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CONTROLLABILITY FOR SEMILINEAR RETARDED CONTROL SYSTEMS IN HILBERT SPACES

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ABSTRACT. This paper deals with well-posedness and L^2-regularity properties for a class of semilinear retarded functional differential equations. A relation between the reachable set of a semilinear system and that of the corresponding linear system is proved. We also show that the Lipschitz continuity and the uniform boundedness of the nonlinear term can be considerably weakened. Finally, a simple example is given, to which our main result can be applied.

1. Introduction

In this paper, we deal with the approximate controllability of the following semilinear equation:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \displaystyle\int_0^t f(t,s,x_s,u(s))ds + Bu(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0, \end{cases} \tag{SE}$$

in Hilbert spaces. Let V and H be complex Hilbert spaces forming Gelfand triple $V \subset H \subset V^*$ with pivot space H. Here A is the operator associated with a sesquilinear form defined on $V \times V$ and satisfying the Gårding inequality. Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H. Let $0 \le h < \infty$ be given and let

$$x_s : [-h, 0) \to H, \quad x_s(r) = x(s+r) \text{ for } -h \le r < 0.$$

Let $f : \mathbb{R}^2 \times L^2(-h,0;V) \times U \to H$ be a nonlinear mapping such that there exist positive constants L_1 and L_2 such that

$$|f(t,s,x,u) - f(t,s,\hat{x},\hat{u})|_H \le L_1\|x-\hat{x}\|_{L^2(-h,0;V)} + L_2\|u-\hat{u}\|_U \tag{1.1}$$

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for all $(t,s) \in \mathbb{R}^2$, $x, \hat{x} \in L^2(-h,0;V)$, and $u, \hat{u} \in U$. The mild solution of (SE) is given by

$$\begin{cases} (x_t)(0) = S(t)\phi^0 + \int_0^t S(t-s)\left\{ \int_0^s f\big(s,\tau,x_\tau,u(\tau)\big)d\tau + Bu(s) \right\} ds, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0, \end{cases}$$

where $S(t)$ is an analytic semigroup generated by A. The solution of (SE) is denoted by $x(T; f, u)$ corresponding to the nonlinear term f and the control u. System (SE) is said to be approximately controllable in the time interval $[0,T]$ if for every given final state x_1 and $\epsilon > 0$, there exists a control function $u \in L^2(0,T;U)$ such that $\|x(T;f,u) - x_1\|_H < \epsilon$. Dauer and Mahmudov [1] considered the approximate controllability of a semilinear control system as a particular case of sufficient conditions for approximate solvability of semilinear equations by assuming that

(1) $S(t)$ is a compact operator for each $t > 0$,
(2) f is continuous and uniformly bounded, and
(3) the corresponding linear system (SE) for $f \equiv 0$ is approximately controllable.

Jeong, Kwun, and Park [3] studied control problems for semilinear retarded functional differential equation under the natural assumption that the embedding $D(A) \subset V$ is compact instead of condition (1). Yamamoto and Park [9] studied the controllability for parabolic equations with uniformly bounded nonlinear terms in the sense of (2). Both for some considerations on the trajectory set of (SE) and that of the corresponding linear system (in the case where $f \equiv 0$) and for matters connected with (3), we refer to [4,10] and references therein. Sukavanam and Nutan Kumar Tomar [5] studied the control problems of (SE) by assuming (1.1), (3) and

(4) there exists a constant $\beta > 0$ such that $\|Bv\| \ge \beta\|v\|$ for all $v \in L^2(0,T;U)$ and $L_1 < \beta$.

In this paper, we no longer require the compactness in (1) and the uniform boundedness and the inequality condition for Lipschitz continuity of f in (4), but instead we need the regularity and a variation of solutions of the given equations. In Sec. 2, we begin to study the well-posedness and L^2-regularity properties for a class of semilinear retarded functional differential equations. In Sec. 3, the relation between the reachable set of the semilinear system and that of the corresponding linear system is found. We also show that the Lipschitz continuity and the uniform boundedness of the nonlinear term can be considerably weakened. Finally, a simple example is given to which our main result can be applied.

2. Semilinear retarded functional equations

Let H and V be complex Hilbert spaces such that $V \subset H \subset V^*$ by identifying the anti-dual of H with H.

Therefore, for brevity, we can assume that $\|u\|_* \le |u| \le \|u\|$ for all $u \in V$, where the notations $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$ denote the norms of H, V, and V^*, respectively, as usual. Let $a(u, v)$ be a bounded sesquilinear form defined on $V \times V$ satisfying the Gårding inequality

$$\operatorname{Re} a(u,u) \ge c_0\|u\|^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0.$$

Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u, v \in V.$$

Then A is a bounded linear operator from V to V^*. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

equipped with the graph norm $\|u\|_{D(A)} = |u| + |Au|$ is also denoted by A. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H, \tag{2.1}$$

where $(V, V^*)_{\frac{1}{2},2}$ denotes the real interpolation space between V and V^*(see [6]). Moreover, for each $T > 0$, by using the interpolation theory, we have

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

The inequalities

$$\begin{aligned} c_0\|u\|^2 \le \operatorname{Re} a(u,u) + c_1|u|^2 &\le C|Au|\,|u| + c_1|u|^2 \\ &\le (C|Au| + c_1|u|)|u| \le C\|u\|_{D(A)}|u| \end{aligned}$$

imply that there exists a constant $C_0 > 0$ such that

$$\|u\| \le C_0\|u\|_{D(A)}^{1/2}|u|^{1/2}. \tag{2.2}$$

It is known that A generates an analytic semigroup $S(t)$ in both H and V^*. For simplicity, we assume that $c_1 = 0$ and hence the closed half-plane $\{\lambda : \operatorname{Re}\lambda \ge 0\}$ is contained in the resolvent set of A.

The following lemma follows from [7, Lemma 3.6.2] or [8].

Lemma 2.1. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^*:*

$$|S(t)x| \le M|x|, \tag{2.3}$$

$$\|S(t)x\|_* \le M\|x\|_*, \tag{2.4}$$

$$|S(t)x| \le Mt^{-1/2}\|x\|_*, \tag{2.5}$$

$$\|S(t)x\| \le Mt^{-1/2}|x|. \tag{2.6}$$

Consider the following initial-value problem for the abstract linear parabolic equation

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + k(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{LE}$$

By virtue of [3, Theorem 2.1] or [2], we have the following result on the corresponding linear equation of (LE).

Proposition 2.1. *Let the assumptions on the principal operator A stated above be satisfied. Then the following properties hold.*

1. *Let $F = (D(A), H)_{1/2,2}$, where $(D(A), H)_{1/2,2}$ is the real interpolation space between $D(A)$ and H (see* [6, Sec. 1.3.3]*). For $x_0 \in F$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of* (LE) *belonging to*
$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; F)$$
and satisfying
$$\|x\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(\|x_0\|_F + \|k\|_{L^2(0,T;H)}), \tag{2.7}$$
where C_1 is a constant depending on T.
2. *Let $x_0 \in H$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of* (LE) *belonging to*
$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$
and satisfying
$$\|x\|_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|x_0| + \|k\|_{L^2(0,T;V^*)}), \tag{2.8}$$
where C_1 is a constant depending on T.

Lemma 2.2. *Suppose that $k \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$\|x\|_{L^2(0,T;D(A))} \le C_1\|k\|_{L^2(0,T;H)}, \tag{2.9}$$

$$\|x\|_{L^2(0,T;H)} \le C_2 T\|k\|_{L^2(0,T;H)}, \tag{2.10}$$

$$\|x\|_{L^2(0,T;V)} \le C_2\sqrt{T}\|k\|_{L^2(0,T;H)}. \tag{2.11}$$

Proof. Estimate (2.9) immediately follows from (2.7). Since

$$\|x\|^2_{L^2(0,T;H)} = \int_0^T \left| \int_0^t S(t-s)k(s)ds \right|^2 dt \le M \int_0^T \left(\int_0^t |k(s)|ds \right)^2 dt$$
$$\le M \int_0^T t \int_0^t |k(s)|^2 ds dt \le M \frac{T^2}{2} \int_0^T |k(s)|^2 ds,$$

it follows that

$$\|x\|_{L^2(0,T;H)} \leq T\sqrt{M/2}\|k\|_{L^2(0,T;H)}.$$

From (2.2), (2.9), and (2.10) it follows that

$$\|x\|_{L^2(0,T;V)} \leq C_0\sqrt{C_1 T}(M/2)^{1/4}\|k\|_{L^2(0,T;H)}.$$

Therefore, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

For each $s \in [0,T]$, we define $x_s : [-h,0] \to H$ as follows:

$$x_s(r) = x(s+r), \quad -h \leq r \leq 0.$$

We set

$$\Pi = L^2(-h,0;V) \quad \text{and} \quad \mathbb{R}^+ = [0,\infty).$$

Lemma 2.3. *Let $x \in L^2(-h,T;V)$. Then the mapping $s \mapsto x_s$ belongs to $C([0,T];\Pi)$ and*

$$\|x_\cdot\|_{L^2(0,T;\Pi)} \leq \sqrt{T}\|x\|_{L^2(-h,T;V)}. \tag{2.12}$$

Proof. The first assertion is easy to verify and (2.12) is a consequence of the estimate

$$\begin{aligned}\|x_\cdot\|^2_{L^2(0,T;\Pi)} &\leq \int_0^T \|x_s\|^2_\Pi ds \leq \int_0^T\int_{-h}^0 \|x(s+r)\|^2 dr ds \\ &\leq \int_0^T ds \int_{-h}^T \|x(r)\|^2 dr \leq T\|x\|^2_{L^2(-h,T;V)}.\end{aligned}$$

□

Consider the following initial-value problem for the abstract semilinear parabolic equation:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \displaystyle\int_0^t f(t,s,x_s,u(s))ds + Bu(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \leq s < 0. \end{cases} \tag{SE}$$

Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H.

Let $f : \mathbb{R}^2 \times \Pi \times U \to H$ be a nonlinear mapping satisfying the following:

(i) for any $x \in \Pi$, $u \in U$ the mapping $f(\cdot,\cdot,x,u)$ is strongly measurable;

(ii) there exist positive constants L_0, L_1, and L_2 such that

$$|f(t,s,x,u) - f(t,s,\hat{x},\hat{u})| \le L_1\|x-\hat{x}\|_\Pi + L_2\|u-\hat{u}\|_U, \tag{F1}$$

$$|f(t,s,0,0)| \le L_0 \tag{F2}$$

for all $(t,s) \in \mathbb{R}^2$, $x, \hat{x} \in \Pi$, and $u, \hat{u} \in U$.

For $x \in L^2(-h,T;V)$ and $u \in L^2(0,T;U)$, $T > 0$, we set

$$G(t,x,u) = \int_0^t f(t,s,x_s,u(s))ds.$$

Lemma 2.4. *Let $x \in L^2(-h,T;V)$ for any $T > 0$. Then $G(\cdot,x,u) \in L^2(0,T;H)$ and*

$$\|G(\cdot,x,u))\|_{L^2(0,T;H)} \le L_0T^{3/2} + T/\sqrt{2}(L_1\sqrt{T}\|x\|_{L^2(-h,T;V)} + L_2\|u\|_{L^2(0,T;U)}). \tag{2.13}$$

Moreover, if $x, \hat{x} \in L^2(-h,T;V)$, then

$$\|G(\cdot,x,u) - G(\cdot,\hat{x},\hat{u})\|_{L^2(0,T;H)} \le T/\sqrt{2}(L_1\sqrt{T}\|x-\hat{x}\|_{L^2(-h,T;V)} + L_2\|u-\hat{u}\|_{L^2(0,T;U)}). \tag{2.14}$$

Proof. From (F1), (F2), and (2.12) it is easily seen that

$$\begin{aligned}
&\|G(\cdot,x,u)\|_{L^2(0,T;H)} \le \|G(\cdot,0,0)\| + \|G(\cdot,x,u) - G(\cdot,0,0)\| \\
&\le L_0T^{3/2} + \left\{\int_0^T \left|\int_0^t (f(t,s,x_s,u(s)) - f(t,s,0,0))ds\right| dt\right\}^{1/2} \\
&\le L_0T^{3/2} + \left\{\int_0^T t\int_0^t (L_1\|x_s\|_\Pi + L_2\|u\|_{L^2(0,T;U)})^2 ds|\right\}^{1/2} \\
&\le L_0T^{3/2} + T/\sqrt{2}(L_1\|x.\|_{L^2(0,T;\Pi)} + L_2\|u\|_{L^2(0,T;U)}) \\
&\le L_0T^{3/2} + T/\sqrt{2}(L_1\sqrt{T}\|x\|_{L^2(-h,T;V)} + L_2\|u\|_{L^2(0,T;U)}).
\end{aligned}$$

The proof of (2.14) is similar. □

Theorem 2.1. *Under assumption* (F) *for the nonlinear mapping f, there exists a unique solution x of* (SE) *such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $(\phi^0,\phi^1) \in H \times \Pi$. Moreover, there exists a constant C_1 such that

$$\|x\|_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|\phi^0| + \|\phi^1\|_\Pi + \|u\|_{L^2(0,T;U)}). \tag{2.15}$$

Proof. We fix $T_0 > 0$ satisfying

$$C_2 L_1 T_0^2/\sqrt{2} < 1 \tag{2.16}$$

with the constant C_2 from Lemma 2.2. Let y be the solution of

$$(y_t)(0) = y(t) = S(t)\phi^0 + \int_0^t S(t-s)\left\{\int_0^s f(s,\tau,x_\tau,u(\tau))d\tau + Bu(s)\right\} ds.$$

We show that $x \mapsto y$ is strictly contractive from $L^2(0,T_0;V)$ to itself if condition (2.12) is satisfied. Let y_t and $\hat{y}_t$ belong to Π with the same initial condition in $[-h,0]$. Then, noting that $x_0(s) - \hat{x}_0(s) = 0$ for $s \in [-h,0]$, we obtain from assumption (F1), (2.11), (2.12), and

$$\begin{aligned}(y_t)(0) &- (\hat{y}_t)(0)\\ &= \int_0^t S(t-s)\left\{\int_0^s \Big(f(s,\tau,x_\tau,u(\tau)) - f(s,\tau,\hat{x}_\tau,u(\tau))\Big)d\tau\right\} ds\end{aligned}$$

that

$$\begin{aligned}\|y-\hat{y}\|_{L^2(0,T_0;V)} &\le C_2\sqrt{T_0}\|G(\cdot,x,u) - G(\cdot,\hat{x},u)\|_{L^2(0,T_0;H)}\\ &\le C_2 L_1 T_0^2/\sqrt{2}\|x(\cdot) - \hat{x}(\cdot)\|_{L^2(0,T_0;V)}.\end{aligned}$$

Therefore, by virtue of condition (2.16), the contraction mapping principle yields that the solution of (SE) exists and is unique in $[0,T_0]$. Let x be a solution of (SE) and $x_0 \in H$. Then there exists a constant C_1 such that

$$\|S(t)\phi_0\|_{L^2(0,T_0;V)} \le C_1|\phi^0| \tag{2.17}$$

in view of Proposition 2.1. Let

$$x_1(t) = \int_0^t S(t-s)\left\{\int_0^s f(s,\tau,x_\tau,u(\tau))d\tau + Bu(s)\right\} ds.$$

Then (2.11) implies

$$\begin{aligned}&\|x_1\|_{L^2(0,T_0;V)} \le C_2\sqrt{T_0}\|G(\cdot,x,u) + Bu\|_{L^2(0,T_0;H)}\\ &\le C_2\sqrt{T}_0(L_1 T_0^{3/2}/\sqrt{2}\|x\|_{L^2(0,T_0;V)} + \|G(\cdot,0,u) + Bu\|_{L^2(0,T_0;H)}).\end{aligned} \tag{2.18}$$

Thus, combining (2.17) with (2.18), we have

$$\begin{aligned}\|x\|_{L^2(0,T_0;V))} \le (1 - C_2 L_1 T_0^2/\sqrt{2})^{-1}(C_1|\phi^0|\\ + C_2\sqrt{T_0}\|G(\cdot,0,u) + Bu\|_{L^2(0,T_0;H)}).\end{aligned}$$

Now we obtain from

$$|x(T_0)| = |S(T_0)\phi^0| + \int_0^{T_0} S(T_0 - s)\left\{\int_0^s f(s,\tau,x_\tau,u(\tau))d\tau + Bu(s)\right\}ds$$
$$\leq M|x_0| + ML_1T_0\|x\|_{L^2(0,T_0;V))} + M\sqrt{T_0}\|G(\cdot,0,u) + Bu\|_{L^2(0,T_0;H)},$$

since condition (2.16) is independent of initial values, that the solution of (SE) can be extended to the interval $[-h, nT_0]$ for every natural number n. An estimate analogous to (2.15) holds for the solution in $[-h, nT_0]$, and hence for the initial value x_{nT_0} in the interval $[nT_0, (n+1)T_0]$. □

3. Approximate controllability of semilinear systems

Let $x(T; f, u)$ be a state value of system (SE) at the time T corresponding to the nonlinear term f and the control u. We define the reachable sets for system (SE) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0, T; U)\}.$$

Definition 3.1. System (SE) is said to be approximately controllable in the time interval $[0, T]$ if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0, T; U)$ such that the solution $x(T; f, u)$ of (SE) satisfies $|x(T; f, u) - x_1| < \epsilon$, i.e., if $\overline{R_T(f)} = H$, where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H, then system (SE) is called approximately controllable at time T.

Let $u \in L^1(0, T; Y)$. Then it is well known that

$$\lim_{h\to 0} h^{-1} \int_0^h \|u(t+s) - u(t)\|_Y ds = 0 \tag{3.1}$$

for almost all points $t \in (0, T)$.

Definition 3.2. The point t, where (3.1) holds, is called the Lebesgue point of u.

First, we consider the approximate controllability of system (SE) in the case where the controller B is the identity operator on H under the Lipschitz conditions (F1) and (F2) on the nonlinear operator f. Therefore, obviously, $H = U$. Consider the linear system

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) + u(t), \\ y(0) = \phi^0, \quad y(s) = \phi^1(s), \quad -h \leq s < 0, \end{cases} \tag{3.2}$$

and the following semilinear control system:

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + \int_0^t f(t,s,x_s,v(s))ds + v(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0. \end{cases} \tag{3.3}$$

Theorem 3.1. *Under assumptions* (F1) *and* (F2), *we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Therefore, if linear system (3.2) *with* $f = 0$ *is approximately controllable, then so is semilinear system* (3.3).

Proof. Let $y(t)$ be a solution of (3.2) corresponding to a control u. First, we show that there exists $v \in L^2(0,T;H)$ such that

$$\begin{cases} v(t) = u(t) - \int_0^t f(t,s,x_s,v(s))ds, \quad 0 < t \le T, \\ v(0) = u(0). \end{cases}$$

Let T_0 be a Lebesgue point of u and v, so that

$$L_2 T_0/\sqrt{2} < 1. \tag{3.4}$$

For given $u \in L^2(0,T;H)$, we define a mapping

$$W : L^2(0,T;H) \to L^2(0,T;H)$$

as follows:

$$(Wv)(t) = u(t) - \int_0^t f(t,s,y_s,v(s))ds, \quad 0 < t \le T_0.$$

It immediately follows from the definition of W that

$$\begin{aligned} &\|Wv_1 - Wv_2\|^2_{L^2(0,T_0;H)} \\ &\quad = \int_0^{T_0} \left| \int_0^t \{f(t,s,y_s,v_1(s)) - f(t,s,y_s,v_2(s))\}ds \right|^2 dt \\ &\quad \le \int_0^{T_0} \left(\int_0^t \Big|\{f(t,s,y_s,v_1(s)) - f(t,s,y_s,v_2(s))\}\Big| ds \right)^2 dt \\ &\le L_2^2 \int_0^{T_0} t \int_0^t |(v_1(s) - v_2(s))|^2 ds dt \le L_2^2 \frac{T_0^2}{2} \int_0^{T_0} |(v_1(s) - v_2(s))|^2 ds, \end{aligned}$$

whence

$$\|Wv_1 - Wv_2\|_{L^2(0,T_0;H)} \le L_2 T_0/\sqrt{2}\|v_1 - v_2\|_{L^2(0,T_0;H)}.$$

By the well-known contraction mapping principle, W has a unique fixed point v in $L^2(0,T_0;H)$ if condition (3.4) is satisfied.

Let

$$v(t) = u(s) - \int_0^t f(t,s,y_s,v(s))ds.$$

Then from (F1), (F2), and Theorem 2.1, it follows

$$\begin{aligned}
\|v\|_{L^2(0,T_0;H)} &\le \|G(\cdot,x,v))\|_{L^2(0,T;H)} + \|u\|_{L^2(0,T_0;H)} \\
&\le T_0/\sqrt{2}(L_1\sqrt{T_0}\|x\|_{L^2(-h,T;V)} + L_2\|v\|_{L^2(0,T;U)}) + \|u\|_{L^2(0,T_0;H)} \\
&\le T_0/\sqrt{2}\{L_1\sqrt{T_0}(C_1(|\phi^0| + \|\phi^1\|_\Pi + \|u\|_{L^2(0,T;U)})) + L_2\|v\|_{L^2(0,T;U)}\} \\
&\quad + \|u\|_{L^2(0,T_0;H)}. \qquad (3.5)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|v\|_{L^2(0,T_0;V))} \le (1 - T_0L_2/\sqrt{2})^{-1}\{T_0/\sqrt{2}\{L_1C_1\sqrt{T_0}(|\phi^0| \\
+ \|\phi^1\|_\Pi + \|u\|_{L^2(0,T;U)}) + \|u\|_{L^2(0,T_0;H)}\}
\end{aligned}$$

and

$$\begin{aligned}
|v(T_0)| &= \left| u(T_0) - \int_0^{T_0} f(T_0,s,y(s),v(s))ds \right| \\
&\le |u(T_0)| + \sqrt{T_0}(L_1\sqrt{T_0}\|y\|_{L^2(-h,T_0;V))} + L_2\|v\|_{L^2(0,T_0;V)}) \\
&\quad + \sqrt{T_0}\|f(\cdot,0,0)\|_{L^2(0,T_0;H)}. \qquad (3.6)
\end{aligned}$$

If $2T_0$ is a Lebesgue point of u and v, then we can solve the equation in $[T_0, 2T_0]$ with the initial value $v(T_0)$ and obtain estimates analogous to (3.5) and (3.6). If not, we can choose $T_1 \in [T_0, 2T_0]$ to be a Lebesgue point of u and v. Since condition (3.4) is independent of initial values, the solution can be extended to the interval $[T_1, T_1 + T_0]$ and, therefore, we have shown that there exists $v \in L^2(0,T;H)$ such that $v(t) = u(t) - \int_0^t f(s,y_s,v(s))ds$. Let y be a solution of (3.2) corresponding to a control u. Consider the following semilinear system:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \displaystyle\int_0^t f(t,s,x_s,v(s))ds + v(t) \\ \qquad = Ax(t) + \displaystyle\int_0^t f(t,s,x_s,v(s))ds + u(t) - \int_0^t f(t,s,y_s,v(s))ds, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0. \end{cases} \tag{3.7}$$

The solutions of (3.2) and (3.7), respectively, can be written as

$$y_t(0) = y(t) = S(t)\phi^0 + \int_0^t S(t-s)u(s)ds$$

and

$$x_t(0) = x(t) = S(t)\phi^0 + \int_0^t S(t-s)u(s)ds + \int_0^t S(t-s)\int_0^s \{f(s,\tau,x_\tau,v(\tau)) - f(s,\tau,y_\tau,v(\tau))\}d\tau ds.$$

Then from Theorem 2.1, it is easily seen that $x.(0) \in C([0,T];H)$, i.e., $x_s(0) \to x_t(0)$ as $s \to t$ in H. Let $\epsilon > 0$ be given. For $t \ge \epsilon$, we set

$$x_t^\epsilon(0) = S(t)\phi^0 + \int_0^{t-\epsilon} S(t-s)u(s)ds + \int_0^{t-\epsilon} S(t-s)\int_0^s \{f(s,\tau,x_\tau^\epsilon,v(\tau)) - f(s,\tau,y_\tau,v(\tau))\}d\tau ds.$$

Then we have

$$x_t(0) - x_t^\epsilon(0) = x(t) - x^\epsilon(t) = \int_{t-\epsilon}^t S(t-s)u(s)ds$$
$$- \int_{t-\epsilon}^t S(t-s)\int_0^s f(s,\tau,y_\tau,v(\tau))d\tau ds + \int_{t-\epsilon}^t S(t-s)\int_0^s f(s,\tau,x_\tau,v(\tau)d\tau ds$$
$$+ \int_0^{t-\epsilon} S(t-s)\int_0^s \{f(s,\tau,x_\tau,v(\tau)) - f(s,\tau,x_\tau^\epsilon,v(\tau))\}d\tau ds.$$

Therefore, as we have seen in the proof of Theorem 2.1, for some constant $T_1 > 0$ satisfying $C_2 L_1 T_1^2/\sqrt{2} < 1$, we easily obtain that

$$\begin{aligned}\|x - x^\epsilon\|_{L^2(0,T_1;V)} &\le C_2\sqrt{\epsilon}\|u\|_{L^2(0,T_1;H)} \\ &\quad + C_2 L_1 \epsilon^2/\sqrt{2}\{\|y\|_{L^2(0,T_1;V)} + \|x\|_{L^2(0,T_1;V)}\} \\ &\quad + C_2 L_1 T_1^2/\sqrt{2}\|x - x^\epsilon\|_{L^2(0,T_1;V)}.\end{aligned}$$

By the step-by-step method, we know that $x^\epsilon \to x$ as $\epsilon \to 0$ in $L^2(0,T;V)$ $(T > 0)$, i.e., $x_t^\epsilon \to x_t$ as $\epsilon \to 0$ in Π for $\epsilon < t < T$. From (2.6), it follows that

$$\begin{aligned}&\|x_t^\epsilon - y_t\|_\Pi^2 \\ &= \int_{-h}^{0} \left| \int_0^{t+s-\epsilon} S(t+s-\tau)\{G(\tau, x^\epsilon, v(\tau)) - G(\tau, y, v(\tau))\}d\tau \right|^2 ds \\ &\le M^2 \int_{-h}^{0} \left(\int_0^{t+s-\epsilon} (t+s-\tau)^{-1/2}\Big\{G(\tau, x^\epsilon, v(\tau)) - G(\tau, y, v(\tau))\Big\}d\tau \right)^2 ds \\ &\le M^2 \int_{-h}^{0} \log\left(\frac{t}{\epsilon}\right) \int_0^{t+s-\epsilon} \Big\{G(\tau, x^\epsilon, v(\tau)) - G(\tau, y, v(\tau))\Big\}^2 d\tau ds \\ &\le hM^2 \log\left(\frac{t}{\epsilon}\right) \int_0^t \left| \int_0^\tau \Big(f\big(\tau, \sigma, x_\sigma^\epsilon, v(\sigma)\big) - f\big(\tau, \sigma, y_\sigma, v(\sigma)\big)\Big) d\sigma \right|^2 d\tau \\ &\le hM^2 \log\left(\frac{t}{\epsilon}\right) \int_0^t \tau \int_0^\tau L_1^2 \|x_\sigma^\epsilon - y_\sigma\|_\Pi^2 d\sigma d\tau \\ &\le h(tML_1)^2/2 \log\left(\frac{t}{\epsilon}\right) \int_0^t \|x_\sigma^\epsilon - y_\sigma\|_\Pi^2 d\sigma.\end{aligned}$$

By using the Gronwall inequality, independently of ϵ, we obtain $x_t^\epsilon = y_t$ in Π for almost all $\epsilon \le t \le T$ and $x_t^\epsilon(0) = y_t(0)$ in H. Therefore, noting that $x_\cdot(0)$, $y_\cdot(0) \in C([0,T;H])$, we see that every solution of the linear system with control u is also a solution of the semilinear system with control w, i.e., we have that $R_T(0) \subset \overline{R_T(f)}$. $\square$

From now on, we consider the initial-value problem for the semilinear parabolic equation (SE). Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H.

Theorem 3.2. *Let a constant* $\beta > 0$ *exist such that* $\|Bu\| \geq \beta\|u\|$ *for all* $u \in L^2(0,T;U)$ *and assumptions* (F1), (F2), *and* $R(f) \subset R(B)$ *be satisfied. Then the semilinear system* (SE) *is approximate controllable.*

Proof. Consider the linear system

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) + u(t), \\ y(0) = \phi^0, \quad y(s) = \phi^1(s), \quad -h \leq s < 0, \end{cases} \tag{3.8}$$

and the following semilinear control system:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \displaystyle\int_0^t f(t,s,x_s,u(s))ds + Bv(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \leq s < 0. \end{cases} \tag{3.9}$$

Let y be a solution of (3.8) corresponding to a control u. We set

$$v(t) = u(t) - B^{-1}\int_0^t f(t,s,y_s,v(s))ds.$$

Then, as we have seen in Theorem 3.1, $v \in L^2(0,T;U)$. Consider the following semilinear system:

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + \int_0^t f(t,s,x_s,u(s))ds + Bv(t) \\ &= Ax(t) + \int_0^t f(t,s,x_s,u(s))ds + Bu(t) - \int_0^t f(t,s,y_s,v(s))ds, \\ x_0(0) &= \phi^0, \quad x_0(s) = \phi^1(s), \quad -h \leq s < 0. \end{aligned}$$

If we define x_t^ϵ, y_t as in the proof of Theorem 3.1, then we obtain

$$x_t^\epsilon(0) - y_t(0) = \int_0^{t-\epsilon} S(t-s)\int_0^s \{f(s,\tau,x_\tau^\epsilon,v(\tau)) - f(s,\tau,y_\tau,v(\tau))\}d\tau ds.$$

Therefore, using the Gronwall inequality in Theorem 3.1, we obtain that $R_T(0) \subset \overline{R_T(f)}$. □

Example 3.1. We consider the semilinear heat equation dealt with by Naito [4] and Zhou [10]. Let

$$H = L^2(0,\pi), \quad V = H_0^1(0,\pi), \quad V^* = H^{-1}(0,\pi),$$
$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx, \quad A = d^2/dx^2,$$
$$D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation:

$$\begin{cases} \dfrac{d}{dt}y(t,x) = Ay(t,x) + \displaystyle\int_0^t f(t,s,y(s-h,x),u(s))ds + Bu(t), \\ y(t,0) = y(t,\pi) = 0, \quad t > 0, \\ y(0,x) = \phi^0(x), \quad y(x,s) = \phi^1(x,s) \quad -h \le s < 0. \end{cases} \tag{SE1}$$

The eigenvalues and eigenfunctions of A are $\lambda_n = -n^2$ and $\phi_n(x) = \sin nx$, respectively. Let

$$U = \left\{ \sum_{n=2}^\infty u_n\phi_n : \sum_{n=2}^\infty u_n^2 < \infty \right\},$$
$$Bu = 2u_2\phi_1 + \sum_{n=2}^\infty u_n\phi_n \quad \text{for} \quad u = \sum_{n=2}^\infty u_n \in U, \quad T > 0.$$

It is easily seen that the operator B is one-to-one and $\mathrm{R}(B)$ is closed. It follows that the operator B satisfies hypothesis (B1). For example, consider the nonlinear term

$$f(t,s,y,u) = \alpha(t,s)(\|D_x y\|\phi_1(y) + \|u\|\phi_2(y)), \quad \alpha(t,s) \in C([0,T]^2).$$

Then f is not uniformly bounded and $\mathrm{R}(f) \subset \mathrm{R}(B)$. For this function f, Theorem 3.2 implies that system (SE1) is approximately controllable. In the case where $B = I$, we obtain the approximate controllability of (SE1) without restrictions such as the uniform boundedness and inequality constraints for the Lipschitz constant of f or compactness of $S(t)$.

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