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STABILITY AND BIFURCATION ANALYSIS ON A RING OF FIVE NEURONS WITH DISCRETE DELAYS

CHANGJIN XU, XIANHUA TANG, and MAOXIN LIAO

ABSTRACT. In this paper, a five-neuron model with discrete delays is considered, where the time delays are regarded as parameters. Its dynamics is studied in terms of local analysis and Hopf bifurcation analysis. By analyzing the associated characteristic transcendental equation, it is found that Hopf bifurcation occurs when these delays pass through a sequence of critical value. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory. Finally, numerical simulations supporting the theoretical analysis are presented.

1. INTRODUCTION

In recent years, the dynamics properties (including stable, unstable, oscillatory and chaotic behavior) of neural networks with delays have become a subject of intense research activity of mathematical fields because of the successful application of neural networks to many fields such as intelligent control, optimization solvers, associative memories (or pattern recognition) etc., and many excellent and interesting results have been obtained (see [1, 3, 8, 11]). It is well known that the dynamic behaviors such as periodic phenomenon, bifurcation and chaos are of great interest and periodic phenomenon has become an important aspect of neural information processing. There are a large number of results about the existence of periodic solutions of neural networks (see [1, 2, 4, 6, 18, 21, 23, 25]) which help in understanding the system's dynamics and are important complements to experimental and numerical investigations using analog circuits and digital computers. It is known that the delayed bidirectional associative memory neural network

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is described by the following system:

$$\begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_{ji})) + I_i, & i = 1, 2, \cdots, n, \\ \dot{y}_j(t) = -\nu_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-\nu_{ij})) + J_j, & j = 1, 2, \cdots, m, \end{cases} \tag{1}$$

where $c_{ji}, d_{ij} (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ are the connection weights through neurons in two layers: the I-layer and J-layer; μ_i and ν_j describe the stability of internal neuron processes on the I-layer and J-layer, respectively. On the I-layer, the neurons whose states are denoted by $x_i(t)$ receive the inputs I_i and the inputs outputted by those neurons in the J-layer via activation functions f_i, while on the J-layer, the neurons whose associated states are denoted by $y_j(t)$ receive the inputs J_j and the inputs outputted by those neurons in the I-layer via activation functions g_j (see [5]). Because neural networks are complex and large-scale nonlinear dynamical systems and the dynamics of the delayed neural networks are even more rich and more complicated [8], most of them deal with the simple delayed neural networks models with two, three or fourth neurons (see [2, 4, 6, 9, 12, 16, 18, 23, 25, 26]). It is expected that we can gain some light for our understanding about the large networks by discussing the dynamics of two, three or four neurons networks (see [2, 4, 6, 9, 12, 16, 18, 23, 25, 26]). But there are inevitably some complicated problems if the simplified networks are carried over to large-scale networks, for example, the characteristic equation and the bifurcating periodic solutions are very complicated. So it is necessary to investigate the large-scale neural networks themselves. In order to obtain a deep and clear understanding of dynamics of the model, some researchers have focused on the studies on Hopf bifurcation of the above neural networks and showed the system exhibits very interesting and rich dynamics. For example, on the above model, Song et al. [20] studied existence and local Hopf bifurcations of a simplified case with three neurons and multiple delays. Huang et al. [10] investigated linear stability and Hopf bifurcation of a two-neuron network with four delays. Cao and Xiao [2] considered stability and Hopf bifurcation of a simplified BAM neural network with two delays. Zou et al. [18] investigated linear stability and Hopf bifurcation in a three-unit neural network with two delays. For more related work on Hopf bifurcation of the delayed bidirectional associative memory neural network, one can see [6, 16, 21, 23, 25] and the references cited therein.

Recently, Haijun Hu and Lihong Huang [12] studied the following differential equations with delay:

$$\begin{cases} \dot{x}_1(t) = -r_1 x_1(t) + g_1(x_1(t)) + f_1(x_4(t-\tau_2)) + f_1(x_2(t-\tau_2)), \\ \dot{x}_2(t) = -r_2 x_2(t) + g_2(x_2(t)) + f_2(x_1(t-\tau_1)) + f_2(x_3(t-\tau_1)), \\ \dot{x}_3(t) = -r_3 x_3(t) + g_3(x_3(t)) + f_3(x_2(t-\tau_2)) + f_3(x_4(t-\tau_2)), \\ \dot{x}_4(t) = -r_4 x_4(t) + g_4(x_4(t)) + f_4(x_3(t-\tau_1)) + f_4(x_1(t-\tau_1)), \end{cases} \tag{2}$$

where $\dot{x} = dx/dt, x_i(t)$ represents the state of the i-th neuron at time t, $r_i \geq 0$ is the internal decay rate, f_i is the connection function between neurons, g_i represents the nonlinear feedback function, $\tau_i \geq 0$ is the connection time delay, $i = 1, 2, 3, 4$. They obtained the condition of the existence of Hopf bifurcation, a formula for determining direction of the Hopf bifurcation and stability of bifurcating periodic solutions.

Motivated by the paper [12] and considering that when the number of neurons is large, the simplified model can reflect the really large neural networks more closely, in this paper, we consider a five dimensional delayed bidirectional associative memory neural network and assume that the information processing between the first neuron and the fifth neuron is instantaneous (i.e., there is no delay of the signals transmission between the first neuron and the fifth neuron). Then we have the following system:

$$\begin{cases} \dot{x}_1(t) = -r_1 x_1(t) + g_1(x_1(t)) + f_1(x_5(t)) + f_1(x_2(t-\tau_2)), \\ \dot{x}_2(t) = -r_2 x_2(t) + g_2(x_2(t)) + f_2(x_1(t-\tau_1)) + f_2(x_3(t-\tau_1)), \\ \dot{x}_3(t) = -r_3 x_3(t) + g_3(x_3(t)) + f_3(x_2(t-\tau_2)) + f_3(x_4(t-\tau_2)), \\ \dot{x}_4(t) = -r_4 x_4(t) + g_4(x_4(t)) + f_4(x_3(t-\tau_1)) + f_4(x_5(t-\tau_1)), \\ \dot{x}_5(t) = -r_5 x_5(t) + g_5(x_5(t)) + f_5(x_4(t-\tau_2)) + f_5(x_1(t)). \end{cases} \tag{3}$$

In order to establish the main results for model (3), it is necessary to make the following assumptions:

(H1) $\quad f_i, g_i \in C^3, \quad f_i(0) = g_i(0) = 0 \quad (i = 1, 2, 3, 4, 5),$

(H2) $\quad \tau_1 + \tau_2 = \tau.$

The purpose of this paper is to discuss stability and properties of Hopf bifurcation of model (3). We would like to mention that there are few papers related to the high dimensional neural networks system with multiple delays. To the best of our knowledge, it is the first time to deal with the dynamical properties of five dimensional neural networks, especially the properties of Hopf bifurcation.

This paper is organized as follows. In Sec. 2, stability of the equilibrium and existence of Hopf bifurcation at the equilibrium are studied. In Sec. 3, direction of Hopf bifurcation and stability and periodic of bifurcating periodic solutions on the center manifold are determined. In Sec. 4, numerical simulations are carried out to illustrate validity of the main results. Some main conclusions are drawn in Sec. 5.

2. Stability of the equilibrium and local Hopf bifurcations

By the hypothesis (H1), it is easy to see that (3) has a unique equilibrium $x_*(0,0,0,0,0)$. Under the hypotheses (H1) and (H2), the linearized equation of (3) at $x_*(0,0,0,0,0)$ takes the form:

$$\begin{cases} \dot{x}_1(t) = -m_1x_1(t) + f_1^{'}(0)x_5(t) + f_1^{'}(0)x_2(t-\tau_2), \\ \dot{x}_2(t) = -m_2x_2(t) + f_2^{'}(0)x_1(t-\tau_1) + f_2^{'}(0)x_3(t-\tau_1), \\ \dot{x}_3(t) = -m_3x_3(t) + f_3^{'}(0)x_2(t-\tau_2) + f_3^{'}(0)x_4(t-\tau_2), \\ \dot{x}_4(t) = -m_4x_4(t) + f_4^{'}(0)x_3(t-\tau_1) + f_4^{'}(0)x_5(t-\tau_1), \\ \dot{x}_5(t) = -m_5x_5(t) + f_5^{'}(0)x_4(t-\tau_2) + f_5^{'}(0)x_1(t), \end{cases} \tag{4}$$

where $m_i = r_i - g_i^{'}(0), (i = 1,2,3,4,5)$. Then the associated characteristic equation of (4) is

$$\begin{aligned} &(\lambda+m_1)(\lambda+m_2)(\lambda+m_3)(\lambda+m_4)(\lambda+m_5) \\ &\quad - [(\lambda+m_1)(\lambda+m_2)(\lambda+m_5)f_3^{'}(0)f_4^{'}(0) \\ &\quad + (\lambda+m_1)(\lambda+m_2)(\lambda+m_3)f_4^{'}(0)f_5^{'}(0) \\ &\quad + (\lambda+m_1)(\lambda+m_4)(\lambda+m_5)f_2^{'}(0)f_3^{'}(0) \\ &\quad + (\lambda+m_3)(\lambda+m_4)(\lambda+m_5)f_1^{'}(0)f_2^{'}(0) \\ &\quad - (\lambda+m_4)f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_5^{'}(0) \\ &\quad - (\lambda+m_2)f_1^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0)]e^{-\lambda\tau} \\ &\quad + [(\lambda+m_1)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) \\ &\quad + (\lambda+m_5)f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) \\ &\quad + (\lambda+m_3)f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0) \\ &\quad - 2f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0)]e^{-2\lambda\tau} = 0. \end{aligned} \tag{5}$$

Let $\lambda = i\omega_0, \tau = \tau_0$, and substituting this into (5), for the sake of simplicity, denote ω_0 and τ_0 by ω, τ, respectively. Separating the real and imaginary parts, we have

$$(a_1+b_1)\cos\omega\tau + (c_1-d_1)\sin\omega\tau = e_1, \tag{6}$$

$$(c_1+d_1)\cos\omega\tau + (a_1-b_1)\sin\omega\tau = e_2, \tag{7}$$

where

$$\begin{aligned} a_1 &= p_1\omega^4 - p_3\omega^2 + p_5, \\ b_1 &= m_1f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) + m_5f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) \\ &\quad + m_3f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0) - 2f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0), \\ c_1 &= \Big[f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) + f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) \end{aligned}$$

$$
\begin{aligned}
&\quad +f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0)\Big]\,\omega,\\
d_1 &= \omega^5 - p_2\omega^3 + p_4\omega,\\
e_1 &= \left[m_1m_2m_5 - (m_1+m_2+m_5)\omega^2\right] f_3^{'}(0)f_4^{'}(0)\\
&\quad + \left[m_1m_2m_3 - (m_1+m_2+m_5)\omega^2\right] f_4^{'}(0)f_5^{'}(0)\\
&\quad + \left[m_1m_4m_5 - (m_1+m_4+m_5)\omega^2\right] f_2^{'}(0)f_3^{'}(0)\\
&\quad + \left[m_3m_4m_5 - (m_3+m_4+m_5)\omega^2\right] f_1^{'}(0)f_2^{'}(0)\\
&\quad - m_4f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_5^{'}(0) - m_2f_1^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0),\\
e_2 &= \left[m_1m_2 + m_1m_5 + m_2m_5)\omega - \omega^3\right] f_3^{'}(0)f_4^{'}(0)\\
&\quad + \left[m_1m_2 + m_1m_3 + m_2m_3)\omega - \omega^3\right] f_4^{'}(0)f_5^{'}(0)\\
&\quad + \left[m_1m_4 + m_1m_5 + m_4m_5)\omega - \omega^3\right] f_2^{'}(0)f_3^{'}(0)\\
&\quad + \left[m_3m_4 + m_3m_5 + m_4m_5)\omega - \omega^3\right] f_1^{'}(0)f_2^{'}(0)\\
&\quad - f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_5^{'}(0)\omega - f_1^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0)\omega,\\
p_1 &= m_1+m_2+m_3+m_4+m_5,\\
p_2 &= m_1m_2+m_1m_3+m_2m_3+m_1m_4+m_2m_4\\
&\quad + m_3m_4+m_1m_5+m_2m_5+m_3m_5+m_41m_5,\\
p_3 &= m_1m_2m_3+m_1m_2m_4+m_1m_3m_4+m_2m_3m_4+m_1m_2m_5\\
&\quad + m_1m_3m_5+m_2m_3m_5+m_1m_4m_5+m_2m_4m_5+m_3m_4m_5,\\
p_4 &= m_1m_2m_3m_4+m_1m_2m_3m_5+m_1m_2m_4m_5+m_1m_3m_4m_5\\
&\quad + m_2m_3m_4m_5,\\
p_5 &= m_1m_2m_3m_4m_5.
\end{aligned}
$$

Thus, we get

$$\cos\omega\tau = \frac{e_1(a_1-b_1)-e_2(c_1-d_1)}{a_1^2-b_1^2-c_1^2+d_1^2}, \tag{8}$$

$$\sin\omega\tau = \frac{e_1(c_1+d_1)-e_2(a_1+b_1)}{a_1^2-b_1^2-c_1^2+d_1^2}. \tag{9}$$

According to $\sin^2\omega\tau + \cos^2\omega\tau = 1$, we obtain

$$[e_1(a_1-b_1)-e_2(c_1-d_1)]^2 + [e_1(c_1+d_1)-e_2(a_1+b_1)]^2 = [a_1^2-b_1^2-c_1^2+d_1^2]^2 \tag{10}$$

which leads to

$$l_1\omega^{16}+l_2\omega^{14}+l_3\omega^{13}+l_4\omega^{12}+l_5\omega^{11}+l_6\omega^{10}+l_7\omega^9+l_8\omega^8 + l_9\omega^7+l_{10}\omega^6+l_{11}\omega^5+l_{12}\omega^4l_{13}\omega^3+l_{14}\omega^2+l_{15}\omega+l_{16}=0, \tag{11}$$

where

$$
\begin{aligned}
l_1 &= e_{22}^2,\\
l_2 &= 2e_{11}e_{12}p_1 - 2e_{11}e_{22} - 2p_2e_{22}^2,\\
l_3 &= -2p_1e_{11}e_{22},\\
l_4 &= e_{12}^2p_1^2 + e_{21}^2 + 4p_2e_{21}e_{22} + e_{22}^2(p_2^2 + 2p_4) + 2e_{11}e_{22}(p_1p_2 + p_3)\\
&\quad + 2ne_{22}^2 - 2p_1^2e_{21}e_{22} - 2p_1p_3e_{22}^2,\\
l_5 &= -2n^2p_1e_{21}e_{22} + 2p_1(e_{12}e_{21} + e_{11}e_{22}) + 2e_{11}e_{22}(p_3 + p_1p_2),\\
l_6 &= -2p_1^2e_{11}e_{12} - 2p_1p_3e_{12}^2 - 2p_2e_{21}^2 + 2e_{21}e_{22}(p_2^2 + 2p_4) + 2p_2p_4e_{22}^2\\
&\quad - 2e_{11}e_{22}(p_1p_4 + p_2p_3 + p_5) - 2n(2e_{21}e_{22} + p_2e_{22}^2) + p_1^2e_{21}^2\\
&\quad + 4p_1p_3e_{21}e_{22} + (p_3^2 + 2p_1p_5)e_{22}^2 + 2ne_{12}^2 + 2b_1p_1e_{22}^2,\\
l_7 &= 2p_1n^2(e_{12}e_{21} + e_{21}e_{22}) + 2n^2(e_{12}e_{21} + e_{11}e_{22}p_1) + 2p_3n^2e_{12}e_{22}\\
&\quad + e_{12}^2 + e_{11}e_{22}(p_5 + p_1p_4 + p_2p_3) + (p_1p_2 + p_3)(e_{12}e_{21} + e_{11}e_{22})\\
&\quad - 2b_1e_{12}e_{22},\\
l_8 &= p_1^2e_{11}^2 + 4p_1p_3e_{11}e_{12} + e_{12}^2(p_3^2 + 2p_1p_5) + ne_{22}^2 + e_{21}^2(p_2^2 + 2p_4)\\
&\quad + 4e_{11}e_{22}(p_1p_2 + p_3) + 2(e_{12}e_{21} + e_{11}e_{22})(p_1p_4 + p_2p_3 + p_5)\\
&\quad + e_{11}e_{22}(p_3p_4 - p_2p_5) + 2n(e_{21}^2 + p_4e_{22}^2 + 2p_2e_{21}e_{22} + p_1^2e_{21}^2\\
&\quad + 4p_1p_3e_{21}e_{22} + e_{22}^2(p_3^2 + 2p_1p_5) - 2p_1p_3e_{21}^2 - 2e_{21}e_{22}(p_3^2 + 2p_1p_5)\\
&\quad - 2p_3p_5e_{22}^2 - 2n(2e_{21}e_{22} + p_2p_{12}^2) - 2b_1(p_3e_{22}^2 + 2p_1e21e_{22},\\
l_9 &= -2n^2p_1e_{11}e_{21} - 2n^2[p_1e_{11}e_{21} + p_3(e_{12}e_{21} + e_{11}e_{22}) + p_2e_{11}e_{22}]\\
&\quad - 2b_1(e_{12}e_{21} + e_{11}e_{22}) - 2e_{11}e_{12} - p_2e_{12}^2 - e_{11}e_{21}(p_3 + p_1p_2)\\
&\quad - (p_5 + p_2p_3 + p_1p_4)(e_{12}e_{21} + e_{11}e_{22}) + (p_2p_5 + p_3p_4)e_{11}e_{22}\\
&\quad + 2b_1p_2e_{11}e_{22},\\
l_{10} &= -2p_1p_3e_{11}^2 - 2e_{11}e_{12}(p_3^2 + 2p_1p_5) - 2p_3p_5e_{12}^2 - 2e_{21}e_{22}k\\
&\quad - 2p_2p_4e_{21}^2 - 2e_{21}e_{22}p_4^2 + 2b_1(2e_{11}e_{21}p_1 + p_3e_{12}^2) + 2p_4p_5e_{11}e_{22}\\
&\quad - 2e_{11}e_{22}(p_1p_4 + p_2p_3 + p_5) - 2(e_{12}e_{21} + e_{11}e_{22})(p_3p_4 - p_2p_5)\\
&\quad - 2n^2(p_2e_{21}^2 + 2e_{11}e_{22}p_4) + n^2e_{12}^2 + e_{12}^2(p_3^2 + 2p_1p_5) + e_{22}^2p_5^2\\
&\quad + e_{21}^2(p_3^2 + 2p_1p_5) + 4p_3p_5e_{21}e_{22} + b_1^2e_{22}^2 + 2n(p_4e_{12}^2 + 2p_2e_{11}e_{12})\\
&\quad - p_3e_{11}e_{12} - p_1(e_{12}e_{21} + e_{11}e_{22}) + 2b_1(p_5e_{22}^2 + p_1e_{21}^2),\\
l_{11} &= 2n^2p_3e_{11}e_{21} + 2n^2[p_3e_{11}e_{21} + p_5(e_{12}e_{21} + e_{11}e_{22}) + 2b_1ne_{11}e_{22}\\
&\quad + 2b_1[e_{11}e_{21} + p_2(e_{12}e_{21} + e_{11}e_{22}) + p_4e_{11}e_{22} + e_{11}^2 + 2p_2e_{11}e_{22}\\
&\quad + p_4e_{11}e_{22}] - 2b_1ne_{12}e_{21} + e_{12}e_{21}(p_5 + p_2p_3 + p_1p_4) - 2b_1(e_{12}e_{21}\\
&\quad + e_{11}e_{22})(p_2p_5 + p_3p_4) + e_{11}e_{22}p_4p_5 - 2b_1[e_{12}e_{21} + p_2(e_{12}e_{21}
\end{aligned}
$$

$$
\begin{aligned}
&+ e_{11}e_{22}) + p_4e_{11}e_{22}],\\
l_{12} &= (p_3^2 + 2p_1p_5)e_{11}^2 + 4e_{11}e_{12}p_3p_5 + p_5^2e_{12}^2 + b_1^2e_{12}^2 + e_{21}^2n\\
&\quad - 2b_1(p_1e_{11}^2 + 2p_3e_{11}e_{12} - p_5e_{12}^2 + e_{11}e_{22}(p_1p_4 + p_2p_3 + p_5) + p_4^2e_{21}^2\\
&\quad + (e_{12}e_{21} + e_{11}e_{22})(p_3p_4 - p_2p_5) + p_4p_5e_{11}e_{22} + 2p_4e_{21}^2 - 2n^2e_{12}e_{21}\\
&\quad - 2p_3p_5e_{21}^2 - 2e_{21}e_{22}p_5^2 - 2b_1^2e_{21}e_{22} - 2n(p_2e_{11}^2 + 2p_4e_{11}e_{22})\\
&\quad \times p_3(e_{12}e_{21} + e_{11}e_{22}) + p_5e_{11}e_{22} + p_1e_{11}e_{22} - 2b_1(p_3^2e_{21}^2 + 2p_5e_{21}e_{22}),\\
l_{13} &= -2n^2p_5e_{11}e_{21} - 2b_1n(e_{12}e_{21} + e_{11}e_{22}) - 2b_1p_2e_{11}e_{21} - p_2e_{11}^2\\
&\quad - 2b_1p_4(e_{12}e_{21} + e_{11}e_{22} + 2p_4e_{11}e_{12} + 2n(p_3e_{11}e_{21} + p_5e_{12}e_{21}\\
&\quad + p_5e_{11}e_{22} + 2b_1(e_{12}e_{21} + e_{11}e_{22}) - p_2p_5 - p_3p_4\\
&\quad + 2e_{11}e_{22}(p_2p_5 + p_3p_4)\times p_4p_5(e_{12}e_{21} + e_{11}e_{22})\\
&\quad + 2b_1[p_2e_{11}e_{21} + p_4(e_{12}e_{21} + e_{11}e_{22})],\\
l_{14} &= -2e_{11}e_{21}p_5^2 - 2e_{11}e_{12}b_1^2 + 2b_1(p_3e_{11}^2 + 2e_{11}e_{12}p_5) + n^2e_{11}^2\\
&\quad - e_{11}e_{21}(p_3p_4 - p_2p_5) - p_4p_5(e_{12}e_{21} + e_{11}e_{22}) + p_5^2e_{21}^2 + b_1^2e_{21}^2\\
&\quad + 2np_4e_{11}^2 + 2b_1p_5e_{21}^2,\\
l_{15} &= 2nb_1e_{11}e_{21} + 2p_4b_1e_{11}e_{21} + p_4e_{11}^2 + 2np_5e_{11}e_{21} - 2b_1ne_{11}e_{21}\\
&\quad - 2p_4p_5e_{11}e_{21} - 2b_1p_4e_{11}e_{21},\\
l_{16} &= p_5^2e_{11}^2 + b_1^2e_{11}^2 - 2b_1p_5e_{11}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
n &= f_1'(0)f_2'(0)f_3'(0)f_4'(0) + f_1'(0)f_2'(0)f_4'(0)f_5'(0)\\
&\quad + f_2'(0)f_3'(0)f_4'(0)f_5'(0),\\
e_{11} &= m_1m_2m_5f_3'(0)f_4'(0) + m_1m_2m_3f_4'(0)f_5'(0)\\
&\quad + m_1m_4m_5f_2'(0)f_3'(0) + m_3m_4m_5f_1'(0)f_2'(0)\\
&\quad - m_4f_1'(0)f_2'(0)'(0)f_3'(0)f_5'(0)'(0) - m_2f_1'(0)f_3'(0)f_4'(0)f_5'(0),\\
e_{12} &= (m_1 + m_2 + m_5)f_3'(0)f_4'(0) + (m_1 + m_2 + m_3)f_4'(0)f_5'(0)\\
&\quad + (m_1 + m_4 + m_5)f_2'(0)f_3'(0) + (m_3 + m_4 + m_5)f_1'(0)f_2'(0),\\
e_{21} &= (m_1m_2 + m_1m_5 + m_2m_5)f_3'(0)f_4'(0)\\
&\quad + (m_1m_2 + m_1m_3 + m_2m_3)f_4'(0)f_5'(0)\\
&\quad + (m_1m_4 + m_1m_5 + m_4m_5)f_2'(0)f_3'(0)\\
&\quad + (m_3m_4 + m_3m_5 + m_4m_5)f_1'(0)f_2'(0)\\
&\quad - f_1'(0)f_2'(0)f_3'(0)f_5'(0) - f_1'(0)f_3'(0)f_4'(0)f_5'(0),
\end{aligned}
$$

$$e_{22} = -f_3'(0)f_4'(0) - f_4'(0)f_5'(0) + f_2'(0)f_3'(0) - f_1'(0)f_2'(0).$$

Suppose that Eq. (11) has positive roots. Without loss of generality, we assume that it has sixteen positive roots, denoted by ω_k, ($k = 1, 2, 3, \cdots, 16$). By (8), we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left[\arccos\frac{e_1(a_1 - b_1) - e_2(c_1 - d_1)}{a_1^2 - b_1^2 - c_1^2 + d_1^2} + 2j\pi\right], \tag{12}$$

where $k = 1, 2, 3, \cdots, 16$ and $j = 0, 1, 2, \cdots$. Then $\pm i\omega_k$ is a pair of purely imaginary roots of Eq. (5) with $\tau = \tau_k^{(j)}$. Obviously, the sequence $\{\tau_k^{(j)}\}_{j=0}^{+\infty}$ is increasing, and

$$\lim_{j\to+\infty}\tau_k^{(j)} = +\infty, \qquad k = 1, 2, 3, \cdots, 16.$$

Then we can define

$$\tau_0 = \tau_{k0}^{(0)} = \min_{1\le k\le 16}\{\tau_k^{(0)}\}, \qquad \omega_0 = \omega_{k0}. \tag{13}$$

Note that when $\tau = 0$, (5) becomes

$$\lambda^5 + p_1\lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda^2 + q_4 = 0, \tag{14}$$

where

$$\begin{aligned}
q_1 &= p_2 - f_3'(0)f_4'(0) - f_4'(0)f_5'(0) - f_2'(0)f_3'(0) - f_1'(0)f_2'(0),\\
q_2 &= p_3 - (m_1 + m_2 + m_5)f_3'(0)f_4'(0) - (m_1 + m_2 + m_3)f_4'(0)f_5'(0)\\
&\quad - (m_1 + m_4 + m_5)f_2'(0)f_3'(0) - (m_3 + m_4 + m_5)f_1'(0)f_2'(0),\\
q_3 &= p_4 - (m_1m_2 + m_1m_5 + m_2m_5)f_3'(0)f_4'(0)\\
&\quad - (m_1m_2 + m_1m_3 + m_2m_3)f_4'(0)f_5'(0)\\
&\quad - (m_1m_4 + m_1m_5 + m_4m_5)f_2'(0)f_3'(0)\\
&\quad - (m_3m_4 + m_3m_5 + m_4m_5)f_1'(0)f_2'(0)\\
&\quad + f_1'(0)f_2'(0)f_3'(0)f_5'(0) + f_1'(0)f_3'(0)f_4'(0)f_5'(0)\\
&\quad + f_2'(0)f_3'(0)f_4'(0)f_5'(0) + f_1'(0)f_2'(0)f_3'(0)f_4'(0)\\
&\quad + f_1'(0)f_2'(0)f_4'(0)f_5'(0),\\
q_4 &= p_5 - m_1m_2m_5f_3'(0)f_4'(0) - m_1m_2m_3f_4'(0)f_5'(0)\\
&\quad - m_1m_4m_5f_2'(0)f_3'(0) - m_3m_4m_5f_1'(0)f_2'(0)\\
&\quad + m_4f_1'(0)f_2'(0)f_3'(0)f_5'(0) - m_2f_1'(0)f_3'(0)f_4'(0)f_5'(0)\\
&\quad + m_1f_2'(0)f_3'(0)f_4'(0)f_5'(0) + m_5f_1'(0)f_2'(0)f_3'(0)f_4'(0)\\
&\quad + m_3f_1'(0)f_2'(0)f_4'(0)f_5'(0) - 2f_1'(0)f_2'(0)f_3'(0)f_4'(0)f_5'(0).
\end{aligned}$$

A set of necessary and sufficient conditions for all roots of (14) to have a negative real part is given by the well-known Routh-Hurwitz criteria in the following form:

$$p_1 > 0, \quad p_1q_1 - q_2 > 0, \quad p_1(q_1q_2 - p_1q_3) - q_2^2 + p_1q_4 > 0, \quad q_4 > 0. \tag{15}$$

In order to obtain the main results in this paper, it is necessary to make the following.

Assumption 1. (H3) If (15) holds. Namely, (5) has sixteen roots with negative real parts when $\tau = 0$, (4) is stable near the equilibrium.

(H4) $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0} \neq 0.$

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq. (5) near $\tau = \tau_k^{(j)}$ satisfying $\alpha\left(\tau_k^{(j)}\right) = 0$, $\omega\left(\tau_k^{(j)}\right) = \omega_k$. Taking the derivative of λ with respect to τ in (5), it is easy to obtain:

$$\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} = \frac{P}{Q} - \frac{\tau}{\lambda}, \tag{16}$$

where

$$\begin{aligned} P = {} & 5\lambda^4 + 4p_1\lambda^3 + 3p_2\lambda^2 + 2p_3\lambda + p_4)e^{\lambda\tau} \\ & - \left[3\lambda^2 + 2(m_1 + m_2 + m_5)\lambda + (m_1m_2 + m_1m_5 + m_2m_5)\right] f_3'(0)f_4'(0) \\ & - 3\left[\lambda^2 + 2(m_1 + m_2 + m_3)\lambda + (m_1m_2 + m_1m_3 + m_2m_3)\right] f_4'(0)f_5'(0) \\ & - 3\left[\lambda^2 + 2(m_1 + m_4 + m_5)\lambda + (m_1m_4 + m_1m_5 + m_4m_5)\right] f_2'(0)f_3'(0) \\ & - 3\left[\lambda^2 + 2(m_3 + m_4 + m_5)\lambda + (m_3m_4 + m_3m_5 + m_4m_5)\right] f_1'(0)f_2'(0) \\ & + f_1'(0)f_2'(0)f_3'(0)f_5'(0) + f_1'(0)f_3'(0)f_4'(0)f_5'(0) + [f_2'(0)f_3'(0)f_4'(0)f_5'(0) \\ & + f_1'(0)f_2'(0)f_3'(0)f_4'(0) + f_1'(0)f_2'(0)f_4'(0)f_5'(0)]e^{-\lambda\tau}, \\ Q \quad & = -\lambda e^{\lambda\tau}(\lambda^5 + p_1\lambda^4 + p_2\lambda^3 + p_3\lambda^2 + p_4\lambda + p_5) \\ & + \lambda e^{-\lambda\tau}\Big[(\lambda + m_1)f_2'(0)f_3'(0)f_4'(0)f_5'(0) \\ & + (\lambda + m_5)f_1'(0)f_2'(0)f_3'(0)f_4'(0) \\ & + (\lambda + m_3)f_1'(0)f_2'(0)f_4'(0)f_5'(0) - 2f_1'(0)f_2'(0)f_3'(0)f_4'(0)f_5'(0)\Big]. \end{aligned}$$

Then we obtain

$$\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1}\Big|_{\tau=\tau_0} = \frac{A_1 + iA_2}{C_1 + iC_2} - \frac{\tau}{\lambda}, \tag{17}$$

where

$$A_1 = 5\omega_0^4 - 3P_2\omega_0^2 + p_4)\cos\omega_0\tau_0 - (2p_3\omega_0 - 4p_1\omega_0^3)\sin\omega_0\tau_0$$

$$
\begin{aligned}
&- \left[-3\omega_0^2 + m_1m_2 + m_1m_5 + m_2m_5\right] f_3^{'}(0)f_4^{'}(0) \\
&- \left[-3\omega_0^2 + m_1m_2 + m_1m_3 + m_2m_3\right] f_4^{'}(0)f_5^{'}(0) \\
&- \left[-3\omega_0^2 + m_1m_4 + m_1m_5 + m_4m_5\right] f_1^{'}(0)f_2^{'}(0) \\
&- \left[-3\omega_0^2 + m_3m_4 + m_3m_5 + m_4m_5\right] f_1^{'}(0)f_2^{'}(0) \\
&+ f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_5^{'}(0) + f_1^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) \\
&+ \Big[f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) + f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) \\
&+ f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0)\Big] \cos\omega_0\tau_0, \\
A_2 = {}& 5\omega_0^4 - 3p_2\omega_0^2 + p_4)\sin\omega_0\tau_0 \\
&- (2p_3\omega_0 - 4p_1\omega_0^3)\cos\omega_0\tau_0 \\
&- 2(m_1 + m_2 + m_5)\omega_0 f_3^{'}(0)f_4^{'}(0) \\
&- 2(m_1 + m_2 + m_3)\omega_0 f_4^{'}(0)f_5^{'}(0) \\
&- 2(m_1 + m_4 + m_5)\omega_0 f_2^{'}(0)f_3^{'}(0) \\
&- 2(m_3 + m_4 + m_5)\omega_0 f_1^{'}(0)f_2^{'}(0) \\
&- [f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) + f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) \\
&+ \Big[f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0)\Big] \sin\omega_0\tau_0, \\
C_1 = {}& (p_1\omega_0^4 - P_3\omega_0^2 + p_5)\omega_0\sin\omega_0\tau_0 - (\omega_0^5 - p_2\omega_0^3 + p_4\omega_0)\omega_0\cos\omega_0\tau_0 \\
&+ \omega_0\sin\omega_0\tau_0 \Big[m_1 f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) + m_5 f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) \\
&+ m_3 f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0) - 2f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0)\Big] \\
&- \Big[f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) + f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) \\
&+ f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0)\Big] \omega_0^2\cos\omega_0\tau_0, \\
C_2 = {}& (\omega_0^5 - P_2\omega_0^3 + p_4\omega_0)\omega_0\sin\omega_0\tau_0 - (p_1\omega_0^4 - p_3\omega_0^2 + p_5)\omega_0\cos\omega_0\tau_0 \\
&+ \omega_0^2\sin\omega_0\tau_0 \Big[f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) + f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) \\
&+ f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0)\Big] + \omega_0\cos\omega_0\tau_0 \Big[m_1 f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0) \\
&+ m_5 f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0) + m_3 f_1^{'}(0)f_2^{'}(0)f_4^{'}(0)f_5^{'}(0) \\
&+ -2f_1^{'}(0)f_2^{'}(0)f_3^{'}(0)f_4^{'}(0)f_5^{'}(0)\Big].
\end{aligned}
$$

Thus we have

$$
\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0} = \frac{A_1C_1 + A_2C_2}{C_1^2 + C_2^2}. \tag{18}
$$

In order to investigate the distribution of roots of the transcendental equation (5), the following lemma that is stated in [19] is useful.

Lemma 1 (See [19]). *For the transcendental equation*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots \\ &+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

From Lemma 1, it is easy to obtain the following results:

Theorem 2. *If* $(H1) - (H4)$ *hold, then*
(I) *For system* (3), *its zero solution is asymptotically stable for* $\tau \in [0, \tau_0)$;
(II) *system* (3) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_0$, *i.e, system* (3) *has a branch of periodic solutions bifurcating from the zero solution near* $\tau = \tau_0$.

3. Direction and stability of the Hopf bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_0$. In this section, we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $x_*(0, 0, 0, 0, 0)$ at these critical value of τ, by using techniques from normal form and center manifold theory [10]. Throughout this section, we always assume that system (5) undergoes Hopf bifurcation at the equilibrium $x_*(0, 0, 0, 0, 0)$ for $\tau = \tau_0$, and then $\pm i\omega_0$ are the corresponding purely imaginary roots of the characteristic equation at the equilibrium $x_*(0, 0, 0, 0, 0)$. Linear part of system (3) at $x_*(0, 0, 0, 0, 0)$ is given by

$$\begin{cases} \dot{x}_1(t) = -m_1 x_1(t) + f_1'(0)x_5(t) + f_1'(0)x_2(t-\tau_2)), \\ \dot{x}_2(t) = -m_2 x_1(t) + f_2'(0)x_1(t-\tau_1) + f_2'(0)x_3(t-\tau_1)), \\ \dot{x}_3(t) = -m_3 x_1(t) + f_3'(0)x_2(t-\tau_2) + f_3'(0)x_4(t-\tau_2)), \\ \dot{x}_4(t) = -m_4 x_1(t) + f_4'(0)x_3(t-\tau_1) + f_4'(0)x_5(t-\tau_1)), \\ \dot{x}_5(t) = -m_1 x_5(t) + f_5'(0)x_4(t-\tau_2) + f_5'(0)x_1(t)) \end{cases} \tag{19}$$

and non-linear part is given by

$$f(\mu, u_t) \quad =$$

$$\begin{pmatrix} \frac{g_1''(0)}{2}x_1^2(t)+\frac{g_1'''(0)}{3!}x_1^3(t)+\frac{f_1''(0)}{2}x_2^2(t-\tau_2)+\frac{f_1'''(0)}{3!}x_2^3(t-\tau_2) \\ \frac{g_2''(0)}{2}x_2^2(t)+\frac{g_2'''(0)}{3!}x_2^3(t)+\frac{f_2''(0)}{2}x_1^2(t-\tau_1)+\frac{f_1'''(0)}{3!}x_1^3(t-\tau_1) \\ \frac{g_3''(0)}{2}x_3^2(t)+\frac{g_3'''(0)}{3!}x_3^3(t)+\frac{f_3''(0)}{2}x_2^2(t-\tau_2)+\frac{f_3'''(0)}{3!}x_2^3(t-\tau_2) \\ \frac{g_4''(0)}{2}x_4^2(t)+\frac{g_4'''(0)}{3!}x_4^3(t)+\frac{f_4''(0)}{2}x_3^2(t-\tau_1)+\frac{f_4'''(0)}{3!}x_3^3(t-\tau_1) \\ \frac{g_5''(0)}{2}x_5^2(t)+\frac{g_5'''(0)}{3!}x_5^3(t)+\frac{f_5''(0)}{2}x_1^2(t)+\frac{f_5'''(0)}{3!}x_1^3(t) \end{pmatrix}$$

$$+\begin{pmatrix} \frac{f_1''(0)}{2}x_5^2(t)+\frac{f_1'''(0)}{3!}x_5^3(t)+\text{h.o.t.} \\ \frac{f_2''(0)}{2}x_3^2(t-\tau_1)+\frac{f_2'''(0)}{3!}x_3^3(t-\tau_1)+\text{h.o.t.} \\ \frac{f_3''(0)}{2}x_4^2(t-\tau_2)+\frac{f_3'''(0)}{3!}x_4^3(t-\tau_2)+\text{h.o.t.} \\ \frac{f_4''(0)}{2}x_5^2(t-\tau_1)+\frac{f_4'''(0)}{3!}x_5^3(t-\tau_1)+\text{h.o.t.} \\ \frac{f_5''(0)}{2}x_4^2(t-\tau_2)+\frac{f_5'''(0)}{3!}x_4^3(t-\tau_2)+\text{h.o.t.} \end{pmatrix}. \tag{20}$$

Denote

$$C^k[-\tau_2^*,0]=\left\{\varphi\,|\,\varphi:[-\tau_2^*,0]\to R^5,\ \text{each component of }\varphi\text{ has } k\text{-order continuous derivative}\right\}.$$

For convenience, denote $C[-\tau_2^*,0]$ by $C^0[-\tau_2^*,0]$.

For $\varphi(\theta)=(\varphi_1(\theta),\varphi_2(\theta),\varphi_3(\theta),\varphi_4(\theta),\varphi_5(\theta))^T\in C\left([-\tau_2^*,0],R^5\right)$, define a family of operators

$$L_\mu\varphi=B\varphi(0)+B_1\varphi(-\tau_1^*-\mu)+B_2(-\tau_2^*) \tag{21}$$

and

$$G(\mu,\varphi)=(k_1,k_2,k_3,k_4,k_5)^T, \tag{22}$$

where

$$B=\begin{pmatrix} -m_1 & 0 & 0 & 0 & f_1'(0) \\ 0 & -m_2 & 0 & 0 & 0 \\ 0 & 0 & -m_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_4 \\ f_5'(0) & 0 & 0 & 0 & -m_5 \end{pmatrix},$$

$$B_1=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ f_2'(0) & 0 & f_2'(0) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_4'(0) & 0 & f_4'(0) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2=\begin{pmatrix} 0 & f_1'(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & f_3'(0) & 0 & f_3'(0) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_5'(0) & 0 \end{pmatrix},$$

$$\begin{aligned}
k_1 &= \frac{g_1^{''}(0)}{2}\varphi_1^2(0) + \frac{g_1^{'''}(0)}{3!}\varphi_1^3(0) + \frac{f_1^{''}(0)}{2}\varphi_2^2(-\tau_2) + \frac{f_1^{'''}(0)}{3!}\varphi_2^3(-\tau_2) \\
&\quad + \frac{f_1^{''}(0)}{2}\varphi_5^2(0) + \frac{f_1^{'''}(0)}{3!}\varphi_5^3(0) + 0\left(||\varphi||)^4\right), \\
k_2 &= \frac{g_2^{''}(0)}{2}\varphi_2^2(0) + \frac{g_2^{'''}(0)}{3!}\varphi_2^3(0) + \frac{f_2^{''}(0)}{2}\varphi_1^2(-\tau_1) + \frac{f_1^{'''}(0)}{3!}\varphi_1^3(-\tau_1) \\
&\quad + \frac{f_2^{''}(0)}{2}\varphi_3^2(-\tau_1) + \frac{f_2^{'''}(0)}{3!}\varphi_3^3(-\tau_1) + 0\left(||\varphi||)^4\right), \\
k_3 &= \frac{g_3^{''}(0)}{2}\varphi_3^2(0) + \frac{g_3^{'''}(0)}{3!}\varphi_3^3(0) + \frac{f_3^{''}(0)}{2}\varphi_2^2(-\tau_2) + \frac{f_3^{'''}(0)}{3!}\varphi_2^3(-\tau_2) \\
&\quad + \frac{f_3^{''}(0)}{2}\varphi_4^2(-\tau_2) + \frac{f_3^{'''}(0)}{3!}\varphi_4^3(-\tau_2) + 0\,(||\varphi||)^4), \\
k_4 &= \frac{g_4^{''}(0)}{2}\varphi_4^2(0) + \frac{g_4^{'''}(0)}{3!}\varphi_4^3(t) + \frac{f_4^{''}(0)}{2}\varphi_3^2(-\tau_1) + \frac{f_4^{'''}(0)}{3!}\varphi_3^3(-\tau_1) \\
&\quad + \frac{f_4^{''}(0)}{2}\varphi_5^2(-\tau_1) + \frac{f_4^{'''}(0)}{3!}\varphi_5^3(-\tau_1) + 0\left(||\varphi||)^4\right), \\
k_5 &= \frac{g_5^{''}(0)}{2}\varphi_5^2(0) + \frac{g_5^{'''}(0)}{3!}\varphi_5^3(0) + \frac{f_5^{''}(0)}{2}\varphi_1^2(0) + \frac{f_5^{'''}(0)}{3!}\varphi_1^3(0) \\
&\quad + \frac{f_5^{''}(0)}{2}\varphi_4^2(-\tau_2) + \frac{f_5^{'''}(0)}{3!}\varphi_4^3(-\tau_2) + 0\left(||\varphi||)^4\right),
\end{aligned}$$

and L_μ is a one-parameter family of bounded linear operators in $C([-\tau_2^*, 0], R^5) \to R^5$. By the Riesz representation theorem, there exists a matric whose components are bounded variation functions $\eta(\theta, \mu)$ in $[-\tau_2^*, 0] \to R^{5^2}$, such that

$$L_\mu \varphi = \int_{-\tau_2^*}^{0} d\eta(\theta, \mu)\varphi(\theta). \tag{23}$$

In fact, choosing

$$\eta(\theta, \mu) = \begin{cases} B, & \theta = 0, \\ B_1\delta(\theta + \tau_1^* + \mu), & \theta \in [-\tau_1^* - \mu, 0), \\ -B_2\delta(\theta + \tau_2^*), & \theta \in [-\tau_2^*, -\tau_1^* - \mu), \end{cases} \tag{24}$$

where $\delta(\theta)$ is Dirac function, then (23) is satisfied. For $(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in (C^1[-\tau_2^*, 0], R^5)$, define

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & -\tau_2^* \le \theta < 0, \\ \displaystyle\int_{-\tau_2^*}^{0} d\eta(s, \mu)\varphi(s), & \theta = 0 \end{cases} \tag{25}$$

and

$$R\varphi = \begin{cases} 0, & -\tau_2^* \le \theta < 0, \\ f(\mu, \varphi), & \theta = 0. \end{cases} \tag{26}$$

Then (3) is equivalent to the abstract differential equation

$$\dot{u_t} = A(\mu)u_t + R(\mu)u_t, \tag{27}$$

where $u = (u_1, u_2, u_3, u_4, u_5)^T,\ u_t(\theta) = u(t+\theta), \theta \in [-\tau_2^*, 0]$.
For $\psi \in C([-\tau_2^*, 0], (R^5)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, \tau_2^*], \\ \displaystyle\int_{-\tau_2^*}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{28}$$

For $\phi \in C([-\tau_2^*, 0], R^5)$ and $\psi \in C([0, \tau_2^*], (R^5)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{29}$$

where $\eta(\theta) = \eta(\theta, 0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3. *$A = A(0)$ and A^* are adjoint operators.*

Proof. Let $\phi \in C^1([-\tau_2^*, 0], R^5)$ and $\psi \in C^1([0, \tau_2^*], (R^5)^*$. It follows from (29) and the definitions of $A = A(0)$ and A^* that

$$\begin{aligned}
&\langle \psi(s), A(0)\phi(\theta) \rangle \\
&= \bar{\psi}(0)A(0)\phi(0) - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \bar{\psi}(0)\int_{-\tau_2^*}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \bar{\psi}(0)\int_{-\tau_2^*}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_2^*}^{0} [\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)]_{\xi=0}^{\theta} \\
&\quad + \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta} \frac{d\bar{\psi}(\xi-\theta)}{d\xi}d\eta(\theta)\phi(\xi)d\xi \\
&= \int_{-\tau_2^*}^{0} \bar{\psi}(-\theta)d\eta(\theta)\phi(0) - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta}\left[-\frac{d\bar{\psi}(\xi-\theta)}{d\xi}\right]d\eta(\theta)\phi(\xi)d\xi \\
&= A * \bar{\psi}(0)\phi(0) - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta} A^*\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \\
&= \langle A^*\psi(s), \phi(\theta) \rangle.
\end{aligned}$$

This shows that $A = A(0)$ and A^* are adjoint operators and the proof is complete. □

By the discussions in Sec. 2, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. We have the following result.

Lemma 4. *The vector*

$$q(\theta) = (1, a_1, a_2, a_3, a_4)^T e^{i\omega_0\theta}, \qquad \theta \in [-\tau_2^*, 0],$$

is the eigenvector of $A(0)$ *corresponding to the eigenvalue* $i\omega_0$*, and*

$$q^*(s) = D(1, a_1^*, a_2^*, a_3^*, a_4^*)e^{i\omega_0 s}, \qquad s \in [0, \tau_2^*],$$

is the eigenvector of A^* *corresponding to the eigenvalue* $-i\omega_0$*, moreover,* $\langle q^*(s), q(\theta)\rangle = 1$*, where*

$$D = \frac{1}{C}, \tag{30}$$

where

$$\begin{aligned} C = 1 + \sum_{i=1}^{4} \bar{a}_i a_1^* + \bar{a}_1 f_1^{'}(0)\tau_2^* e^{i\omega_0\tau_2^*} + a_1^*\tau_1^* f_2^{'}(0)e^{i\omega_0\tau_1^*}(1 + \bar{a}_2) \\ + a_2^*\tau_1^* f_3^{'}(0)e^{i\omega_0\tau_2^*}(\bar{a}_1 + \bar{a}_3) + a_3^*\tau_1^* f_4^{'}(0)e^{i\omega_0\tau_1^*}(\bar{a}_2 + \bar{a}_4) \\ + a_4^*\tau_2^* f_5^{'}(0)e^{i\omega_0\tau_2^*}\bar{a}_3. \end{aligned}$$

Proof. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0$, namely, $A(0)q(\theta) = i\omega_0 q(\theta)$ and $A^*q(s) = -i\omega_0 q^*(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q(s) = -dq^*(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\theta}$ and $q^*(s) = q(0)e^{i\omega_0 s}$. In addition,

$$\int_{-\tau_2^*}^{0} d\eta(\theta)q(\theta) = \begin{pmatrix} -m_1 & 0 & 0 & 0 & f_1^{'}(0) \\ 0 & -m_2 & 0 & 0 & 0 \\ 0 & 0 & -m_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_4 \\ f_5^{'}(0) & 0 & 0 & 0 & -m_5 \end{pmatrix} q(0)$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ f_2^{'}(0) & 0 & f_2^{'}(0) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_4^{'}(0) & 0 & f_4^{'}(0) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} q(-\tau_1^*)$$

$$+\begin{pmatrix} 0 & f_1^{'}(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & f_3^{'}(0) & 0 & f_3^{'}(0) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_5^{'}(0) & 0 \end{pmatrix} q(-\tau_2^*)$$
$$= A(0)q(0) = i\omega_0 q(0). \tag{31}$$

That is

$$\begin{pmatrix} -m_1 + a_4 f_1^{'}(0) + a_1 f_1^{'}(0)e^{-i\omega_0\tau_2^*} \\ -m_2 a_1 + f_2^{'}(0)e^{-i\omega_0\tau_1^*} + a_2 f_2^{'}(0)e^{-i\omega_0\tau_1^*} \\ -m_3 a_2 + f_3^{'}(0)a_1 e^{-i\omega_0\tau_2^*} + a_3 f_3^{'}(0)e^{-i\omega_0\tau_2^*} \\ -m_4 a_3 + f_4^{'}(0)a_2 e^{-i\omega_0\tau_1^*} + a_4 f_4^{'}(0)e^{-i\omega_0\tau_1^*} \\ f_5^{'}(0) - m_5 a_4 + a_3 f_5^{'}(0)e^{-i\omega_0\tau_2^*} \end{pmatrix} = \begin{pmatrix} i\omega_0 \\ ia_1\omega_0 \\ ia_2\omega_0 \\ ia_3\omega_0 \\ ia_4\omega_0 \end{pmatrix}. \tag{32}$$

Therefore, we can easily obtain

$$\begin{aligned} a_1 &= \frac{(m_3 + i\omega_0)f_2^{'}(0)e^{-i\omega_0\tau_1^*} + f_2^{'}(0)f_3^{'}(0)e^{-i\omega_0(\tau_1^*+\tau_2^*)}}{(m_2 + i\omega_0)(m_3 + i\omega_0) - f_2^{'}(0)f_3^{'}(0)e^{-i\omega_0(\tau_1^*+\tau_2^*)}}, \\ a_2 &= \frac{(m_2 + i\omega_0)f_3^{'}(0)e^{-i\omega_0\tau_1^*} + f_2^{'}(0)f_3^{'}(0)e^{-i\omega_0(\tau_1^*+\tau_2^*)}}{(m_2 + i\omega_0)(m_3 + i\omega_0) - f_2^{'}(0)f_3^{'}(0)e^{-i\omega_0(\tau_1^*+\tau_2^*)}}, \\ a_3 &= \frac{(m_5 + i\omega_0)a_4 - f_5^{'}(0)}{f_5^{'}(0)e^{-i\omega_0\tau_2^*}}, \\ a_4 &= i\omega_0 + m_1 - a_1 f_1^{'}(0)e^{-i\omega_0\tau_2^*}. \end{aligned}$$

On the other hand,

$$\int_{-1}^{0} q^*(-t)d\eta(t) = \begin{pmatrix} -m_1 & 0 & 0 & 0 & f_1^{'}(0) \\ 0 & -m_2 & 0 & 0 & 0 \\ 0 & 0 & -m_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_4 \\ f_5^{'}(0) & 0 & 0 & 0 & -m_5 \end{pmatrix} q^*(0)$$

$$+\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ f_2^{'}(0) & 0 & f_2^{'}(0) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_4^{'}(0) & 0 & f_4^{'}(0) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} q^*(-\tau_1^*)$$

$$
\begin{aligned}
&+\begin{pmatrix} 0 & f_1'(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & f_3'(0) & 0 & f_3'(0) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_5'(0) & 0 \end{pmatrix} q^*(-\tau_2^*) \\
&= A^* q^*(0) = -i\omega_0 q^*(0).
\end{aligned} \tag{33}
$$

Namely,

$$
\begin{pmatrix} -i\omega_0 + m_1 - f_2'(0)e^{i\omega_0\tau_1^*}a_1^* - f_5'(0)a_4^* \\ -f_1'(0)e^{i\omega_0\tau_1^*} + (-i\omega_0 + m_2)a_1^* - f_3'(0)a_2^* e^{i\omega_0\tau_2^*} \\ (-i\omega_0 + m_3)a_2^* - f_4^*(0)e^{i\omega_0\tau_1^*}a_3^* \\ -f_4^*(0)e^{i\omega_0\tau_1^*}a_2^* + (-i\omega_0 + m_4)a_3^* - f_5'(0)e^{i\omega_0\tau_2^*}a_4^* \\ -f_1^*(0) + (-i\omega_0 + m_5)a_4^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{34}
$$

Therefore, we can easily obtain

$$
\begin{aligned}
a_1^* &= \frac{(-i\omega_0 + m_2)(-i\omega_0 + m_5) - f_1'(0)f_5'(0)}{(-i\omega_0 + m_5)f_2'(0)e^{i\omega_0\tau_1^*}}, \\
a_2^* &= \frac{(-i\omega_0 + m_2)a_1^* - f_1'(0)e^{i\omega_0\tau_2^*}}{f_3'(0)e^{i\omega_0\tau_2^*}}, \\
a_3^* &= \frac{(-i\omega_0 + m_3)a_2^*}{f_4'(0)e^{i\omega_0\tau_1^*}}, \\
a_4^* &= \frac{f_1'(0)}{-i\omega_0 + m_5}.
\end{aligned}
$$

□

In the sequel, we shall verify that $\langle q^*(s), q(\theta)\rangle = 1$. In fact, from (29), we have

$$
\begin{aligned}
\langle q^*(s), q(\theta)\rangle &= \bar{D}(1, \bar{a_1^*}, \bar{a_2^*}, \bar{a_3^*}, \bar{a_4})(1, a_1, a_2, a_3, a_4)^T \\
&\quad - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{a_1^*}, \bar{a_2^*}, \bar{a_3^*}, \bar{a_4})e^{-i\omega_0(\xi-\theta)}d\eta(\theta)(1, a_1, a_2, a_3, a_4)^T e^{i\omega_0\xi}d\xi \\
&= \bar{D}\left[1 + \sum_{i=1}^{4} a_i\bar{a_1^*} - \int_{-\tau_2^*}^{0}(1, \bar{a_1^*}, \bar{a_2^*}, \bar{a_3^*}, \bar{a_4})\theta e^{i\omega_0\theta}d\eta(\theta)(1, a_1, a_2, a_3, a_4)^T\right] \\
&= \bar{D}\left\{1 + \sum_{i=1}^{4} a_i\bar{a_1^*} + (1, \bar{a_1^*}, \bar{a_2^*}, \bar{a_3^*}, \bar{a_4})\left[-\tau_1^* e^{-i\omega_0\tau_1^*}B_1 - \tau_2^* e^{-i\omega_0\tau_1^*}B_2\right]\right. \\
&\quad \left.\times (1, a_1, a_2, a_3, a_4)^T\right\}
\end{aligned}
$$

$$= \bar{D}\Bigg[1 + \sum_{i=1}^{4} a_i \bar{a}_i^* + a_1 f_1^{'}(0)\tau_2^* e^{-i\omega_0\tau_2^*} + \bar{a}_1^*\tau_1^* f_2^{'}(0)e^{-i\omega_0\tau_1^*}(1+a_2)$$
$$+ \bar{a}_2^*\tau_1^* f_2^{'}(0)e^{-i\omega_0\tau_1^*}(a_1+a_3) + \bar{a}_3^*\tau_1^* f_2^{'}(0)e^{-i\omega_0\tau_1^*}(a_2+a_4)$$
$$+ \bar{a}_4^*\tau_2^* f_5^{'}(0)e^{-i\omega_0\tau_2^*}a_3\Bigg] = 1.$$

Next, we use the same notations as those in Hassard, Kazarinoff and Wan [10], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of Eq. (3) when $\mu = 0$.

Define

$$z(t) = \langle q^*, x_t \rangle, \quad W(t,\theta) = x_t(\theta) - 2Re\{z(t)q(\theta)\} \tag{35}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{36}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z,\bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{37}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (3),

$$\begin{aligned}
\dot{z}(t) &= \langle q^*(s), \dot{x}_t \rangle = \langle q^*(s), A(0)u_t + R(0)u_t \rangle \\
&= \langle q^*(s), A(0)x_t \rangle + \langle q^*(s), R(0)x_t \rangle \\
&= \langle A^* q^*(s), x_t \rangle + \bar{q^*}(0)R(0)x_t \\
&\quad - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta} \bar{q^*}(\xi-\theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\
&= \langle i\omega_0 q^*(s), x_t \rangle + \bar{q^*}(0)f(0, x_t(\theta) \\
&\overset{\text{def}}{=} i\omega_0 z(t) + \bar{q^*}(0)f_0(z(t), \bar{z}(t)).
\end{aligned} \tag{38}$$

That is

$$\dot{z}(t) = i\omega_0 z + g(z,\bar{z}), \tag{39}$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{40}$$

Hence, we have

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) = f(0,x_t) \\
&= \bar{D}(1, \bar{a}_1^*, \bar{a}_2^*, \bar{a}_3^*, \bar{a}_4^*) \\
&\quad \times (f_1(0,x_t), f_2(0,x_t), f_3(0,x_t), f_4(0,x_t), f_5(0,x_t))^T,
\end{aligned} \tag{41}$$

where

$$
\begin{aligned}
f_1(0,x_t) &= \frac{g_1''(0)}{2}x_{1t}^2(0) + \frac{g_1'''(0)}{3!}x_{1t}^3(0) + \frac{f_1''(0)}{2}x_{2t}^2(-\tau_2) \\
&\quad + \frac{f_1'''(0)}{3!}x_{2t}^3(-\tau_2) + \frac{f_1''(0)}{2}x_{5t}^2(0) + \frac{f_1'''(0)}{3!}x_{5t}^3(0) + \text{h.o.t.}, \\
f_2(0,x_t) &= \frac{g_2''(0)}{2}x_{2t}^2(0) + \frac{g_2'''(0)}{3!}x_{2t}^3(0) + \frac{f_2''(0)}{2}x_{1t}^2(-\tau_1) \\
&\quad + \frac{f_1'''(0)}{3!}x_{1t}^3(-\tau_1) + \frac{f_2''(0)}{2}x_{3t}^2(-\tau_1) \\
&\quad + \frac{f_2'''(0)}{3!}x_{3t}^3(-\tau_1) + \text{h.o.t.}, \\
f_3(0,x_t) &= \frac{g_3''(0)}{2}x_{3t}^2(0) + \frac{g_3'''(0)}{3!}x_{3t}^3(0) + \frac{f_3''(0)}{2}x_{2t}^2(-\tau_2) \\
&\quad + \frac{f_3'''(0)}{3!}x_{2t}^3(-\tau_2) + \frac{f_3''(0)}{2}x_{4t}^2(-\tau_2) \\
&\quad + \frac{f_3'''(0)}{3!}x_{4t}^3(-\tau_2) + \text{h.o.t.}, \\
f_4(0,x_t) &= \frac{g_4''(0)}{2}x_{4t}^2(0) + \frac{g_4'''(0)}{3!}x_{4t}^3(0) + \frac{f_4''(0)}{2}x_{3t}^2(-\tau_1) \\
&\quad + \frac{f_4'''(0)}{3!}x_{3t}^3(-\tau_1) + \frac{f_4''(0)}{2}x_{5t}^2(-\tau_1) \\
&\quad + \frac{f_4'''(0)}{3!}x_{5t}^3(-\tau_1) + \text{h.o.t.}, \\
f_5(0,x_t) &= \frac{g_5''(0)}{2}x_{5t}^2(0) + \frac{g_5'''(0)}{3!}x_{5t}^3(0) + \frac{f_5''(0)}{2}x_{1t}^2(0) \\
&\quad + \frac{f_5'''(0)}{3!}x_{1t}^3(0) + \frac{f_5''(0)}{2}x_{4t}^2(-\tau_2) \\
&\quad + \frac{f_5'''(0)}{3!}x_{4t}^3(-\tau_2) + \text{h.o.t.}
\end{aligned}
$$

Noticing that

$$x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta), x_{4t}(\theta), x_{5t}(\theta))^T = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}$$

and

$$q(\theta) = (1, a_1, a_2, a_3, a_4)^T e^{i\omega_0\theta},$$

we have

$$
\begin{aligned}
x_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots, \\
x_{2t}(0) &= a_1 z + \bar{a}_1\bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
x_{3t}(0) &= a_2 z + \bar{a}_2 \bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0) z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
x_{4t}(0) &= a_3 z + \bar{a}_3 \bar{z} + W_{20}^{(4)}(0)\frac{z^2}{2} + W_{11}^{(4)}(0) z\bar{z} + W_{02}^{(4)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
x_{5t}(0) &= a_4 z + \bar{a}_4 \bar{z} + W_{20}^{(5)}(0)\frac{z^2}{2} + W_{11}^{(5)}(0) z\bar{z} + W_{02}^{(5)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
x_{2t}(-\tau_2) &= a_1 e^{-i\omega_0\tau_2} z + \bar{a}_1 e^{i\omega_0\tau_2}\bar{z} + W_{20}^{(2)}(-\tau_2)\frac{z^2}{2}\\
&\quad + W_{11}^{(2)}(-\tau_2) z\bar{z} + W_{02}^{(2)}(-\tau_2)\frac{\bar{z}^2}{2} + \cdots,\\
x_{4t}(-\tau_2) &= a_3 e^{-i\omega_2\tau_0} z + \bar{a}_3 e^{i\omega_0\tau_2}\bar{z} + W_{20}^{(4)}(-\tau_2)\frac{z^2}{2}\\
&\quad + W_{11}^{(4)}(-\tau_2) z\bar{z} + W_{02}^{(4)}(-\tau_2)\frac{\bar{z}^2}{2} + \cdots,\\
x_{1t}(-\tau_1) &= e^{-i\omega_0\tau_1} z + e^{i\omega_0\tau_1}\bar{z} + W_{20}^{(1)}(-\tau_1)\frac{z^2}{2}\\
&\quad + W_{11}^{(1)}(-\tau_1) z\bar{z} + W_{02}^{(1)}(-\tau_1)\frac{\bar{z}^2}{2} + \cdots.\\
x_{3t}(-\tau_1) &= a_2 e^{-i\omega_1\tau_0} z + \bar{a}_2 e^{i\omega_0\tau_1}\bar{z} + W_{20}^{(3)}(-\tau_1)\frac{z^2}{2}\\
&\quad + W_{11}^{(3)}(-\tau_1) z\bar{z} + W_{02}^{(3)}(-\tau_1)\frac{\bar{z}^2}{2} + \cdots,\\
x_{5t}(-\tau_1) &= a_4 e^{-i\omega_0\tau_1} z + \bar{a}_4 e^{i\omega_0\tau_1}\bar{z} + W_{20}^{(5)}(-\tau_1)\frac{z^2}{2} + W_{11}^{(5)}(-\tau_1) z\bar{z}\\
&\quad + W_{02}^{(5)}(-\tau_1)\frac{\bar{z}^2}{2} + \cdots.
\end{aligned}
$$

From (40) and (41), we have

$$
\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0) f_0(z,\bar{z})\\
&= \bar{D}\left[f_1(0,x_t) + \bar{a}_1^* f_2(0,x_t) + \bar{a}_2^* f_3(0,x_t)\right.\\
&\quad \left. + \bar{a}_3^* f_4(0,x_t) + \bar{a}_4^* f_5(0,x_t)\right]\\
&= \frac{1}{2}\bar{D}\Big\{ \left[g_1^{''}(0) + f_1^{''}(0) a_1^2 e^{-2i\omega_0\tau_2} + f_1^{''}(0) a_4^2\right]\\
&\quad + \bar{a}_1^* \left[g_2^{''}(0) a_1^2 + f_2^{''}(0) e^{-2i\omega_0\tau_2} + f_2^{''}(0) a_2^2 e^{-2i\omega_0\tau_2}\right]\\
&\quad + \bar{a}_2^* \left[g_3^{''}(0) a_1^2 + f_3^{''}(0) e^{-2i\omega_0\tau_2} + f_3^{''}(0) a_3^2 e^{-2i\omega_0\tau_2}\right]\\
&\quad + \bar{a}_3^* \left[g_4^{''}(0) a_3^2 + f_4^{''}(0) a_2^2 e^{-2i\omega_0\tau_2} + f_4^{''}(0) a_4^2 e^{-2i\omega_0\tau_2}\right]\\
&\quad + \bar{a}_4^* \left[g_5^{''}(0) a_4^2 + f_5^{''}(0) + f_5^{''}(0) a_3 e^{-2i\omega_0\tau_2}\right] \Big\} z^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \bar{D}\Big\{ \Big[g_1^{''}(0) + f_1^{''}(0)a_1^2 e^{-2i\omega_0\tau_2} + f_1^{''}(0)a_4\bar{a}_4 + g_2^{''}(0)a_1\bar{a}_1\Big] \\
&+ \bar{a}_1^* \Big[g_2^{''}(0)a_1\bar{a}_1 + f_2^{''}(0) + f_2^{''}(0)a_2\bar{a}_2\Big] \\
&+ \bar{a}_2^* \Big[g_2^{''}(0)a_1\bar{a}_1 + f_3^{''}(0)a_1\bar{a}_1 + f_3^{''}(0)a_3\bar{a}_3\Big] \\
&+ \bar{a}_3^* \Big[g_4^{''}(0)a_3\bar{a}_3 + f_4^{''}(0)a_2\bar{a}_2 + f_4^{''}(0)a_4\bar{a}_4\Big] \\
&+ \bar{a}_4^* \Big[g_5^{''}(0)a_4\bar{a}_3 + f_5^{''}(0) + f_4^{''}(0)a_3\bar{a}_3\Big] \Big\} z\bar{z}, \\
&+ \frac{1}{2}\bar{D}\Big\{ \Big[g_1^{''}(0) + f_1^{''}(0)a_1\bar{a}_1 e^{2i\omega_0\tau_2} + f_1^{''}(0)\bar{a}_4^2 + g_2^{''}(0)\bar{a}_1^2\Big] \\
&+ \bar{a}_1^* \Big[g_2^{''}(0)\bar{a}_1^2 + f_2^{''}(0)e^{2i\omega_0\tau_1} + f_2^{''}(0)a_2^2 e^{2i\omega_0\tau_1}\Big] \\
&+ \bar{a}_2^* \Big[g_2^{''}(0)\bar{a}_1^2 + f_3^{''}(0)\bar{a}_1^2 e^{2i\omega_0\tau_2} + f_3^{''}(0)\bar{a}_3^2 e^{2i\omega_0\tau_2}\Big] \\
&+ \bar{a}_3^* \Big[g_4^{''}(0)\bar{a}_3^2 + f_4^{''}(0)\bar{a}_2^2 e^{2i\omega_0\tau_1} + f_4^{''}(0)\bar{a}_4^2 e^{2i\omega_0\tau_1}\Big] \\
&+ \bar{a}_4^* \Big[g_5^{''}(0)\bar{a}_4^2 + f_5^{''}(0) + f_5^{''}(0)\bar{a}_3^2 e^{2i\omega_0\tau_2}\Big] \Big\} \bar{z}^2, \\
&+ \frac{1}{2}\bar{D}\Big\{ g_1^{''}(0)\Big[W_{20}^{(1)}(0)) + 2W_{11}^{(1)}(0)\Big] + g_1^{'''}(0) \\
&+ f_1^{'''}(0)\Big[W_{20}^{(2)}(-\tau_2))\bar{a}_1 e^{i\omega_0\tau_2} + 2W_{11}^{(2)}(-\tau_2)a_1 e^{-i\omega_0\tau_2}\Big] \\
&+ f_1^{'''}(0)a_1^2\bar{a}_1 e^{-i\omega_0\tau_2} + f_1^{''}(0)\Big[W_{50}^{(5)}(0)\bar{a}_4 + 2W_{11}^{(5)}(0)a_4\Big] \\
&+ f_1^{'''}(0)a_2^4\bar{a}_4 + g_2^{''}(0)\Big[W_{20}^{(2)}(0)\bar{a}_1 + 2W_{11}^{(2)}(0)a_1\Big] \\
&+ \bar{a}_1^* \Big[g_2^{''}(0)\Big(W_{20}^{(2)}(0)\bar{a}_1 + 2W_{11}^{(2)}(0)a_1\Big) + g_2^{'''}(0)a_1^2\bar{a}_1 \\
&+ f_2^{'''}(0)\Big(W_{20}^{(1)}(-\tau_1)e^{i\omega_0\tau_1} + 2W_{11}^{(1)}(-\tau_1)e^{-i\omega_0\tau_1}\Big) \\
&+ f_2^{'''}(0)e^{-i\omega_0\tau_1} + f_2^{''}(0)\Big(W_{20}^{(3)}\ (-\tau_1)\bar{a}_2 e^{i\omega_0\tau_1} \\
&+\ 2W_{11}^{(3)}(-\tau_1)a_2 e^{-i\omega_0\tau_1}\Big)\Big] + \bar{a}_2^* \Big[g_1^{''}(0)\Big(W_{20}^{(2)}\ (0)\bar{a}_1 \\
&+ 2W_{11}^{(2)}(0)a_1\Big) + g_3^{'''}(0)a_2^2\bar{a}_2 + f_3^{'''}(0)\Big(W_{20}^{(2)}(-\tau_2)\bar{a}_1 e^{i\omega_0\tau_2} \\
&+ 2W_{11}^{(2)}(-\tau_2)a_1 e^{-i\omega_0\tau_2}\Big) + f_3^{'''}(0)a_1^2\bar{a}_1 e^{i\omega_0\tau_2} \\
&+ f_3^{''}(0)\Big(W_{20}^{(4)}(-\tau_2)\bar{a}_3 e^{i\omega_0\tau_2} + 2W_{11}^{(4)}(-\tau_2)a_3 e^{-i\omega_0\tau_2}\Big) \\
&+\ f_3^{'''}(0)a_3^2\bar{a}_3 e^{-i\omega_0\tau_2}\Big] + \bar{a}_3^* \Big[g_4^{''}(0)\Big(W_{20}^{(4)}(0)\bar{a}_3 + 2W_{11}^{(4)}(0)a_3\Big) \\
&+ g_4^{'''}(0)a_3^2\bar{a}_3 + f_4^{''}(0)\Big(W_{20}^{(3)}(-\tau_1)\bar{a}_2 e^{i\omega_0\tau_1}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2W_{11}^{(3)}(-\tau_1)a_2e^{-i\omega_0\tau_1}\Big) \\
&\times f_4^{'''}(0)a_2^2\bar{a}_2e^{-i\omega_0\tau_1} + f_4^{''}(0)\left(W_{20}^{(5)}(-\tau_1)\bar{a}_4e^{i\omega_0\tau_1}\right. \\
&+ \left. 2W_{11}^{(5)}(-\tau_1)a_4e^{-i\omega_0\tau_1}\right) + f_4^{'''}(0)a_4^2\bar{a}_4e^{-i\omega_0\tau_1}\Big] \\
&+ \bar{a}_4^*\left[g_5^{''}(0)\left(W_{20}^{(5)}(0)\bar{a}_4 + 2W_{11}^{(5)}(0)a_4\right) + g_5^{'''}(0)a_4^2\bar{a}_4\right. \\
&+ f_5^{''}(0)\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) \\
&+ f_5^{'''}(0) + f_5^{''}(0)\left(W_{20}^{(4)}(-\tau_2)\bar{a}_3e^{i\omega_0\tau_2}\right. \\
&+ \left.\left. 2W_{11}^{(4)}(-\tau_2)a_3e^{-i\omega_0\tau_2} + f_5^{'''}(0)a_3^2\bar{a}_3e^{-i\omega_0\tau_2}\right)\right]\Big\}z^2\bar{z} + \cdots,
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
g_{20} = \bar{D}\Big\{ &\left[g_1^{''}(0) + f_1^{''}(0)a_1^2e^{-2i\omega_0\tau_2} + f_1^{''}(0)a_4^2\right] \\
&+ \bar{a}_1^*\left[g_2^{''}(0)a_1^2 + f_2^{''}(0)e^{-2i\omega_0\tau_2} + f_2^{''}(0)a_2^2e^{-2i\omega_0\tau_2}\right] \\
&+ \bar{a}_2^*\left[g_3^{''}(0)a_1^2 + f_3^{''}(0)e^{-2i\omega_0\tau_2} + f_3^{''}(0)a_3^2e^{-2i\omega_0\tau_2}\right] \\
&+ \bar{a}_3^*\left[g_4^{''}(0)a_3^2 + f_4^{''}(0)a_2^2e^{-2i\omega_0\tau_2} + f_4^{''}(0)a_4^2e^{-2i\omega_0\tau_2}\right] \\
&+ \bar{a}_4^*\left[g_5^{''}(0)a_4^2 + f_5^{''}(0) + f_5^{''}(0)a_3e^{-2i\omega_0\tau_2}\right]\Big\}, \\
g_{11} = \bar{D}\Big\{ &\left[g_1^{''}(0) + f_1^{''}(0)a_1^2e^{-2i\omega_0\tau_2} + f_1^{''}(0)a_4\bar{a}_4 + g_2^{''}(0)a_1\bar{a}_1\right] \\
&+ \bar{a}_1^*\left[g_2^{''}(0)a_1\bar{a}_1 + f_2^{''}(0) + f_2^{''}(0)a_2\bar{a}_2\right] \\
&+ \bar{a}_2^*\left[g_2^{''}(0)a_1\bar{a}_1 + f_3^{''}(0)a_1\bar{a}_1 + f_3^{''}(0)a_3\bar{a}_3\right] \\
&+ \bar{a}_3^*\left[g_4^{''}(0)a_3\bar{a}_3 + f_4^{''}(0)a_2\bar{a}_2 + f_4^{''}(0)a_4\bar{a}_4\right] \\
&+ \bar{a}_4^*\left[g_5^{''}(0)a_4\bar{a}_3 + f_5^{''}(0) + f_4^{''}(0)a_3\bar{a}_3\right]\Big\}, \\
g_{02} = \bar{D}\Big\{ &\left[g_1^{''}(0) + f_1^{''}(0)a_1\bar{a}_1e^{2i\omega_0\tau_2} + f_1^{''}(0)\bar{a}_4^2 + g_2^{''}(0)\bar{a}_1^2\right] \\
&+ \bar{a}_1^*\left[g_2^{''}(0)\bar{a}_1^2 + f_2^{''}(0)e^{2i\omega_0\tau_1} + f_2^{''}(0)a_2^2e^{2i\omega_0\tau_1}\right] \\
&+ \bar{a}_2^*\left[g_2^{''}(0)\bar{a}_1^2 + f_3^{''}(0)\bar{a}_1^2e^{2i\omega_0\tau_2} + f_3^{''}(0)\bar{a}_3^2e^{2i\omega_0\tau_2}\right] \\
&+ \bar{a}_3^*\left[g_4^{''}(0)\bar{a}_3^2 + f_4^{''}(0)\bar{a}_2^2e^{2i\omega_0\tau_1} + f_4^{''}(0)\bar{a}_4^2e^{2i\omega_0\tau_1}\right] \\
&+ \bar{a}_4^*\left[g_5^{''}(0)\bar{a}_4^2 + f_5^{''}(0) + f_5^{''}(0)\bar{a}_3^2e^{2i\omega_0\tau_2}\right]\Big\}, \\
g_{21} = \bar{D}\Big\{ &g_1^{''}(0)\left[W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right] + g_1^{'''}(0)
\end{aligned}
$$

$$
\begin{aligned}
&+ f_1^{'''}(0)\left[W_{20}^{(2)}(-\tau_2)\bar{a}_1 e^{i\omega_0\tau_2} + 2W_{11}^{(2)}(-\tau_2)a_1 e^{-i\omega_0\tau_2}\right] \\
&+ f_1^{'''}(0)a_1^2\bar{a}_1 e^{-i\omega_0\tau_2} + f_1^{''}(0)\left[W_{20}^{(5)}(0)\bar{a}_4 + 2W_{11}^{(5)}(0)a_4\right] \\
&+ f_1^{'''}(0)a_2^4\bar{a}_4 + g_2^{''}(0)\left[W_{20}^{(2)}(0)\bar{a}_1 + 2W_{11}^{(2)}(0)a_1\right] \\
&+ \bar{a}_1^*\Big[g_2^{''}(0)\left(W_{20}^{(2)}(0)\bar{a}_1 + 2W_{11}^{(2)}(0)a_1\right) + g_2^{'''}(0)a_1^2\bar{a}_1 \\
&+ f_2^{'''}(0)\left(W_{20}^{(1)}(-\tau_1)e^{i\omega_0\tau_1} + 2W_{11}^{(1)}(-\tau_1)e^{-i\omega_0\tau_1}\right) \\
&+ f_2^{'''}(0)e^{-i\omega_0\tau_1} + f_2^{''}(0)\Big(W_{20}^{(3)}\ (-\tau_1)\bar{a}_2 e^{i\omega_0\tau_1} \\
&+\ 2W_{11}^{(3)}(-\tau_1)a_2 e^{-i\omega_0\tau_1}\Big)\Big] + \bar{a}_2^*\Big[g_1^{''}(0)\Big(W_{20}^{(2)}(0)\bar{a}_1 \\
&+ 2W_{11}^{(2)}(0)a_1\Big) + g_3^{'''}(0)a_2^2\bar{a}_2 + f_3^{'''}(0)\Big(W_{20}^{(2)}(-\tau_2)\bar{a}_1 e^{i\omega_0\tau_2} \\
&+ 2W_{11}^{(2)}(-\tau_2)a_1 e^{-i\omega_0\tau_2}\Big) + f_3^{'''}(0)a_1^2\bar{a}_1 e^{i\omega_0\tau_2} \\
&+ f_3^{''}(0)\left(W_{20}^{(4)}(-\tau_2)\bar{a}_3 e^{i\omega_0\tau_2} + 2W_{11}^{(4)}(-\tau_2)a_3 e^{-i\omega_0\tau_2}\right) \\
&+\ f_3^{'''}(0)a_3^2\bar{a}_3 e^{-i\omega_0\tau_2}\,] + \bar{a}_3^*\Big[g_4^{''}(0)\left(W_{20}^{(4)}(0)\bar{a}_3 + 2W_{11}^{(4)}(0)a_3\right) \\
&+ g_4^{'''}(0)a_3^2\bar{a}_3 + f_4^{''}(0)\left(W_{20}^{(3)}(-\tau_1)\bar{a}_2 e^{i\omega_0\tau_1} + 2W_{11}^{(3)}(-\tau_1)a_2 e^{-i\omega_0\tau_1}\right) \\
&\times f_4^{'''}(0)a_2^2\bar{a}_2 e^{-i\omega_0\tau_1} + f_4^{''}(0)\Big(W_{20}^{(5)}(-\tau_1)\bar{a}_4 e^{i\omega_0\tau_1} \\
&+\ 2W_{11}^{(5)}(-\tau_1)a_4 e^{-i\omega_0\tau_1}\Big) + f_4^{'''}(0)a_4^2\bar{a}_4 e^{-i\omega_0\tau_1}\Big] \\
&+ \bar{a}_4^*\Big[g_5^{''}(0)\left(W_{20}^{(5)}(0)\bar{a}_4 + 2W_{11}^{(5)}(0)a_4\right) + g_5^{'''}(0)a_4^2\bar{a}_4 \\
&+ f_5^{''}(0)\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) + f_5^{'''}(0) + f_5^{''}(0)\Big(W_{20}^{(4)}(-\tau_2)\bar{a}_3 e^{i\omega_0\tau_2} \\
&+\ 2W_{11}^{(4)}(-\tau_2)a_3 e^{-i\omega_0\tau_2} + f_5^{'''}(0)a_3^2\bar{a}_3 e^{-i\omega_0\tau_2}\Big)\Big]\Big\}.
\end{aligned}
$$

For unknown

$$
\begin{gathered}
W_{20}^{(1)}(0), W_{10}^{(1)}(0), W_{20}^{(1)}(-\tau_1), W_{11}^{(1)}(-\tau_1), \\
W_{20}^{(4)}(0), W_{20}^{(4)}(-\tau_2), W_{11}^{(4)}(0), W_{11}^{(4)}(-\tau_2), \\
W_{20}^{(2)}(0), W_{20}^{(2)}(-\tau_2), W_{11}^{(2)}(0), W_{11}^{(2)}(-\tau_2), \\
W_{20}^{(5)}(0), W_{11}^{(5)}(0), W_{20}^{(3)}(-\tau_1), W_{11}^{(3)}(-\tau_1)
\end{gathered}
$$

in g_{21}, we still need to compute them.

From (27) and (28), we have

$$
W' = \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -\tau_2^* \le \theta < 0, \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\} + \bar{f}, & \theta = 0. \end{cases}
$$

$$\stackrel{\text{def}}{=} \quad AW + H(z,\bar{z},\theta), \tag{42}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots . \tag{43}$$

Comparing the coefficients, we obtain

$$(A - 2i\omega_0)W_{20} = -H_{20}(\theta), \tag{44}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{45}$$

And we know that for $\theta \in [-\tau_2^*, 0)$

$$H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \tag{46}$$

Comparing the coefficients of (43) with (46) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{47}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{48}$$

From (44), (47) and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \tag{49}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta}, \tag{50}$$

where E_1 is a constant vector. Similarly, from (45), (48) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \tag{51}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2, \tag{52}$$

where E_2 is a constant vector.

In what follows, we shall seek appropriate E_1, E_2 in (50), (52), respectively. It follows from the definition of A and (47), (48) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0W_{20}(0) - H_{20}(0) \tag{53}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{54}$$

where $\eta(\theta) = \eta(0,\theta)$.

From (48), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + (H_1, H_2, H_3, H_4, H_5)^T, \tag{55}$$

where

$$\begin{aligned} H_1 &= g_1^{''}(0) + f_1^{''}(0)a_1^2e^{-2i\omega_0\tau_2} + f_1^{''}(0)a_4^2, \\ H_2 &= g_2^{''}(0)a_1^2 + f_2^{''}(0)e^{-2i\omega_0\tau_1} + f_2^{''}(0)a_2e^{-2i\omega_0\tau_1}, \end{aligned}$$

$$
\begin{aligned}
H_3 &= g_2^{''}(0)a_1^2 + f_3^{''}(0)a_1^2e^{-2i\omega_0\tau_2} + f_3^{''}(0)a_3^2e^{-2i\omega_0\tau_2},\\
H_4 &= g_4^{''}(0)a_3^2 + f_4^{''}(0)a_2^2e^{-2i\omega_0\tau_1} + f_4^{''}(0)a_4^2e^{-2i\omega_0\tau_1},\\
H_5 &= g_5^{''}(0)a_4^2 + f_5^{''}(0) + f_5^{''}(0)a_3e^{-2i\omega_0\tau_2}.
\end{aligned}
$$

From (49), we have

$$
H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + (P_1, P_2, P_3, P_4, P_5)^T, \tag{56}
$$

where

$$
\begin{aligned}
P_1 &= g_1^{''}(0) + f_1^{''}(0)a_1^2e^{-2i\omega_0\tau_2} + f_1^{''}(0)a_4\bar{a}_4 + g_2^{''}(0)a_1\bar{a}_1,\\
P_2 &= g_2^{''}(0)a_1\bar{a}_1 + f_2^{''}(0) + f_2^{''}(0)a_2\bar{a}_2,\\
P_3 &= g_2^{''}(0)a_1\bar{a}_1 + f_3^{''}(0)a_1\bar{a}_1 + f_3^{''}(0)a_3\bar{a}_3,\\
P_4 &= g_4^{''}(0)a_3\bar{a}_3 + f_4^{''}(0)a_2\bar{a}_2 + f_4^{''}(0)a_4\bar{a}_4,\\
P_5 &= g_5^{''}(0)a_4\bar{a}_4 + f_5^{''}(0) + f_5^{''}(0)a_3\bar{a}_3.
\end{aligned}
$$

From (44), (45) and the definition of A, we have

$$
\begin{cases}
BW_{20}(0) + B_1W_{20}(-\tau_1^*) + B_2(-\tau_2^*) = 2i\omega_0W_{20} - H_{20}(0),\\
BW_{11}(0) + B_1W_{11}(-\tau_1^*) + B_2(-\tau_2^*) = -H_{11}(0).
\end{cases} \tag{57}
$$

Noting that

$$
\left(i\omega_0 I - \int_{-\tau_2^*}^{0} e^{i\omega_0\theta} d\eta(\theta)\right) q(0) = 0, \tag{58}
$$

$$
\left(-i\omega_0 I - \int_{-\tau_2^*}^{0} e^{-i\omega_0\theta} d\eta(\theta)\right) \bar{q}(0) = (H_1, H_2, H_3, H_4, H_5)^T \tag{59}
$$

and substituting (54) and (59) into (57), we have

$$
\left(2i\omega_0 I - \int_{-\tau_2^*}^{0} e^{2i\omega_0\theta} d\eta(\theta)\right) E_1 = (H_1, H_2, H_3, H_4, H_5)^T. \tag{60}
$$

Thus the equality given in Fig. 1 follows.

Hence,

$$
E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1},\ E_1^{(4)} = \frac{\Delta_{14}}{\Delta_1},\ E_1^{(5)} = \frac{\Delta_{15}}{\Delta_1}, \tag{61}
$$

where the determinants Δ_1 and Δ_{ij} are given in Figs. 2–4.

Similarly, substituting (55) and (60) into (58), we have

$$
\left(\int_{-\tau_2^*}^{0} d\eta(\theta)\right) E_2 = (P_1, P_2, P_3, P_4, P_5)^T. \tag{62}
$$

$$\begin{pmatrix} 2i\omega_0+m_1 & -f_1^{'}(0)e^{-2i\omega_0\tau_2} & 0 & 0 & 0 \\ -f_2^{'}(0)e^{-2i\omega_0\tau_1} & 2i\omega_0+m_2 & f_2^{'}(0)e^{-2i\omega_0\tau_1} & 0 & 0 \\ 0 & f_3^{'}(0)e^{-2i\omega_0\tau_2} & 2i\omega_0+m_3 & f_3^{'}(0)e^{-2i\omega_0\tau_2} & 0 \\ 0 & 0 & f_4^{'}(0)e^{-2i\omega_0\tau_1} & 2i\omega_0+m_4 & f_4^{'}(0)e^{-2i\omega_0\tau_1} \\ -f_5^{'}(0) & 0 & 0 & f_5^{'}(0)e^{-2i\omega_0\tau_2} & 2i\omega_0+m_5 \end{pmatrix}$$
$$\times (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)}, E_1^{(5)})^T$$
$$= (H_1, H_2, H_3, H_4, H_5)^T. \qquad (61)$$

Fig. 1

That is

$$\begin{pmatrix} -m_1 & f_1^{'}(0) & 0 & 0 & f_1^{'}(0) \\ f_2^{'}(0) & -m_2 & f_2^{'}(0) & 0 & 0 \\ 0 & f_3^{'}(0) & -m_3 & f_3^{'}(0) & 0 \\ 0 & 0 & -f_4^{'}(0) & -m_4 & -f_4^{'}(0) \\ f_5^{'}(0) & 0 & 0 & f_5^{'}(0) & -m_5 \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \\ E_2^{(4)} \\ E_2^{(5)} \end{pmatrix} = \begin{pmatrix} -P_1 \\ -P_2 \\ -P_3 \\ -P_4 \\ -P_5 \end{pmatrix}. \qquad (63)$$

$$\Delta_1 = \det \begin{pmatrix} 2i\omega_0 + m_1 & -f_1'(0)e^{-2i\omega_0\tau_2} & 0 & 0 & 0 \\ -f_2'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0 + m_2 & f_2'(0)e^{-2i\omega_0\tau_1} & 0 & 0 \\ 0 & f_3'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0 + m_3 & f_3'(0)e^{-2i\omega_0\tau_2} & 0 \\ 0 & 0 & f_4'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0 + m_4 & f_4'(0)e^{-2i\omega_0\tau_1} \\ -f_5'(0) & 0 & 0 & f_5'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0 + m_5 \end{pmatrix},$$

$$\Delta_{11} = \det \begin{pmatrix} H_1 & -f_1'(0)e^{-2i\omega_0\tau_2} & 0 & 0 & 0 \\ H_2 & 2i\omega_0 + m_2 & f_2'(0)e^{-2i\omega_0\tau_1} & 0 & 0 \\ H_3 & f_3'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0 + m_3 & f_3'(0)e^{-2i\omega_0\tau_2} & 0 \\ H_4 & 0 & f_4'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0 + m_4 & f_4'(0)e^{-2i\omega_0\tau_1} \\ H_5 & 0 & 0 & f_5'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0 + m_5 \end{pmatrix},$$

Fig. 2

$$\Delta_{12} = \det\begin{pmatrix} 2i\omega_0 + m_1 & H_1 & 0 & 0 & 0 \\ -f_2'(0)e^{-2i\omega_0\tau_1} & H_2 & f_2'(0)e^{-2i\omega_0\tau_1} & 0 & 0 \\ 0 & H_3 & 2i\omega_0 + m_3 & f_3'(0)e^{-2i\omega_0\tau_2} & 0 \\ 0 & H_4 & f_4'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0 + m_4 & f_4'(0)e^{-2i\omega_0\tau_1} \\ -f_5'(0) & H_5 & 0 & f_5'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0 + m_5 \end{pmatrix},$$

$$\Delta_{13} = \det\begin{pmatrix} 2i\omega_0 + m_1 & -f_1'(0)e^{-2i\omega_0\tau_2} & H_1 & 0 & 0 \\ -f_2'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0 + m_2 & H_2 & 0 & 0 \\ 0 & f_3'(0)e^{-2i\omega_0\tau_2} & H_3 & f_3'(0)e^{-2i\omega_0\tau_2} & 0 \\ 0 & 0 & H_4 & 2i\omega_0 + m_4 & f_4'(0)e^{-2i\omega_0\tau_1} \\ -f_5'(0) & 0 & H_5 & f_5'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0 + m_5 \end{pmatrix},$$

Fig. 3

$$\Delta_{14} = \det\begin{pmatrix} 2i\omega_0+m_1 & -f_1'(0)e^{-2i\omega_0\tau_2} & 0 & H_1 & 0 \\ -f_2'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0+m_2 & f_2'(0)e^{-2i\omega_0\tau_1} & H_2 & 0 \\ 0 & f_3'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0+m_3 & H_3 & 0 \\ 0 & 0 & f_4'(0)e^{-2i\omega_0\tau_1} & H_4 & f_4'(0)e^{-2i\omega_0\tau_1} \\ -f_5'(0) & 0 & 0 & H_5 & 2i\omega_0+m_5 \end{pmatrix},$$

$$\Delta_{15} = \det\begin{pmatrix} 2i\omega_0+m_1 & -f_1'(0)e^{-2i\omega_0\tau_2} & 0 & 0 & H_1 \\ -f_2'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0+m_2 & f_2'(0)e^{-2i\omega_0\tau_1} & 0 & H_2 \\ 0 & f_3'(0)e^{-2i\omega_0\tau_2} & 2i\omega_0+m_3 & f_3'(0)e^{-2i\omega_0\tau_2} & H_3 \\ 0 & 0 & f_4'(0)e^{-2i\omega_0\tau_1} & 2i\omega_0+m_4 & H_4 \\ -f_5'(0) & 0 & 0 & f_5'(0)e^{-2i\omega_0\tau_2} & H_5 \end{pmatrix}.$$

Fig. 4

Hence,

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},\ E_2^{(4)} = \frac{\Delta_{24}}{\Delta_2},\ E_2^{(5)} = \frac{\Delta_{25}}{\Delta_2}, \quad (64)$$

where

$$\begin{aligned}
\Delta_2 &= \det\begin{pmatrix} -m_1 & f_1'(0) & 0 & 0 & f_1'(0) \\ f_2'(0) & -m_2 & f_2'(0) & 0 & 0 \\ 0 & f_3'(0) & -m_3 & f_3'(0) & 0 \\ 0 & 0 & -f_4'(0) & -m_4 & -f_4'(0) \\ f_5'(0) & 0 & 0 & f_5'(0) & -m_5 \end{pmatrix}, \\
\Delta_{21} &= \det\begin{pmatrix} -P_1 & f_1'(0) & 0 & 0 & f_1'(0) \\ -P_2 & -m_2 & f_2'(0) & 0 & 0 \\ -P_3 & f_3'(0) & -m_3 & f_3'(0) & 0 \\ -P_4 & 0 & -f_4'(0) & -m_4 & -f_4'(0) \\ -P_5 & 0 & 0 & f_5'(0) & -m_5 \end{pmatrix}, \\
\Delta_{22} &= \det\begin{pmatrix} -m_1 & -P_1 & 0 & 0 & f_1'(0) \\ f_2'(0) & -P_2 & f_2'(0) & 0 & 0 \\ 0 & -P_3 & -m_3 & f_3'(0) & 0 \\ 0 & -P_4 & -f_4'(0) & -m_4 & -f_4'(0) \\ f_5'(0) & -P_5 & 0 & f_5'(0) & -m_5 \end{pmatrix}, \\
\Delta_{23} &= \det\begin{pmatrix} -m_1 & f_1'(0) & -P_1 & 0 & f_1'(0) \\ f_2'(0) & -m_2 & -P_2 & 0 & 0 \\ 0 & f_3'(0) & -P_3 & f_3'(0) & 0 \\ 0 & 0 & -P_4 & -m_4 & -f_4'(0) \\ f_5'(0) & 0 & -P_5 & f_5'(0) & -m_5 \end{pmatrix}, \\
\Delta_{24} &= \det\begin{pmatrix} -m_1 & f_1'(0) & 0 & -P_1 & f_1'(0) \\ f_2'(0) & -m_2 & f_2'(0) & -P_2 & 0 \\ 0 & f_3'(0) & -m_3 & -P_3 & 0 \\ 0 & 0 & -f_4'(0) & -P_4 & -f_4'(0) \\ f_5'(0) & 0 & 0 & -P_5 & -m_5 \end{pmatrix}, \\
\Delta_{25} &= \det\begin{pmatrix} -m_1 & f_1'(0) & 0 & 0 & -P_1 \\ f_2'(0) & -m_2 & f_2'(0) & 0 & -P_2 \\ 0 & f_3'(0) & -m_3 & f_3'(0) & -P_3 \\ 0 & 0 & -f_4'(0) & -m_4 & -P_4 \\ f_5'(0) & 0 & 0 & f_5'(0) & -P_5 \end{pmatrix}.
\end{aligned}$$

From (50), (52), (61), (64), we can calculate g_{21} and derive the following values:

$$
\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}},\\
\beta_2 &= 2\mathrm{Re}(c_1(0)),\\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2 \mathrm{Im}\{\lambda^{'}(\tau_0)\}}{\omega_0}.
\end{aligned}
$$

These formulas give a description of the Hopf bifurcation periodic solutions of (3) at $\tau = \tau_0$, on the center manifold. From the discussion above, we have the following result.

Theorem 5. *For system* (3)*, if* $(H1) - (H4)$ *hold, the periodic solution is supercritical (subcritical) if* $\mu_2 > 0$ $(\mu_2 < 0)$*; The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if* $\beta_2 < 0$ $(\beta_2 > 0)$*. The periodic of the bifurcating periodic solutions increase (decrease) if* $T_2 > 0$ $(T_2 < 0)$.

4. Numerical examples

In this section, we present some numerical results of system (3) to verify the analytical predictions obtained in the previous section. From Sec. 3, we can determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Consider the following special case of system (3):

$$
\begin{cases}
\dot{x}_1(t) = -\ 0.6x_1(t) - 0.6\tanh(x_1(t)) + 0.3\tanh(x_5(t))\\
\qquad\quad + \ 0.5\tanh(x_2(t-\tau_2)),\\
\dot{x}_2(t) = -\ 1.7x_2(t) + 0.5\tanh(x_2(t)) - 1.4\tanh(x_1(t-\tau_1))\\
\qquad\quad - \ 1.3\tanh(x_3(t-\tau_1)),\\
\dot{x}_3(t) = -\ 0.8x_3(t) - 0.5\tanh(x_3(t)) + 0.6\tanh(x_2(t-\tau_2))\\
\qquad\quad + \ 0.4\tanh(x_4(t-\tau_2)),\\
\dot{x}_4(t) = -\ 1.8x_4(t) + 0.8\tanh(x_4(t)) - 0.8\tanh(x_3(t-\tau_1))\\
\qquad\quad - \ 0.8\tanh(x_5(t-\tau_1)),\\
\dot{x}_5(t) = -\ 1.5x_5(t) + 0.7\tanh(x_5(t)) - \ 0.7\tanh(x_4(t-\tau_2))\\
\qquad\quad + \ 0.6\tanh(x_1(t)).
\end{cases}
\tag{65}
$$

By some complicated computation by means of Matlab 7.0, we get $\omega_0 \approx 0.8541, \tau_0 \approx 6.2, \lambda^{'}(\tau_0) \approx 1.2437 - 3.4122i$. Noting that $\tanh^{''}(0) = 0$, we can easily obtain $g_{20} = g_{02} = 0, g_{11} \approx -4.2832 + 4.2139i$. Thus we can calculate the following values: $c_1(0) \approx -2.9542 - 22.2355i, \mu_2 \approx 0.5642, \beta_2 \approx -4.4636, T_2 \approx 22.1327$. We obtain that the conditions indicated in Theorem 2 are satisfied. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$.

Choose $\tau_1 = 3, \tau_2 = 2.5$, then $\tau = \tau_1 + \tau_2 = 5.5 < \tau_0 \approx 6.2$. Thus, the equilibrium $x_*(0,0,0,0,0)$ is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figs. 5–7). When τ passes through the critical value $\tau_0 \approx 6.2$, the equilibrium $x_*(0,0,0,0,0)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the equilibrium $x_*(0,0,0,0,0)$. Choose $\tau_1 = 4, \tau_2 = 3.5$, then $\tau = \tau_1 + \tau_2 = 7.5 < \tau_0 \approx 6.2$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $x_*(0,0,0,0,0)$ at τ_0 are stable, which are depicted in Figs. 8–10.

5. Conclusions

In this paper, we have investigated local stability of the equilibrium $x_*(0,0,0,0,0)$ and local Hopf bifurcation in a ring of five-neuron model with discrete delays. we have showed that if the conditions (H1), (H2), (H3) and (H4) hold, the equilibrium $x_*(0,0,0,0,0)$ of system (3) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. We have also showed that, if the condition (H1), (H2), (H3) and (H4) hold, as the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occurs at $x_*(0,0,0,0,0)$, i.e., a family of periodic orbits bifurcates from the the positive equilibrium $x_*(0,0,0,0,0)$. At last, direction of Hopf bifurcation and stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem.

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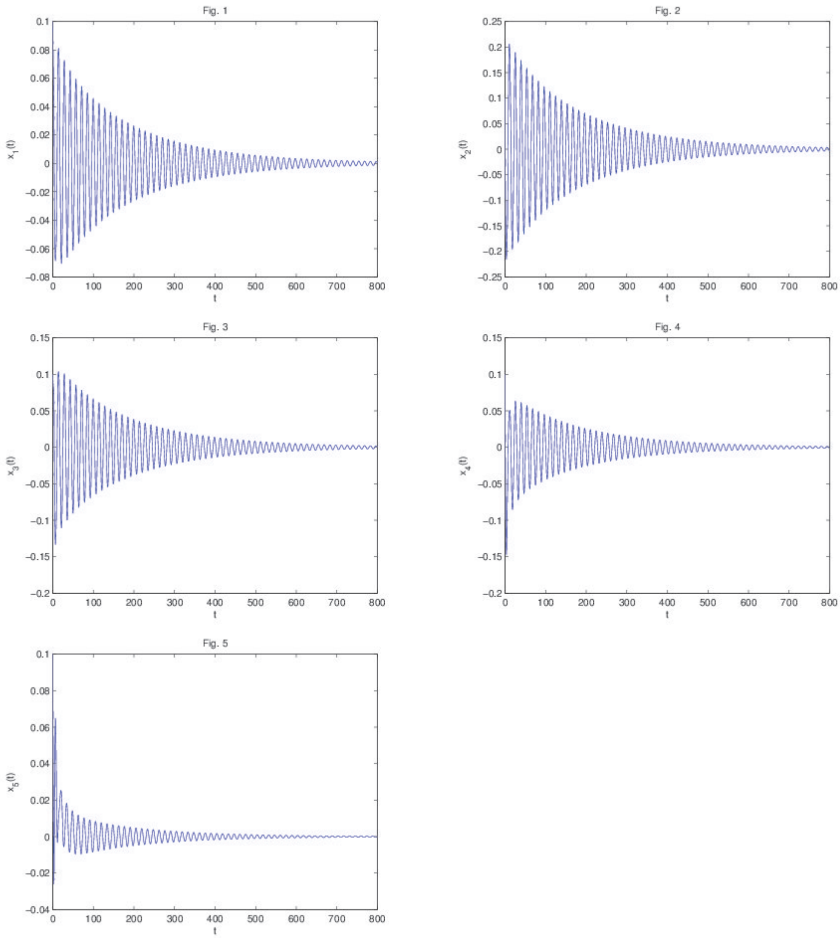

Fig. 5. Dynamic behavior of system (65): times series of $x_i (i = 1, 2, 3, 4, 5)$. A Matlab simulation of the asymptotically stable origin to system (65) with $\tau_1 = 3, \tau_2 = 2.5$ and $\tau_1 + \tau_2 = \tau = 5.5 < \tau_0 \approx 6.2$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1).

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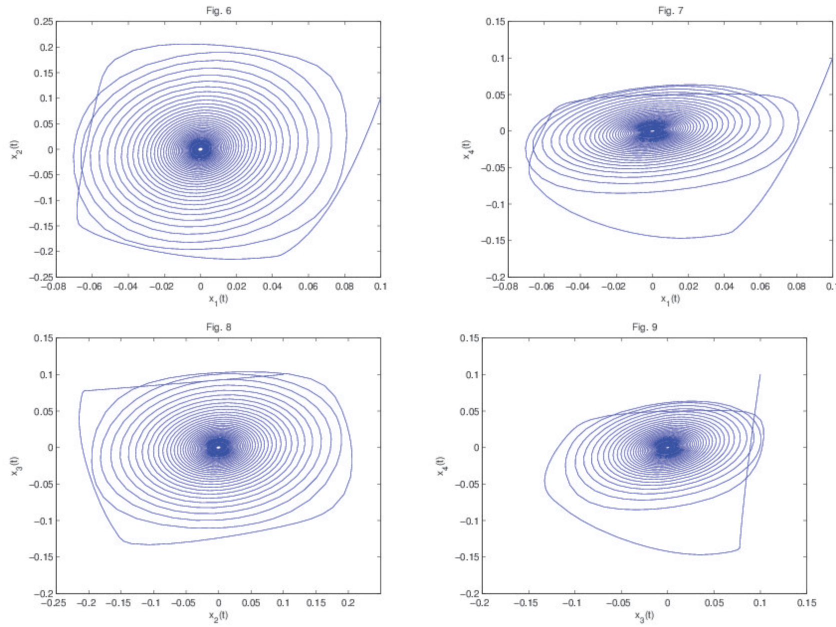

Fig. 6. Dynamic behavior of system (65): projection on $x_1 - x_2; x_1 - x_4; x_2 - x_3; x_3 - x_4$ plane. A Matlab simulation of the asymptotically stable origin to system (65) with $\tau_1 = 3, \tau_2 = 2.5$ and $\tau_1 + \tau_2 = \tau = 5.5 < \tau_0 \approx 6.2$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1).

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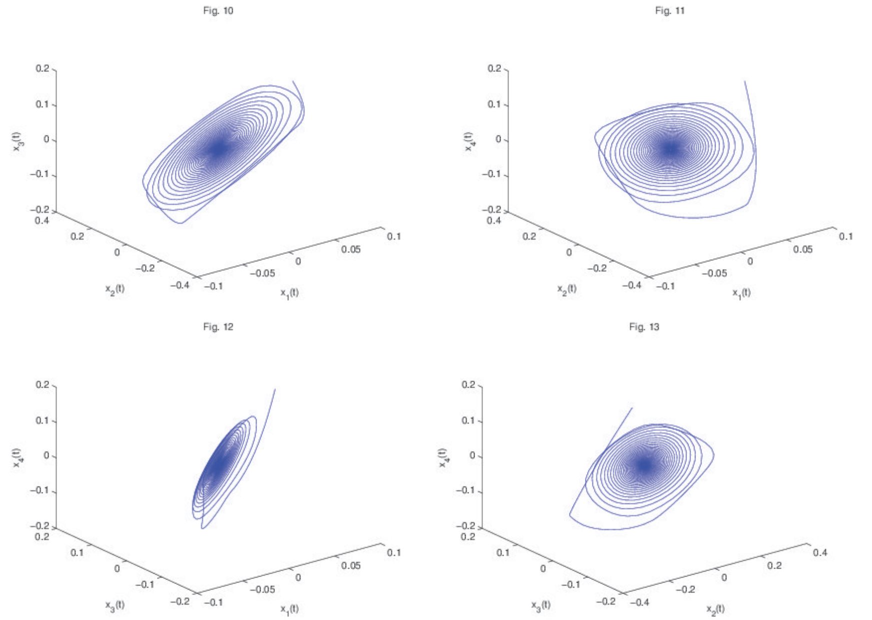

Fig. 7. Dynamic behavior of system (65): projection on $x_1 - x_2 - x_3; x_1 - x_2 - x_4; x_1 - x_3 - x_4; x_2 - x_3 - x_4$ space, respectively. A Matlab simulation of the asymptotically stable origin to system (65) with $\tau_1 = 3, \tau_2 = 2.5$ and $\tau_1 + \tau_2 = \tau = 5.5 < \tau_0 \approx 6.2$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1).

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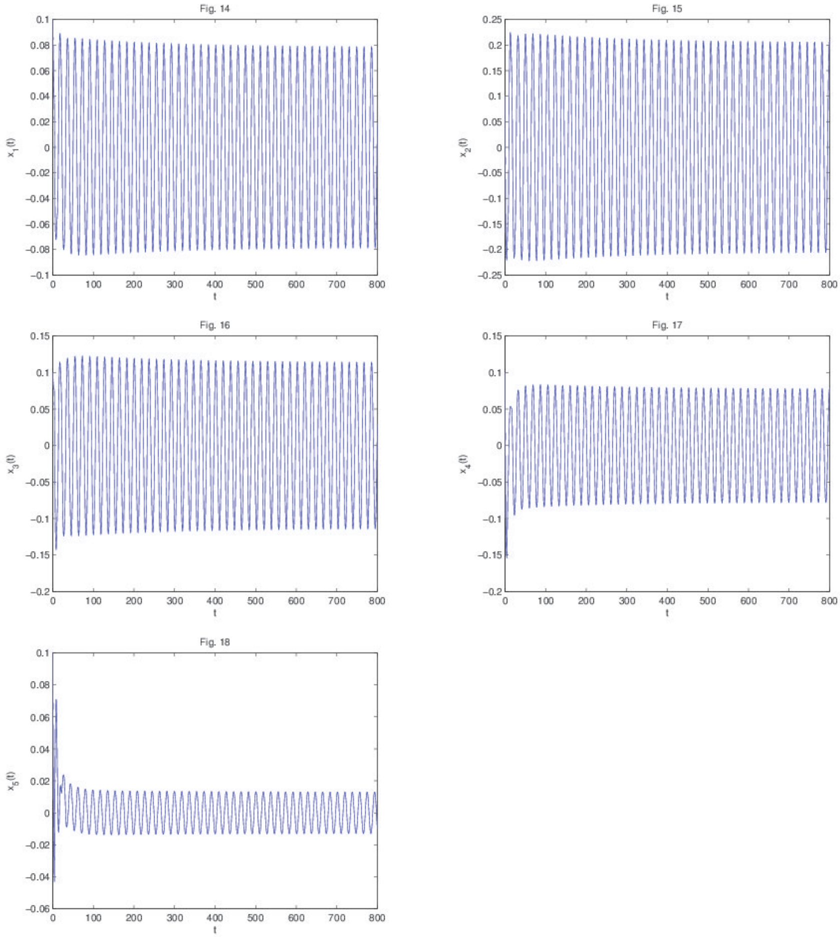

Fig. 8. Dynamic behavior of system (65): times series of $x_i (i = 1, 2, 3, 4, 5)$. A Matlab simulation of a periodic solution to system (65) with $\tau_1 = 4, \tau_2 = 3.5$ and $\tau_1 + \tau_2 = \tau = 7.5 > \tau_0 \approx 6.2$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1).

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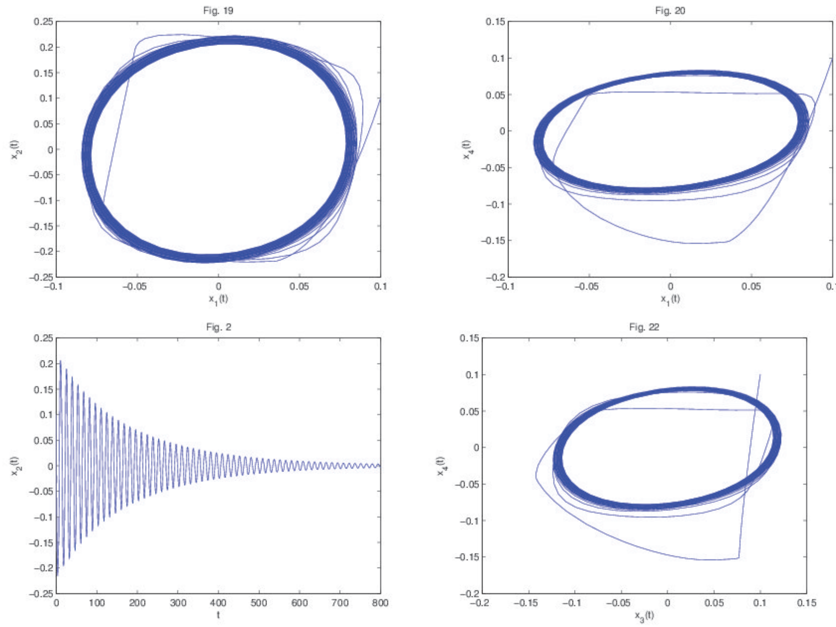

Fig. 9. Dynamic behavior of system (65): projection on $x_1 - x_2; x_1 - x_4; x_2 - x_3; x_3 - x_4$ plane, respectively. A Matlab simulation of a periodic solution to system (65) with $\tau_1 = 4, \tau_2 = 3.5$ and $\tau_1 + \tau_2 = \tau = 7.5 > \tau_0 \approx 6.2$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1).

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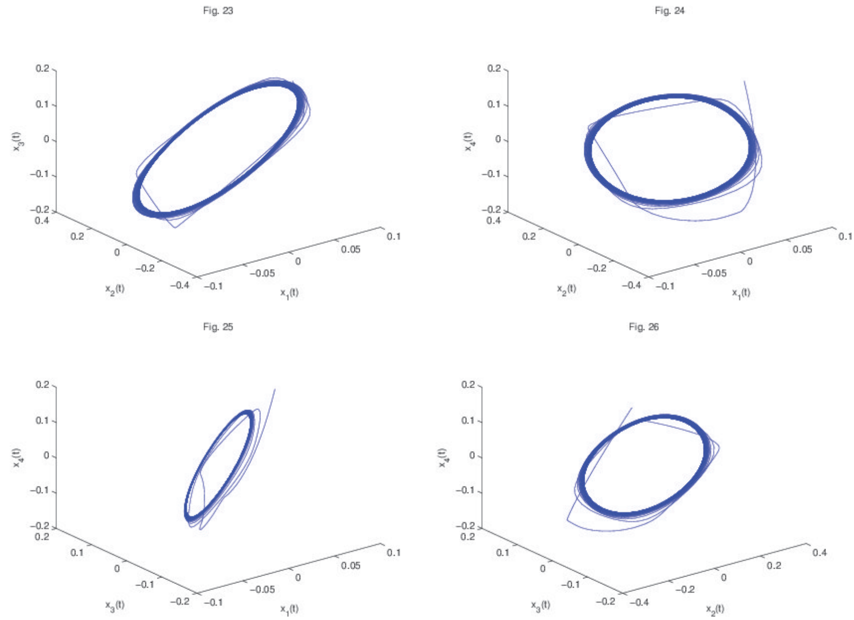

Fig. 10. Dynamic behavior of system (65): projection on $x_1 - x_2 - x_3; x_1 - x_2 - x_4; x_1 - x_3 - x_4; x_2 - x_3 - x_4$ space, respectively. A Matlab simulation of a periodic solution to system (65) with $\tau_1 = 4, \tau_2 = 3.5$ and $\tau_1 + \tau_2 = \tau = 7.5 > \tau_0 \approx 6.2$. The initial value is (0.1, 0.1, 0.1, 0.1, 0.1).

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Authors' addresses:

Changjin Xu
Guizhou Key Laboratory of Economics System Simulation,
Guizhou University of Finance and Economics, Guiyang,
Guizhou 550004, China
E-mail: xcj403@126.com

Xianhua Tang
School of Mathematical Science and Computing Technology,
Central South University, Changsha,
Hunan 410083, China
E-mail: tangxh@mail.csu.edu.cn

Maoxin Liao
School of Mathematics and Physics,
Nanhua University,
Hengyang, 421001, China
E-mail: maoxinliao@163.com