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SUB-RIEMANNIAN GEODESICS ON THE THREE-DIMENSIONAL SOLVABLE NON-NILPOTENT LIE GROUP SOLV^-

A.D. MAZHITOVA

ABSTRACT. In this paper we study geodesics of a left-invariant sub-Riemannian metric on a three-dimensional solvable Lie group. A system of differential equations for geodesics is derived from Pontryagin maximum principle and by using Hamiltonian structure. In a generic case the normal geodesics are described by elliptic functions, and their qualitative behavior is quite complicated.

1. INTRODUCTION

In this paper we describe geodesics of a left-invariant sub-Riemannian metric on a three-dimensional solvable Lie group. This group is widely known in geometry, because it allows compact quotient-spaces and it gives one of the Thurston three-dimensional geometries [1]. By the classification theorem of Agrachev–Barilari [2] there are invariant sub-Riemannian geometries realized on four solvable non-nilpotent Lie groups: SE(2), SH(2), SOLV^-, and SOLV^+.

In this classification, our geometry corresponds to the case SOLV^-.

The case of SOLV^+ we shall consider separately.

Various aspects of the integration of geodesic flows on sub-Riemannian manifolds have been widely studied (see, for example, [3–6, 11]). Note, that the geodesics of other three-dimensional nonsolvable or nilpotent sub-Riemannian geometries have been described recently in terms of elementary functions [7, 8]. In our situation it is necessary to use elliptic functions.

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2. Basic definitions

2.1. **Geodesics of sub-Riemannian manifolds.** Let M^n be a smooth n-dimensional manifold. A smooth family of k-dimensional subspaces in the tangent spaces at points of M^n

$$\Delta = \{\Delta(q) : \quad \Delta(q) \subset T_q M^n \quad \forall q \in M^n, \quad \dim \Delta(q) = k\}$$

is called completely nonintegrable, if the vector fields tangent to Δ, and all their iterated commutators generate the tangent bundle TM^n:

$$\operatorname{span}\Big\{[f_1, [\dots [f_{m-1}, f_m] \dots]](q) : \\ f_i(q) \in \Delta(q) \forall q \in M^n, m = 1, \dots\Big\} = T_q M^n.$$

Sometimes this distribution is called completely nonholonomic.

A two-dimensional distribution on a three-dimensional manifold is completely nonholonomic if and only if

$$\operatorname{span}\{f_1(q), f_2(q), [f_1(q), f_2(q)]\} = T_q M^3,$$

where at every point q the vectors $f_1(q)$ and $f_2(q)$ form a basis in $\Delta(q)$.

Let g_{ij} be a complete Riemannian metric on M^n. A triple (M^n, Δ, g_{ij}) is called a sub-Riemannian manifold. A Lipschitz continuous curve $\gamma : [0, T] \to M^n$ is called admissible if $\dot{\gamma}(t) \in \Delta(\gamma(t))$ for almost all $t \in [0, T]$. The length of this curve is equal to

$$l(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right)} dt.$$

The distance between two points on the manifold is defined by the formula

$$d(q_0, q_1) = \inf_{\gamma \in \Omega_{q_0, q_1}} l(\gamma),$$

where Ω_{q_0, q_1} is the set of all admissible curves connecting points q_0 and q_1. This function $d(\cdot, \cdot)$ is called the sub-Riemannian metric on M^n. A geodesic of this metric is an admissible curve $\gamma : [0, T] \to M^n$, which locally minimizes the length functional $l(\gamma)$.

Geodesics of sub-Riemannian metrics satisfy the Pontryagin maximum principle (see, for instance, [5]), which we formulate below. Let $f_1, \dots, f_k$ be vector fields which are tangent to Δ and span Δ at every point of M^n (or of a domain of M^n).

The Pontryagin maximum principle is stated as follows:

Theorem 1. *Let M^n be a smooth n-dimensional manifold. Let us consider for Lipschitz continuous curves the following minimum problem:*

$$\dot q = \sum_{i=1}^{k} u_i f_i(q), \quad u_i \in \mathbb{R}, \quad \int_0^T \sum_{i=1}^{k} u_i^2(t)dt \longrightarrow \min, \quad q(0) = q_0, \quad q(T) = q_1$$

with a fixed T. Let us consider the mapping $\mathcal{H} : T^*M^n \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$, given by the function

$$\mathcal{H}(q, \lambda, p_0, u) := \langle \lambda, \sum_{i=1}^{k} u_i f_i(q)\rangle + p_0 \sum_{i=1}^{k} u_i^2.$$

If a curve $q(\cdot) : [0, T] \to M^n$ with a control $u(\cdot) : [0, T] \to \mathbb{R}^k$ is optimal, then there exists Lipshitzian covector function $\lambda(\cdot) : t \in [0, T] \mapsto \lambda(t) \in T^*_{q(t)}M^n$, $(\lambda(t), p_0) \neq 0$ and a constant $p_0 \leq 0$ such that

i) $\dot q(t) = \dfrac{\partial \mathcal{H}}{\partial \lambda}(q(t), \lambda(t), p_0, u(t))$,

ii) $\dot\lambda(t) = -\dfrac{\partial \mathcal{H}}{\partial q}(q(t), \lambda(t), p_0, u(t))$,

iii) $\dfrac{\partial \mathcal{H}}{\partial u}(q(t), \lambda(t), p_0, u(t)) = 0$.

A curve $q(\cdot) : [0, T] \to M^n$, satisfying the Pontryagin maximum principle is called an extremal (curve). To such a curve there corresponds a set of pairs $(\lambda(\cdot), p_0)$. The type (normal or abnormal) of an extremal depends on the value of p_0:

- if $p_0 \neq 0$, then the extremal is called *normal*;
- if $p_0 = 0$, then the extremal is called *abnormal*;
- extremal is called *strictly abnormal* if it is not projected (on M^n) onto a normal extremal.

For a normal extremal we can put $p_0 = -\frac{1}{2}$.

Normal extremals are geodesics [5]. In the contact case, when at every point the distribution Δ coincides with the annihilator of the contact form on M^n, there are no nontrivial abnormal extremals (this fact is indicated in [7]). In the case, when the space of vector fields on a manifold is generated by vector fields tangent to the nonholonomic distribution and their commutators, there are no strictly abnormal extremals [5]. Both of the above statements apply to three-dimensional sub-Riemannian manifolds M^3.

By iii), $u_i = \langle \lambda(t), f_i(t)\rangle$ and a curve $q(\cdot) : [0, T] \to M^n$ is geodesic if and only if it is the projection onto M^n of a solution $(\lambda(t), q(t))$ of the Hamiltonian system on T^*M^n with the following Hamiltonian function:

$$H(\lambda, q) = \frac{1}{2}\left(\sum_{i=1}^{k} \langle \lambda, f_i\rangle^2\right), \quad q \in M^n, \quad \lambda \in T^*_q M^n.$$

The Hamiltonian H is constant along any solution of the Hamiltonian system. Moreover, $H = \frac{1}{2}$ if and only if the geodesic is length parameterized.

2.2. **Elliptic functions. Jacobi functions.** Let us recall some necessary facts on Jacobi elliptic functions. The integrals

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

and

$$\int_0^x \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx,$$

are called elliptic integrals of the first and second kind, respectively, in the normal Legendre form (see [10,12]), where k $(0 < k < 1)$ is the modulus of these integrals, $k' = \sqrt{1-k^2}$ is the additional modulus. By the substitution $x = \sin\varphi$ these integrals reduce to the normal trigonometric form

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \tag{1}$$

$$\mathrm{E}(\varphi, k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\alpha}\;\; d\alpha = \int_0^{\sin\varphi} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx. \tag{2}$$

Consider an integral of the first kind in the normal trigonometric form

$$v = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

Now consider the upper limit as a function of v. This function is denoted by

$$\varphi = \mathrm{am}\,(v,k) = \mathrm{am}\, v$$

and is called the amplitude, and this process is called inversion of the integral. Thus, the following functions:

$$\sin\varphi = \sin(\mathrm{am}\, v) = \mathrm{sn}\, v,$$

$$\cos\varphi = \cos(\mathrm{am}\, v) = \mathrm{cn}\, v,$$

$$\Delta\mathrm{am}\, v = \sqrt{1-k^2\sin^2\varphi} = \sqrt{1-k^2\mathrm{sn}^2 v} = \mathrm{dn}\, v$$

are called Jacobi functions and are related by

$$\mathrm{sn}^2 v + \mathrm{cn}^2 v = 1, \quad \mathrm{dn}^2 v + k^2\mathrm{sn}^2 v = 1.$$

By derivation, we obtain

$$\frac{d\;\mathrm{sn}\, v}{dv} = \mathrm{cn}\, v\;\mathrm{dn}\, v,$$

$$\frac{d\,\mathrm{cn}\,v}{dv} = -\mathrm{sn}\,v\ \mathrm{dn}\,v,$$
$$\frac{d\,\mathrm{dn}\,v}{dv} = -k^2\ \mathrm{sn}\,v\ \mathrm{cn}\,v$$

and conclude that

$$\left(\frac{d\,\mathrm{sn}\,v}{dv}\right)^2 = (1-\mathrm{sn}^2 v)(1-k^2\mathrm{sn}^2 v),$$
$$\left(\frac{d\,\mathrm{cn}\,v}{dv}\right)^2 = (1-\mathrm{cn}^2 v)(k'^2+k^2\mathrm{cn}^2 v), \tag{3}$$
$$\left(\frac{d\,\mathrm{dn}\,v}{dv}\right)^2 = (1-\mathrm{dn}^2 v)(\mathrm{dn}^2 v-k'^2).$$

The first equation of (2) implies that $\mathrm{sn}\,v$ is the inversion of the elliptic integral of the first kind in the normal Legendre form

$$v = \int\limits_0^{\mathrm{sn}\,v} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{4}$$

From the second and third equations we obtain that $\mathrm{cn}\,v$ and $\mathrm{dn}\,v$ are the result of inversion of the following functions

$$v = \int\limits_1^{\mathrm{cn}\,v} \frac{dx}{\sqrt{(1-x^2)(k'^2+k^2x^2)}}, \tag{5}$$

$$v = \int\limits_1^{\mathrm{dn}\,v} \frac{dx}{\sqrt{(1-x^2)(x^2-k'^2)}}. \tag{6}$$

All Jacobi functions are periodic. Note that the function $\mathrm{sn}\,v$ is odd, but $\mathrm{cn}\,v$ and $\mathrm{dn}\,v$ are even, therefore we assume, what in the two last integrals, when the functions $\mathrm{cn}\,v$ and $\mathrm{dn}\,v$ pass through the critical points, respectively the radical changes sign of.

3. Sub-Riemannian problem on the group SOLV$^-$

Us consider the three-dimensional Lie group SOLV$^-$ formed by all matrices of the form

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

Its Lie algebra is spanned by the vectors

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

meeting the following commutation relations:

$$[e_1, e_2] = 0; \quad [e_1, e_3] = e_1; \quad [e_2, e_3] = -e_2.$$

We take a new basis

$$a_1 = e_1 + e_2; \quad a_2 = e_1 - e_2; \quad a_3 = e_3, \tag{7}$$

in which the commutation relations take the form

$$[a_1, a_2] = 0, \quad [a_1, a_3] = a_2, \quad [a_2, a_3] = a_1.$$

Consider the left-invariant metric on SOLV^- defined by its values at the identity of the group:

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

The Lie group SOLV^- is diffeomorphic to the space $\mathbb{R}^3$. Indeed, x, y, z are the global coordinates on SOLV^- and they also can be considered as global coordinates on $\mathbb{R}^3$. The tangent space at each point of SOLV^- is spanned by matrices of the form

$$\partial_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \partial_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \partial_z = \begin{pmatrix} -e^{-z} & 0 & 0 \\ 0 & e^z & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which are left translations of the basic vectors:

$$L_{q*}(e_1) = e^{-z}\partial_x, \quad L_{q*}(e_2) = e^z \partial_y, \quad L_{q*}(e_3) = \partial_z.$$

Since the metric is left-invariant, we have

$$g_{ij}(x, y, z) = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the basis a_1, a_2, a_3 we have

$$L_{q*}(a_1) = e^{-z}\partial_x + e^z\partial_y, \quad L_{q*}(a_2) = e^{-z}\partial_x - e^z\partial_y, \quad L_{q*}(a_3) = \partial_z.$$

The inner product takes the form

$$\langle L_{q*}(a_i), L_{q*}(a_j) \rangle = \langle a_i, a_j \rangle = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

In this paper we study the sub-Riemannian problem on the three-dimensional Lie group SOLV^- defined by the distribution $\Delta = \mathrm{span}\{a_1, a_3\}$ with metric (8).

Let $G = \mathrm{SOLV}^-$, $\mathcal{G}$ be its Lie algebra with the basic vectors a_1, a_2, a_3 (7). We split the Lie algebra $\mathcal{G}$ into the sum $p \bigoplus k$, where $p = \mathrm{span}\{a_1, a_3\}$, $k = \mathrm{span}\{a_2\}$.

Consider the two-dimensional left-invariant distribution $\Delta = \mathrm{span}\{a_1, a_3\}$ in TG, and the left-invariant Riemannian metric (8) for which the spaces p and k are orthogonal, i.e., the metric tensor splits as follows:

$$g = (g_{ij}) = g_p + g_k.$$

Introduce a parameter τ and consider the metrics

$$g_\tau = g_p + \tau g_k.$$

Every such a metric together with Δ defines the same sub-Riemannian manifold because only the restriction of the metric onto Δ is important.

However the Hamiltonian function for the geodesic flows of these metrics depends on τ:

$$H(x, p, \tau) = \frac{1}{2} g_\tau^{ij}(x) p_i p_j,$$

where $g_{ij} g^{jk} = \delta_i^k$. We have

$$g_{\tau,ij} = \begin{pmatrix} \frac{1+\tau}{2} e^{2z} & \frac{1-\tau}{2} & 0 \\ \frac{1-\tau}{2} & \frac{1+\tau}{2} e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_\tau^{ij} = \begin{pmatrix} \frac{1+\tau}{2\tau} e^{-2z} & -\frac{1-\tau}{2} & 0 \\ -\frac{1-\tau}{2} & \frac{1+\tau}{2\tau} e^{2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Hamiltonian function H for the normal geodesic flow of the sub-Riemannian metric is obtained from $H(x, p, \tau)$ in the limit

$$\tau \to \infty,$$

and we derive

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{4}\, e^{-2z} p_x^2 + \frac{1}{2}\, p_x p_y + \frac{1}{4}\, e^{2z} p_y^2 + \frac{1}{2} p_z^2. \tag{9}$$

The Hamiltonian equations $\dot{x}^i = \dfrac{\partial H}{\partial p_i}, \dot{p}_i = -\dfrac{\partial H}{\partial x^i}$ take the form

$$\begin{aligned} \dot{x} &= \frac{1}{2}\, e^{-2z} p_x + \frac{1}{2}\, p_y, & \dot{p_x} &= 0, \\ \dot{y} &= \frac{1}{2}\, e^{2z} p_y + \frac{1}{2}\, p_x, & \dot{p_y} &= 0, \\ \dot{z} &= p_z, & \dot{p_z} &= \frac{1}{2}\, e^{-2z} p_x^2 - \frac{1}{2}\, e^{2z} p_y^2. \end{aligned} \tag{10}$$

These differential equations can be derived from the Pontryagin maximum principle as well. The corresponding Hamiltonian takes the form

$$\begin{aligned} &H(x, y, z, p_x, p_y, p_z, p_0, u_1, u_3) \\ &\qquad = \frac{1}{\sqrt{2}} \left(u_1 p_x e^{-z} + u_1 p_y e^{z}\right) + u_3 p_z + p_0(u_1^2 + u_3^2), \end{aligned}$$

where $p_0 = -\dfrac{1}{2}$, u_1, u_3 are control functions.

The system (10) has three first integrals:

$$I_1 = H, \quad I_2 = p_x, \quad I_3 = p_y,$$

which are functionally independent almost everywhere, and therefore the system is completely integrable.

Since the flow is left-invariant as well as the distribution Δ and the metric, without loss of generality we assume, that all geodesics originate at the identity of group, that is, we have the following initial conditions for the system (10):

$$x(0) = 0, \quad y(0) = 0, \quad z(0) = 0. \tag{11}$$

In the sequel, we put

$$H = \frac{1}{2}, \quad \frac{p_x}{\sqrt{2}} = a, \quad \frac{p_y}{\sqrt{2}} = b.$$

By substituting these expressions into (9), we obtain

$$1 = \left(e^{-z}a + e^{z}b\right)^2 + p_z^2, \tag{12}$$

which implies

$$p_z = \pm\sqrt{1 - \left(e^{-z}\, a + e^{z}\, b\right)^2}.$$

By substituting this expression to the third equation of (10) we obtain equation for the temporal variable t for positive values of p_z

$$t = \int \frac{dz}{\sqrt{1 - \left(e^{-z}\, a + e^{z}\, b\right)^2}}. \tag{13}$$

If $p_z < 0$, then all calculations will be similar, but with the opposite sign.

Make the change of variables

$$u = e^{z},$$

and rewrite (13) as

$$t = \int \frac{du}{\sqrt{u^2 - \left(a + bu^2\right)^2}}. \tag{14}$$

The last expression is not integrable in terms of elementary functions and defines an elliptic integral, except of special cases, when this elliptic integral degenerates. These cases will be discussed below.

Consider first the generic case $a \neq 0$ and $b \neq 0$.

The subradical expression in (14) has discriminant $D = 1 - 4ab \geq 0$ accordin to (12).

$D = 0$ if and only if $p_z = 0$ according to system (10) and equation (12). That case is degeneraet.

Thus, if $D > 0\left(ab < \dfrac{1}{4}\right)$, then there exist σ_1^2 and σ_2^2, such that the following holds:

$$\begin{aligned} u^2 - \left(a + bu^2\right)^2 &= -b^2u^4 + (1-2ab)u^2 - a^2 \\ &= -b^2(u^2-\sigma_1^2)(u^2-\sigma_2^2) = \sigma_1^4 b^2\left(1 - \frac{u^2}{\sigma_1^2}\right)\left(\frac{u^2}{\sigma_1^2} - \frac{\sigma_2^2}{\sigma_1^2}\right), \end{aligned}$$

and

$$\sigma_{1,2}^2 = \frac{1 - 2ab \pm \sqrt{1-4ab}}{2b^2}. \tag{15}$$

Put

$$w = \frac{u}{\sigma_1} \tag{16}$$

and rewrite (14) in the following form:

$$t = \frac{1}{\sigma_1 b}\int \frac{dw}{\sqrt{(1-w^2)\left(w^2 - \dfrac{\sigma_2^2}{\sigma_1^2}\right)}}. \tag{17}$$

We apply the Jacobi elliptic function (6) in order to inverse this integral:

$$\sigma_1 b\, t = \int\limits_1^{\mathrm{dn}\,(\sigma_1 bt)} \frac{dw}{\sqrt{(1-w^2)\left(w^2 - \frac{\sigma_2^2}{\sigma_1^2}\right)}},$$

where $k'^2 = \dfrac{\sigma_2^2}{\sigma_1^2}$. Therefore

$$w = \mathrm{dn}\,(\sigma_1 bt, k),$$

where

$$k^2 = 1 - \frac{\sigma_2^2}{\sigma_1^2}. \tag{18}$$

By inverting (16), putting $u = e^z$, and keeping in mind the initial condition (11) and the equality $\mathrm{dn}\,(0,k) = 1$, we obtain

$$z(t) = \ln \mathrm{dn}\,(\sigma_1 bt, k).$$

By substituting this expression into the first equation of (10) and integrating it in elliptic functions (see [10]), we derive:

$$x(t) = \frac{1}{\sqrt{2}}\left[\frac{a}{\sigma_1 b}\left(-\frac{k^2 \mathrm{sn}\,(\sigma_1 bt)\mathrm{cn}\,(\sigma_1 bt)}{k'^2\sqrt{1 - k^2\mathrm{sn}^2(\sigma_1 bt)}} + \frac{1}{k'^2}\mathrm{E}(\mathrm{am}\,(\sigma_1 bt), k)\right) + bt\right] + C,$$

where $\mathrm{E}(x,k)$ is the elliptic integral of the second kind (2).

Since $\operatorname{sn}(0,k) = 0, \operatorname{cn}(0,k) = 1, \operatorname{am}(0,k) = 0$ and $\mathrm{E}(0,k) = 0$, we have $C = 0$. From the second equation of this system we conclude that

$$y(t) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sigma_1} \mathrm{E}(\operatorname{am}(\sigma_1 bt),\, k) + at \right) + Q,$$

with $Q = \text{const}$. By (11), we compute that $Q = 0$.

Let us now consider the cases, where the elliptic integral (14) degenerates:

1. $a = 0, \quad b = 0;$
2. $a = 0, \quad b \neq 0;$
3. $b = 0, \quad a \neq 0;$
4. $D = 0 \left(ab = \dfrac{1}{4} \right).$

We consider them successively.

1) $a = 0, \quad b = 0$. From the equations (10), (11) and (13) it is clear that

$$x(t) = 0, \quad y(t) = 0, \quad z(t) = t. \tag{19}$$

2) $a = 0, \quad b \neq 0$. We have $p_x = 0, \quad p_y = \sqrt{2}b$.
The equation (14) is rewritten as

$$t = \int \frac{du}{\sqrt{u^2 - b^2u^4}}.$$

By integration and transformation the resulting expression by the inverse change of variable, we obtain $u = e^z$:

$$e^t = \frac{Cbe^z}{1 + \sqrt{1 - b^2e^{2z}}},$$

where $C = \text{const}$ and $C > 0$. The last expression together with the initial condition (11) implies

$$C = \frac{1 + \sqrt{1 - b^2}}{b},$$

and we derive that

$$e^z = \frac{2Ce^t}{b\,(C^2 + e^{2t})},$$

i.e.,

$$z(t) = \ln \frac{2Ce^t}{b\,(C^2 + e^{2t})}, \tag{20}$$

which after substituting the formula for C takes the form

$$z(t) = \ln \frac{2(1 + \sqrt{1 - b^2})\, e^t}{2(1 + \sqrt{1 - b^2}) - b^2 + b^2e^{2t}}.$$

By the first equation of (10), we have

$$x(t) = \frac{b}{\sqrt{2}}\, t,$$

and the second equation of (10) together with (20) and (11) implies

$$y(t) = -\frac{\sqrt{2}C^2}{b(C^2+e^{2t})} + \frac{\sqrt{2}C^2}{b(C^2+1)}.$$

Finally in the case **2)** we have the explicit formulas for solutions:

$$\begin{aligned} x(t) &= \frac{b}{\sqrt{2}}\,t, \\ y(t) &= -\frac{\sqrt{2}\left(2(1+\sqrt{1-b^2})-b^2\right)}{2b(1+\sqrt{1-b^2})-b^3+b^3e^{2t}} + \frac{\sqrt{2}\left(2(1+\sqrt{1-b^2})-b^2\right)}{2b(1+\sqrt{1-b^2})}, \\ z(t) &= \ln\frac{2(1+\sqrt{1-b^2})\,e^t}{2(1+\sqrt{1-b^2})-b^2+b^2e^{2t}}. \end{aligned} \tag{21}$$

3) $b=0,\quad a\neq 0$. We have $p_y=0,\quad p_x=\sqrt{2}a$.
The equation (14) takes the form

$$t = \int\frac{du}{\sqrt{u^2-a^2}}.$$

We put $u=e^z$ and derive

$$e^t = (e^z+\sqrt{e^{2z}-a^2})\,C,$$

where $C=\text{const}$ and $C>0$. The last expression together with (11) implies

$$C = \frac{1}{1+\sqrt{1-a^2}}, \tag{22}$$

from which we obtain

$$z(t) = \ln\frac{C^2a^2+e^{2t}}{2Ce^t},$$

where C is given by (22). As in the case 1) we derive from the first two equations of (10) that

$$x(t) = -\frac{\sqrt{2}aC^2}{e^{2t}+C^2a^2} + \frac{\sqrt{2}aC^2}{1+C^2a^2},\quad y(t) = \frac{a}{\sqrt{2}}\,t. \tag{23}$$

Finally we obtain

$$\begin{aligned} x(t) &= -\frac{\sqrt{2}\,a}{e^{2t}\left[2(1+\sqrt{1-a^2})-a^2\right]+a^2} + \frac{\sqrt{2}\,a}{2(1+\sqrt{1-a^2})}, \\ y(t) &= \frac{a}{\sqrt{2}}\,t, \\ z(t) &= \ln\left(\frac{a^2}{2(1+\sqrt{1-a^2})\,e^t} + \frac{(1+\sqrt{1-a^2})\,e^t}{2}\right). \end{aligned} \tag{24}$$

4) $D=0\left(ab=\dfrac{1}{4}\right)$.

Note that in wiew of (11) the formula (12) is rewritten as

$$(a+b)^2 + p_z^2 = 1, \tag{25}$$

which means that

$$|a+b| \le 1. \tag{26}$$

Then it is clear that $a = b = \frac{1}{2}$ or $a = b = -\frac{1}{2}$, and for these values the equation (25) implies that $p_z = 0$. Therefore solutions to (10) in the case **4)** are linear:

$$\begin{aligned} x(t) &= \frac{t}{\sqrt{2}}, \qquad y(t) = \frac{t}{\sqrt{2}}, \quad z(t) = 0, \quad a = b = \frac{1}{2}; \\ x(t) &= -\frac{t}{\sqrt{2}}, \quad y(t) = -\frac{t}{\sqrt{2}}, \quad z(t) = 0, \quad a = b = -\frac{1}{2}. \end{aligned} \tag{27}$$

Thus we have the following

Theorem 2. *In a generic case the normal geodesics* (*with the initial condition* (11)) *are described by the formulas* (*for* $p_z > 0$)*:*

$$\begin{aligned} x(t) &= -\frac{ak^2 \operatorname{sn}(\sigma_1 bt)\operatorname{cn}(\sigma_1 bt)}{\sqrt{2}\sigma_1 bk'^2\sqrt{1-k^2\operatorname{sn}^2(\sigma_1 bt)}} + \frac{a\,\mathrm{E}(\operatorname{am}(\sigma_1 bt),\,k)}{\sqrt{2}\sigma_1 bk'^2} + \frac{b}{\sqrt{2}}t, \\ y(t) &= \frac{\mathrm{E}(\operatorname{am}(\sigma_1 bt),\,k)}{\sqrt{2}\sigma_1} + \frac{a}{\sqrt{2}}t, \\ z(t) &= \ln \operatorname{dn}(\sigma_1 bt), \end{aligned} \tag{28}$$

where the parameters σ_1 *and* k *are determined by* a *and* b $\left(ab < \frac{1}{4}\right)$ *via* (15) *and* (18).

In the degenerate cases 1)–4) *the normal geodesics* (*with the initial condition* (11)) *are described in terms of elementary functions by the formulas* (19), (21), (24), *and* (27).

Notice that normal geodesics in the theorem are parameterized by a, b. The constants k, σ_1 are defined by a, b as explained before.

The qualitative behavior of generic normal geodesic is quite complicated. Fig. 1 and Fig. 2 show parts of the geodesic spheres of radius of 0.15 and 0.25 (a scale on each figure itself; axis z is exponentially scaled).

A grid on the spheres corresponds to two parameters θ and μ, where θ is angle of the initial vector of the geodesic with respect to the axis x and μ is the initial acceleration value $x+y$ along the geodesic, i.e. μ can be interpreted as the acceleration with which the geodesic is drawn out of the starting point. In Figures θ varies from $\pi/6$ to $5\pi/6$ (part of the sphere of $-\pi/6$ to $-5\pi/6$ is obtained as a mirror). Parameter μ varies from -45 to 45.

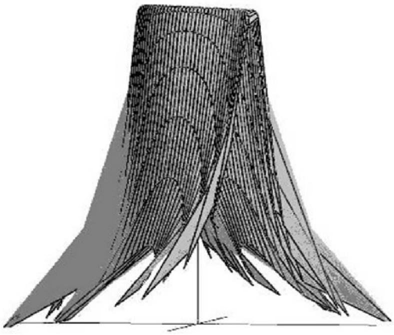

Fig. 1. Part of the geodesic sphere (of radius 0.15)

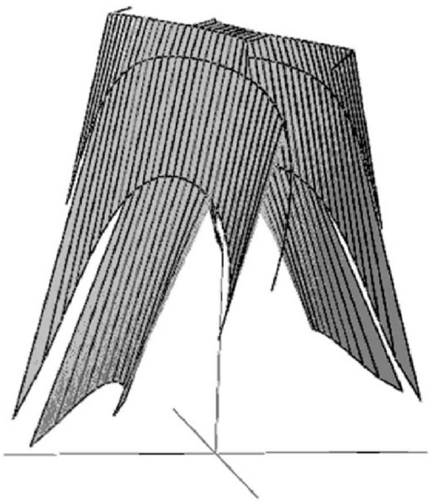

Fig. 2. Part of the geodesic sphere (of radius 0.25)

On this grid it can be seen only the qualitative behavior of a sphere with increasing radius. The figures practically do not show parts of spheres, which are too fast going to infinity, as well as those, which coincide to the geodesics, changing too quickly the direction. We can see that part of geodesics starting at small angle to plane x, y goes to large values of coordinates x, y very quickly, even for not large values of parameter μ. For sufficiently large θ and mean values of $|\mu|$ geodesics deviate not too much from plane $x = y$, but if $|\mu|$ increases the deviation from this plane begins. Assuming exponential scale of axis z we see that coordinate z increases much more slowly than x and y.

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Author's address:
Department of mathematics and mechanics,
Al-Farabi Kazakh National University,
Al-Farabi ave. 71, Almaty 050038, Kazakhstan
E-mail: Akmaral.Mazhitova@kaznu.kz