s,k))^2},$$

$$\psi_3(s) = \frac{E(2s,k) - E(2s,k)cn\ (2s,k)}{1-cn\ (2s,k)}, \qquad F(\varphi,k) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2 sin^2\ t}},$$

$$k^2 = \frac{\sqrt{C_1^2+\xi_4^2} - C_1}{2\sqrt{C_1^2+\xi_4^2}}, \quad \mathfrak{g} = \frac{1}{2\left(C_1^2+\xi_4^2\right)^{\frac{1}{4}}}, \quad \varphi_1 = cos^{-1}\left(\frac{\xi_3^2 - 2\sqrt{C_1+\xi_4^2}}{\xi_3^2 + 2\sqrt{C_1+\xi_4^2}}\right)$$

we get the expressions of $B_i(s)$ as following:

$$B_1(s) = \int_0^s \beta(t)dt = \mathfrak{g}\psi_1(\tilde{s}) + D_1,$$

$$B_2(s) = \int_0^s \beta^2(t)dt = \frac{\sqrt{2}}{\mathfrak{g}}[-3\tilde{s} + (1-cn\ (2\tilde{s},k))\psi_2(\tilde{s}) + \psi_3(\tilde{s})] + D_2,$$

$$B_3(s) = \int_0^s \beta^3(t)dt = \frac{1}{2\mathfrak{g}^2 k'^2}\left[k'^2 cs^2\ (\tilde{s},k) + k'^2(2k^2-1)\psi_1(\tilde{s}) - (2k^4 - k^6 - k^2)e^{\psi_1(\tilde{s})}\right] + D_3,$$

$$B_4(s) = \int_0^s \beta^4(t)dt = \frac{\sqrt{2}}{3\mathfrak{g}^3}\left[-\frac{3}{2}\tilde{s} + (3-4k^2)\psi_2(\tilde{s}) + 4k'^2(1+k^2)\tilde{s} - 2k'^2 E(2\tilde{s},k)\right] + D_4,$$

where $\tilde{s} = \frac{1}{2}\left(F \pm \frac{s}{\sqrt{2}\mathfrak{g}}\right)$, $F = F(\varphi_1,k)$, D_1, D_2, D_3, D_4 are constants, and

$$D_1 = -\mathfrak{g}\psi_1\left(\frac{F}{2}\right),$$

$$D_2 = \frac{\sqrt{2}}{\mathfrak{g}}\left[\frac{3F}{2} - (1-cn\ (F,k))\psi_2\left(\frac{F}{2}\right) - \psi_3\left(\frac{F}{2}\right)\right],$$

$$D_3 = \frac{1}{2\mathfrak{g}^2 k'^2}\left[-k'^2 cs^2\ \left(\frac{F}{2},k\right) - k'^2(2k^2-1)\psi_1\left(\frac{F}{2}\right) + (2k^4-k^6-k^2)e^{\psi_1\left(\frac{F}{2}\right)}\right],$$

$$D_4 = \frac{\sqrt{2}}{3\mathfrak{g}^3}\left[\frac{3F}{4} - (3-4k^2)\psi_2\left(\frac{F}{2}\right) - 2k'^2(1+k^2)F + 2k'^2 E(F,k)\right].$$

References

1. Agrachev AA, Chakir El-HEl-A, Gauthier JP. Sub-Riemannian metrics on $\mathbb{R}^3$. Proc Conf Canad Math Soc. 1998;25:29–78.
2. Agrachev AA, Barilari D, Boscain U. On the Hausdorff volume in sub-Riemannian geometry. Calc Var. 2012;43:355–388.
3. Agrachev AA, Boscain U, Gauthier JP, Rossi F. The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular lie groups. J Funct Anal. 2009;256(6):2621–2655.
4. Agrachev AA, Chariot G, Gauthier JP, Zakalyukin V. On sub-Riemannian caustics and wave fronts for contact distributions in the three space. J Dynam Control Systems. 2000;6(3):365–395.
5. Ardentov AA, Sachkov YL. Extremal trajectories in the nilponent sub-Riemannian problem on the Engel group. Matematicheskii Sbornik. 2011;202(11):31–54.
6. Ardentov AA, Sachkov YL. Conjugate points in nilpotent sub-Riemannian problem on the Engel group. J Math Sci. 2013;195(3):369–390.
7. Ardentov AA, Sachkov YL. Cut time in sub-Riemannian problem on Engel group. ESAIM: Control, Optimisation and Calculus of Variations, accepted.

8. Cartan E. Sur quelques quadratures dont l'élént différentiel contient des fonctions arbitraires. Bull Soc Math France. 1901;29:118–130.
9. Beals R, Gaveau B, Greiner PC. Hamilton-Jacobi theory and the heat Kernal on Heisenberg groups. J Math Pures Appl. 2000;79(7):633–689.
10. Chang DC, Markina I, Vasilev A. Sub-Lorentzian geometry on anti-de Sitter space. J Math Pures Appl. 2008;90(1):82–110.
11. Grochowski M. Geodesics in the sub-Lorentzian geometry. Bull Polish Acad Sci Math. 2002;2:161–178.
12. Grochowski M. Normal forms of germs of contact sub-Lorentzian structures on $\mathbb{R}^3$, differentiability of the sub-Lorentzian distance function. J Dynam Control Systems. 2003;9(4):531–547.
13. Grochowski M. Reachable sets for the Heisenberg sub-Lorentzian structure on $\mathbb{R}^3$, an estimate for the distance function. J Dynam Control Systems. 2006;12(2):145–160.
14. Grochowski M. On the Heisenberg sub- Lorentzian metric on $\mathbb{R}^3$. Geometric Singularity Theory, Banach Center Publications. 2004;65:57–65.
15. Gromov M. Carnot-Carathéodory spaces seen from within. Sub-Riemannian Geometry. 144. Birkhauser, Basel. 1996.
16. Huang TR, Yang XP. Geodesics in the Heisenberg Group H_n with a Lorentzian metric. J Dynam Control Systems. 2012;18(1):21–40.
17. O'Neill B, Vol. 103. Semi-Riemannian geometry: with applications to relativity, pure and applied mathematics. New York: Academic Press; 1983.
18. Korolko A, Markina I. Nonholonomic Lorentzian geometry on some $\mathbb{H}$-type groups. J Geom Anal. 2008;19(4):864–889.
19. Montgomery R. A tour of sub-Riemannian geometries, their geodesics and applications. Mathematical Surveys and Monographs 91, American Mathematical Society, Providence. 2002.