Journal of Dynamical and Control Systems, Vol. 17, No. 1, January 2011, 49–75 (©2011)
DOI: 10.1007/s10883-011-9110-7

NORMAL FORMS AND REACHABLE SETS FOR ANALYTIC MARTINET SUB-LORENTZIAN STRUCTURES OF HAMILTONIAN TYPE

M. GROCHOWSKI

ABSTRACT. In this paper, we construct normal forms for sub-Lorentzian structures (H, g), where H is an analytic Martinet-type distribution on $\mathbb{R}^3$ and g is an analytic Lorentzian metric on H, under the assumption that abnormal curves foliating the Martinet surface for H are timelike Hamiltonian geodesics. As an application, we compute reachable sets from a point for such structures. It turns out that such sets are described by four analytic functions and, consequently, they are semi-analytic. We also compute future null conjugate and cut loci, and the image under the exponential mapping for abovementioned structures.

1. INTRODUCTION

1.1. **Statement of main results.** In this paper, we investigate reachable sets from a point for a class of sub-Lorentzian metrics on $\mathbb{R}^3$. In order to state our results, we introduce basic facts and notions from the sub-Lorentzian geometry. For more details, the reader is referred to [5, 8]. By a *sub-Lorentzian structure* (or a *metric*) on a manifold M we mean a couple (H, g), where H is a smooth bracket generating distribution of constant rank on M and g is a Lorentzian metric on H. For $q \in M$, a vector $v \in H_q$ is said to be *timelike* if $g(v, v) < 0$, *nonspacelike* if $g(v, v) \leq 0$, and *null* if $g(v, v) = 0$ but $v \neq 0$. A *time orientation* of (H, g) is, by definition, a continuous timelike vector field on M. Assume that (H, g) is time-oriented by a vector field X. A nonspacelike vector $v \in H_q$ is called *future directed* if $g(v, X(q)) < 0$. In what follows, we assume that (M, H, g) is time-oriented. As in our previous papers, the following abbreviations are used: t. means "timelike," nspc. means "nonspacelike," and f.d. means "future directed." So, for example, t.f.d. means "timelike future directed."

2000 *Mathematics Subject Classification.* 53B30, 53C30.
Key words and phrases. Sub-Lorentzian manifolds, geodesics, reachable sets, geometric optimality, Martinet distributions.

By a *curve* we mean an absolutely continuous curve $\gamma : [a,b] \to M$ such that $\dot\gamma(t) \in H_{\gamma(t)}$ a.e. on $[a,b]$. A curve γ defined on an interval $[a,b]$ is said to be t.f.d. (respectively, nspc.f.d., null f.d.) if so is its tangent vector $\dot\gamma(t)$ a.e. on $[a,b]$. Fix a point $q_0 \in M$ and a neighborhood U of it. By $I^+(q_0,U)$ (respectively, $J^+(q_0,U)$, $N^+(q_0,U)$) we denote the set of all points $q \in U$ which can be reached by a t.f.d. (respectively, nspc.f.d., null f.d.) curve starting at q_0 and contained in U. $I^+(q_0,U)$ (respectively, $J^+(q_0,U)$, $N^+(q_0,U)$) is called the (*future*) *timelike* (respectively, *nonspacelike*, *null*) *reachable set from* q_0 *relative to* U. In order to use some results concerning reachable sets, we must make certain assumptions about a neighborhood U of q_0. Note that if U is sufficiently small, then our sub-Lorentzian metric can be extended to a Lorentzian metric, say $\tilde g$, on U. So U is said to be a *normal neighborhood* of q_0 if it is a convex normal neighborhood of q_0 with respect to $\tilde g$, and $\overline{U}$ is contained in some other convex normal neighborhood of q_0 with respect to $\tilde g$ (see Sec. 5.2 for an alternative definition of normal neighborhoods). For example, it can be proved (see [8]) that if U is a normal neighborhood of q_0, then $J^+(q_0,U)$ is closed with respect to U. Also,

$$\operatorname{int} I^+(q_0,U) = \operatorname{int} J^+(q_0,U) = \operatorname{int} N^+(q_0,U)$$

and

$$\tilde\partial I^+(q_0,U) = \tilde\partial J^+(q_0,U) = \tilde\partial N^+(q_0,U),$$

where $\tilde\partial$ is the boundary with respect to U.

In determining reachable sets from q_0 it is important to know nspc.f.d. curves starting at q_0 and contained in the boundary of reachable sets. Such curves are called *geometrically optimal*.

Let U be an open subset of M. Assume that $X_0,\dots,X_k$ is an orthonormal basis of (H,g) defined on U and such that X_0 is timelike. On T^*U, we define the so-called *geodesic Hamiltonian* $\mathcal{H} : T^*U \to \mathbb{R}$ associated with the structure (H,g):

$$\mathcal{H}(q,p) = -\frac{1}{2}\langle p, X_0(q)\rangle^2 + \frac{1}{2}\sum_{j=1}^{k}\langle p, X_j(q)\rangle^2$$

(note that $\mathcal{H}$ also admits a global definition, which is independent of a local orthonormal frame; see [8]). Denote by $\overrightarrow{\mathcal{H}}$ the Hamiltonian vector field on T^*U corresponding to the Hamiltonian $\mathcal{H}$, and let Φ_t stand for the (local) flow of $\overrightarrow{\mathcal{H}}$. A curve $\gamma : [0,T] \to M$ is called a *Hamiltonian geodesic* if it has the form $\gamma(t) = \pi \circ \Phi_t(\lambda)$, where $\lambda \in T^*_{\gamma(0)}M$ and $\pi : T^*M \to M$ is the canonical projection. It can be proved (see [5,8]) that each sufficiently short sub-arc of a nspc.f.d. Hamiltonian geodesic is a length maximizer.

In [9], we showed how to compute reachable sets from a point for contact sub-Lorentzian structures on $\mathbb{R}^3$. In this paper, we generalize this method to a class of noncontact distributions. Let H be a rank-two bracket generating distribution defined in a neighborhood U of zero in $\mathbb{R}^3$. For sufficiently small U, $H = \ker\omega$, ω is a 1-form on U. Clearly, $\omega \wedge d\omega = f\Omega$, where $f \in C^\infty(U)$ and Ω is a volume form. If $f(0) \neq 0$, then H is a contact distribution. On the other hand, assume that $f(0) = 0$ but $d_0 f \neq 0$. Let $S = \{f = 0\}$. For a generic point $q \in S$, we have $(\omega \wedge df)_q \neq 0$. A distribution H satisfying the condition $(\omega \wedge df)_q \neq 0$ for every $q \in S$ is called a *Martinet-type distribution* (or simply a *Martinet distribution*), and the surface S is called the *Martinet surface* for H. In what follows, we assume that H is a Martinet distribution. By the definition, H induces a nonsingular horizontal line field on S: $S \ni q \to T_qS \cap H_q$. We assume that this field is timelike. Curves on S that are trajectories of this line field are abnormal curves for H; suitably parameterized, they become t.f.d. Using Proposition 5.2, one can prove that these curves are in fact length maximizers. They are also geometrically optimal.

Sub-Lorentzian structures (H, g), where H is a Martinet distribution and the mentioned line field is timelike, are called *Martinet sub-Lorenztian structures*. In this paper, we consider Martinet sub-Lorentzian structures satisfying one additional condition: abnormal curves foliating S are, up to a change of parameter, t.f.d. Hamiltonian geodesics. Such structures are referred to as *Martinet sub-Lorentzian structures of Hamiltonian type*.

In [3], the authors computed normal forms for contact sub-Riemannian structures on $\mathbb{R}^3$. By adapting their reasoning to the sub-Lorentzian case, normal forms for contact sub-Lorentzian structures on $\mathbb{R}^3$ were found in [6]. Similar considerations in the Martinet case lead to the following theorem.

Theorem 1.1. *Assume that (H, g) is an analytic, time-oriented, Martinet sub-Lorentzian structure defined in a neighborhood U of the zero in $\mathbb{R}^3$. Assume that (H, g) is of Hamiltonian type. Then, possibly after shrinking U, there are coordinates x, y, z defined on U in which (H, g) admits an orthonormal frame in the following normal form*:

$$\begin{aligned} X &= \frac{\partial}{\partial x} + y\varphi\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + \frac{1}{2}y^2(1+\psi)\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y} - x\varphi\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - \frac{1}{2}xy(1+\psi)\frac{\partial}{\partial z} \end{aligned} \tag{1.1}$$

with a time orientation X, where φ and ψ are analytic functions on U, $\psi(0,0,z) = 0$, and the Martinet surface S for H is given by $S = \{y = 0\}$.

Using these normal forms, one can investigate the structure of reachable sets from the origin.

Theorem 1.2. *Let (H, g) be an analytic, time-oriented, sub-Lorentzian structure given by an orthonormal frame in the normal form* (1.1)*, defined on a neighborhood U of the zero. Assume that U is a sufficiently* *small normal neighborhood of the origin. Then there exist analytic functions* $\eta_1, \dots, \eta_4$ *on* U *and a* 2*-dimensional semi-analytic set* Σ *such that* $U \cap \{x \geq 0\} \setminus \Sigma$ *has two connected components, say* Σ^+ *and* Σ^-*, and*

$$\begin{aligned} J^+(0,U) &= A_1 \cup A_2 \cup A_3 \cup A_4, \\ I^+(0,U) &= \operatorname{int}(A_1 \cup A_2 \cup A_3 \cup A_4) \cup A_5, \\ N^+(0,U) &= \operatorname{int}(A_1 \cup A_2 \cup A_3 \cup A_4) \cup \tilde{\partial}(A_1 \cup A_2) \setminus A_5, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \{(x,y,z) \in U : \eta_1(x,y,z) \leq 0\} \cap \Sigma^+ \cap \{z \geq 0\}, \\ A_2 &= \{(x,y,z) \in U : \eta_2(x,y,z) \leq 0\} \cap \Sigma^- \cap \{z \geq 0\}, \\ A_3 &= \{(x,y,z) \in U : \eta_3(x,y,z) \leq 0\} \cap \{y \geq 0,\ x \geq 0\} \cap \{z \leq 0\}, \\ A_4 &= \{(x,y,z) \in U : \eta_4(x,y,z) \leq 0\} \cap \{y \leq 0,\ x \geq 0\} \cap \{z \leq 0\}, \\ A_5 &= \{(0,0,x) : x > 0\}; \end{aligned}$$

A_5 *is the set of points of the abnormal t.f.d. geodesic starting at* 0*. In particular, the three sets* $I^+(0,U)$*,* $J^+(0,U)$*, and* $N^+(0,U)$ *are semi-analytic.*

1.2. **Structure of the paper.** In Sec. 2, reachable sets for the flat Martinet sub-Lorentzian structure are computed (see Proposition 2.1).

In Sec. 3, we prove the main results of the paper, Theorems 1.1 and 1.2.

Section 4 presents some corollaries of Theorems 1.1 and 1.2. In Sec. 4.1, the future null conjugate locus is computed. Then we draw attention to an interesting feature of the future conjugate locus of a point belonging to the Martinet surface: it contains unique maximizing geodesics (null and timelike) starting at this point. The future null cut locus for a Martinet sub-Lorentzian structure of Hamiltonian type is computed in Sec. 4.2. In Sec. 4.3, we examine the image under the exponential mapping and continuity of the sub-Lorentzian distance function. Finally, in Sec. 4.4, we show how to apply our results to computing reachable sets from a point for control affine systems with a scalar input u, and with constraints $|u| \leq \delta$.

Section 5.1 presents two other normal forms for Martinet sub-Lorentzian structures; by the way we prove Proposition 5.2 asserting that the abnormal curve initiating at a point q is geometrically optimal, i.e., it is contained in the boundary of the reachable set from q. In Sec. 5.2, we give a constructive definition of normal neighborhoods.

2. Reachable sets in the Martinet flat case

In this section, we compute reachable sets in the so-called Martinet flat case. To be more precise, let

$$\hat{X} = \frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial}{\partial z}, \quad \hat{Y} = \frac{\partial}{\partial y} - \frac{1}{2}xy\frac{\partial}{\partial z}$$

be two vector fields on $\mathbb{R}^3$. Consider the sub-Lorentzian structure $(\hat{H}, \hat{g})$, where $\hat{H} = \mathrm{Span}\{\hat{X}, \hat{Y}\}$. Let $\hat{X}, \hat{Y}$ be an orthonormal basis for $\hat{g}$ with a time orientation $\hat{X}$. The Martinet surface for $\hat{H}$ is $S = \{y = 0\}$. S is foliated by t.f.d. abnormal curves $t \to (t, 0, z)$. It can be easily proved that in the case under consideration all abnormal curves are Hamiltonian geodesics. Using [8, Theorems 3.1 and 3.2], we see that a curve $t \to (t, 0, z)$, which is a t.f.d. Hamiltonian geodesic, is contained in the boundary of the reachable set. The sub-Lorentzian structure $(\hat{H}, \hat{g})$ is referred to as the *flat Martinet sub-Lorentzian structure*. This case is very important because, as we shall soon see, the flat case gives an approximation to the general case considered in the next section.

Our aim is to prove Proposition 2.1 below. By $\hat{J}^+(0, U)$, $\hat{I}^+(0, U)$, and $\hat{N}^+(0, U)$ we denote the corresponding reachable sets from the origin for $(\hat{H}, \hat{g})$. We also use the following notation:

$$\hat{J}^+(0) = \hat{J}^+(0, \mathbb{R}^3), \quad \hat{I}^+(0) = \hat{I}^+(0, \mathbb{R}^3), \quad \hat{N}^+(0) = \hat{N}^+(0, \mathbb{R}^3).$$

Proposition 2.1. *Reachable sets from zero for the flat Martinet sub-Lorentzian structure have the form*

$$\hat{J}^+(0) = \hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_3 \cup \hat{A}_4, \tag{2.1}$$

$$\hat{I}^+(0) = \mathrm{int}\left(\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_3 \cup \hat{A}_4\right) \cup A_5, \tag{2.2}$$

$$\hat{N}^+(0) = \mathrm{int}\left(\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_3 \cup \hat{A}_4\right) \cup \tilde{\partial}\left(\hat{A}_1 \cup \hat{A}_2\right) \setminus A_5, \tag{2.3}$$

where

$$\begin{aligned}
\hat{A}_1 &= \left\{(x,y,z) : z - \frac{1}{16}(x^2 - y^2)(x + 3y) \le 0,\ x \ge 0, y \ge 0,\ z \ge 0\right\},\\
\hat{A}_2 &= \left\{(x,y,z) : z - \frac{1}{16}(x^2 - y^2)(x - 3y) \le 0,\ x \ge 0,\ y \le 0,\ z \ge 0\right\},\\
\hat{A}_3 &= \left\{(x,y,z) : -z - \frac{1}{4}(xy^2 - y^3) \le 0,\ x \ge 0,\ y \ge 0,\ z \le 0\right\},\\
\hat{A}_4 &= \left\{(x,y,z) : -z - \frac{1}{4}(xy^2 + y^3) \le 0,\ x \ge 0,\ y \le 0,\ z \le 0\right\},\\
A_5 &= \{(x,0,0) : x > 0\}.
\end{aligned}$$

Moreover, if U is a normal neighborhood of 0, then

$$\begin{gathered}
\hat{I}^+(0, U) = \hat{I}^+(0) \cap U, \quad J^+(0, U) = \hat{J}^+(0) \cap U,\\
\hat{N}^+(0, U) = \hat{N}^+(0) \cap U.
\end{gathered} \tag{2.4}$$

Before starting the proof, we recall a definition of one more notion that will be used throughout the paper. Let (M, H, g) be a sub-Lorentzian manifold. Let $U \subset M$ be an open subset and $f \in C^\infty(U)$. By the *horizontal* *gradient of the function* f we mean the vector field $\nabla_H f$ defined by the condition

$$d_q f(v) = g(v, \nabla_H f(q))$$

for every‘ $q \in U$ and $v \in H_q$. If $X_0, \dots, X_k$ is an orthonormal basis for (H, g) defined on U with a time orientation X_0, then

$$\nabla_H f = -X_0(f)X_0 + X_1(f)X_1 + \cdots + X_k(f)X_k.$$

Clearly, if, for example, $\gamma : [a, b] \to U$ is a t.f.d. curve and $\nabla_H f$ is null f.d., then the function $t \to f(\gamma(t))$ is decreasing.

Proof. In order to prove Proposition 2.1, we first note that, due to the behavior of the fields $\hat{X}$ and $\hat{Y}$ on $\{y = \pm x\}$, the two inclusions

$$\hat{J}^+(0) \subset \{|y| \leq x,\ x \geq 0\}, \quad \hat{I}^+(0) \subset \{|y| < x,\ x > 0\} \tag{2.5}$$

hold. It is also clear that

$$\hat{J}^+(0) \cap \{z = 0\} = \{|y| \leq x,\ x \geq 0\} \cap \{z = 0\} \tag{2.6}$$

since the half-lines $\{(x, ax, 0) : x \geq 0\}$ are t.f.d. Hamiltonian geodesics for $|a| < 1$ and null f.d. Hamiltonian geodesics for $|a| = 1$.

Let us define two hypersurfaces

$$\Gamma_1 = \{(x, x, z) : x, z \in \mathbb{R}\}, \tag{2.7}$$

$$\Gamma_2 = \{(x, -x, z) : x, z \in \mathbb{R}\}. \tag{2.8}$$

We will construct four analytic functions $\hat{\eta}_1, \dots, \hat{\eta}_4$, where $\hat{\eta}_1$ is the solution of the Cauchy problem

$$(\hat{X} - \hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_1}(x, x, z) = z,$$

$\hat{\eta}_2$ is the solution of the Cauchy problem

$$(\hat{X} + \hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_2}(x, -x, z) = z,$$

$\hat{\eta}_3$ is the solution of the Cauchy problem

$$(\hat{X} + \hat{Y})(\eta) = 0, \quad \eta_{|S}(x, 0, z) = -z,$$

and $\hat{\eta}_4$ is the solution to the Cauchy problem

$$(\hat{X} - \hat{Y})(\eta) = 0, \quad \eta_{|S}(x, 0, z) = -z.$$

After easy computations we obtain

$$\hat{\eta}_1(x, y, z) = z - \frac{1}{16}(x^2 - y^2)(x + 3y),$$

$$\hat{\eta}_2(x, y, z) = z - \frac{1}{16}(x^2 - y^2)(x - 3y),$$

$$\hat{\eta}_3(x, y, z) = -z - \frac{1}{4}(xy^2 - y^3),$$

$$\hat{\eta}_4(x, y, z) = -z - \frac{1}{4}(xy^2 + y^3)$$

(they are the functions appearing in the hypothesis of Proposition 2.1). Moreover,

$$\begin{aligned}
\nabla_{\hat{H}}\hat{\eta}_1(x,y,z) &= \frac{3}{16}(x-y)(x+3y)(X-Y),\\
\nabla_{\hat{H}}\hat{\eta}_2(x,y,z) &= \frac{3}{16}(x+y)(x-3y)(X+Y),\\
\nabla_{\hat{H}}\hat{\eta}_3(x,y,z) &= \frac{3}{4}y^2(X+Y),\\
\nabla_{\hat{H}}\hat{\eta}_4(x,y,z) &= \frac{3}{4}y^2(X-Y).
\end{aligned} \tag{2.9}$$

Thus, we see that $\nabla_{\hat{H}}\hat{\eta}_1$ is null f.d. on $\left\{-\frac{1}{3}x < y < x,\ x > 0\right\}$, $\nabla_{\hat{H}}\hat{\eta}_2$ is null f.d. on $\left\{-x < y < \frac{1}{3}x,\ x > 0\right\}$, while $\nabla_{\hat{H}}\hat{\eta}_3$ and $\nabla_{\hat{H}}\hat{\eta}_4$ are null f.d. on $\{y \neq 0,\ x > 0\}$.

Now take a nspc.f.d. curve $\gamma : [0,T] \to \mathbb{R}^3$, $\gamma(0) = 0$. In view of (2.5), to prove that $\hat{J}^+(0)$ is contained in the right-hand side of (2.1), it suffices to note that the function $t \to \hat{\eta}_1(\gamma(t))$ is nonincreasing on every connected component of

$$\{t \in [0,T] : \gamma(t) \in \{0 \le y \le x,\ x \ge 0\}\},$$

the function $t \to \hat{\eta}_2(\gamma(t))$ is nonincreasing on every connected component of

$$\{t \in [0,T] : \gamma(t) \in \{-x \le y \le 0,\ x \ge 0\}\},$$

the function $t \to \hat{\eta}_3(\gamma(t))$ is nonincreasing on every connected component of

$$\{t \in [0,T] : \gamma(t) \in \{0 \le y \le x,\ x \ge 0\}\},$$

and, finally, the function $t \to \hat{\eta}_4(\gamma(t))$ is nonincreasing on every connected component of

$$\{t \in [0,T] : \gamma(t) \in \{-x \le y \le 0,\ x \ge 0\}\}.$$

In order to prove the inverse inclusion in (2.1), first let us state a lemma which easily follows from properties of Lorentzian metrics.

Lemma 2.1 (cf. [8]). *Let U be an open subset of a sub-Lorentzian manifold (M,H,g). Assume that $f : U \to \mathbb{R}$ is a smooth function such that $\nabla_H f$ is everywhere null f.d. Then every level surface $\{f = \text{const}\}$ is invariant with respect to $\nabla_H f$. Moreover, for every $q \in \{f = \text{const}\}$, the tangent space $T_q\{f = \text{const}\}$ contains the unique nonspacelike direction, namely the nonspacelike direction of $\nabla_H f(q)$. It also follows that trajectories of $\nabla_H f$ are geometrically optimal and are unique (up to a change of parameter) maximizing U-geodesics.*

Take a point $q = (x_0, y_0, z_0)$ belonging to the right-hand side of (2.1). If $z_0 = 0$, then, by (2.6), there is nothing to do, so assume that $z_0 \neq 0$. To proceed further, let us note that, by Lemma 2.1, the set $\tilde{\partial}\hat{A}_1 \cap \{y > 0,\ z > 0\}$

is formed by null f.d. curves starting at $\{y = x,\ x \geq 0,\ z = 0\}$; they are reparametrizations of trajectories of the horizontal gradient $\nabla_{\hat{H}}\hat{\eta}_1$. Similar reasonings show that $\tilde{\partial}\hat{A}_2 \cap \{y < 0,\ z > 0\}$ is formed by null f.d. curves that start at $\{y = -x,\ x \geq 0,\ z = 0\}$ and that are reparametrizations of trajectories of $\nabla_{\hat{H}}\hat{\eta}_2$. Also $\tilde{\partial}\hat{A}_3 \cap \{y > 0,\ z < 0\}$ (respectively, $\tilde{\partial}\hat{A}_4 \cap \{y < 0,\ z < 0\}$) is made up of null f.d. curves starting at $\{y = z = 0,\ x \geq 0\}$ which are reparametrizations of trajectories of $\nabla_{\hat{H}}\hat{\eta}_3$ (respectively, $\nabla_{\hat{H}}\hat{\eta}_4$). Now, if $q \in \tilde{\partial}\left(\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_3 \cup \hat{A}_4\right)$, then, by the above argument, $q \in \hat{J}^+(0)$. On the other hand, if $q \in \operatorname{int}\left(\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_3 \cup \hat{A}_4\right)$, let $\dot{\sigma}(t) = -\hat{X}(\sigma(t))$, $\sigma(0) = q_0$; σ is timelike past directed. Clearly,

$$q \in \operatorname{int}\left(\hat{A}_1 \cup \hat{A}_2 \cup \hat{A}_3 \cup \hat{A}_4\right) \cap \hat{A}_i$$

for suitable i. Obviously, $\hat{\eta}_i(q) < 0$ and the function $t \to \hat{\eta}_i(\sigma(t))$ increases. It follows that σ intersects first either $\{z = 0\}$ or $\tilde{\partial}\left(\hat{A}_1 \cup \cdots \cup \hat{A}_4\right)$ (the y-coordinate of $\sigma(t)$ remains constant during the whole motion), in both cases leading us, again by the above argument, to conclusion that $q \in \hat{J}^+(0)$ which proves (2.1).

Next let us recall that $\operatorname{int}\hat{I}^+(0) = \operatorname{int}\hat{J}^+(0)$ and $\tilde{\partial}\hat{I}^+(0) = \tilde{\partial}\hat{J}^+(0)$ (see [8]). Thus, to obtain $\hat{I}^+(0)$, it suffices to take $\hat{J}^+(0)$ and remove from it all geometrically optimal curves which are not t.f.d. Using (2.1), Lemma 2.1, and the above remark, we can see that the abnormal curve $t \to (t, 0, 0)$ is the only t.f.d. geometrically optimal curve starting at 0. Thus, (2.2) is proved.

Now (2.3) is easily proved. Finally, to obtain (2.4), we use the same argument as in [7]. The proof of Proposition 2.1 is complete. □

3. Reachable sets in the general Hamiltonian Martinet case

3.1. **Normal forms.** Assume that (H, g) is an analytic Martinet sub-Lorentzian structure of Hamiltonian type defined in a neighborhood U of the zero in $\mathbb{R}^3$. Denote by S the Martinet surface for H. In this section, we prove Theorem 1.1. During the proof, we will assume that U is as small as is needed for our purposes. Let Γ be a curve which is contained in S, passes through the origin and is transverse to H.

Lemma 3.1. *There are coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ on U with the following properties*:

1. $S = \{\tilde{y} = 0\}$, $\Gamma = \{\tilde{x} = \tilde{y} = 0\}$;
2. $H_{|S} = \ker d\tilde{z}$;

3. $\frac{\partial}{\partial \tilde{x}}_{|S}$ *and* $\frac{\partial}{\partial \tilde{y}}_{|S}$ *form an orthonormal basis for* $H_{|S}$ *with a time orientation* $\frac{\partial}{\partial \tilde{x}}_{|S}$.

Proof. Assume that $\gamma = \gamma(\tilde{z})$ is a parametrization of a curve Γ, $\gamma(0) = 0$. Let X, Y be an orthonormal frame for (H, g) defined on U, such that X is a time orientation and $X_{\ |S}$ is tangent to S. If g^t (respectively, h^t) is the flow of X (respectively, Y), then our coordinates are given by the mapping $(\tilde{x}, \tilde{y}, \tilde{z}) \to h^{\tilde{y}} g^{\tilde{x}} \gamma(\tilde{z})$. □

Denote by $\mathcal{H}$ the geodesic Hamiltonian for our structure. If $\tilde{p}, \tilde{q}, \tilde{r}$ are dual coordinates to $\tilde{x}, \tilde{y}, \tilde{z}$, then

$$\mathcal{H}_{|T^*_S \mathbb{R}^3} = -\frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{q}^2,$$

where

$$T^*_S \mathbb{R}^3 = \bigcup_{q \in S} T^*_q \mathbb{R}^3.$$

It follows that

$$\mathcal{H}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}) = -\frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{q}^2 + \tilde{y}\mathcal{G}_1(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r})$$

for an analytic function $\mathcal{G}_1$. Using the assumption that the curves $t \to (t, 0, z_0)$ are Hamiltonian geodesics, one in fact has

$$\mathcal{H}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}) = -\frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{q}^2 + \tilde{y}^2\mathcal{G}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}) \tag{3.1}$$

with an analytic function $\mathcal{G}$. Let

$$A_\Gamma = \{(0, 0, \tilde{z}, \tilde{p}, \tilde{q}, 0) : \tilde{z} \in (-\varepsilon, \varepsilon)\} \subset T^*\mathbb{R}^3,$$

$\varepsilon > 0$ is sufficiently small. Note that the set A_Γ is simply the set of initial conditions for Hamiltonian geodesics satisfying the transversality condition of the Pontryagin maximum principle with respect to Γ. Define the mapping $\mu : A_\Gamma \to M$ by the formula

$$\mu(\tilde{z}, \tilde{p}, \tilde{q}) = \pi \circ \Phi_1(0, 0, \tilde{z}, \tilde{p}, \tilde{q}, 0)$$

(recall that Φ_t is the flow of $\overrightarrow{\mathcal{H}}$). One easily proves that if N is a sufficiently small neighborhood of the set $\{(0, 0, \tilde{z}, 0, 0, 0) : \tilde{z} \in (-\varepsilon, \varepsilon)\}$ in A_Γ, then $\mu_{|N}$ is a diffeomorphism onto its image. Thus we can assume that $U = \mu(N)$. By the way we have proved that every t.f.d. Hamiltonian geodesic $t \to \pi \circ \Phi_t(0, 0, \tilde{z}, \tilde{p}, \tilde{q}, 0)$, with $(0, 0, \tilde{z}, \tilde{p}, \tilde{q}, 0) \in N$, is length-maximizing between Γ and its endpoint. The coordinates defined by the composition

$$\mathbb{R}^3 \xleftarrow{\alpha} \mathbb{R}^3 \xleftarrow{(\tilde{z}, -\tilde{p}, \tilde{q})} N \xleftarrow{\mu^{-1}} U,$$

$\alpha(a,b,c) = (b,c,a)$, are called *normal coordinates* and are denoted by x,y,z. To be precise, a point q has normal coordinates x,y,z if and only if $q = \pi \circ \Phi_1(0,0,z,-x,y,0)$. Using (3.1) together with our assumption that the abnormal curves are t.f.d. Hamiltonian geodesics, it is clear that $S = \{y = 0\}$. Moreover, $X_{|S} = \dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial x}_{|\Gamma}$ and $\dfrac{\partial}{\partial y}_{|\Gamma}$ form an orthonormal frame for $H_{|\Gamma}$.

Next, we proceed exactly as in [6] finally obtaining an orthonormal frame

$$\begin{aligned} X &= \frac{\partial}{\partial x} - yB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - yA\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y} + xB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + xA\frac{\partial}{\partial z} \end{aligned} \tag{3.2}$$

with a time orientation X, where A and B are analytic on U.

By the definition of a Martinet-type distribution, $[X,Y](q) \in H_q$ for every point $q \in S$. An easy computation yields

$$[X,Y]_{|S} = x\left(3B + x\frac{\partial B}{\partial x} + x^2B^2\right)\frac{\partial}{\partial y} + \left(2A + x\frac{\partial A}{\partial x} + x^2BA\right)\frac{\partial}{\partial z}; \tag{3.3}$$

all quantities in the right-hand side of (3.3) should be taken at $(x,0,z)$. Thus, there exists an analytic function $r = r(x,z)$ such that

$$\begin{cases} x\left(3B + x\dfrac{\partial B}{\partial x} + x^2B^2\right) = r(1 + x^2B), \\ 2A + x\dfrac{\partial A}{\partial x} + x^2BA = rxA \end{cases} \tag{3.4}$$

(again all quantities in (3.4) are to be taken at $(x,0,z)$). We solve the first equation in (3.4) with respect to r and substitute the result into the second equation in (3.4). After simplifying we obtain

$$2A + x\frac{\partial A}{\partial x} + x^3\left(\frac{\partial A}{\partial x}B - A\frac{\partial B}{\partial x}\right) = 0, \tag{3.5}$$

which implies that $A(x,0,z)$ vanishes identically. Indeed, assume that we can find z such that the function $x \to A(x,0,z)$ does not vanish identically. Then the Taylor formula gives

$$A(x,0,z) = a_k(z)x^k + o(x^k), \quad x \to 0,$$

where $a_k(z) \neq 0$, $k > 0$. Substituting it into (3.5), we obtain

$$(k+2)a_k(z)x^k = o(x^k).$$

Therefore, $a_k(z) = 0$; a contradiction.

It follows that A can be replaced by yA (for some other A) in Eqs. (3.2). As a result, we obtain an orthonormal frame in the form

$$\begin{aligned} X &= \frac{\partial}{\partial x} - yB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - y^2 A\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y} + xB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + xyA\frac{\partial}{\partial z}. \end{aligned} \tag{3.6}$$

Now we will compute brackets $[X,Y]$, $[X,[X,Y]]$, and $[Y,[X,Y]]$. Again, by the definition of Martinet-type distributions, for every $q \in S$ we have $[X,Y](q) \in H_q$ and

$$\operatorname{Span}\{X(q), Y(q), [X,[X,Y]](q), [Y,[X,Y]](q)\} = T_q\mathbb{R}^3. \tag{3.7}$$

Our computations are carried out on $S = \{y = 0\}$, so we can ignore terms of sufficiently high order with respect to y:

$$\begin{aligned} [X,Y] &= \left(3yB + xy\frac{\partial B}{\partial x} + x^2 yB^2\right)\frac{\partial}{\partial x} \\ &\quad + \left(3xB + x^2\frac{\partial B}{\partial x} + xy\frac{\partial B}{\partial y} + x^3 B^2\right)\frac{\partial}{\partial y} \\ &\quad + \left(3yA + xy\frac{\partial A}{\partial x} + x^2 yAB\right)\frac{\partial}{\partial z} + O(y^2), \\ [X,[X,Y]] &= 3\left(B + x\frac{\partial B}{\partial x} + 3x^2B^2 + 2x^3 B\frac{\partial B}{\partial x}\right)\frac{\partial}{\partial y} + O(y), \\ [Y,[X,Y]] &= (\dots)\frac{\partial}{\partial x} + (\dots)\frac{\partial}{\partial y} \\ &\quad + \left(3A + x\frac{\partial A}{\partial x} + x^2 AB + x^3\left(\frac{\partial A}{\partial x}B - A\frac{\partial B}{\partial x}\right)\right)\frac{\partial}{\partial z} + O(y); \end{aligned}$$

the terms in parentheses do not play any role since

$$H_{|S} = \operatorname{Span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}.$$

Using the last expression, we find that the fact that (3.7) holds on U is equivalent to the inequality $A(0,0,z) \neq 0$ on U.

As a final step towards obtaining (1.1), we will normalize coefficients of $\partial/\partial z$ in (3.6) by making the following change of coordinates: $x \to x$, $y \to y$, $z \to \alpha(z)z$, where α is an analytic function, which is a solution to the equation

$$\alpha(z) + z\frac{d\alpha}{dz} = \frac{1}{2A(0,0,z)}.$$

As a result, we obtain exactly the same formulas as in (3.6) but additionally with the condition $A(0,0,z) = 1/2$ for every $(0,0,z) \in U$. To complete the proof of Theorem 1.1, now it suffices to set $\varphi = -B$ and $\psi = 2A - 1$.

3.2. **Reachable sets.** Let (H, g) be a sub-Lorentzian structure defined on a neighborhood U of the origin in $\mathbb{R}^3$. Assume that $H = \mathrm{Span}\{X, Y\}$, where X and Y are in the normal form (1.1) and g is a Lorentzian metric on H defined by declaring the frame X, Y to be orthonormal with a time orientation X. In this section, we show how to compute reachable sets from the origin for such structures. As in the previous section, we assume that U is as small as we need. The procedure is almost the same as in the flat case from Sec. 2. To start with, let us note that by the construction of normal forms, (2.5) and (2.6) remain true in the general case.

Let Γ_1 be defined as in (2.7) and Γ_2 as in (2.8). We construct four analytic functions $\eta_1, \dots, \eta_4$, where η_1 is the solution of the Cauchy problem

$$(X - Y)(\eta) = 0, \quad \eta_{|\Gamma_1}(x, x, z) = z,$$

η_2 is the solution of the Cauchy problem

$$(X + Y)(\eta) = 0, \quad \eta_{|\Gamma_2}(x, -x, z) = z,$$

η_3 is the solution of the Cauchy problem

$$(X + Y)(\eta) = 0, \quad \eta_{|S}(x, 0, z) = -z,$$

and η_4 is the solution of the Cauchy problem

$$(X - Y)(\eta) = 0, \quad \eta_{|S}(x, 0, z) = -z.$$

Let us write

$$X = \hat{X} + X_1, \quad Y = \hat{Y} + Y_1, \quad \eta_i = \hat{\eta}_i + R_i, \quad i = 1, \dots, 4.$$

One can see that finding, for example, η_1 is equivalent to solving the Cauchy problem

$$(X - Y)(R_1) = -(X_1 - Y_1)(\hat{\eta}_1), \quad R_{1|\Gamma_1}(x, x, z) = 0$$

from which it follows that

$$R_1(x, y, z) = O(r^4), \quad r = \sqrt{x^2 + y^2 + z^2},$$

i.e., R_1 has order (at zero) greater than $\hat{\eta}_1$. Since similar arguments are valid for η_2, η_3, and η_4, we conclude that every η_i can be regarded as a perturbation of $\hat{\eta}_i$, $i = 1, \dots, 4$.

Now we will compute the horizontal gradient of each η_i. First, we take η_1. By the definition of η_1,

$$\nabla_H \eta_1 = -X(\eta_1)(X - Y).$$

Next,

$$X(\eta_1) = \frac{1}{2}((X - Y) + (X + Y))(\eta_1) = \frac{1}{2}(X + Y)(\eta_1).$$

But $(X + Y)_{|\Gamma_1}$ is tangent to Γ_1 and is equal to $\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$; so, again by the definition of η_1, $X(\eta_1)$ is divisible by $x - y$. Using this fact, (2.9), and the above remark, we obtain the first equation in (3.8) below. Similar considerations prove the second equality in (3.8). Further, take η_3 into considerations. Clearly,

$$\nabla_H \eta_3 = -X(\eta_3)(X+Y).$$

Since $\eta_{3|y=0} = -z$, $\eta_3 = -z + y\zeta_3$ with analytic ζ_3. By definition, $(X+Y)(\eta_3) = 0$. Writing this and substituting $y = 0$, we obtain the condition $(1 - x^2\beta)\zeta_{3|y=0} = 0$ in a neighborhood of zero. Thus, $\eta_3 + z$ is divisible by y^2, and so is $X(\eta_3)$. In this way, we obtain the third equation in (3.8). The fourth equation is obtained similarly. To sum up, we list just obtained results:

$$\begin{aligned}
\nabla_H \eta_1(x,y,z) &= \frac{3}{16}(x-y)(x+3y+O(r^2))(X-Y),\\
\nabla_H \eta_2(x,y,z) &= \frac{3}{16}(x+y)(x-3y+O(r^2))(X+Y),\\
\nabla_H \eta_3(x,y,z) &= \frac{3}{4}y^2(1+O(r))(X+Y),\\
\nabla_H \eta_4(x,y,z) &= \frac{3}{4}y^2(1+O(r))(X-Y)
\end{aligned} \tag{3.8}$$

as $r \to 0$.

Take a sufficiently small $\varepsilon > 0$. Using the expressions (3.8), we see that $\nabla_H \eta_1$ is null f.d. on

$$\left\{\left(-\frac{1}{3}+\varepsilon\right)x < y < x,\ x > 0\right\} \cap U,$$

$\nabla_H \eta_2$ is null f.d. on

$$\left\{-x < y < \left(\frac{1}{3}-\varepsilon\right)x,\ x > 0\right\} \cap U.$$

Moreover, $\eta_1 < \eta_2$, i.e., $\{\eta_2 \le 0\} \subset \{\eta_1 \le 0\}$, on

$$\{\varepsilon x < y < x,\ x > 0\} \cap U$$

and $\eta_2 < \eta_1$, i.e., $\{\eta_1 \le 0\} \subset \{\eta_2 \le 0\}$, on

$$\{-x < y < -\varepsilon x,\ x > 0\} \cap U.$$

Let

$$Z = \{\eta_1 = \eta_2 = 0\} \cap \{-\varepsilon x < y < \varepsilon x,\ x \ge 0\} \cap U.$$

Clearly, looking at the horizontal gradients of η_1 and η_2, Z is a 1-dimensional semi-analytic set (see [10] for properties of semi-analytic sets). Moreover, it is seen that

$$(X-Y)(\eta_1 - \eta_2) = -2X(\eta_2) > 0$$

on $\{-\varepsilon x < y < \varepsilon x,\ x > 0\}$. It follows that Z is formed by a single analytic curve entering the origin. Let us set $\Sigma = \rho^{-1}(\rho(Z)) \cap U$, where $\rho : \mathbb{R}^3 \ni (x,y,z) \to (x,y) \in \mathbb{R}^2$ is the projection onto the (x,y)-plane. The set Σ is semi-analytic since $\rho(Z)$ is semi-analytic, and $\rho(Z)$ is semi-analytic since it can be considered as the image of the (semi-analytic) set Z under $\rho_{|\{\eta_1=0\}} : \{\eta_1 = 0\} \to \mathbb{R}^2$, which is a bi-analytic diffeomorphism (by the implicit function theorem $\{\eta_1 = 0\}$ is of the the form $\{z = \Phi(x, y)\}$ with an analytic function Φ). Now $U \cap \{x \geq 0\} \backslash \Sigma$ consists of two connected components (which are also semi-analytic sets), say Σ^+ and Σ^-.

Now we define four sets $A_1, \ldots, A_4$ as in the hypothesis of Theorem 1.2. The above considerations yield that $\nabla_H \eta_i$ is either null f.d. or equal to zero on A_i, $i = 1, \ldots, 4$; therefore, using the same arguments as in Sec. 2, one proves that

$$J^+(0, U) \subset A_1 \cup \cdots \cup A_4.$$

To prove the inverse inclusion, we note that, similarly as in the flat case, the boundary $\tilde{\partial}(A_1 \cup \cdots \cup A_4)$ consists of nspc.f.d. curves starting at the origin. Indeed, $\tilde{\partial} A_1 \cap \Sigma^+ \cap \{z > 0\}$ is formed by null f.d. curves starting at $\{y = x,\ x \geq 0,\ z = 0\}$. By Lemma 2.1, they are reparametrizations of trajectories of the horizontal gradient $\nabla_H \eta_1$. Similarly, we find that $\tilde{\partial} A_2 \cap \Sigma^- \cap \{z > 0\}$ is formed by null f.d. curves which start at $\{y = -x,\ x \geq 0,\ z = 0\}$ and which are reparametrizations of trajectories of $\nabla_H \eta_2$. Finally, $\tilde{\partial} A_3 \cap \{y > 0\} \cap \{z < 0\}$ (respectively, $\tilde{\partial} A_4 \cap \{y < 0\} \cap \{z < 0\}$) is formed by null f.d. curves that start at $\{y = z = 0,\ x \geq 0\}$ and are reparametrizations of trajectories of $\nabla_H \eta_3$ (respectively, $\nabla_H \eta_4$). Now, to complete the proof of Theorem 1.2 we proceed exactly as in the flat case.

3.3. **Example.** The first example of the exact computation of reachable sets was given in Sec. 2; here we give another example. Consider the sub-Lorentzian structure given by

$$X = \frac{\partial}{\partial x} + \frac{1}{2} y^2 (1 + xy) \frac{\partial}{\partial z},$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2} xy (1 + xy) \frac{\partial}{\partial z},$$

i.e., $\varphi = 0$ and $\psi = xy$ in (1.1). Using the procedure described above we obtain

$$\eta_1(x, y, z) = \hat{\eta}_1 - \frac{1}{32}(x+y)^4(x-y) + \frac{1}{96}(x+y)^2(x-y)^3 + \frac{1}{64}(x+y)^3(x-y)^2 - \frac{1}{128}(x+y)(x-y)^4,$$

$$\eta_2(x, y, z) = \hat{\eta}_2 + \frac{1}{32}(x-y)^4(x+y) - \frac{1}{96}(x-y)^2(x+y)^3 - \frac{1}{64}(x-y)^3(x+y)^2 + \frac{1}{128}(x-y)(x+y)^4,$$

$$\eta_3(x,y,z) = \hat{\eta}_3(x,y,z) - \frac{1}{6}(x-y)^2 y^3 - \frac{1}{8}(x-y)y^4,$$
$$\eta_4(x,y,z) = \hat{\eta}_4(x,y,z) - \frac{1}{6}(x+y)^2 y^3 + \frac{1}{8}(x+y)y^4.$$

After some computations we obtain

$$\eta_1(x,y,z) - \eta_2(x,y,z) = -\frac{1}{192}\,(x-y)\,(x+y)\left(5x^3 + 55xy^2 + 72y\right).$$

The equation $5x^3 + 55xy^2 + 72y = 0$ has one solution passing through zero:

$$y_1(x) = \frac{72}{110x}\left(-1 + \sqrt{1 - \frac{275}{1296}x^4}\right).$$

Clearly,

$$y_1(x) = \frac{1}{110x}\left(-72 \cdot \frac{1}{2} \cdot \frac{275}{1296}x^4 + O(x^8)\right)$$

as $x \to 0$; thus, $y_1(0) = y_1'(0) = 0$, and in this case

$$\Sigma = \left\{(x,y,z) : y - \frac{1}{x}\left(1 - \sqrt{1 - \frac{275}{1296}x^4}\right) = 0,\ x \geq 0\right\}.$$

Finally, we obtain

$$\begin{aligned}
A_1 &= \{(x,y,z) \in U : \eta_1(x,y,z) \leq 0\} \\
&\qquad \cap \left\{y \geq \frac{1}{x}\left(1 - \sqrt{1 - \frac{275}{1296}x^4}\right),\ x \geq 0\right\} \cap \{z \geq 0\}, \\
A_2 &= \{(x,y,z) \in U : \eta_2(x,y,z) \leq 0\} \\
&\qquad \cap \left\{y \leq \frac{1}{x}\left(1 - \sqrt{1 - \frac{275}{1296}x^4}\right),\ x \geq 0\right\} \cap \{z \geq 0\}, \\
A_3 &= \{(x,y,z) \in U : \eta_3(x,y,z) \leq 0\} \cap \{y \geq 0,\ x \geq 0\} \cap \{z \leq 0\}, \\
A_4 &= \{(x,y,z) \in U : \eta_4(x,y,z) \leq 0\} \cap \{y \leq 0,\ x \geq 0\} \cap \{z \leq 0\},
\end{aligned}$$

where U is a sufficiently small neighborhood of 0.

4. Some further results

In this section, we prove certain results which are mainly straightforward corollaries of Theorems 1.1 and 1.2. First, we state necessary definitions.

Let (M,H,g) be a sub-Lorentzian manifold. Let D_q be the set of all covectors $\lambda \in T_q^*M$ such that the curve $t \to \pi \circ \Phi_t(\lambda)$ is defined on $[0,1]$. By the *exponential mapping with the pole at* q we mean the map

$$\exp_q : D_q \to M, \quad \exp_q(\lambda) = \pi \circ \Phi_1(\lambda).$$

Clearly, D_q is an open set and $\exp_q$ is smooth (respectively, analytic) for smooth (respectively, analytic) (M, H, g). It is easy to see that a Hamiltonian geodesic $\gamma(t) = \pi \circ \Phi_t(\lambda)$, $\lambda \in T_q^*M$, has the form $\gamma(t) = \exp_q(t\lambda)$.

4.1. **Future conjugate loci.** Let (M, H, g) be a time-oriented, sub-Lorentzian manifold. A point $q \in M$ is said to be *conjugate* to a point $q_0 \in M$ if there is $\lambda \in T_{q_0}^*M$ such that $\exp_{q_0}(\lambda) = q$ and $d_\lambda \exp_{q_0}$ is singular. Then we say that *q is conjugate to q_0 along a geodesic* $\gamma(t) = \exp_{q_0}(t\lambda)$. By the *future null* (respectively, *timelike, nonspacelike*) *conjugate locus of a point* q_0 we mean the set of all points conjugate to q_0 along null f.d. (respectively, t.f.d., nspc.f.d.) Hamiltonian geodesics; we denote it by $\mathrm{Conj}_{q_0}^{null}$ (respectively, $\mathrm{Conj}_{q_0}^{t}$, $\mathrm{Conj}_{q_0}^{nspc}$). Obviously,

$$\mathrm{Conj}_{q_0}^{nspc} = \mathrm{Conj}_{q_0}^{null} \cup \mathrm{Conj}_{q_0}^{t}$$

since Hamiltonian geodesics preserve their casual character.

Proposition 4.1. *Assume that (H, g) is an analytic time-oriented Martinet sub-Lorentzian structure of Hamiltonian type defined on a sufficiently small neighborhood U of a point $q_0 \in \mathbb{R}^3$. Then $\mathrm{Conj}_{q_0}^{null}$ is equal to the union of the two null f.d. Hamiltonian geodesics starting at q_0.*

Proof. We know from [8] that there are exactly two null f.d. Hamiltonian geodesics γ_+, γ_- starting at q_0 and that they are geometrically optimal. We must show that γ_+ (respectively, γ_-) is formed by critical values for $\exp_{q_0}$. To this end, it suffices to show that γ_+ (respectively, γ_-) lies on the boundary of the set $\exp_{q_0}(D_{q_0})$. This becomes obvious when we transform our structure to the "prenormal" form (3.2). Noting that a curve $\gamma(t)$ is a spacelike Hamiltonian geodesisc for the structure (H, g) if and only if $\gamma(t)$ is a timelike Hamiltonian geodesic for the structure $(H, -g)$, we arrive at the conclusion that all spacelike geodesics starting at q_0 are, with respect to the coordinates as in (3.2), contained in the set $\{|y| > |x|\}$. This and the proof of Theorem 1.2 show that $\gamma_+ \cup \gamma_- \subset \partial \exp_{q_0}(D_{q_0})$. □

Note that the above result holds in fact for arbitrary smooth sub-Lorentzian structures in $\mathbb{R}^3$.

Proposition 4.1 allows to observe a phenomenon which is highly contradictory to the intuition. Namely, γ_+ (respectively, γ_-) is a unique maximizing geodesic (see [8] for a definition), which is entirely contained in $\mathrm{Conj}_{q_0}^{null}$. Similarly, in the situation described by Theorem 1.2, the curve $t \to (t, 0, 0)$ is a unique maximizing geodesic which is contained in $\mathrm{Conj}_{q_0}^{t}$.

4.2. **Future null cut locus.** Again, we are given a time-oriented sub-Lorentzian manifold (M, H, g). Fix a point $q_0 \in M$. Let us define the *future null cut locus* $\mathrm{Cut}_{q_0}^{null}(M)$ *of* q_0 as the set of all points $q \in M$ such that there exists a null f.d. geodesic $\gamma : [0, T] \to M$ with the following properties: $\gamma(0) = q_0$, $\gamma(t) = q$, $0 < t < T$, $\gamma_{|[0,t]}$ is a maximizing geodesic, and $\gamma_{|[0,t+\varepsilon]}$ is not a maximizing geodesic for any $\varepsilon > 0$, $t + \varepsilon \le T$.

Proposition 4.2. *Let* (H, g) *be an analytic, time-oriented, sub-Lorentzian structure defined on a neighborhood* U *of the origin, where* H *is a Martinet distribution. Assume that* (H, g) *is given on* U *as in Theorems* 1.1 *and* 1.2. *Then the future null cut locus* $\mathrm{Cut}_0^{null}(U)$ *of the origin is equal to*

$$\tilde{\partial} J^+(0, U) \cap \{z > 0\} \cap \Sigma \backslash \{0\}.$$

Proof. We know from Sec. 3 all null f.d. geometrically optimal curves starting at the origin. There are two possibilities. First is to move along $t \to (t, t, 0)$ and then along a reparametrization of a trajectory of $\nabla_H \eta_1$. By the construction, such a curve remains on the boundary $\tilde{\partial} J^+(0, U)$ until it reaches Σ and, by Lemma 2.1, is a unique length maximizer. After crossing Σ it enters $\mathrm{int}\, J^+(0, U) = \mathrm{int}\, I^+(0, U)$ and from this time is no longer neither geometrically optimal nor length maximizing.

The second possibility is first to travel along $t \to (t, -t, 0)$ and then along a reparametrization of a trajectory of $\nabla_H \eta_2$. To complete the proof, we use the similar reasoning as in the previous case. □

By the way we see that $\mathrm{Cut}_0^{null}(U)$ is a semi-analytic set. We also have

$$\mathrm{Cut}_0^{null}(U) \cap \mathrm{Conj}_0^{null} = \varnothing.$$

4.3. **Image of the set of "nonspacelike" covectors under the exponential mapping.** First, we prove the following proposition.

Proposition 4.3. *Let* (H, g) *be an analytic, time-oriented, sub-Lorentzian structure given as in Theorem* refthm1.1 *on a normal neighborhood* U *of the origin. Then*

$$\exp_0 (\{\lambda \in D_0 : \mathcal{H}(\lambda) < 0,\ \langle \lambda, X(0) \rangle < 0\}) \cap U = I^+(0, U) \tag{4.1}$$

and

$$\exp_0 (\{\lambda \in D_0 : \mathcal{H}(\lambda) \le 0,\ \langle \lambda, X(0) \rangle < 0\}) \cap U = I^+(0, U) \cup \gamma_+ \cup \gamma_-, \tag{4.2}$$

where γ_+ *and* γ_- *are the two null f.d. Hamiltonian geodesics in* U *starting at the zero.*

Proof. Our aim is to find maximizing geodesics. Using the reasoning conducted in Secs. 3.2 and 4.2 we already know all maximizing geodesics initiating at 0 and lying on the boundary $\tilde{\partial} I^+(0, U)$. As described above, these are up to a reparametrization:

- (i) concatenations of a segment of the trajectory of $X + Y$ starting at 0 and a segment of a trajectory of $X - Y$;
- (ii) concatenations of a segment of the trajectory of $X - Y$ starting at 0 and a segment of a trajectory of $X + Y$;

(iii) concatenations of a segment of the trajectory of X starting at 0 and a segment of a trajectory of $X+Y$;
(iv) concatenations of a segment of the trajectory of X starting at 0 and a segment of a trajectory of $X-Y$.

Now we find about geodesics joining 0 to a point $q_1 \in \operatorname{int} I^+(0,U)$. To this end, we consider the following optimization problem: among all curves satisfying

$$\dot{q} = v_0 X(q) + v_1 Y(q), \quad q(0) = 0, \quad q(T) = q_1,$$

where the final time $T > 0$ is not fixed, controls are measurable and bounded, $-v_0^2(t) + v_1^2(t) \leq 0$, $v_0(t) > 0$ a.e., find ones that maximize the length functional

$$\int_0^T \sqrt{v_0^2(t) - v_1^2(t)} dt.$$

This problem does not depend on the choice of parameterization, so we can change it as follows: among all curves satisfying

$$\dot{q} = X(q) + uY(q), \quad q(0) = 0, \quad q(T) = q_1,$$

where the final time $T > 0$ is not fixed, controls are measurable, and $|u| \leq 1$ a.e., find ones that maximize the length functional

$$\int_0^T \sqrt{1 - u^2(t)} dt.$$

As we know (see [5]), this problem has a solution, and we use the Pontryagin maximum principle (PMP for brevity; see, e.g., [1]) to solve it.

Assume that $\gamma : [0,T] \to U$ is a maximizing geodesic. By the PMP, it is a projection of either an abnormal or normal bi-extremal. Consider first the abnormal case. In this case, the Hamiltonian of the PMP has the form

$$h_u(q,p) = \langle p, X(q) \rangle + u \langle p, Y(q) \rangle.$$

Let $(\gamma(t), p(t))$ be an abnormal bi-extremal projecting onto γ, and let $u(t)$ be a control generating γ. By the PMP,

$$u(t) \langle p(t), Y(\gamma(t)) \rangle = \max_{|u| \leq 1} u \langle p(t), Y(\gamma(t)) \rangle$$

a.e. and $h_{u(t)}(\gamma(t), p(t)) = 0$. If $\langle p(t), Y(\gamma(t)) \rangle$ does not vanish, then $u(t) = \pm 1$ and γ is null f.d. On the other hand, if $\langle p(t), Y(\gamma(t)) \rangle$ vanishes on some interval Δ (or $u_{|\Delta} = 0$), then γ is the abnormal (t.f.d.) geodesics starting at 0. Thus, maximizing abnormal extremals are either timelike abnormal, or null, or their suitable concatenations as described at the begining of the proof (any additional switching causes a given extremal to cease to be maximizing).

Now assume that $(\gamma(t), p(t))$ is a normal bi-extremal. This time the PMP Hamiltonian has the form

$$h_u(q,p) = \langle p, X(q)\rangle + u\,\langle p, Y(q)\rangle + \sqrt{1-u^2}.$$

If $u(t)$ is a control generating γ, then

$$u(t)\,\langle p(t), Y(\gamma(t))\rangle + \sqrt{1-u(t)^2} = \max_{|u|\le 1}\left(u\,\langle p(t), Y(\gamma(t))\rangle + \sqrt{1-u^2}\right) \tag{4.3}$$

a.e. It can be seen that the maximum in the right-hand side of (4.3) is attained for

$$u(t) = \frac{\langle p(t), Y(\gamma(t))\rangle}{\sqrt{\langle p(t), Y(\gamma(t))\rangle^2 + 1}},$$

thus normal extremals are timelike. It follows (see [5]) that normal extremals are, up to a change of parameter, t.f.d. Hamiltonian geodesics. This proves (4.1) and (4.2). □

Let $f[U]$ be the sub-Lorentzian distance function (see [8]) for the sub-Lorentzian structure considered in Proposition 4.3. We will prove the following corollary.

Corollary 4.1. *The function $f[U]$ is not continuous at every point of the set $\tilde{\partial} J^+(0,U) \cap \{z \le 0\} \backslash \{y = \pm x\}$.*

Proof. Indeed, having proved Theorem 1.2, owing to Proposition 4.3 together with [5] we know that $f[U]$ is continuous at all points of $\operatorname{int} I^+(0,U)$. At the same time, the continuity of $f[U]$ on $U\backslash J^+(0,U)$ is obvious. On the other hand, by [8] we know that the continuity of $f[U]$ at a point $q \in \tilde{\partial} J^+(0,U)$ is equivalent to the condition $f[U](q) = 0$. To complete the proof, it suffices to note that $f[U]$ restricted to $\tilde{\partial} J^+(0,U) \cap \{z \le 0\} \backslash \{y = \pm x\}$ is positive. □

4.4. **Application to control affine systems with constraints $|u| \le \delta$.** All what we have said above can be applied to study reachable sets from a point for affine control systems

$$\dot{q} = X(q) + uY(q), \quad |u| \le \delta, \tag{4.4}$$

where $\delta > 0$, X and Y are vector fields defined on an open subset of $\mathbb{R}^3$ such that $\mathrm{Span}\{X, Y\}$ is a Martinet distribution. Indeed, assume that we are given such a system. We define a time-oriented sub-Lorentzian structure (H, g) where $H = \mathrm{Span}\{X, Y\}$ and $\frac{1}{\delta}X, Y$ is an orthonormal frame for g with a time orientation X. Clearly, any trajectory of (4.4) is nspc.f.d. for (H, g). Conversely, let $\gamma : [0, T] \to \mathbb{R}^3$ be nspc.f.d. for (H, g), i.e.,

$$\dot{\gamma}(t) = u_0(t)\left(\frac{1}{\delta}X\right)(\gamma(t)) + u_1(t)Y(\gamma(t))$$

with $-u_0^2(t)+u_1^2(t)\le 0$, $u_0(t)>0$ a.e. Let $\beta:[0,T]\to[0,T_1]$ be defined by

$$\beta(t)=\int_0^t \frac{1}{\delta}u_0(s)ds,\quad T_1=\int_0^T \frac{1}{\delta}u_0(s)ds.$$

The function β is invertible; let $\alpha:[0,T_1]\to[0,T]$ be its inverse. Finally, let $\tilde\gamma:[0,T_1]\to\mathbb{R}^3$, $\tilde\gamma(t)=\gamma(\alpha(t))$,

$$(\tilde\gamma(t))\dot{} = X(\tilde\gamma(t))+\delta\frac{u_1(\alpha(t))}{u_0(\alpha(t))}Y(\tilde\gamma(t)),$$

so this is a desired reparametrization of γ. Therefore, if $q_0\in U$ for an open $U\subset\mathbb{R}^3$, the reachable set from a point q_0 for (4.4) in U is just the set $J^+(q_0,U)$ computed for the structure (H,g) defined above. This has one more implication—the passage to normal forms does not change the reachble sets for (4.4).

4.5. **Example.** To give an idea how to compute reachable sets for (4.4), we will compute one example corresponding to flat case studied in Sec. 2. In fact, a control affine analogue of Theorem 1.2 can be proved. For simplicity, we will prove a control affine analogue of Proposition 2.1.

Consider the affine control system

$$\dot q=\hat X+u\hat Y,\quad |u|\le\delta, \tag{4.5}$$

where

$$\hat X=\frac{\partial}{\partial x}+\frac{1}{2}y^2\frac{\partial}{\partial z},\quad \hat Y=\frac{\partial}{\partial y}-\frac{1}{2}xy\frac{\partial}{\partial z},$$

and δ is a given positive number. Let

$$\Gamma_{1,\delta}=\{(x,\delta x,z):x,z\in\mathbb{R}\},\quad \Gamma_{2,\delta}=\{(x,-\delta x,z):x,z\in\mathbb{R}\}.$$

We define four analytic functions $\hat\eta_{1,\delta},\dots,\hat\eta_{4,\delta}$, where $\hat\eta_{1,\delta}$ is the solution of the Cauchy problem

$$\left(\frac{1}{\delta}\hat X-\hat Y\right)(\eta)=0,\quad \eta_{|\Gamma_{1,\delta}}(x,\delta x,z)=z,$$

$\hat\eta_2$ is the solution of the Cauchy problem

$$\left(\frac{1}{\delta}\hat X+\hat Y\right)(\eta)=0,\quad \eta_{|\Gamma_{2,\delta}}(x,-\delta x,z)=z,$$

$\hat\eta_3$ is the solution of the Cauchy problem

$$\left(\frac{1}{\delta}\hat X+\hat Y\right)(\eta)=0,\quad \eta_{|S}(x,0,z)=-z,$$

and $\hat\eta_4$ is the solution of the Cauchy problem

$$\left(\frac{1}{\delta}\hat X-\hat Y\right)(\eta)=0,\quad \eta_{|S}(x,0,z)=-z.$$

We obtain

$$\hat{\eta}_{1,\delta}(x,y,z) = z - \frac{1}{16\delta}(\delta^2x^2 - y^2)(\delta x + 3y),$$
$$\hat{\eta}_{2,\delta}(x,y,z) = z - \frac{1}{16\delta}(\delta^2x^2 - y^2)(\delta x - 3y),$$
$$\hat{\eta}_{3,\delta}(x,y,z) = -z - \frac{1}{4\delta}(\delta xy^2 - y^3),$$
$$\hat{\eta}_{4,\delta}(x,y,z) = -z - \frac{1}{4\delta}(\delta xy^2 + y^3).$$

Let $\mathcal{A}_{\leq\delta}(0)$ (respectively, $\mathcal{A}_{<\delta}(0)$, $\mathcal{A}_{=\delta}(0)$) be the set of endpoints of all trajectories of (4.5) that start at 0 and are generated by controls with values in $[-\delta,\delta]$ (respectively, in $(-\delta,\delta)$, $\{+\delta,-\delta\}$).

Proposition 4.4. *We have*

$$\mathcal{A}_{\leq\delta}(0) = \hat{A}_{1,\delta} \cup \hat{A}_{2,\delta} \cup \hat{A}_{3,\delta} \cup \hat{A}_{4,\delta},$$
$$\mathcal{A}_{<\delta}(0) = \operatorname{int}\left(\hat{A}_{1,\delta} \cup \hat{A}_{2,\delta} \cup \hat{A}_{3,\delta} \cup \hat{A}_{4,\delta}\right) \cup A_5,$$
$$\mathcal{A}_{=\delta}(0) = \operatorname{int}\left(\hat{A}_{1,\delta} \cup \hat{A}_{2,\delta} \cup \hat{A}_{3,\delta} \cup \hat{A}_{4,\delta}\right) \cup \tilde{\partial}\left(\hat{A}_{1,\delta} \cup \hat{A}_{2,\delta}\right) \backslash A_5,$$

where

$$\hat{A}_{1,\delta} = \{(x,y,z) : \hat{\eta}_{1,\delta}(x,y,z) \leq 0,\ x \geq 0,\ y \geq 0,\ z \geq 0\},$$
$$\hat{A}_{2,\delta} = \{(x,y,z) : \hat{\eta}_{2,\delta}(x,y,z) \leq 0,\ x \geq 0,\ y \leq 0,\ z \geq 0\},$$
$$\hat{A}_{3,\delta} = \{(x,y,z) : \hat{\eta}_{3,\delta}(x,y,z) \leq 0,\ x \geq 0,\ y \geq 0,\ z \leq 0\},$$
$$\hat{A}_{4,\delta} = \{(x,y,z) : \hat{\eta}_{4,\delta}(x,y,z) \leq 0, x \geq 0,\ y \leq 0,\ z \leq 0\},$$
$$A_5 = \{(x,0,0) : x > 0\}.$$

In particular, $\mathcal{A}_{\leq\delta}(0)$, $\mathcal{A}_{<\delta}(0)$, and $\mathcal{A}_{=\delta}(0)$ are semi-analytic.

5. Appendices

5.1. Some other normal forms for Lorentzian metrics on Martinet-type distributions. In this section, we provide two other normal forms for Lorentzian metrics on Martinet type distributions, which are based on different ideas than those presented in Sec. 3.

The first idea of constructing normal forms comes from [2], where Riemannian metrics on Martinet distributions are studied. Arguments used in [2] can be easily adopted to sub-Lorentzian situation. As a result, we obtain the following proposition.

Proposition 5.1. *Let (H,g) be an analytic, time-oriented, Martinet sub-Lorentzian structure defined in a neighborhood of the origin in $\mathbb{R}^3$. Then there exist coordinates x, y, z defined near zero in which*

$$H = \ker\left(dz - \frac{1}{2}y^2dx\right)$$

and g possesses an orthonormal frame X, Y, where X is a time orientation, in the form

$$X = (1+F)\left(\frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial}{\partial z}\right),$$
$$Y = (1+G)\frac{\partial}{\partial y},$$

where F and G are analytic functions such that $F_{|y=0} = 0$ and $G(0) = 0$. In particular, denoting by S the Martinet surface for H, $S = \{y = 0\}$, we see that the abnormal curves foliating S have the form $t \to (t, 0, z_0)$.

Proof. The proof is almost the same as in the sub-Riemannian case, with obvious differences concerning indefiniteness of the metric. Consider, for example, the isoperimetric case. We will use the same notation as in [2]. So, [2, Lemma 2.11] remains unchanged, [2, Proposition 2.13] requires slight modification, namely, in our case $g = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2$ with $a(0,0) < 0$. Then it follows from [2, (2.8)] that also $A(0) = a(0)\sigma_X^2(0) < 0$. Now suitably rewriting the equation

$$3\sigma_y + y\sigma_{yy} = F(y, \sigma, \sigma_X, y\sigma_y, y\sigma_{XX}, y\sigma_{Xy}),$$

one can prove the convergence of σ by the direct application of [4, Theorem 6.3.1] (one easily checks that the Poincaré condition is satisfied). In [2, Lemma 2.15] we need to have $A_{|Y=0} = -1$. It can be achieved by assuming that σ_0 satisfies the differential equation

$$\sigma_0' = (-a_0(\sigma_0(X)))^{-1/2}.$$

Finally, note that in [2, Lemma 2.17] we have $\det\left(a_0c_0 - b_0^2\right) < 0$ and $a_0 < 0$. □

Note that here we do not have to assume that abnormal curves are Hamiltonian geodesics.

Normal forms from Proposition 5.1 are not very convenient to investigate reachable sets, however they immediately give the following result.

Proposition 5.2. *Let (H, g) be a time-oriented, Martinet, sub-Lorentzian structure. Then for every $q_0 \in S$, the Martinet surface S for H, and every sufficiently small neighborhood U of q_0, the abnormal curve initiating at q_0 is contained in $\tilde{\partial} J^+(q_0, U)$.*

Proof. We can assume that our structure is given as in Proposition 5.1 with $q_0 = 0$. Now, if $\gamma(t) = (x(t), y(t), z(t))$ is a nspc.f.d. curve in U, where U is a sufficiently small neighborhood of 0, then $\dot{x} > 0$ a.e. On the other hand, the equation for horizontal curves is

$$\dot{z} = y^2\dot{x}.$$

Thus, if $\gamma(t) = (x(t), y(t), z(t))$ is a nspc.f.d. curve defined on $[0, T]$ such that $\gamma(0) = (0,0,0)$ and $\gamma(T) = (a, 0, 0)$, then

$$z(T) = \int_0^T y^2(t)\dot{x}(t)dt = 0.$$

It follows that $y(t) = 0$ on $[0, T]$. We conclude that γ is a reparametrization of the t.f.d. abnormal curve $t \to (t, 0, 0)$. Now it suffices to recall [8] that $\operatorname{int} I^+(0, U) = \operatorname{int} J^+(0, U) = \operatorname{int} N^+(0, U)$. □

Normal forms from Theorem 1.1 and Proposition 5.1 are suitable for problems that are "measured" in some sense by the distance to a point. Below we present normal forms which can be useful when considering problems which are measured by the distance to the Martinet surface. Their idea comes from [3], where similar normal forms for Riemannian metrics on Martinet distributions were suggested.

Proposition 5.3. *Let (H, g) ba a smooth, time-oriented, Martinet, sub-Lorentzian structure defined in a neighborhood U of the origin in $\mathbb{R}^3$. Assume that (H, g) is of Hamiltonian type. Let S be the Martinet surface for H. Then, provided that U is sufficiently small, there exist coordinates x, y, z on U in which $S = \{y = 0\}$, and (H, g) admits an orthonormal frame in the following normal form:*

$$\begin{aligned} X &= (1 + ya)\frac{\partial}{\partial x} + y^2(1 + yb)\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y}, \end{aligned} \tag{5.1}$$

where X is a time orientation and $a, b \in C^\infty(U)$.

Proof. We assume that we are enclosed in a neighborhood U of the origin, which is assumed to be as small as we need.

Let us note that there exist coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ on U such that $S = \{\tilde{y} = 0\}$ and $\frac{\partial}{\partial \tilde{x}}_{|S}$ and $\frac{\partial}{\partial \tilde{y}}_{|S}$ form an orthonormal basis of $H_{|S}$ with a time orientation on S equal to $\frac{\partial}{\partial \tilde{x}}_{|S}$. Let $\tilde{p}, \tilde{q}, \tilde{r}$ be the dual coordinates. As in Sec. 3.1, the geodesic Hamiltonian has the form

$$\mathcal{H}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}) = -\frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{q}^2 + \tilde{y}^2\mathcal{G}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r})$$

with a smooth function $\mathcal{G}$. Let

$$A_S = \{(\tilde{x}, 0, \tilde{z}, 0, \tilde{q}, 0)\} \subset T^*\mathbb{R}^3;$$

define the mapping

$$\mu : A_S \to \mathbb{R}^3, \quad \mu(\tilde{x}, \tilde{z}, \tilde{q}) = \exp_{(\tilde{x},0,\tilde{z})}(0, \tilde{q}, 0).$$

Note that A_S is the set of the initial conditions of spacelike Hamiltonian geodesics satisfying the transversality condition with respect to S. Clearly, there exists a neighborhood N of the set $\{(\tilde{x}, 0, \tilde{z}, 0, 0, 0)\}$ in A_S such that $\mu_{|N}$ is a diffeomorphism onto its image, and we can assume that $U = \mu(N)$. Next, we similarly define the *normal coordinates* by the composition

$$\mathbb{R}^3 \xleftarrow{\alpha} N \xleftarrow{\mu^{-1}} U,$$

$\alpha(a, b, c) = (a, c, b)$. Thus, a point q has the normal coordinates (x, y, z) if and only if $q = \exp_{(x,0,z)}(0, y, 0)$.

By the definition of normal coordinates, spacelike Hamiltonian geodesics with initial conditions from A_S have the form $\gamma(s) = (x, s, z)$. Now

$$\left(\frac{\partial}{\partial y}\right)_{\gamma(s)} = \dot{\gamma}(s);$$

in the other words, $\partial/\partial y$ is a unit spacelike vector field. Let $\lambda = (x, 0, z, 0, 1, 0)$. Then

$$\gamma(s) = \pi \circ \Phi_1(x, 0, z, 0, s, 0) = \pi \circ \Phi_s(\lambda),$$

and $s \to \Phi_s(\lambda)$ is the Hamiltonian lift of γ. Using the definition of the geodesic Hamiltonian [8], we see that

$$g\left(\dot{\gamma}(s), v\right) = \langle \Phi_s(\lambda), v\rangle$$

for every $v \in H_{\gamma(s)}$. Let $\mathcal{C} = \{\mathcal{H} = 1/2\}$. Of course, $\Phi_s(\lambda) \in \mathcal{C}$ for every s (for which $\Phi_s(\lambda)$ is defined). Recall that, since $\mathcal{H}(q, p)$ is homogeneous with respect to p, the Liouville form α restricted to $\mathcal{C}$ is invariant with respect to Φ_s. Finally, let ξ be the vector field on $\mathcal{C}$ defined by

$$\pi_* \circ \Phi_{s*}\xi = \frac{\partial}{\partial x}$$

(ξ can be identified with $\partial/\partial x$, where x is treated as a coordinate on A_S). Taking all what we have said together we obtain a sequence of equalities:

$$\left\langle \Phi_s(\lambda), \frac{\partial}{\partial x} \right\rangle = \langle \Phi_s(\lambda), \pi_* \circ \Phi_{s*}\xi\rangle = \left\langle \alpha_{\Phi_s(\lambda)}, \Phi_{s*}\xi\right\rangle$$

$$= \left\langle \left(\Phi_s^*\alpha\right)_\lambda, \xi\right\rangle = \langle \alpha_\lambda, \xi\rangle = \langle \lambda, \pi_*\xi\rangle = \left\langle \lambda, \frac{\partial}{\partial x}\right\rangle = 0,$$

where, since $\pi \circ \Phi_s(\lambda) = (x, s, z)$, it is clear that

$$\pi_*\xi = \frac{\partial}{\partial x}.$$

Similarly, we show that

$$\left\langle \Phi_s(\lambda), \frac{\partial}{\partial z} \right\rangle = 0.$$

In this way, we have proved that there exists an orthonormal basis for (H, g), which, in normal coordinates, has the form

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}, \tag{5.2}$$

where X is a time orientation and a and b are smooth functions. By the construction, $X_{|S} = \partial/\partial x$, which implies that $a_{|y=0} = 1$ and $b_{|y=0} = 0$. Thus, (5.2) can be written as follows:

$$X = (1 + ya)\frac{\partial}{\partial x} + yb\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} \tag{5.3}$$

with some other smooth functions a and b. Further, note that the field $[X, Y]_{|S}$ is horizontal. We have

$$[X, Y] = -\left(a + y\frac{\partial a}{\partial y}\right)\frac{\partial}{\partial x} - \left(b + y\frac{\partial a}{\partial y}\right)\frac{\partial}{\partial z},$$

which yields $b_{|y=0} = 0$. Thus, b in (5.3) can be replaced by yb, for some other smooth b, and we obtain

$$X = (1 + ya)\frac{\partial}{\partial x} + y^2 b\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}. \tag{5.4}$$

We know that

$$\operatorname{Span}\{X(q), Y(q), [X, [X, Y]](q), [Y, [X, Y]](q)\} = T_q\mathbb{R}^3$$

for every $q \in S$. We obtain that the coefficient of $\partial/\partial z$ in $[X, [X, Y]]_{|S}$ vanishes and

$$[Y, [X, Y]]_{|S} = -2\frac{\partial a}{\partial y}_{|y=0}\frac{\partial}{\partial x} - 2b_{|y=0}\frac{\partial}{\partial z}.$$

Thus, $b(x, 0, z) \neq 0$. Normalizing z-axis as in Sec. 3.1, we can assume that $b(x, 0, z) = 1$. This allows one to replace b in (5.4) by $1 + yb$ for yet some other smooth b. This completes the proof. □

5.2. **Alternative definition of sub-Lorentzian normal neighborhoods.** Some theorems on reachable sets $J^+(q_0, U)$, $I^+(q_0, U)$, and $N^+(q_0, U)$ and also some results on the existence of maximizing geodesics that were proved in previous papers by the author require a special type of neighborhoods U of q_0; indeed, for example, in the Heisenberg case (see [7]), the set $J^+(0, U)$ is not closed relative to U, where

$$U = \Big\{(x, y, z) : -\frac{1}{2}x - \delta < y < -\frac{1}{2}x + \delta,$$
$$\frac{1}{2}x - \delta < y < \frac{1}{2}x + \delta,\ -\delta^2 < z < \delta^2\Big\},$$

$\delta > 0$, is a convex neighborhood of the origin. The definition of normal neighborhoods stated in the introduction is not constructive. Here we propose another, constructive definition of neighborhoods, which also allows one to prove theorems on reachable sets.

Assume that (M, H, g) is a time-oriented, sub-Lorentzian manifold and $q_0 \in M$. Let $X_0, \dots, X_k$ be an orthonormal frame for (H, g) defined in a neighborhood $\hat{U}$ of q_0, with a time orientation X_0. Denote by $f : \hat{U} \to \mathbb{R}$ a smooth function satisfying the following conditions: $f(q_0) = 0$, $X_0(f)(q_0) = 1$, and $X_j(f)(q_0) = 0$, $j = 1, \dots, k$. Shrinking $\hat{U}$, if necessary, we can assume that

$$\inf \left\{ X_0(f)(q) : q \in \hat{U} \right\} > \sqrt{k} \sup \left\{ |X_j(f)(q)| : q \in \hat{U},\ j = 1, \dots, k \right\}. \tag{5.5}$$

Inequality (5.5) ensures that the horizontal gradient $\nabla_H f$ is timelike past directed (i.e., $-\nabla_H f$ is t.f.d.). Of course, this implies that f is increasing along nspc.f.d. curves contained in $\hat{U}$. Now let $U \subset \hat{U}$ be a neighborhood of q_0 such that $J^+(q_0, \hat{U}) \cap \partial U = f^{-1}(c)$ for $c > 0$. Every set U defined like this possesses the following property: let $\gamma : [0, T] \to U$ be a nspc.f.d. curve starting at q_0; if $\gamma(t) \in \partial U$ for $t \in [0, T)$, then there exists $\varepsilon > 0$, $t + \varepsilon < T$, such that $\gamma(s) \notin \overline{U}$ for $t < s < t + \varepsilon$. This property is sufficient to prove [5, Lemma 5.1] and, in what follows, to prove [8, Theorems 3.1 and 3.2] concerning reachable sets relative to U.

Acknowledgement. This work was partially supported by the Ministry of Research and Higher Education of Polish government (project No. 2703/H03/2007/32).

References

1. A. Agrachev and Yu. Sachkov, Control Theory from Geometric Viewpoint. *Encycl. Math. Sci.* **87**, *Springer-Verlag* (2004).
2. A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka, Sub-Riemannian sphere in Martinet flat case. *ESAIM Control Optim. Calc. Var.* **2** (1997), 377–448.
3. A. Agrachev, El-A. El-H. Chakir, and J. P. Gauthier, Sub-Riemannian metrics on $\mathbb{R}^3$. *Can. Math. Soc. Conf. Proc.* **25** (1998), 29–78.
4. R. Gérard and H. Tahora, Singular Nonlinear Partial Differential Equations. *Vieweg-Verlag* (1996).
5. M. Grochowski, Geodesics in the sub-Lorentzian Geometry. *Bull. Pol. Acad. Sci.* **50** (2002), No. 2, 161–178.
6. ———, Normal forms of germs of contact sub-Lorentzian structures on $\mathbb{R}^3$. Differentiability of the sub-Lorentzian distance. *J. Dynam. Control Systems* **9** (2003), No. 4, 531–547.

7. ______, Reachable sets for the Heisenberg sub-Lorentian metric on $\mathbb{R}^3$. An estimate for the distance function. *J. Dynam. Control Systems* **12** (2006), No. 2, 145–160.
8. ______, Properties of reachable sets in the sub-Lorentzian geometry. *J. Geom. Phys.* **59** (2009), 885–900.
9. ______, Reachable sets for contact sub-Lorentzian structures on $\mathbb{R}^3$. Application to control affine systems on $\mathbb{R}^3$ with a scalar input. To appear in *J. Math. Sci.*
10. S. Łojasiewicz, Ensembles semi-analytiques. *Inst. Hautes Études Sci., Bures-sur-Yvette, France* (1964).

(Received November 13 2009, received in revised form April 08 2010)

Author's address:
Faculty of Mathematics and Science,
Cardinal Stefan Wyszyński University, Waszawa, Poland;
Institute of Mathematics,
Polish Academy of Sciences, Warszawa, Poland
E-mail: mgrochow@impan.gov.pl, m.grochowski@uksw.edu.pl