Journal of Dynamical and Control Systems, Vol. 16, No. 2, April 2010, 163–188 (©2010)
DOI: 10.1007/s10883-010-9088-6

NONLINEAR NEUMANN BOUNDARY STABILIZATION OF THE WAVE EQUATION USING ROTATED MULTIPLIERS

P. CORNILLEAU, J.-P. LOHÉAC, and A. OSSES

ABSTRACT. We study the boundary stabilization of the wave equation by means of a linear or nonlinear Neumann feedback. The rotated multiplier method leads to new geometrical cases concerning the active part of the boundary where the feedback is applied. Due to mixed boundary conditions, these cases generate singularities. Under a simple geometrical condition concerning the orientation of the boundary, we obtain stabilization results in both cases.

INTRODUCTION

In this paper, we are concerned with the stabilization of the wave equation in a multi-dimensional body $\Omega \subset \mathbb{R}^n$ by using a feedback law applied on some part of its boundary. The problem can be written as follows:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = F & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0 & \text{in } \Omega, \\ u'(0) = u_1 & \text{in } \Omega, \end{cases}$$

where we denote by u', u'', Δu, and $\partial_\nu u$ the first time-derivative of the scalar function u, the second time-derivative of u, the standard Laplacian of u, and the normal outward derivative of u on $\partial\Omega$, respectively; $(\partial\Omega_D, \partial\Omega_N)$ is a partition of $\partial\Omega$ and F is the feedback function which may depend on the state (u, u'), the position $\boldsymbol{x}$ and time t.

Our purpose here is to choose the feedback function F and the active part of the boundary, $\partial\Omega_N$, so that for every initial data, the energy function

$$E(u,t) = \frac{1}{2} \int_\Omega (|u'(t)|^2 + |\nabla u(t)|^2)\, d\boldsymbol{x},$$

2000 *Mathematics Subject Classification.* 93D15, 35L05, 35J25.
Key words and phrases. Wave equation, boundary stabilization, multiplier method.

is decreasing with respect to time t and vanishes as $t \to \infty$. Formally, we can write the time derivative of E as follows:

$$E'(u,t) = \int\limits_{\partial\Omega_N} F u' \, d\sigma,$$

and a sufficient condition so that E is nonincreasing is $Fu' \leq 0$ on $\partial\Omega_N$.

In the two-dimensional case and in the framework of the Hilbert uniqueness method [12], it can be shown that the energy function is uniformly decreasing as $t \to \infty$, by choosing $\boldsymbol{m} : \boldsymbol{x} \mapsto \boldsymbol{x} - \boldsymbol{x}_0$, where $\boldsymbol{x}_0$ is some given point in $\mathbb{R}^n$ and

$$\partial\Omega_N = \{\boldsymbol{x} \in \partial\Omega \mid m(\boldsymbol{x}) \cdot \nu(\boldsymbol{x}) > 0\}, \quad F = -\boldsymbol{m} \cdot \boldsymbol{\nu} u',$$

where $\boldsymbol{\nu}$ is the normal unit vector pointing outward of Ω. This method has been performed by many authors (see, e.g., [10] and the references therein). Here we extend the above result for rotated multipliers [15] by following [4], i.e., we take in account singularities which can appear when changing boundary conditions along the interface $\Gamma = \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$.

1. Notation and main results

Let Ω be a bounded open connected set of $\mathbb{R}^n (n \geq 2)$ such that

$$\partial\Omega \text{ is of class } C^2 \text{ in the sense of Nečas [14].} \tag{1}$$

Let $\boldsymbol{x}_0$ be a fixed point in $\mathbb{R}^n$. We denote by I the $(n \times n)$ identity matrix, by A a real $(n \times n)$ skew-symmetric matrix, and by d a positive real number such that $d^2 + \|A\|^2 = 1$, where $\|\cdot\|$ stands for the usual 2-norm of matrices. Now we define the following vector function:

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{m}(\boldsymbol{x}) = (dI + A)(\boldsymbol{x} - \boldsymbol{x}_0).$$

We consider a partition $(\partial\Omega_N, \partial\Omega_D)$ of $\partial\Omega$ such that

$$\begin{cases} \Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \text{ is a } C^3\text{-manifold of dimension } n-2, \\ \boldsymbol{m} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma, \\ \partial\Omega \cap \omega \text{ is a } C^3\text{-manifold of dimension } n-1, \\ \mathcal{H}^{n-1}(\partial\Omega_D) > 0, \end{cases} \tag{2}$$

where ω is a suitable neighborhood of Γ and $\mathcal{H}^{n-1}$ denotes the usual $(n-1)$-dimensional Hausdorff measure.

Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function such that

$$g \text{ is nondecreasing and } \exists K > 0\text{: } |g(s)| \leq K|s| \text{ a.e.} \tag{3}$$

Now let us consider the following wave problem:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = -\boldsymbol{m}\cdot\boldsymbol{\nu}\, g(u') & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0 & \text{in } \Omega, \\ u^{'}(0) = u_1 & \text{in } \Omega, \end{cases} \tag{S}$$

for some initial data

$$(u_0, u_1) \in \mathrm{H}_D^1(\Omega) \times \mathrm{L}^2(\Omega) := \mathrm{H},$$

where

$$\mathrm{H}_D^1(\Omega) = \left\{ v \in \mathrm{H}^1(\Omega) \;\middle|\; v = 0 \text{ on } \partial\Omega_D \right\}.$$

This problem is well-posed in H. Indeed, following Komornik [9], we define the nonlinear operator $\mathcal{W}$ by

$$\mathcal{W}(u, v) = (-v, -\Delta u),$$

where

$$D(\mathcal{W}) = \Big\{ (u,v) \in \mathrm{H}_D^1(\Omega) \times \mathrm{H}_D^1(\Omega) \;\Big|\; \Delta u \in \mathrm{L}^2(\Omega) \text{ and } \partial_\nu u = -\boldsymbol{m}\cdot\boldsymbol{\nu}\, g(v) \text{ on } \partial\Omega_N \Big\},$$

so that (S) can be written in the form

$$\begin{cases} (u,v)' + \mathcal{W}(u,v) = 0, \\ (u,v)(0) = (u_0, u_1). \end{cases}$$

It is a classical fact that $\mathcal{W}$ is a maximal-monotone operator on H and that $D(\mathcal{W})$ is dense in H for the usual norm (see, e.g., [1]). Hence, for any initial data $(u_0, v_0) \in D(\mathcal{W})$, there is a unique strong solution (u, v) such that $u \in \mathrm{W}^{1,\infty}(\mathbb{R}; \mathrm{H}_D^1(\Omega))$ and $\Delta u \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^2(\Omega))$. Moreover, for two initial data, the corresponding solutions satisfy

$$\|(u^1(t), v^1(t)) - (u^2(t), v^2(t))\|_{\mathrm{H}} \le C\|(u_0^1, v_0^1) - (u_0^2, v_0^2)\|_{\mathrm{H}} \quad \forall t \ge 0.$$

Using the density of $D(\mathcal{W})$, one can extend the map

$$D(\mathcal{W}) \to \mathrm{H}, \quad (u_0, v_0) \mapsto (u(t), v(t))$$

to a strongly continuous semigroup of contractions $(S(t))_{t\ge 0}$ and define for $(u_0, v_0) \in \mathrm{H}$ the weak solution $(u(t), u'(t)) = S(t)(u_0, u_1)$ with the regularity $u \in C(\mathbb{R}_+; \mathrm{H}_D^1(\Omega)) \cap C^1(\mathbb{R}_+; \mathrm{L}^2(\Omega))$. We hence define the energy function of solutions by

$$E(u,0) = \frac{1}{2} \int_{\Omega} (|u_1|^2 + |\nabla u_0|^2)\, d\boldsymbol{x},$$

$$E(u,t) = \frac{1}{2} \int_{\Omega} (|u'(t)|^2 + |\nabla u(t)|^2)\, d\boldsymbol{x}$$

if $t > 0$. In order to obtain stabilization results, we need further assumptions concerning the feedback function g:

$$\exists p \geq 1,\ \exists k > 0, \quad |g(s)| \geq k \min\{|s|, |s|^p\},\ \text{a.e.} \tag{4}$$

Concerning the boundary, we assume that

$$\begin{aligned} \partial\Omega_N &\subset \Big\{ \boldsymbol{x} \in \partial\Omega \ \Big|\ \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \geq 0 \Big\}, \\ \partial\Omega_D &\subset \Big\{ \boldsymbol{x} \in \partial\Omega \ \Big|\ \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \leq 0 \Big\}, \end{aligned} \tag{5}$$

and the additional geometric assumption

$$\boldsymbol{m} \cdot \boldsymbol{\tau} \leq 0 \quad \text{on } \Gamma, \tag{6}$$

where $\boldsymbol{\tau}(\boldsymbol{x})$ is the normal unit vector pointing outward of $\partial\Omega_N$ at a point $\boldsymbol{x} \in \Gamma$ when considering $\partial\Omega_N$ as a sub-manifold of $\partial\Omega$.

Remark 1. It is worth observing that it is not necessary to assume that

$$\mathcal{H}^{n-1}\left(\Big\{ \boldsymbol{x} \in \partial\Omega_N \ \Big|\ \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) > 0 \Big\} \right) > 0$$

to obtain stabilization. In fact, our choice of $\boldsymbol{m}$ implies such properties (see examples in Sec. 4) whether the energy tends to zero.

Since the pioneering work [8], it is now a well-known fact that Rellich type relations [16] are very useful for the study of control and stabilization of the wave problem. As we said before, Komornik and Zuazua [10] have shown how these relations can also help us to stabilize the wave problem. In order to generalize it for dimensions higher than 3, the key-problem is to show the existence of a decomposition of the solution in regular and singular parts [6, 11] which can be applied to stabilization problems or control problems. The first results in this direction are due to Moussaoui [13] and Bey, Lohéac, and Moussaoui [4] who also have established a Rellich type relation in any dimension.

In this new case of Neumann feedback deduced from [15], our goal is to generalize those Rellich relations to get stabilization results about (S) under assumptions (5), (6).

As well as in [9], we shall prove here two results of uniform boundary stabilization.

Exponential boundary stabilization. We here consider the case where $p = 1$ in (4). This is satisfied when g is linear,

$$\exists \alpha > 0 : g(s) = \alpha s \quad \forall s \in \mathbb{R}.$$

In these cases, the energy function is exponentially decreasing.

Theorem 1. *Assume that geometrical conditions* (2) *and* (5) *hold and that the feedback function g satisfies* (3) *and* (4) *with $p = 1$. Under the further geometrical assumption* (6), *there exist $C > 0$ and $T > 0$ (independent of d) such that for all initial data in* H, *the energy of the solution u satisfies*

$$E(u, t) \leq E(u, 0) \exp\left(1 - \frac{d}{C} t\right) \quad \forall t > \frac{T}{d}.$$

The constants C and T depend only on the geometry.

Rational boundary stabilization. Now we consider the general case and we get rational boundary stabilization.

Theorem 2. *Assume that geometrical conditions* (2) *and* (5) *hold and that the feedback function g satisfies* (3) *and* (4) *with $p > 1$.*

Then under the further geometrical assumption (6), *there exist $C > 0$ and $T > 0$ (independent of d) such that for all initial data in* H, *the energy of the solution u of satisfies*

$$E(u, t) \leq C t^{2/(1-p)} \quad \forall t > \frac{T}{d},$$

where C depends on the initial energy $E(u, 0)$.

Remark 2. Taking advantage of the works of Banasiak–Roach [2] who generalized Grisvard's results [6] in the piecewise regular case, we will see that Theorems 1 and 2 remain true in the bi-dimensional case, where assumption (1) is replaced by following:

$$\begin{aligned}&\partial\Omega \text{ is a curvilinear polygon of class } C^2, \text{ each component of}\\ &\partial\Omega \setminus \Gamma \text{ is a } C^2\text{-manifold of dimension } 1,\end{aligned} \tag{7}$$

and condition (6) is replaced by

$$0 \leq \varpi_{\boldsymbol{x}} \leq \pi \quad \forall \boldsymbol{x} \in \Gamma \quad \text{and} \quad \text{if } \varpi_{\boldsymbol{x}} = \pi, \text{ then } \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{\tau}(\boldsymbol{x}) \leq 0, \tag{8}$$

where $\varpi_{\boldsymbol{x}}$ is the angle at the boundary in the point $\boldsymbol{x}$.

These two results are obtained by estimating some integral of the energy function as well as in [9]. This specific estimates are obtained thanks to an adapted Rellich relation.

2. Rellich relations

2.1. **A regular case.** We can easily construct a Rellich relation corresponding to the above vector field m when considered functions are sufficiently smooth.

Proposition 3. *Assume that Ω is an open set of $\mathbb{R}^n$ with boundary of class C^2 in the sense of Nečas. If u belongs to $\mathrm{H}^2(\Omega)$, then*

$$2\int_\Omega \Delta u\,(\boldsymbol{m}\cdot\nabla u)\,d\boldsymbol{x} = d(n-2)\int_\Omega |\nabla u|^2\,d\boldsymbol{x} + \int_{\partial\Omega}\big(2\partial_\nu u\,(\boldsymbol{m}\cdot\nabla u) - \boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2\big)\,d\sigma.$$

Proof. Using the Green–Riemann identity, we obtain

$$2\int_\Omega \Delta u(\boldsymbol{m}\cdot\nabla u)\,d\boldsymbol{x} = \int_{\partial\Omega} 2\partial_\nu u\,(\boldsymbol{m}\cdot\nabla u)\,d\sigma - 2\int_\Omega \nabla u.\nabla(\boldsymbol{m}\cdot\nabla u)\,d\boldsymbol{x}.$$

So, observing that

$$\nabla u\cdot\nabla(\boldsymbol{m}\cdot\nabla u) = \frac{1}{2}\boldsymbol{m}\cdot\nabla(|\nabla u|^2) + d|\nabla u|^2 + (A\nabla u)\cdot\nabla u$$

and since A is skew-symmetric, we obtain

$$2\int_\Omega \Delta u(\boldsymbol{m}\cdot\nabla u)\,d\boldsymbol{x} = \int_{\partial\Omega} 2\partial_\nu u(\boldsymbol{m}\cdot\nabla u)\,d\sigma - 2d\int_\Omega |\nabla u|^2\,d\boldsymbol{x} - \int_\Omega \boldsymbol{m}\cdot\nabla(|\nabla u|^2)\,d\boldsymbol{x}.$$

With another use of the Green–Riemann formula, we obtain the required result for $\mathrm{div}(\boldsymbol{m}) = nd$. □

Now we try to extend this result to the case of an element u belonging less regular when Ω is sufficiently smooth.

2.2. **Bi-dimensional case.** We begin by the plane case. It is the simplest case from the point of view of singularity theory and its understanding dates from Shamir [17].

Theorem 4. *Assume that $n = 2$. Under geometrical conditions* (2) *and* (7)*, let $u \in \mathrm{H}^1(\Omega)$ be such that*

$$\Delta u \in \mathrm{L}^2(\Omega), \quad u|_{\partial\Omega_D} \in \mathrm{H}^{3/2}(\partial\Omega_D), \quad \partial_\nu u|_{\partial\Omega_N} \in \mathrm{H}^{1/2}(\partial\Omega_N). \qquad (9)$$

Then $2\partial_\nu u(\boldsymbol{m}\cdot\nabla u)-\boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2$ *belongs to* $\mathrm{L}^1(\partial\Omega)$ *and there exist coefficients* $(c_{\boldsymbol{x}})_{\boldsymbol{x}\in\Gamma}$ *such that*

$$2\int_\Omega \Delta u(\boldsymbol{m}\cdot\nabla u)\,d\boldsymbol{x} = \int_{\partial\Omega}(2\partial_\nu u(\boldsymbol{m}\cdot\nabla u) - \boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2)\,d\sigma + \frac{\pi}{4}\sum_{\boldsymbol{x}\in\Gamma/\varpi_{\boldsymbol{x}}=\pi} c_{\boldsymbol{x}}^2(\boldsymbol{m}\cdot\boldsymbol{\tau})(\boldsymbol{x}).$$

Proof. We first begin by some considerations which will also be used in the general case. It is a classical result that $u\in\mathrm{H}^2(\omega)$ for every open domain ω such that $\omega\Subset\overline{\Omega}\setminus\Gamma$. For completeness, we recall the proof.

A trace result shows that there exists $u_R\in\mathrm{H}^2(\omega)$ such that $u_R=u$ on $\partial\Omega_D$ and $\partial_\nu u_R=\partial_\nu u$ on $\partial\Omega_N$. Hence, setting $f=\Delta u_R-\Delta u\in\mathrm{L}^2(\Omega)$, we see that $u_S=u-u_R$ satisfies

$$\begin{cases} -\Delta u_S = f & \text{in } \Omega,\\ u_S = 0 & \text{on } \partial\Omega_D,\\ \partial_\nu u_S = 0 & \text{on } \partial\Omega_N. \end{cases} \tag{10}$$

Now, if $\omega\Subset\Omega\setminus\Gamma\cup\partial\Omega_D$ and ξ is a cut-off function such that $\xi=1$ on ω and $\operatorname{supp}(\xi)\subset\Omega$, then for suitable $g\in\mathrm{L}^2(\Omega)$, $u_\omega=u_S\xi$ satisfies the Dirichlet problem

$$\begin{cases} \Delta u_\omega = g & \text{on } \Omega,\\ u_\omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and using the classical method of difference quotients (see [6]), we conclude that $u_\omega\in\mathrm{H}^2(\Omega)$, hence $u_S\in\mathrm{H}^2(\omega)$.

Else, if $\omega\Subset\Omega\setminus\Gamma\cup\partial\Omega_N$ and ξ is a cut-off function such that $\xi=1$ on ω and $\operatorname{supp}(\xi)\subset\Omega$, then for a suitable $g\in\mathrm{L}^2(\Omega)$, $u_\omega=u_S\xi$ satisfies the Neumann problem

$$\begin{cases} -\Delta u_\omega + u_\omega = g, & \text{on } \Omega,\\ \partial_\nu u_\omega = 0, & \text{on } \partial\Omega, \end{cases}$$

and, using similar argument, we obtain $u_S\in\mathrm{H}^2(\omega)$.

Let

$$\Omega_\varepsilon = \Big\{\boldsymbol{x}\in\Omega \;\Big|\; d(\boldsymbol{x},\Gamma)>\varepsilon\Big\}.$$

By the compactness of Ω_ε, we have $u\in\mathrm{H}^2(\Omega_\varepsilon)$. An application of Proposition 3 to our particular situation gives us the following relation:

$$2\int_{\Omega_\varepsilon}\Delta u(\boldsymbol{m}\cdot\nabla u)\,d\boldsymbol{x} = \int_{\partial\Omega_\varepsilon}\Big(2\partial_\nu u(\boldsymbol{m}\cdot\nabla u)-\boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2\Big)\,d\sigma,$$

and we set $\varepsilon \to 0$. Using the derivatives with respect to $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$, we obtain

$$2 \int_{\Omega_\varepsilon} \Delta u(\boldsymbol{m} \cdot \nabla u)\, d\boldsymbol{x}$$
$$= \int_{\partial\Omega_\varepsilon} \boldsymbol{m} \cdot \boldsymbol{\nu} \left((\partial_\nu u)^2 - (\partial_\tau u)^2\right) d\sigma + 2 \int_{\partial\Omega_\varepsilon} \boldsymbol{m} \cdot \boldsymbol{\tau}\, (\partial_\nu u)(\partial_\tau u)\, d\sigma.$$

First, since $\Delta u \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{H}^1(\Omega)$, the Lebesgue dominated convergence theorem implies

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Delta u(\boldsymbol{m} \cdot \nabla u)\, d\boldsymbol{x} = \int_{\Omega} \Delta u(\boldsymbol{m} \cdot \nabla u) d\boldsymbol{x}.$$

Now we consider work boundary terms. Let us introduce the following partition of $\partial\Omega_\varepsilon$:

$$\widetilde{\partial\Omega_\varepsilon} = \partial\Omega_\varepsilon \cap \partial\Omega, \quad \partial\Omega_\varepsilon^* = \partial\Omega_\varepsilon \cap \Omega$$

and use the decomposition result due to Banasiak and Roach [2]: every variational solution of (10) can be represented as the sum of singular functions. There exist some coefficients $(c_{\boldsymbol{x}})_{\boldsymbol{x}\in\Gamma}$ and $u_R \in \mathrm{H}^2(\Omega)$ such that

$$u = u_R + \sum_{\boldsymbol{x}\in\Gamma} c_{\boldsymbol{x}} U_S^{\boldsymbol{x}} =: u_R + u_S, \tag{11}$$

where $U_S^{\boldsymbol{x}}$ are singular functions which, in some neighborhood of $\boldsymbol{x} \in \Gamma$, are defined in local polar coordinates (see Fig. 1) by

$$U_S^{\boldsymbol{x}}(r, \theta) = \rho(r)\, r^{\pi/2\varpi_{\boldsymbol{x}}} \sin\left(\frac{\pi}{2\varpi_{\boldsymbol{x}}}\theta\right),$$

where ρ is a cut-off function.

Using the density of $C^1(\overline{\Omega})$ in $\mathrm{H}^2(\Omega)$, we can assume that $u_R \in C^1(\overline{\Omega})$.

First, consider boundary terms on $\widetilde{\partial\Omega_\varepsilon}$. We claim that for some constant $C > 0$,

$$|\boldsymbol{m} \cdot \boldsymbol{\nu}| \le C d(\cdot, \Gamma).$$

In fact, if $\boldsymbol{x} \in \Omega$ and $\boldsymbol{x}_1 \in \Gamma$ satisfy $|\boldsymbol{x} - \boldsymbol{x}_1| = d(\boldsymbol{x}, \Gamma)$, then we have

$$\boldsymbol{m} \cdot \boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{m}(\boldsymbol{x}) \cdot (\boldsymbol{\nu}(\boldsymbol{x}) - \boldsymbol{\nu}(\boldsymbol{x}_1)) + (\boldsymbol{m}(\boldsymbol{x}) - \boldsymbol{m}(\boldsymbol{x}_1)) \cdot \boldsymbol{\nu}(\boldsymbol{x}_1)$$

observing that $\boldsymbol{m} \cdot \boldsymbol{\nu}(\boldsymbol{x}_1) = 0$. Hence, using the fact that $\boldsymbol{\nu}$ is a piecewise C^1-function (see Fig. 2), we obtain

$$|\boldsymbol{m} \cdot \boldsymbol{\nu}(\boldsymbol{x})| \le \left(\|m\|_\infty \|\boldsymbol{\nu}'\|_\infty + 1\right) d(\boldsymbol{x}, \Gamma).$$

Now, working in local coordinates, we have

$$d(\boldsymbol{x}, \Gamma)\, |\nabla u|^2 \in \mathrm{L}^\infty(\partial\Omega).$$

Hence the Lebesgue theorem implies

$$\lim_{\varepsilon \to 0} \int_{\widetilde{\partial\Omega_\varepsilon}} \boldsymbol{m}\cdot\boldsymbol{\nu}\, \left((\partial_\nu u)^2 - (\partial_\tau u)^2\right)\, d\sigma = \int_{\partial\Omega} \boldsymbol{m}\cdot\boldsymbol{\nu}\, \left((\partial_\nu u)^2 - (\partial_\tau u)^2\right)\, d\sigma.$$

On the other hand, assumptions (9) give

$$\partial_\nu u_{/\partial\Omega_N} \in \mathrm{H}^{1/2}(\partial\Omega_N), \quad \partial_\tau u_{/\partial\Omega_N} \in \mathrm{H}^{-1/2}(\partial\Omega_N),$$
$$\partial_\nu u_{/\partial\Omega_D} \in \mathrm{H}^{-1/2}(\partial\Omega_D), \quad \partial_\tau u_{/\partial\Omega_D} \in \mathrm{H}^{1/2}(\partial\Omega_D).$$

Hence we obtain

$$\int_{\widetilde{\partial\Omega_\varepsilon}} \boldsymbol{m}\cdot\boldsymbol{\tau}\, (\partial_\nu u)(\partial_\tau u)\, d\sigma \longrightarrow \int_{\partial\Omega} \boldsymbol{m}\cdot\boldsymbol{\tau}\, (\partial_\nu u)(\partial_\tau u)\, d\sigma$$

as $\varepsilon \to 0$. Now we must consider the boundary term on $\partial\Omega_\varepsilon^*$, $I_\varepsilon(\nabla u)$. It is a quadratic form with respect to ∇u; using (11), one can decompose it as follows:

$$I_\varepsilon(\nabla u_R) + 2J_\varepsilon(\nabla u_R, \nabla u_S) + I_\varepsilon(\nabla u_S),$$

where J_ε is the corresponding bilinear form.

Concerning $I_\varepsilon(\nabla u_R)$, the regularity of $\boldsymbol{m}$ yields the estimate

$$|I_\varepsilon(\nabla u_R)| \leq C \int_{\partial\Omega_\varepsilon^*} |\nabla u_R|^2\, d\sigma.$$

This term is $O(\varepsilon)$ since ∇u_R is bounded on Ω.

For the term $I_\varepsilon(\nabla u_S)$, we first observe that, adjusting the cut-off functions, the supports of $u_S^{\boldsymbol{x}}$ and $u_S^{\boldsymbol{y}}$ are disjoint, provided that $\boldsymbol{x} \neq \boldsymbol{y}$. Hence, using decomposition (11), we can write

$$I_\varepsilon(\nabla u_S) = \sum_{\boldsymbol{x}\in\Gamma} c_{\boldsymbol{x}}^2 \int_{C_\varepsilon(\boldsymbol{x})} \left(2\partial_\nu u_S^{\boldsymbol{x}}(\boldsymbol{m}\cdot\nabla u_S^{\boldsymbol{x}}) - \boldsymbol{m}\cdot\boldsymbol{\nu}\, |\nabla u_S^{\boldsymbol{x}}|^2\right) d\sigma.$$

If $\varpi_{\boldsymbol{x}} < \pi$, one gets

$$2\partial_\nu u_S^{\boldsymbol{x}}(\boldsymbol{m}\cdot\nabla u_S^{\boldsymbol{x}}) - \boldsymbol{m}\cdot\boldsymbol{\nu}\, |\nabla u_S^{\boldsymbol{x}}|^2 = O(\varepsilon^{\frac{\pi}{\varpi_{\boldsymbol{x}}}-2})$$

on $C_\varepsilon(\boldsymbol{x})$. Hence, after integrating on $C_\varepsilon(\boldsymbol{x})$, we obtain

$$\lim_{\varepsilon\to 0} I_1^{\boldsymbol{x}}(\varepsilon) = 0.$$

If $\varpi_{\boldsymbol{x}} = \pi$, we will need the following identity:

$$2\partial_\nu u_S^{\boldsymbol{x}}(\boldsymbol{m}\cdot\nabla u_S^{\boldsymbol{x}}) - \boldsymbol{m}\cdot\boldsymbol{\nu}\, |\nabla u_S^{\boldsymbol{x}}|^2 = \frac{1}{4\varepsilon}(\boldsymbol{m}\cdot\boldsymbol{\tau})(\boldsymbol{x}) \quad \text{on } C_\varepsilon(\boldsymbol{x}).$$

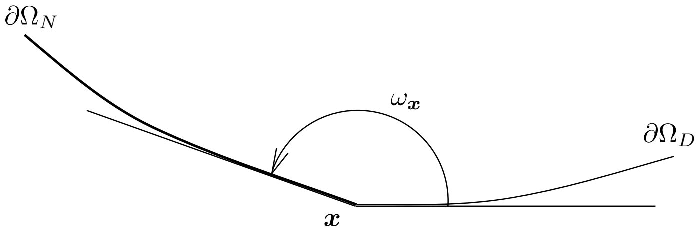

Fig. 1. Shape of the boundary near an angular point $\boldsymbol{x}$

One can observe that $C_\varepsilon(\boldsymbol{x})$ behaves as a half-circle when $\varepsilon \to 0$. Integration gives

$$\lim_{\varepsilon\to 0} \int\limits_{C_\varepsilon(\boldsymbol{x})} \left(2(\nu \cdot \nabla u_S^{\boldsymbol{x}})(\boldsymbol{m} \cdot \nabla u_S^{\boldsymbol{x}}) - \boldsymbol{m} \cdot \boldsymbol{\nu}\, |\nabla u_S^{\boldsymbol{x}}|^2\right) d\sigma = \frac{\pi}{4}(\boldsymbol{m} \cdot \boldsymbol{\tau})(\boldsymbol{x}).$$

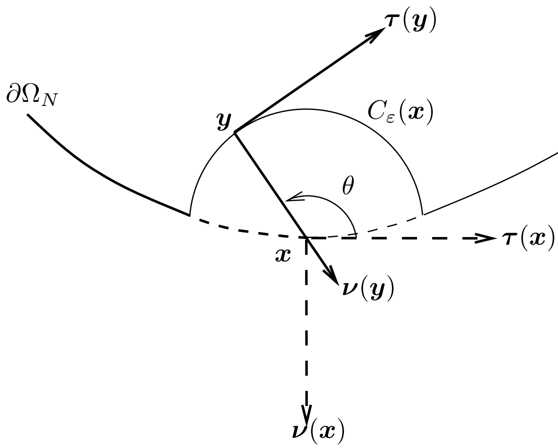

Fig. 2. Unit vectors $\boldsymbol{\nu}(\boldsymbol{x})$, $\boldsymbol{\tau}(\boldsymbol{x})$, $\boldsymbol{\nu}(\boldsymbol{y})$ and $\boldsymbol{\tau}(\boldsymbol{y})$ when $\partial\Omega$ is regular at $\boldsymbol{x}$

Finally, the bilinear term $J_\varepsilon(\nabla u_R, \nabla u_S)$ can be written entirely

$$\int\limits_{\partial\Omega_\varepsilon^*} \partial_\nu u_R\, (\boldsymbol{m}.\nabla u_S)\, d\sigma + \int\limits_{\partial\Omega_\varepsilon^*} \partial_\nu u_S\, (\boldsymbol{m}.\nabla u_R)\, d\sigma - \int\limits_{\partial\Omega_\varepsilon^*} (\boldsymbol{m} \cdot \boldsymbol{\nu})\, (\nabla u_R \cdot \nabla u_S)\, d\sigma.$$

Using the regularity of $\boldsymbol{m}$ and the Cauchy–Schwarz inequality, we obtain an estimate of the form

$$|J_\varepsilon(\nabla u_R, \nabla u_S)| \leq C \left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_R|^2 \, d\sigma \right)^{1/2} \left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_S|^2 \, d\sigma \right)^{1/2} .$$

We have seen that the first term in this inequality vanishes as $\varepsilon \to 0$. For the second term, we now observe that, if ε is sufficiently small, then

$$\partial\Omega_\varepsilon^* = \bigsqcup_{\boldsymbol{x}\in\Gamma} C_\varepsilon(\boldsymbol{x}),$$

where $C_\varepsilon(\boldsymbol{x})$ is an arc of circle of radius ε centered at $\boldsymbol{x}$. Then we can write

$$\int_{\partial\Omega_\varepsilon^*} |\nabla u_S|^2 \, d\sigma \leq 2 \sum_{\boldsymbol{x},\boldsymbol{y}\in\Gamma} c_{\boldsymbol{y}}^2 \int_{C_\varepsilon(\boldsymbol{x})} |\nabla U_S^{\boldsymbol{y}}|^2 \, d\sigma.$$

A similar calculation shows that, for $\boldsymbol{x} \in \Gamma$,

$$\int_{C_\varepsilon(\boldsymbol{x})} |\nabla U_S^{\boldsymbol{x}}|^2 \, d\sigma = O(1).$$

Moreover, if $\boldsymbol{x} \neq \boldsymbol{y}$, $U_S^{\boldsymbol{y}}$ is bounded near $\boldsymbol{x}$, we obtain

$$\int_{C_\varepsilon(\boldsymbol{x})} |\nabla U_S^{\boldsymbol{y}}|^2 \, d\sigma = O(\varepsilon).$$

This completes the proof. ☐

Remark 3. The assumption $\mathcal{H}^1(\partial\Omega_D) > 0$ is not necessary in the above proof. We will now see why we need this assumption on the Dirichlet part in higher dimension.

2.3. **General case.** Now we state the result in general dimension.

Theorem 5. *Assume that* $n \geq 3$. *Under geometrical conditions* (1) *and* (2), *let* $u \in \mathrm{H}^1(\Omega)$ *be such that*

$$\Delta u \in \mathrm{L}^2(\Omega), \quad u|_{\partial\Omega_D} \in \mathrm{H}^{3/2}(\partial\Omega_D), \quad \partial_\nu u|_{\partial\Omega_N} \in \mathrm{H}^{1/2}(\partial\Omega_N). \tag{12}$$

Then $2\partial_\nu u(\boldsymbol{m} \cdot \nabla u) - \boldsymbol{m} \cdot \boldsymbol{\nu} \, |\nabla u|^2$ *belongs to* $\mathrm{L}^1(\partial\Omega)$ *and there exists* $\zeta \in \mathrm{H}^{1/2}(\Gamma)$ *such that*

$$2 \int_\Omega \Delta u(\boldsymbol{m} \cdot \nabla u) \, d\boldsymbol{x} = d(n-2) \int_\Omega |\nabla u|^2 \, d\boldsymbol{x} + \int_{\partial\Omega} (2\partial_\nu u(\boldsymbol{m} \cdot \nabla u) - \boldsymbol{m} \cdot \boldsymbol{\nu} \, |\nabla u|^2) \, d\sigma + \int_\Gamma \boldsymbol{m}.\boldsymbol{\tau} \, |\zeta|^2 \, d\gamma \, .$$

Proof. We will essentially follow [4]. As in the plane case, we set

$$\Omega_\varepsilon = \{\boldsymbol{x} \in \Omega;\ d(\boldsymbol{x}, \Gamma) > \varepsilon\}.$$

For any given $\varepsilon > 0$, we can apply the identity of Proposition 3:

$$2 \int_{\Omega_\varepsilon} \Delta u(\boldsymbol{m} \cdot \nabla u)\, d\boldsymbol{x} = d(n-2) \int_{\Omega_\varepsilon} |\nabla u|^2\, d\boldsymbol{x} + \int_{\partial\Omega_\varepsilon} (2\partial_\nu u\,(\boldsymbol{m} \cdot \nabla u) - \boldsymbol{m} \cdot \boldsymbol{\nu}\, |\nabla u|^2)\, d\sigma,$$

and we will again analyze the behavior of each term as $\varepsilon \to 0$.

First, since $\Delta u \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{H}^1(\Omega)$, the Lebesgue dominated convergence theorem immediately implies

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Delta u(\boldsymbol{m} \cdot \nabla u)\, d\boldsymbol{x} = \int_{\Omega} \Delta u(\boldsymbol{m} \cdot \nabla u)\, d\boldsymbol{x},$$

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u|^2\, d\boldsymbol{x} = \int_{\Omega} |\nabla u|^2\, d\boldsymbol{x}.$$

Below, we will consider boundary terms. We define $\widetilde{\partial\Omega_\varepsilon} = \partial\Omega_\varepsilon \cap \partial\Omega$ and $\partial\Omega_\varepsilon^* = \partial\Omega_\varepsilon \cap \Omega$ (see Fig. 3).

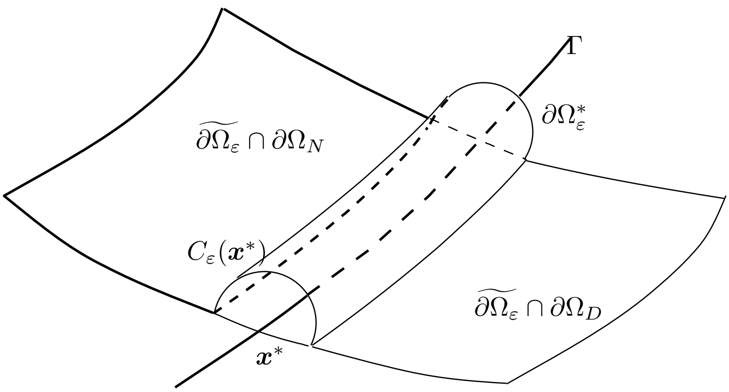

Fig. 3. Picture of $\partial\Omega_\varepsilon^*$ and $\widetilde{\partial\Omega_\varepsilon}$

Consider boundary integral terms on $\widetilde{\partial\Omega_\varepsilon}$. As well as in the plane case, there exists some constant $C > 0$ such that $|\boldsymbol{m} \cdot \boldsymbol{\nu}| \le C\, d(\cdot, \Gamma)$. Thus, using the fact that

$$d(\cdot, \Gamma)|\nabla u|^2 \in \mathrm{L}^1(\partial\Omega)$$

(see [4, Proposition 3]), we can apply the Lebesgue theorem to conclude that

$$\int_{\widetilde{\partial\Omega_\varepsilon}} \boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2\,d\sigma \to \int_{\partial\Omega} \boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2\,d\sigma$$

as $\varepsilon \to 0$. For the second integral, denoting by $\nabla_{\partial\Omega}$ the tangential gradient along $\partial\Omega$, we write

$$\partial_\nu u(\boldsymbol{m}\cdot\nabla u) = \boldsymbol{m}\cdot\boldsymbol{\nu}\,|\partial_\nu u|^2 + \partial_\nu u(\boldsymbol{m}\cdot\nabla_{\partial\Omega} u).$$

The first term is integrable. The second term is, on $\partial\Omega_N$, the product of a $\mathrm{H}^{1/2}$ term by a $\mathrm{H}^{-1/2}$ one and, on $\partial\Omega_D$, the product of a $\mathrm{H}^{-1/2}$ term by a $\mathrm{H}^{1/2}$ one. Hence, the Lebesgue theorem yields

$$\int_{\widetilde{\partial\Omega_\varepsilon}} \partial_\nu u(\boldsymbol{m}\cdot\nabla u)\,d\sigma \to \int_{\partial\Omega} \partial_\nu u(\boldsymbol{m}\cdot\nabla u)\,d\sigma$$

as $\varepsilon \to 0$.

Now let us consider boundary integral terms on $\partial\Omega_\varepsilon^*$. We assume that $\varepsilon \leqslant \varepsilon_0$ and we define

$$\omega_{\varepsilon_0} := \Omega\backslash\Omega_{\varepsilon_0}.$$

As well as in the plane case, we can write

$$u = u_R + u_S, \tag{13}$$

where u_S is the variational solution of some homogeneous mixed boundary problem and u_R belongs to $\mathrm{H}^2(\omega_{\varepsilon_0})$. Working by approximation if necessary, we can assume that $u_R \in C^1(\overline{\omega_{\varepsilon_0}})$. Considering the same quadratic form as in the bi-dimensional case, this leads to the following splitting

$$\int_{\partial\Omega*_\varepsilon} \left(2\partial_\nu u(\boldsymbol{m}\cdot\nabla u) - \boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2\right)\,d\sigma$$
$$= I_\varepsilon(\nabla u_R) + I_\varepsilon(\nabla u_S) + 2J_\varepsilon(\nabla u_R, \nabla u_S).$$

Since $\nabla u_R \in \mathrm{L}^\infty(\omega_{\varepsilon_0})$ and $\mathcal{H}^{n-1}(\partial\Omega_\varepsilon^*) \to 0$ as $\varepsilon \to 0$, the first term $I_\varepsilon(\nabla u_R)$ clearly vanishes.

As above, the bilinear term $J_\varepsilon(\nabla u_R, \nabla u_S)$ is

$$\int_{\partial\Omega_\varepsilon^*} \partial_\nu u_R(\boldsymbol{m}\cdot\nabla u_S)\,d\sigma + \int_{\partial\Omega_\varepsilon^*} \partial_\nu u_S(\boldsymbol{m}\cdot\nabla u_R)\,d\sigma - \int_{\partial\Omega_\varepsilon^*} (\boldsymbol{m}\cdot\boldsymbol{\nu})\,(\nabla u_R\cdot\nabla u_S)\,d\sigma.$$

Using the regularity of $\boldsymbol{m}$ and the Cauchy–Schwarz inequality, we obtain an estimate of the form

$$|J_\varepsilon(\nabla u_R, \nabla u_S)| \leqslant C\left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_R|^2\,d\sigma\right)^{1/2}\left(\int_{\partial\Omega_\varepsilon^*} |\nabla u_S|^2\,d\sigma\right)^{1/2}. \tag{14}$$

As above, it is clear that the first term vanishes as $\varepsilon \to 0$.

Before analyzing $I_\varepsilon(\nabla u_S)$, we introduce some notation.

Every $\boldsymbol{x} \in \partial\Omega^*_\varepsilon$ belongs to a unique plane $\boldsymbol{x}^* + \langle \boldsymbol{\tau}^*, \boldsymbol{\nu}^* \rangle$ (setting: $\boldsymbol{\tau}^* = \boldsymbol{\tau}(\boldsymbol{x}^*)$, $\boldsymbol{\nu}^* = \boldsymbol{\nu}(\boldsymbol{x}^*)$) and, more precisely, to an arc-circle $C_\varepsilon(\boldsymbol{x}^*)$ of radius ε centered at $\boldsymbol{x}^* \in \Gamma$ (the figure is similar to Fig. 2 in the plane $\boldsymbol{x}^* + \langle \boldsymbol{\tau}^*, \boldsymbol{\nu}^* \rangle$). We define

$$D_\varepsilon(\boldsymbol{x}^*) := \omega_\varepsilon \cap (\boldsymbol{x}^* + \langle \boldsymbol{\tau}^*, \boldsymbol{\nu}^* \rangle) .$$

For any $\boldsymbol{x} \in D_{\varepsilon_0}(\boldsymbol{x}^*)$, we separate the derivatives of u along the sub-manifold $\boldsymbol{x} - \boldsymbol{x}^* + \Gamma$ with the co-normal derivatives:

$$\nabla u(\boldsymbol{x}) = \nabla_\Gamma u(\boldsymbol{x}) + \nabla_2 u(\boldsymbol{x}), \quad \nabla_\Gamma u(\boldsymbol{x}) \in T_{\boldsymbol{x}^*}\Gamma, \quad \nabla_2 u(\boldsymbol{x}) \in \langle \boldsymbol{\tau}^*, \boldsymbol{\nu}^* \rangle . \tag{15}$$

Using methods of difference quotients (see, e.g., [4, Theorem 4]), we obtain $\nabla_\Gamma u \in \mathrm{H}^1(\omega_{\varepsilon_0})$, i.e., $\nabla_\Gamma u_S \in \mathrm{H}^1(\omega_{\varepsilon_0})$. We also need the following result concerning the behavior of boundary integrals.

Lemma 6. *Let $\varepsilon_0 > 0$. Assume that u is such that $u = 0$ on $\partial\omega_{\varepsilon_0} \cap \partial\Omega_D$,*

$$u(\boldsymbol{x}^*, \cdot) \in \mathrm{H}^1(D_{\varepsilon_0}(\boldsymbol{x}^*)) \quad \forall \boldsymbol{x}^* \in \Gamma,$$

and

$$\left(\boldsymbol{x}^* \mapsto \|u(\boldsymbol{x}^*, \cdot)\|_{\mathrm{H}^1(D_{\varepsilon_0(\boldsymbol{x}^*)})}\right) \in \mathrm{L}^2(\Gamma).$$

Then there exists $C > 0$, depending only on Ω, such that, for any sufficiently small ε,

$$\int_\Gamma \|u(\boldsymbol{x}^*, \cdot)\|^2_{\mathrm{L}^2(\mathrm{C}_\varepsilon(\boldsymbol{x}^*))} \, d\gamma(\boldsymbol{x}^*) \le C\varepsilon \int_\Gamma \|u(\boldsymbol{x}^*, \cdot)\|^2_{\mathrm{H}^1(\mathrm{D}_\varepsilon(\boldsymbol{x}^*))} \, d\gamma(\boldsymbol{x}^*).$$

Proof. We begin by changing coordinates as well as in [4]. For every $\boldsymbol{x}^*_0 \in \Gamma$, there exist $\rho_0 > 0$ and a C^2-diffeomorphism Θ from an open neighborhood W of $\boldsymbol{x}^*_0$ to $B(\rho_0) := B_{n-2}(\rho_0) \times B_2(\rho_0)$ (see Fig. 5) such that

$$\begin{aligned}
\Theta(\boldsymbol{x}^*_0) &= 0, \\
\Theta(W \cap \Omega) &= \{\boldsymbol{y} \in B(\rho_0) \mid y_n > 0\}, \\
\Theta(W \cap \partial\Omega_D) &= \{\boldsymbol{y} \in B(\rho_0) \mid y_{n-1} > 0,\ y_n = 0\}, \\
\Theta(W \cap \partial\Omega_N) &= \{\boldsymbol{y} \in B(\rho_0) \mid y_{n-1} < 0,\ y_n = 0\}, \\
\Theta(W \cap \Gamma) &= \{\boldsymbol{y} \in B(\rho_0) \mid y_{n-1} = 0,\ y_n = 0\} := \gamma(\rho_0).
\end{aligned}$$

Reducing ε_0 if necessary, we may assume that $D_{\varepsilon_0}(\boldsymbol{x}^*_0) \subset W$. Then we obtain, writing for $\boldsymbol{x} \in W$, $\Theta(\boldsymbol{x}) = (Y, \widetilde{y}) \in \mathbb{R}^{n-2} \times \mathbb{R}^2$, and $v := u \circ \Theta^{-1}$,

$$\int_{W\cap\Gamma} \int_{C_\varepsilon(\boldsymbol{x}^*)} u^2 \, d\ell \, d\gamma(\boldsymbol{x}*) = \int_{\gamma(\rho_0)} \int_{\Theta(C_\varepsilon(\boldsymbol{x}^*))} v^2 \, d\ell(\widetilde{y}) \, dY.$$

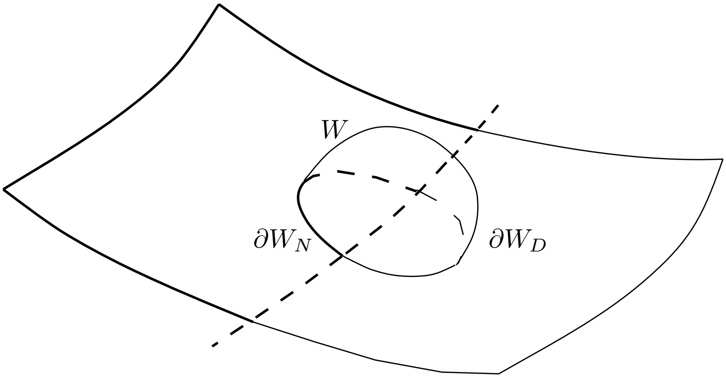

Fig. 4. The set W

Setting

$$B_2^+(\rho) := \{\widetilde{y} = (y_{n-1}, y_n) \in B_2(\rho) \mid y_n > 0\},$$
$$C_2^+(\rho) := \{\widetilde{y} = (y_{n-1}, y_n) \in \partial B_2(\rho) \mid y_n > 0\},$$

we first observe that we can choose $\rho_{\boldsymbol{x}^*}$ such that

$$\{Y\} \times B_2^+(\rho) \subset \Theta(D_\varepsilon(\boldsymbol{x}^*)).$$

Hence denoting by π_2 the projection on $\{0_{\mathbb{R}^{n-2}}\} \times \mathbb{R}^2$, the change of variables

$$\pi_2(\Theta(C_\varepsilon(\boldsymbol{x}^*))) \to C_2^+(\rho), \quad \widetilde{y} \mapsto z = \rho \frac{\widetilde{y}}{|\widetilde{y}|}$$

gives the estimate

$$\int_{\Theta(C_\varepsilon(\boldsymbol{x}^*))} v(Y, \widetilde{y})^2 \, d\ell(\widetilde{y}) \leq C \int_{C_2^+(\rho)} v(Y, z)^2 \, d\ell(z) \tag{16}$$

for a constant C depending only on $\boldsymbol{x}_0^*$.

Now we estimate this latter integral in terms of $\|\nabla_2 v\|_{\mathrm{L}^2(\{y'\} \times B_2^+(\rho))}$. Setting $v_\rho(\widetilde{y}) := v(Y, \widetilde{y})$, we obtain $\nabla v_\rho \in \mathrm{L}^2(B_2^+(1))$ and

$$\|\nabla v_\rho\|_{\mathrm{L}^2(B_2^+(1))} = \|\nabla_2 v\|_{\mathrm{L}^2(\{y'\} \times B_2^+(\rho))},$$
$$\|v_\rho\|_{\mathrm{L}^2(C_2^+(1))} = \rho^{-\frac{1}{2}} \|v\|_{\mathrm{L}^2(\{y'\} \times C_2^+(\rho))}.$$

Observing that $v_\rho = 0$ on $B_2^{++}(1) := \{(y_{n-1}, y_n) \in B_2^+(1) \mid y_n > 0\}$, the trace theorem and the Poincaré inequality yield, for some universal constant $C > 0$, the estimate

$$\int_{C_2^+(\rho)} v^2(y', \widetilde{y}) \, d\ell(\widetilde{y}) \leq C\rho \|\nabla_2 v\|^2_{\mathrm{L}^2(\{Y\} \times B_2^+(\rho))}.$$

Hence, owing to (16), we have

$$\int_{\Theta(C_\varepsilon(\boldsymbol{x}^*))} v^2(Y,\widetilde{y})\,d\ell(\widetilde{y}) \le C\rho_{\boldsymbol{x}^*}\|\nabla_2 v\|^2_{\mathrm{L}^2(\{Y\}\times B_2^+(\rho_{\boldsymbol{x}^*}))}.$$

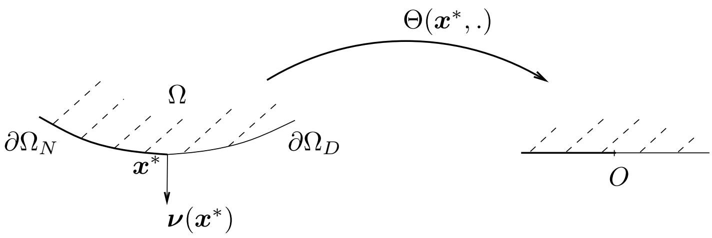

Fig. 5. The C^2-diffeomorphism $\Theta(\boldsymbol{x}^*,\cdot)$ in the plane $\boldsymbol{x}^* + \langle\boldsymbol{\tau}^*,\boldsymbol{\nu}^*\rangle$

Observing that $\rho_{\boldsymbol{x}^*}$ is $O(\varepsilon)$ uniformly on $W\cap\Gamma$ and the diffeomorphism $\Theta(\boldsymbol{x}^*,.)$ (see Fig. 6), we can conclude that, for some constant $C_{\boldsymbol{x}_0^*}$ depending only on $\boldsymbol{x}_0^*$,

$$\int_{\Theta(C_\varepsilon(\boldsymbol{x}*))} v^2(Y,\widetilde{y})\,d\ell(\widetilde{y}) \le C_{\boldsymbol{x}_0^*}\varepsilon\|u(\boldsymbol{x}^*,\cdot)\|^2_{\mathrm{H}^1(\Theta^{-1}(\{Y\}\times B_2^+(\rho)))}$$
$$\le C_{\boldsymbol{x}_0^*}\varepsilon\|u(\boldsymbol{x}^*,\cdot)\|^2_{\mathrm{H}^1(D_\varepsilon(\boldsymbol{x}^*))}.$$

Hence, after an integration on $W\cap\Gamma$,

$$\int_{W\cap\Gamma}\int_{C_\varepsilon(\boldsymbol{x}^*)} u^2\,d\ell\,d\gamma(\boldsymbol{x}^*) \le C_{\boldsymbol{x}_0^*}\varepsilon\int_{W\cap\Gamma}\|u(\boldsymbol{x}^*,\cdot)\|^2_{\mathrm{H}^1(D_\varepsilon(\boldsymbol{x}^*))}\,d\gamma(\boldsymbol{x}^*).$$

Finally, we complete the proof by using a partition of unity on the open sets $(W_{\boldsymbol{x}_0^*})_{\boldsymbol{x}_0^*\in\Gamma}$. Lemma 6 is proved. □

Let us come back to our problem. Using (15) for u_S, by the Pythagoras theorem we have

$$\int_{\partial\Omega_\varepsilon^*}|\nabla u_S|^2\,d\sigma = \int_{\partial\Omega_\varepsilon^*}|\nabla_\Gamma u_S|^2\,d\sigma + \int_{\partial\Omega_\varepsilon^*}|\nabla_2 u_S|^2\,d\sigma.$$

Applying Lemma 6 to $\nabla_\Gamma u_S$, we obtain that the first term vanishes as $\varepsilon\to 0$. As well as in the bi-dimensional case, we will see that the second term above is bounded, using more information on u_S.

Owing to [4, Theorem 4] and the Borel–Lebesgue theorem, we can write

$$u_S(\boldsymbol{x}) = \eta(\boldsymbol{x}^*)U_S(\boldsymbol{x}-\boldsymbol{x}^*) := \eta(\boldsymbol{x}^*)U_S^{\boldsymbol{x}^*}(\boldsymbol{x}) \quad \text{on } \omega_{\varepsilon_0}, \tag{17}$$

where U_S is locally diffeomorphic to the Shamir function and $\eta \in H^{1/2}(\Gamma)$. We then have, owing to the Fubini theorem,

$$\int_{\partial\Omega_\varepsilon^*} |\nabla_2 u_S|^2 \, d\sigma = \int_\Gamma \eta(\boldsymbol{x}^*)^2 \int_{C_\varepsilon(\boldsymbol{x}^*)} |\nabla_2 U_S^{\boldsymbol{x}^*}|^2 \, d\ell \, d\gamma(\boldsymbol{x}^*),$$

and, as well as in the bi-dimensional case, we show that this term is bounded by $O(1)\,\|\eta\|^2_{L^2(\Gamma)}$. Now we have proved that the second term in (14) is bounded, that is

$$J_\varepsilon(\nabla u_R) \to 0 \quad \text{as } \varepsilon \to 0.$$

To treat the last term $I_\varepsilon(\nabla u_S)$, we use similar tools. Splitting (13) for u_S gives us

$$I_\varepsilon(\nabla u_S) = I_\varepsilon(\nabla_2 u_S) + I_\varepsilon(\nabla_\Gamma u_S) + 2J_\varepsilon(\nabla_2 u_S, \nabla_\Gamma u_S).$$

As above, the term $I_\varepsilon(\nabla_\Gamma u_S)$ is estimated by

$$\int_{\partial\Omega_\varepsilon^*} |\nabla_\Gamma u_S|^2 \, d\sigma.$$

It then vanishes as $\varepsilon \to 0$.

The bilinear term is estimated by

$$\left(\int_{\partial\Omega_\varepsilon^*} |\nabla_2 u_S|^2 \, d\sigma \right)^{1/2} \left(\int_{\partial\Omega_\varepsilon^*} |\nabla_\Gamma u_S|^2 \, d\sigma \right)^{1/2};$$

it then tends to zero since the first term is bounded and the second one vanishes as $\varepsilon \to 0$.

For the last term $I_\varepsilon(\nabla_2 u_S)$, we use (17) and the Fubini theorem and have

$$\int_\Gamma \eta(\boldsymbol{x}^*)^2 \int_{C_\varepsilon} \Big((\boldsymbol{x}^*) 2(\boldsymbol{\nu} \cdot \nabla_2 U_S^{\boldsymbol{x}^*})(\boldsymbol{m} \cdot \nabla_2 U_S^{\boldsymbol{x}^*}) - \boldsymbol{m} \cdot \boldsymbol{\nu} \, |\nabla_2 U_S^{\boldsymbol{x}^*}|^2 \Big) \, d\ell \, d\gamma(\boldsymbol{x}^*).$$

First, we work in the plane $\boldsymbol{x}^* + \langle \boldsymbol{\tau}^*, -\boldsymbol{\nu}^* \rangle$ and, as above, we obtain

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon(\boldsymbol{x}^*)} (2(\boldsymbol{\nu} \cdot \nabla_2 U_S^{\boldsymbol{x}^*})(\boldsymbol{m} \cdot \nabla_2 U_S^{\boldsymbol{x}^*}) - \boldsymbol{m} \cdot \boldsymbol{\nu} \, |\nabla_2 U_S^{\boldsymbol{x}^*}|^2) \, d\ell = \frac{\pi}{4} \boldsymbol{m}(\boldsymbol{x}^*) \cdot \boldsymbol{\tau}(\boldsymbol{x}^*).$$

Moreover, for any $\varepsilon > 0$, this integral term on $C_\varepsilon(\boldsymbol{x}^*)$ is dominated by $\frac{\pi}{2}\|\boldsymbol{m}\|_\infty \in L^1(\Gamma)$. So, the dominated convergence theorem can be applied and, finally,

$$\lim_{\varepsilon \to 0} I_\varepsilon(\nabla_2 u_S) = \frac{\pi}{4} \int_\Gamma \eta^2 \boldsymbol{m} \cdot \boldsymbol{\tau} \, d\gamma.$$

The proof is now complete with $\zeta = \dfrac{\sqrt{\pi}}{2}\eta$. $\square$

Now we apply the Rellich relation to the stabilization of solutions of (S).

3. Proof of linear and nonlinear stabilization

We begin by writing the following consequence of Sec. 2.

Corollary 7. *Assume that $t \mapsto (u(t), u'(t))$ is a strong solution of (S) and that the geometrical additional assumption* (5) *if $n \geq 3$ or* (6) *if $n = 2$ holds. Then for every time t, $u(t)$ satisfies*

$$2\int_\Omega \Delta u(\boldsymbol{m}\cdot\nabla u)\,d\boldsymbol{x} \leq d(n-2)\int_\Omega |\nabla u|^2\,d\boldsymbol{x} + \int_{\partial\Omega}(2\partial_\nu u\,(\boldsymbol{m}\cdot\nabla u) - \boldsymbol{m}\cdot\boldsymbol{\nu}\,|\nabla u|^2)\,d\sigma.$$

Proof. Indeed, under theses hypotheses, for each time t, $(u(t), u'(t)) \in D(\mathcal{W})$ so that $u(t)$ satisfies (9) or (12). The corollary is then an application of Theorem 4 or 5. □

We will be able to prove Theorems 1 and 2 showing that, for $\alpha = (p-1)/2$, one can apply the following result (see [9]).

Proposition 8. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function such that there exists $\alpha \geq 0$ and $C > 0$ such that*

$$\int_t^\infty E^{\alpha+1}(s)\,ds \leq CE(t) \quad \forall t \geq 0.$$

Then, setting $T = CE^\alpha(0)$, we have

$$\text{if } \alpha = 0, \qquad \forall t \geq T, \qquad E(t) \leq E(0)\exp\left(1 - \frac{t}{T}\right),$$

$$\text{if } \alpha > 0, \qquad \forall t \geq T, \qquad E(t) \leq E(0)\left(\frac{T+\alpha T}{T+\alpha t}\right)^{1/\alpha}.$$

We come back to our proof now.

Proof. Following [5, 9], we prove the estimates for $(u_0, u_1) \in D(\mathcal{W})$, which, using the density of the domain, will be sufficient to obtain get the result for all solutions. Setting $Mu = 2\boldsymbol{m}\cdot\nabla u + d(n-1)u$, we prove the following result.

Lemma 9. *For any $0 \le S < T < \infty$, we have*

$$2d \int_S^T E^{\frac{p+1}{2}}\, dt \le -\left[E^{\frac{p-1}{2}} \int_\Omega u' M u \, d\boldsymbol{x}\right]_S^T + \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}} E' \int_\Omega u' M u \, d\boldsymbol{x}\, dt + \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} \boldsymbol{m}\cdot\boldsymbol{\nu} \left((u')^2 - |\nabla u|^2 - g(u')Mu\right) d\sigma\, dt.$$

Proof. Using the fact that u satisfies (S) and observing that $u''Mu = (u'Mu)' - u'Mu'$, integration by parts gives

$$0 = \int_S^T E^{\frac{p-1}{2}} \int_\Omega (u'' - \Delta u) M u \, d\boldsymbol{x}\, dt = \left[E^{\frac{p-1}{2}} \int_\Omega u' M u \, d\boldsymbol{x}\right]_S^T - \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}} E' \int_\Omega u' M u \, d\boldsymbol{x}\, dt - \int_S^T E^{\frac{p-1}{2}} \int_\Omega (u'Mu' + \Delta u M u)\, d\boldsymbol{x}\, dt.$$

Corollary 7 now yields

$$\int_\Omega \Delta u\, Mu\, d\boldsymbol{x} \le d(n-1)\int_\Omega \Delta u\, u\, d\boldsymbol{x} + d(n-2)\int_\Omega |\nabla u|^2\, d\boldsymbol{x} + \int_{\partial\Omega} (2\partial_\nu u\, (\boldsymbol{m}\cdot\nabla u) - \boldsymbol{m}\cdot\boldsymbol{\nu}\, |\nabla u|^2)\, d\sigma.$$

Hence, the Green–Riemann formula leads to

$$\int_\Omega \Delta u\, Mu\, d\boldsymbol{x} \le -d\int_\Omega |\nabla u|^2\, d\boldsymbol{x} + \int_{\partial\Omega} (\partial_\nu u\, Mu - \boldsymbol{m}\cdot\boldsymbol{\nu}\, |\nabla u|^2)\, d\sigma.$$

Using the boundary conditions and the fact that $\nabla u = \partial_\nu u\, \boldsymbol{\nu}$ on $\partial\Omega_D$, we obtain

$$\int_\Omega \Delta u\, Mu\, d\boldsymbol{x} \le -d\int_\Omega |\nabla u|^2\, d\boldsymbol{x} - \int_{\partial\Omega_N} \boldsymbol{m}\cdot\boldsymbol{\nu}\, \left(g(u')\, Mu + |\nabla u|^2\right) d\sigma.$$

On the other hand, using $\operatorname{div}(\boldsymbol{m}) = nd$, another use of the Green formula yields

$$\int_\Omega u'\, Mu'\, d\boldsymbol{x} = -d \int_\Omega |u'|^2\, d\boldsymbol{x} + \int_{\partial\Omega_N} \boldsymbol{m}.\boldsymbol{\nu}\, |u'|^2\, d\sigma.$$

□

Coming back to our problem, the Young inequality gives

$$\left| \int_\Omega u'\, Mu\, d\boldsymbol{x} \right| \leq CE(t).$$

Lemma 9 shows that

$$2d \int_S^T E^{\frac{p+1}{2}}\, dt \leq C\left(E^{\frac{p+1}{2}}(T) + E^{\frac{p+1}{2}}(S)\right) + C \int_S^T E^{\frac{p-1}{2}} E'\, dt$$
$$+ \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} \boldsymbol{m}\cdot\boldsymbol{\nu}\left(|u'|^2 - |\nabla u|^2 - g(u')Mu\right) d\sigma\, dt.$$

For simplicity, let $d\sigma_m = \boldsymbol{m}\cdot\boldsymbol{\nu}\, d\sigma$. Observing that

$$E'(t) = -\int_{\partial\Omega_N} g(u')u'\, d\sigma_m \leq 0,$$

we obtain, for a constant $C > 0$ independent of $E(0)$ if $p = 1$,

$$2d \int_S^T E^{\frac{p+1}{2}}\, dt \leq CE(S) + \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} \left(|u'|^2 - |\nabla u|^2 - g(u')Mu\right) d\sigma_m\, dt.$$

Using the definition of Mu and the Young inequality, we obtain, for any $\varepsilon > 0$,

$$2d \int_S^T E^{\frac{p+1}{2}}\, dt \leq CE(S)$$
$$+ \int_S^T E^{\frac{p-1}{2}} \int_{\partial\Omega_N} \left(|u'|^2 + \left(\|\boldsymbol{m}\|_\infty^2 + \frac{d^2(n-1)^2}{4\varepsilon}\right) g(u')^2 + \varepsilon u^2\right) d\sigma_m\, dt.$$

Now, using the Poincaré inequality, we can choose $\varepsilon > 0$ such that

$$\varepsilon \int_{\partial\Omega_N} \boldsymbol{m}\cdot\boldsymbol{\nu}\, u^2\, d\sigma \leq \frac{d}{2} \int_\Omega |\nabla u|^2\, d\boldsymbol{x} \leq dE.$$

We conclude

$$d\int_S^T E^{\frac{p+1}{2}}\,dt \le CE(S) + C\int_S^T E^{\frac{p-1}{2}}\int_{\partial\Omega_N}\big((u')^2+g(u')^2\big)\,d\sigma_m\,dt.$$

We split $\partial\Omega_N$ to bound the last term of the above estimate

$$\partial\Omega_N^1 = \{\boldsymbol{x}\in\partial\Omega_N;\ |u'(\boldsymbol{x})|>1\},\quad \partial\Omega_N^2 = \{\boldsymbol{x}\in\partial\Omega_N;\ |u'(\boldsymbol{x})|\le 1\}.$$

Using (3) and (4), we obtain

$$\int_S^T E^{\frac{p-1}{2}}\int_{\partial\Omega_N^1}\big(|u'|^2+g(u')^2\big)\,d\sigma_m\,dt$$
$$\le C\int_S^T E^{\frac{p-1}{2}}\int_{\partial\Omega_N} u'g(u')\,d\sigma_m\,dt \le CE(S),$$

where C neither depend on $E(0)$ if $p=1$.

On the other hand, using (3) and (4), the Jensen inequality, and the boundedness of $\boldsymbol{m}$, we successively obtain

$$\int_{\partial\Omega_N^2}\big((u')^2+g(u')^2\big)\,d\sigma_m \le C\int_{\partial\Omega_N^2}(u'g(u'))^{2/(p+1)}\,d\sigma_m$$
$$\le C\left(\int_{\partial\Omega_N^2} u'g(u')\,d\sigma_m\right)^{\frac{2}{p+1}} \le C(-E')^{\frac{2}{p+1}}.$$

Hence, using the Young inequality again, we obtain for every $\varepsilon>0$

$$\int_S^T E^{\frac{p-1}{2}}\int_{\partial\Omega_N^2}\big((u')^2+g(u')^2\big)\,d\sigma_m\,dt \le \int_S^T(\varepsilon E^{\frac{p+1}{2}} - C(\varepsilon)E')\,dt$$
$$\le \varepsilon\int_S^T E^{\frac{p+1}{2}}\,dt + C(\varepsilon)E(S).$$

Finally, we obtain, for some $C(\varepsilon)$ and C independent of $E(0)$ if $p=1$,

$$d\int_S^T E^{\frac{p+1}{2}}\,dt \le C(\varepsilon)E(S) + \varepsilon C\int_S^T E^{\frac{p+1}{2}}\,dt.$$

Now choosing $\varepsilon C\le d/2$, Theorems 1 and 2 follow from Proposition 8. □

4. Examples and numerical results

4.1. **Examples.** We consider the case where Ω is a plane convex polygonal domain. The normal unit vector pointing outward of Ω is piecewise constant and the nature of boundary conditions involved by the multiplier method can be determined on each edge, independently of other edges.

Along each edge, vector $\boldsymbol{\nu}$ is constant and the boundary conditions are defined by the sign of

$$\boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) = (R_\theta(\boldsymbol{x} - \boldsymbol{x}_0)) \cdot \boldsymbol{\nu}(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{x}_0) \cdot R_{-\theta}(\boldsymbol{\nu}(\boldsymbol{x})).$$

Hence we construct $\boldsymbol{\nu}$, $R_{-\theta}(\boldsymbol{\nu})$, and we can determine the sign of above coefficient with respect to the position of $\boldsymbol{x}_0$. To this end, we construct two straight lines, orthogonal with respect to $R_{-\theta}(\boldsymbol{\nu})$ so that each of them contains one vertex of the considered edge. This determines a belt and if $\boldsymbol{x}_0$ belongs to this belt, we obtained mixed boundary conditions along this edge, if $\boldsymbol{x}_0$ does not belong to this belt, then we get Dirichlet or Neumann boundary conditions along whole the edge (see Fig. 6).

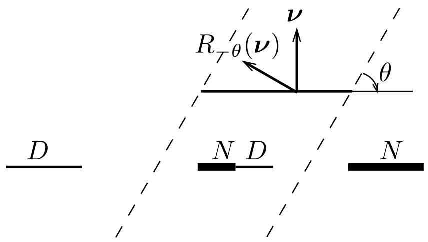

Fig. 6. Boundary conditions along some edge depending on the position of $\boldsymbol{x}_0$

Performing this method for a square, $\Omega = (0,1)^2$, we show in Fig. 7 the different cases of boundary conditions depending on the position of $\boldsymbol{x}_0$. Three main cases are considered:

1. $0 < \theta < \pi/4$: above belts controlling opposite edges have a nonempty intersection, which is a belt of positive thickness,
2. $\theta = \pi/4$: this intersection is a straight line,
3. $\pi/4 < \theta < \pi/2$: the intersection is empty.

The case where θ is negative can be easily deduced by symmetry.

In the three above cases, there are four angular sectors (shaded areas in Fig. 7) such that if $\boldsymbol{x}_0$ belongs to one of them, then geometrical condition (6) is satisfied.

4.2. **Numerical results.** We perform numerical experiments by considering the following case:

$$\Omega = (0,1)^2, \quad \partial\Omega_D = \big(\{0\} \times [0,1/2]\big) \cup \big([0,1] \times \{0\}\big), \quad \partial\Omega_N = \partial\Omega \setminus \partial\Omega_D,$$

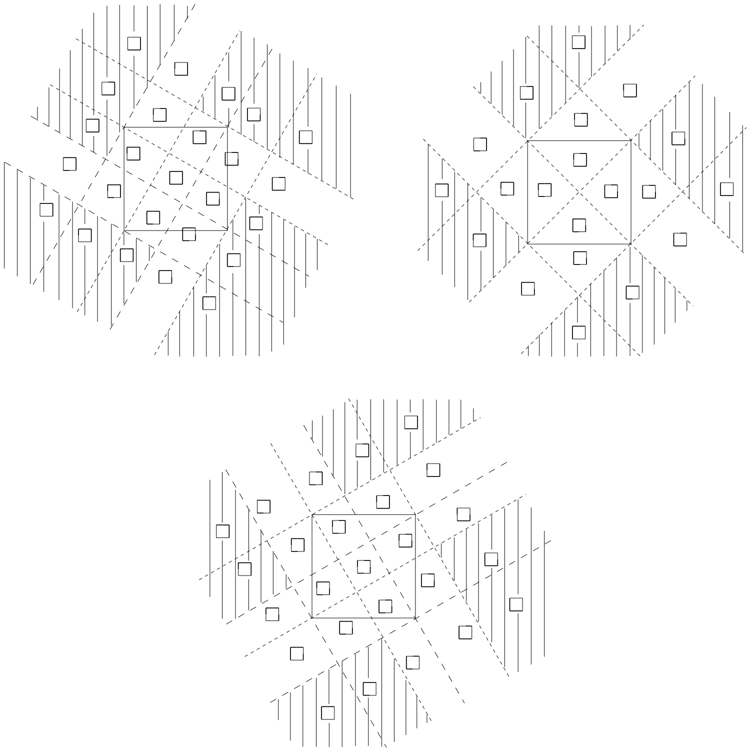

Fig. 7. Shape of boundary data with respect to $\boldsymbol{x}_0$ (cases 1, 2, and 3)

and using above vector field

$$\boldsymbol{m}(\boldsymbol{x}) = R_\theta(\boldsymbol{x} - \boldsymbol{x}_0).$$

We only consider the case of a linear feedback. The problem is as follows:

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+^*, \\ \partial_\nu u = -\boldsymbol{m} \cdot \boldsymbol{\nu}\, u' & \text{on } \partial\Omega_N \times \mathbb{R}_+^*, \\ u(0) = u_0 & \text{in } \Omega, \\ u^{'}(0) = u_1 & \text{in } \Omega. \end{cases}$$

We will investigate the cases where θ varies in $[0, \arctan(2)]$. A particular case is given in Fig. 8.

Our aim is to study numerically the variations of the speed of stabilization with respect to the position of $\boldsymbol{x}_0$ and the value of θ. To this end, we have constructed a finite differences scheme (in the space). This leads to a linear

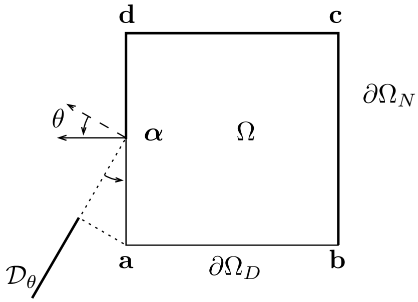

Fig. 8. When $\boldsymbol{x}_0$ belongs to $\mathcal{D}_\theta$, geometrical condition (6) is satisfied at $\boldsymbol{\alpha}$

second order differential equation

$$U'' + BU' + KU = 0, \tag{18}$$

where B is the feedback matrix and $-K$ is the discretized Laplace operator.

Let us define $V = K^{1/2}U$. The above differential equation can be rewritten as follows:

$$\begin{pmatrix} V \\ U' \end{pmatrix}' = \begin{pmatrix} 0 & K^{1/2} \\ -K^{1/2} & -B \end{pmatrix} \begin{pmatrix} V \\ U' \end{pmatrix}$$

and the energy function can be approximated by

$$\frac{1}{2}(\langle U, KU\rangle + \|U'\|^2) = \frac{1}{2}(\|V\|^2 + \|U'\|^2).$$

The decreasing rate is given by the highest eigenvalue of above matrix. The results of our computations are shown in Fig. 9, where we constructed the decreasing rate as a function depending on θ and the position of $\boldsymbol{x}_0$ represented by the abscissa λ along $\mathcal{D}_\theta$.

It can be observed that in this case, the decreasing rate is increasing with θ and the best position for $\boldsymbol{x}_0$ is the origin of half-line $\mathcal{D}_\theta$.

Acknowledgments. This work was partially supported by FONDECYT 1061263 and ECOS-CONICYT CO4E08.

References

1. H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. *Math. Stud.* **5** (1973).
2. J. Banasiak and G.-F. Roach, On mixed boundary value problems of Dirichlet oblique-derivative type in plane domains with piecewise differentiable boundary. *J. Differ. Equations* **79** (1989), No. 1, 111–131.
3. C. Bardos, G. Lebeau, and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30** (1992), No. 5, 1024–1065.

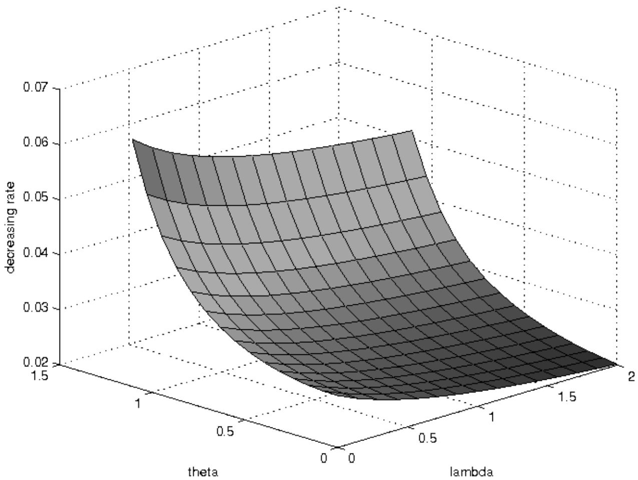

Fig. 9. Dependence of the decreasing rate with respect to θ and λ

4. R. Bey, J.-P. Lohéac, and M. Moussaoui, Singularities of the solution of a mixed problem for a general second order elliptic equation and boundary stabilization of the wave equation. *J. Math. Pures Appl.* **78** (1999), 1043–1067.
5. F. Conrad and B. Rao, Decay of solutions of the wave equation in a star-shaped domain with nonlinear boundary feedback. *Asympt. Anal.* **7** (1993), No. 1, 159–177.
6. P. Grisvard, Elliptic problems in nonsmooth domains. *Pitman, London* (1985).
7. ———, Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités. *J. Math. Pures Appl.* **68** (1989), 215–259.
8. L. F. Ho, Observabilité frontière de l'équation des ondes. *C. R. Acad. Sci. Paris, Sér.* I *Math.* **302** (1986), 443-446.
9. V. Komornik, Exact controllability and stabilization. The multiplier method. *Masson–John Wiley, Paris* (1994).
10. V. Komornik and E. Zuazua, A direct method for the boundary stabilization of the wave equation. *J. Math. Pures Appl.* **69** (1990), 33–54.
11. V. A. Kozlov, V. G. Maz'ya, and J. Rossmann, Elliptic boundary-value problems in domains with point singularities. *Amer. Math. Soc., Providence* (1997).
12. J.-L. Lions, Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués. *Masson, Paris* (1988).

13. M. Moussaoui, Singularités des solutions du problème mêlé, contrôlabilité exacte et stabilisation frontière. *ESAIM Proc. Élasticité, Viscoélasticité et Contrôle Optimal, Huitièmes Entretiens du Centre Jacques Cartier* (1996), 157–168.
14. J. Nečas, Les méthodes directes en théorie des équations elliptiques. *Masson, Paris* (1967).
15. A. Osses, A rotated multiplier applied to the controllability of waves, elasticity and tangential Stokes control. *SIAM J. Control Optim.* **40** (2001), No. 3, 777–800.
16. F. Rellich, Darstellung der Eigenwerte von $\Delta u + \lambda u$ durch ein Randintegral. *Math. Z.* **46** (1940), 635–636.
17. E. Shamir, Regularity of mixed second order elliptic problems. *Isr. Math. J.* **6** (1968), 150–168.

(Received August 03 2008)

Authors' addresses:

P. Cornilleau
Université de Lyon, École centrale de Lyon, D.M.I.,
Institut Camille-Jordan (C.N.R.S. U.M.R. 5208),
36 avenue Guy-de-Collongue, 69134 Écully cedex, France

J.-P. Lohéac
Independent Moscow University,
Laboratoire J.-V. Poncelet (C.N.R.S. U.M.I. 2615),
Bol. Vlasyevsky Per. 11, 119002 Moscow, Russia
E-mail: Jean-Pierre.Loheac@math.cnrs.fr

A. Osses
Universidad de Chile, D.I.M.,
Centro de Modelamiento Matemático,
(C.N.R.S. U.M.I. 2807 CNRS-Uchile),
Casilla 170/3–Correo 3, Santiago, Chile