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LOCAL REGULARITY OF OPTIMAL TRAJECTORIES FOR CONTROL PROBLEMS WITH GENERAL BOUNDARY CONDITIONS

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ABSTRACT. Let f and g be two smooth vector fields on a manifold M. Given a submanifold S of M, we study the local structure of time-optimal trajectories for the single-input control-affine system $\dot{q} = f(q) + ug(q)$ with the initial condition $q(0) \in S$. When the codimension s of S in M is small ($s \leq 4$) and the system has a small codimension singularity at a point $q_0 \in S$, we prove that all time-optimal trajectories contained in a sufficiently small neighborhood of q_0 are finite concatenations of bang and singular arcs. The proof is based on an extension of the index theory to the case of general boundary conditions.

1. INTRODUCTION

Consider the single-input control-affine system

$$\dot{q} = f(q) + u\,g(q), \quad q \in M, \quad u \in [-1, 1], \tag{1.1}$$

on a smooth manifold M. Given a smooth submanifold S of M, we study the regularity of trajectories of (1.1) connecting in minimum time S with points of M close to S. The regularity can be expressed in terms of upper bounds on the number of bang and singular arcs of time-optimal trajectories, and on restrictions on the possible concatenations of such arcs.

We call the *point-to-point problem* the case where S reduces to one point, while we refer to the general case as to the *manifold-to-point problem*. The analysis of generic regularity properties of time-minimizing point-to-point trajectories has initiated a rich literature (see, e.g., [16,17] for the case where M is two-dimensional and [4,14,18] for the three-dimensional situation). For local time-optimal syntheses, we refer to [9] and [15] for dimensions two and three, respectively.

Less results are available, up to our knowledge, for the manifold-to-point problem. The main contributions are given by [7] and [11]. These two

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papers furnish a classification of time-optimal syntheses for the manifold-to-point problem, when S has codimension one and M has dimension two or three, for generic analytic systems. In [11], the results are extended in the following way: no restriction is imposed on the dimension of M and the synthesis is obtained near all points of S, where the system has a singularity of codimension smaller than three. Moreover, the cited articles, together with [8], study in detail the situation where g is everywhere tangent to S, since it is motivated by applications to control problems for batch reactors.

The results obtained in the present paper hold for source manifolds S of codimension less than or equal to four, without restrictions on the dimension of M. The systems are assumed to be smooth but not necessarily analytic. The case of codimension four is new also when S reduces to a point, i.e., for the point-to-point problem in dimension four. In general, if s is the codimension of S, we give a local description of the time-optimal trajectories near all points of S, where the singularity of the system has codimension smaller than $5 - s$.

The index theory for bang-bang controls, developed by Agrachev and Gamkrelidze [4], can be reformulated in order to be applied to the manifold-to-point problem (Theorem 2). The upper bounds on the number of arcs which are obtained in this way turn out to be the same as the ones which hold for the point-to-point problem in an s-dimensional manifold.

Sections 2–5 contain the formulation of the problem and obtaining various properties of optimal trajectories. In particular, the notion of extremal trajectory is recalled, as well as the transversality condition associated with the general boundary condition $q(0) \in S$. In Sec. 6, we study the cases of lower codimension, for which the Pontryagin maximum principle is sufficient to characterize time-optimal trajectories. In the remaining cases, index theory is applied to bound the number of arcs of bang-bang trajectories (Proposition 1). The general case of trajectories which contain also singular arcs is treated by arguments which can be developed independently of dimension considerations (see Sec. 8). Sections 9 and 10 constitute the core of the computational machinery. Various asymptotic expressions of quadratic forms corresponding to second-order variations of the switching times are obtained.

2. Formulation of the problem

Let M be a finite-dimensional manifold and S be an embedded submanifold of M of codimension s. Denote by $n + s$ the dimension of M. All manifolds and vector fields considered in this paper are assumed to be of class C^∞.

Consider the control problem

$$\begin{cases} \dot{q} = f(q) + u\,g(q), & q \in M,\ u \in [-1,1], \\ q(0) \in S, \end{cases} \tag{2.1}$$

where f and g are two vector fields on M. We say that S is the *source* of trajectories of the system. Admissible controls are measurable functions $u : t \mapsto u(t) \in [-1,1]$. For each $T > 0$, the *attainable set from S at time T* is the set of all endpoints of trajectories of (2.1) corresponding to admissible controls evaluated at time T.

By analogy with what was done in [6] for the point-to-point problem, we introduce a notion of optimality, which includes time-minimality, appropriate for formulating general regularity results.

Definition 1. An admissible trajectory $q : [0,T] \to M$ of (2.1) is said to be *quasi-optimal* if for every neighborhood W of the pair (f,g), with respect to the C^1 product topology on the space of pairs of vector fields, there exists (f',g') in W such that $q(T)$ is not in the interior of the attainable set from S at time T for the system corresponding to (f',g').

In particular, all trajectories which minimize or maximize the time needed to join S to a point q of M are quasi-optimal.

Our main result is the following.

Theorem 1. *Let M be a finite-dimensional manifold and S be a submanifold of M. For a generic pair (f,g) of vector fields on M, there exists a stratified set A of codimension five in M such that, for every q_0 in $S \setminus A$, the following condition holds: there exist $T > 0$ and a neighborhood U of q_0 such that a quasi-optimal solution of (2.1) contained in U and of time-length less than T is the concatenation of at most seven between bang and singular arcs.*

For the definition of bang and singular arcs, see Sec. 3. Note that if the codimension of S in M is greater than four, then the theorem gives no information. On the other hand, if the dimension of M is less than or equal to four, then A is necessarily empty.

From now on, q_0 will denote a fixed point in S. By the Thom transversality theorem, the proof of Theorem 1 is obtained by furnishing a classification of the singularities of the Lie bracket configuration of (f,g) at q_0 up to order $4-s$, and by showing that, for all such configurations, the bound holds for quasi-optimal solutions of (2.1) contained in a neighborhood of q_0.

In order to classify concisely these singularities, we introduce the notation

$$X_{\pm} = [f \pm g, [f,g]], \quad X_{\pm\mp} = [f \pm g, X_{\mp}].$$

We also find it useful to define

$$V = \operatorname{span}\{g(q_0), [f,g](q_0)\} + T_{q_0}S. \tag{2.2}$$

We use the symbol $\pitchfork$ to denote the transversal intersection of vector subspaces of $T_{q_0}S$, and we say that a vector intersects transversally a subspace if the span that it generates does.

We label the singularities of the classification by means of the codimension of the source and the codimension of the singularity: an (s,d)-point is a codimension-d singularity on a codimension-s source.

- **(1,0)** $g(q_0) \pitchfork T_{q_0}S$;
- **(1,1)** $g(q_0) \in T_{q_0}S$ and $[f,g](q_0) \pitchfork T_{q_0}S$;
- **(1,2)** $g(q_0), [f,g](q_0) \in T_{q_0}S$ and $X_+(q_0), X_-(q_0) \pitchfork T_{q_0}S$;
- **(1,3)** $g(q_0), [f,g](q_0), X_+(q_0) \in T_{q_0}S$ and $X_-(q_0), X_{++}(q_0) \pitchfork T_{q_0}S$;
- **(2,0)** $\mathrm{span}\{g(q_0),\ [f,g](q_0)\} \pitchfork T_{q_0}S$;
- **(2,1)** $\mathrm{codim}\, V = 1$ and $\mathrm{span}\{g(q_0),\ X_+(q_0)\}$, $\mathrm{span}\{g(q_0),\ X_-(q_0)\} \pitchfork T_{q_0}S$;
- **(2,2a)** $\mathrm{codim}\, V = 1$, $X_+(q_0) \in V$, and the intersection of $\mathrm{span}\{g(q_0),\ X_-(q_0)\}$, $\mathrm{span}\{g(q_0),\ X_{++}(q_0)\}$ with $T_{q_0}S$ is transversal;
- **(2,2b)** $g(q_0) \in T_{q_0}S$ and $\mathrm{span}\{[f,g](q_0),\ X_+(q_0)\}$, $\mathrm{span}\{[f,g](q_0),\ X_-(q_0)\} \pitchfork T_{q_0}S$;
- **(3,0)** $\mathrm{span}\{g(q_0),\ [f,g](q_0),\ X_+(q_0)\}$, $\mathrm{span}\{g(q_0),\ [f,g](q_0),\ X_-(q_0)\} \pitchfork T_{q_0}S$;
- **(3,1)** $\mathrm{codim}\, V = 1$, $X_+(q_0) \in V$, and $X_-(q_0), X_{++}(q_0) \pitchfork V$;
- **(4,0)** $\mathrm{span}\{g(q_0),\ [f,g](q_0),\ X_+(q_0),\ X_-(q_0)\}$, $\mathrm{span}\{g(q_0),\ [f,g](q_0),\ X_+(q_0),\ X_{++}(q_0)\}$, and $\mathrm{span}\{g(q_0),\ [f,g](q_0),\ X_-(q_0),\ X_{--}(q_0)\}$ intersect $T_{q_0}S$ transversally.

The classification does not consider singularities which can be obtained from those listed above by performing a transposition between $+$ and $-$. This is justified, since the substitution $-g$ instead of g in (1.1) preserves the control system, reversing the formal roles of $+$ and $-$.

The general upper bound of seven arcs stated in Theorem 1 can be refined when we deal, one by one, with the different cases. Table 1 collects all local upper bounds on the number of arcs of quasi-optimal trajectories, as they will be obtained in the paper. Each row corresponds to one class of (s,d)-points. The second column contains the corresponding local bound for trajectories which have at least one singular arc of positive length, and accounts for the maximal non-bang-bang concatenations which are candidate to be quasi-optimal. In the third column, the bound is presented which applies to purely bang-bang trajectories.

As was said in the introduction, the cases $(1,0)$, $(1,1)$, and $(1,2)$ have already been studied, at least in the analytic case, in [7, 11], with different methods and in a different perspective. In fact, these works deal with the whole local synthesis of the problem and the upper bound comes as a preliminary step. We prefer to treat these cases here as well, since they give us the opportunity to proceed more gradually and, we hope, to clarify our reasoning.

(s,d)	non-bang-bang bound		bang-bang bound	general bound
$(1,0)$	0	/	1	1
$(1,1)$	0	/	2	2
$(1,2)$	3	BSB	3	3
$(1,3)$	4	$BSBB$, $BBSB$	3	4
$(2,0)$	0	/	2	2
$(2,1)$	3	BSB	3	3
$(2,2\text{a})$	4	$BSBB$, $BBSB$	3	4
$(2,2\text{b})$	3	BSB	3	3
$(3,0)$	3	BSB	3	3
$(3,1)$	4	$BSBB$, $BBSB$	4	4
$(4,0)$	4	$BSBB$, $BBSB$	7	7

TABLE 1

3. First-order optimality conditions

By the Pontryagin maximum principle [13], if $q(\cdot)$ is a quasi-optimal solution of problem (2.1), then there exist $c \in \mathbb{R}$ and an absolutely continuous covector trajectory $\lambda(\cdot)$ in T^*M such that

$$\lambda(t) \in T^*_{q(t)}M \setminus \{0\} \quad \text{for every } t \in [0,T], \tag{3.1}$$

$$\lambda(0) \perp T_{q(0)}S, \tag{3.2}$$

which satisfy for almost every t the equation

$$\dot{\lambda}(t) = \vec{h}_{u(t)}(\lambda(t)) \tag{3.3}$$

and the relation

$$h_{u(t)}(\lambda(t)) = \min_{v\in[-1,1]} h_v(\lambda(t)) \equiv c, \tag{3.4}$$

where the family of Hamiltonians h_v is defined by

$$h_v(\lambda(t)) = \langle \lambda(t), (f+vg)(q(t))\rangle \tag{3.5}$$

and $\vec{h}_v$ denotes the Hamiltonian vector field on T^*M associated with h_v. If a coordinate system is fixed and covectors are identified with row vectors, then (3.3) can be written as

$$\dot{\lambda}(t) = -\lambda(t)\big(Df(q(t)) + u(t)Dg(q(t))\big). \tag{3.6}$$

An *extremal pair* is a pair $(\lambda(\cdot), q(\cdot))$ of absolutely continuous trajectories, where $q(\cdot)$ is admissible for (1.1) and $\lambda(\cdot)$ satisfied (3.1), (3.3), and (3.4). A trajectory of (1.1) is *extremal* if it admits an extremal lift in T^*M.

Definition 2. If an extremal pair $(\lambda(\cdot), q(\cdot))$ satisfies (3.2), then we say that it is an *S-extremal pair* and $q(\cdot)$ is called an *S-extremal trajectory*.

With any extremal pair $(\lambda(\cdot), q(\cdot))$, it is classically associated the *switching function*

$$\varphi(t) = \langle \lambda(t), g(q(t)) \rangle , \tag{3.7}$$

which has the property, easily deducible from (3.4), that

$$u(t) = -\operatorname{sign}(\varphi(t))$$

for almost all t such that $\varphi(t) \neq 0$.

Given a vector field X on M, the function $t \mapsto \langle \lambda(t), X(q(t)) \rangle$ is almost everywhere differentiable and, for almost all t in $[0, T]$,

$$\frac{d}{dt} \langle \lambda(t), X(q(t)) \rangle = \langle \lambda(t), [f + u(t)g, X](q(t)) \rangle , \tag{3.8}$$

which can be deduced from (3.3). It follows that the switching function is C^1, its derivative is given by

$$\dot\varphi(t) = \langle \lambda(t), [f, g](q(t)) \rangle$$

and that, for almost all t, there exists

$$\ddot\varphi(t) = \langle \lambda(t), [f + u(t)g, [f, g]](q(t)) \rangle . \tag{3.9}$$

An *arc* is a piece of trajectory $q|_{[\tau_1,\tau_2]}$ such that the corresponding control function $u|_{[\tau_1,\tau_2]}$ is C^∞ up to modifications on a set of measure zero. We will always assume that an arc is maximal in the sense that for every subinterval J of the domain of definition of $q(\cdot)$ which contains $[\tau_1, \tau_2]$ strictly, no representative of $u|_J$ is C^∞. We will use the word "arc" also to refer to the interval $[\tau_1, \tau_2]$ itself. We say that two distinct arcs $[\tau_1, \tau_2]$ and $[t_1, t_2]$ are *concatenated* if $\tau_2 = t_1$ or $\tau_1 = t_2$. Particular arcs are the so-called *bang arcs*, for which $u(\cdot)$ is constantly equal (a.e.) to -1 or $+1$. A trajectory which is a finite concatenation of bang arcs is called a *bang-bang trajectory*. An arc which is not bang is said to be *singular*. A finite concatenation of arcs is described by juxtaposition of the letters S or B, each of them corresponding to a singular or, respectively, a bang arc. For example, a BSB trajectory is the concatenation of bang, singular, and bang arcs. The letter B is sometimes replaced by one of the symbols $+$ and $-$, depending on the sign of the corresponding control.

In the interior of bang and singular arcs, the right-hand side of (3.8) is absolutely continuous with respect to t, and the formula can be iterated infinitely many times. In particular, φ is C^∞ along any arc.

We complete this section by recalling some notions from chronological calculus (see [3]). Here and in the sequel, all the vector fields under consideration are assumed to be complete. This is justified by the fact that our attention is focused on local results only. The convenience of such assumption relies on the fact that, with any complete vector field X on M and with any time $t \in \mathbb{R}$, we can associate the flow of X at time t, which we denote by

$$e^{tX} : q \mapsto e^{tX}(q).$$

Similarly, with any complete nonautonomous vector field $t \mapsto X_t$, measurable with respect to t, and with any two time instants $t_1, t_2 \in \mathbb{R}$, we can associate the flow $\overrightarrow{\exp} \int_{t_1}^{t_2} X_t dt$, which maps $q \in M$ to the solution at time t_2 of the nonautonomous Cauchy problem

$$\dot{q}(t) = X_t(q(t)), \quad q(t_1) = q.$$

Note that

$$\overrightarrow{\exp} \int_{t_1}^{t_2} X_t dt = \left(\overrightarrow{\exp} \int_{t_2}^{t_1} X_t dt \right)^{-1}.$$

A vector field X has a natural interpretation in the space of linear operators on $C^\infty(M)$: given a smooth function a on M and a point $q \in M$, $(Xa)(q)$ is defined as the derivative of a in the direction $X(q)$ at the point q. From this point of view, the Lie bracket of two vector fields X_1 and X_2 is given by the commutator between them, i.e.,

$$((\operatorname{ad} X_1)X_2)a = [X_1, X_2]a = X_1(X_2 a) - X_2(X_1 a).$$

Any diffeomorphism P acts on the space of vector fields, associating with X the vector field $\operatorname{Ad} P(X)$, by the formula

$$\operatorname{Ad} P(X)(q) = (P^{-1})_*(X(P(q))).$$

One easily obtains

$$\operatorname{Ad} P([X_1, X_2]) = [\operatorname{Ad} P(X_1), \operatorname{Ad} P(X_2)]. \tag{3.10}$$

The formula

$$\frac{d}{dt} \left(\left(\operatorname{Ad} \overrightarrow{\exp} \int_{t_1}^{t} X_\tau d\tau \right) Y \right)(q) = \left(\left(\operatorname{Ad} \overrightarrow{\exp} \int_{t_1}^{t} X_\tau d\tau \right) [X_t, Y] \right)(q),$$

which holds at q fixed, for almost all t, justifies the notation

$$\overrightarrow{\exp} \int_{t_1}^{t} \operatorname{ad} X_\tau d\tau = \operatorname{Ad} \overrightarrow{\exp} \int_{t_1}^{t} X_\tau d\tau.$$

For a particular case of autonomous vector fields, we write

$$e^{t \operatorname{ad} X} = \operatorname{Ad} e^{tX}.$$

Let $(\lambda(\cdot), q(\cdot))$ be an extremal pair and $u(\cdot)$ be the corresponding control function. Fix a vector field X and two time instants t_1 and t_2 in the domain of definition of $q(\cdot)$. From Eq. (3.3), it follows that

$$\langle \lambda(t_1), X(q(t_1)) \rangle = \left\langle \lambda(t_2), \left(\overrightarrow{\exp} \int_{t_2}^{t_1} \operatorname{ad}(f + u(\tau)g) d\tau X \right)(q(t_2)) \right\rangle \tag{3.11}$$

(see [5, Proposition 11.3]).

4. Second-order optimality conditions

Our aim is to derive necessary conditions for a bang-bang trajectory to be quasi-optimal, in the spirit of results of [4].

Theorem 2. *Let $(\lambda(\cdot), q(\cdot))$ be an S-extremal pair for (2.1) and $u(\cdot)$ be the corresponding control function. Assume that $u(\cdot)$ is bang-bang on a subinterval $[\tau_0, \tau_{K+1}]$ of the domain of definition of $q(\cdot)$, with K switching times $(\tau_0 <)\tau_1 < \tau_2 < \cdots < \tau_K (< \tau_{K+1})$. Denote by ν the value of u on (τ_0, τ_1). Assume that $\lambda(\cdot)$ is the unique S-extremal lift of $q(\cdot)$, up to multiplication by a positive scalar. Fix $\overline{\tau}$ in the domain of definition of $q(\cdot)$ and set*

$$h_i = \left(\overrightarrow{\exp} \int_{\overline{\tau}}^{\tau_i} \mathrm{ad}(f + u(\tau)g) d\tau \right) (f + (-1)^i \nu g), \quad i = 0, \dots, K. \tag{4.1}$$

Fix a system of coordinates $(x_1, \dots, x_{n+s})$ in a neighborhood of $q(\overline{\tau})$ so that

$$\left(\overrightarrow{\exp} \int_0^{\overline{\tau}} (f + u(\tau)g) d\tau \right) (S) = \{(x_1, \dots, x_{n+s}) \mid x_{n+1} = 0, \dots, x_{n+s} = 0\}. \tag{4.2}$$

Associate with every vector field X its horizontal and vertical parts X^{h} and X^{v} satisfying at every point of the neighborhood the relations

$$X^{\mathrm{h}} \in \mathrm{span}\{\partial_{x_1}, \dots, \partial_{x_n}\}, \quad X^{\mathrm{v}} \in \mathrm{span}\{\partial_{x_{n+1}}, \dots, \partial_{x_{n+s}}\}, \tag{4.3}$$

respectively. For every $i = 0, \dots, K$ and every $j = 1, \dots, n$, let $H^{\mathrm{h}}_{i,j} \in \mathbb{R}$ be defined by

$$h_i^{\mathrm{h}}(q(\overline{\tau})) = \sum_{j=1}^n H^{\mathrm{h}}_{i,j} \partial_{x_j}.$$

Let Q be the quadratic form

$$\begin{aligned} Q(\alpha) = & \sum_{0 \le i < j \le K} \alpha_i \alpha_j \langle \lambda(\overline{\tau}), [h_i, h_j](q(\overline{\tau})) \rangle \\ & - \sum_{j=1}^n \left(\sum_{i=0}^K \alpha_i H^{\mathrm{h}}_{i,j} \right) \left(\sum_{i=0}^K \alpha_i \left\langle \lambda(\overline{\tau}), \partial_{x_j} h_i^{\mathrm{v}}(q(\overline{\tau})) \right\rangle \right) \end{aligned} \tag{4.4}$$

defined on the space

$$\begin{aligned} H = \Bigg\{ \alpha = (\alpha_0, \dots, \alpha_K) \in \mathbb{R}^{K+1} \ \Bigg| \ & \sum_{i=0}^K \alpha_i = 0, \\ \sum_{i=0}^K \alpha_i h_i(q(\overline{\tau})) \in & \left(\overrightarrow{\exp} \int_0^{\overline{\tau}} (f + u(\tau)g) d\tau \right)_* (T_{q_0} S) \Bigg\}. \end{aligned} \tag{4.5}$$

If $q(\cdot)$ is quasi-optimal, then Q is nonnegative definite.

Proof. Let $(\lambda(\cdot), q(\cdot))$ and the notation be as in the statement of the theorem. Note that the choice of h_i is made so that if $u_s^i(\cdot)$ denotes the control function obtained from $u(\cdot)$ by modifying the length of the $(i+1)$th bang arc, replacing $\tau_{i+1} - \tau_i$ by $\tau_{i+1} - \tau_i + s$, then

$$\frac{d}{ds}\bigg|_{s=0} \left(\overrightarrow{\exp} \int_{\overline{\tau}}^{\tau_{K+1}} (f + u(\tau)g) d\tau \right)^{-1} \left(\overrightarrow{\exp} \int_0^{\tau_{K+1}+s} (f + u_s^i(\tau)g) d\tau (q(0)) \right) = h_i(q(\overline{\tau})).$$

Choose n vector fields $Y_1, \ldots, Y_n$ which form a basis of TS in a neighborhood of $q(0)$ on S. We introduce a family of admissible trajectories for problem (2.1), obtained from $q(\cdot)$ by variation of both the starting point and control. For every pair (α, β) in

$$W = \left\{ (\alpha = (\alpha_0, \ldots, \alpha_K), \beta = (\beta_1, \ldots, \beta_n)) \in \mathbb{R}^{K+1} \times \mathbb{R}^n \;\middle|\; \sum_{i=0}^{K} \alpha_i = 0 \right\}$$

and for every $s \in \mathbb{R}$, let

$$G(\alpha, \beta, s) = \left(\overrightarrow{\exp} \int_{\overline{\tau}}^{\tau_{K+1}} (f + u(\tau)g) d\tau \right)^{-1}$$
$$\circ e^{(\tau_{K+1} - \tau_K + s\alpha_K)(f + (-1)^K \nu g)} \circ \cdots \circ e^{(\tau_1 - \tau_0 + s\alpha_0)(f + \nu g)}$$
$$\circ \overrightarrow{\exp} \int_0^{\tau_0} (f + u(\tau)g) d\tau \circ e^{s\beta_n Y_n} \circ \cdots \circ e^{s\beta_1 Y_1}(q(0)).$$

For small $|s|$,

$$s \mapsto \overrightarrow{\exp} \int_{\overline{\tau}}^{\tau_{K+1}} (f + u(\tau)g) d\tau (G(\alpha, \beta, s))$$

is a curve in M passing through $q(\tau_{K+1})$ at $s = 0$. Each point of this curve is the evaluation of an admissible trajectory of (2.1) at time τ_{K+1}. Let

$$P = \left(\overrightarrow{\exp} \int_0^{\overline{\tau}} (f + u(\tau)g) d\tau \right)^{-1}.$$

The tangent vector to $s \mapsto G(\alpha, \beta, s)$ at $s = 0$ is given by

$$V_1(\alpha, \beta) = \sum_{j=1}^{n} \beta_j \operatorname{Ad} P(Y_j)(q(\overline{\tau})) + \sum_{i=0}^{K} \alpha_i h_i(q(\overline{\tau})).$$

If $(\alpha, \beta) \in W$ is such that $V_1(\alpha, \beta) = 0$, then the tangent vector $\frac{d^2}{ds^2} G(\alpha, \beta, s)\Big|_{s=0}$ at $q(\overline{\tau})$ is intrinsically defined and, roughly speaking, points toward an attainable direction. It turns out that a necessary condition for $q(\cdot)$ to be quasi-optimal is that there exist no $(\alpha,\beta)\in W\cap\ker V_1$ such that

$$\left\langle \lambda(\overline{\tau}), \frac{d^2}{ds^2}G(\alpha,\beta,s)\bigg|_{s=0}\right\rangle < 0 \tag{4.6}$$

(see, e.g., the proof of Theorem 2.1 in [6].)

The second-order derivative of G with respect to s evaluated at $s=0$ is given on the kernel of V_1 by

$$\begin{aligned} V_2(\alpha,\beta) = &\sum_{1\le i<j\le n} \beta_i\beta_j \operatorname{Ad} P([Y_i,Y_j])(q(\overline{\tau})) \\ &+\sum_{j=1}^{n}\sum_{i=0}^{K}\alpha_i\beta_j[\operatorname{Ad} P(Y_j),h_i](q(\overline{\tau})) + \sum_{0\le i<j\le K}\alpha_i\alpha_j[h_i,h_j](q(\overline{\tau})). \end{aligned}$$

Since S is an integral leaf for the distribution generated by the Y_i, then $[Y_i,Y_j](q(0))\in T_{q(0)}S$ for every i,j; therefore, due to the transversality condition (3.2), the first term of the expression of V_2 is annihilated by $\lambda(\overline{\tau})$, which can be deduced from (3.11). Thus,

$$\begin{aligned} \langle\lambda(\overline{\tau}),V_2(\alpha,\beta)\rangle = &\sum_{j=1}^{n}\sum_{i=0}^{K}\alpha_i\beta_j\left\langle\lambda(\overline{\tau}),[\operatorname{Ad} P(Y_j),h_i](q(\overline{\tau}))\right\rangle \\ &+\sum_{0\le i<j\le K}\alpha_i\alpha_j\left\langle\lambda(\overline{\tau}),[h_i,h_j](q(\overline{\tau}))\right\rangle. \end{aligned} \tag{4.7}$$

Let

$$\overline{Q}(\alpha) = \sum_{0\le i<j\le K}\alpha_i\alpha_j\left\langle\lambda(\overline{\tau}),[h_i,h_j](q(\overline{\tau}))\right\rangle, \tag{4.8}$$

and

$$R = \sum_{j=1}^{n}\sum_{i=0}^{K}\alpha_i\beta_j\left\langle\lambda(\overline{\tau}),[\operatorname{Ad} P(Y_j),h_i](q(\overline{\tau}))\right\rangle.$$

Due to the freedom in the choice of Y_i, we can assume that $\operatorname{Ad} P(Y_i) = \partial_{x_i}$, $i=1,\dots,n$. If (α,β) is in the kernel of V_1, then

$$\sum_{j=1}^{n}\beta_j \operatorname{Ad} P(Y_j)(q(\overline{\tau})) = -\sum_{i=0}^{K}\alpha_i h_i(q(\overline{\tau})),$$

and, therefore,

$$\beta_j = -\sum_{i=0}^{K}\alpha_i H^{\mathrm{h}}_{i,j}. \tag{4.9}$$

Note that (4.9) defines an identification between $W\cap\ker V_1$ and the space H introduced in (4.5).

We have

$$
\begin{aligned}
R = R(\alpha) &= \sum_{j=1}^{n} \beta_j \left(\sum_{i=0}^{K} \alpha_i \Big\langle \lambda(\overline{\tau}), \partial_{x_j} h_i^{\mathrm{v}}(q(\overline{\tau})) \Big\rangle \right) \\
&= -\sum_{j=1}^{n} \left(\sum_{i=0}^{K} \alpha_i H_{i,j}^{\mathrm{h}} \right) \left(\sum_{i=0}^{K} \alpha_i \Big\langle \lambda(\overline{\tau}), \partial_{x_j} h_i^{\mathrm{v}}(q(\overline{\tau})) \Big\rangle \right).
\end{aligned}
$$

The theorem follows since the existence of a variation (α, β) as in (4.6) is equivalent to the positivity of the index of the quadratic form given in (4.7). □

Remark 1. The choice of $\overline{\tau}$ does not modify the nature of the second-order necessary condition stated in Theorem 2. Varying $\overline{\tau}$, we obtain a family of equivalent formulations, one of which will be chosen, in the applications, for its computational convenience.

Remark 2. Let $q : [0, T] \to M$ be an admissible trajectory of (1.1) such that $q(T) \in S$. The trajectory $t \mapsto q'(t) = q(T - t)$ obtained from $q(\cdot)$ by the reversion of time, is admissible for the control system

$$
\dot{q} = -f(q) - ug(q), \quad u \in [-1, 1]. \tag{4.10}
$$

There is a one-to-one correspondence between extremal lifts of $q(\cdot)$ and of $q'(\cdot)$. Indeed, $(\lambda(\cdot), q(\cdot))$ is an extremal pair for (1.1) if and only if $t \mapsto (-\lambda(T - t), q'(t))$ is an extremal pair for (4.10).

Assume that $q'(\cdot)$ is quasi-optimal for (4.10). In particular, there exists an extremal lift $\lambda(\cdot)$ of $q(\cdot)$ satisfying

$$
\lambda(T) \perp T_{q(T)} S. \tag{4.11}
$$

Let $q(\cdot)$ be bang-bang on $[\tau_0, \tau_{K+1}] \subset [0, T]$, and denote by $(\tau_0 <)\tau_1 < \tau_2 < \cdots < \tau_K (< \tau_{K+1})$ its K switching times. In addition, assume that $q(\cdot)$ has a unique extremal lift $\lambda(\cdot)$ which satisfies (4.11), up to multiplication by a positive scalar. Fix $\overline{\tau} \in [0, T]$ and define h_i as in (4.1), $i = 0, \ldots, K$. If h'_i is the vector field corresponding to the switching time $T - \tau_i$ for system (4.10), then $h'_i = -h_i$. Define Q and H as in (4.4) and (4.5), where the horizontal-vertical splitting is given by a system of coordinates near $q(\overline{\tau})$ which rectifies $\overrightarrow{\exp} \int_T^{\overline{\tau}} (f + u(\tau)g) d\tau (S)$. Since $q'(\cdot)$ is quasi-optimal, it follows from the above considerations that Q is a nonnegative-definite quadratic form on H.

Theorem 3 (generalized Legendre condition)**.** *Let $(\lambda(\cdot), q(\cdot))$ be an S-extremal pair of (2.1). Assume that $\lambda(\cdot)$ is uniquely defined (up to multiplication by a positive scalar) and let I be a singular arc contained in the domain of definition of $q(\cdot)$ such that $\varphi|_I \equiv 0$. Then*

$$
\Big\langle \lambda(t), [g, [f, g]](q(t)) \Big\rangle \leq 0
$$

for every $t \in I$.

A proof of Theorem 3 for extremal trajectories can be found in [5, Chap. 20]. (The first formulation and mathematical proof of the generalized Legendre condition go back, anyhow, to [2,10].) The proof fits in the case of S-extremal trajectories, where $\lambda(\cdot)$ may not be unique as extremal lift, but it is when also (3.2) is taken into account.

We point out that the sign condition in Theorem 3 is formulated in the opposite way than in [5]. This is due to the fact that, in the present statement of the Pontryagin maximum principle, condition (3.4) is given in terms of minimization of the Hamiltonian, whereas in [5] it has a more standard maximization form.

5. Preliminary results

Lemma 1. *Let U be an open, relatively compact subset of M. Fix an Euclidean structure on the cotangent bundle $T^*\overline{U}$, i.e., associate with each $q \in \overline{U}$ an Euclidean structure on T_q^*M, smooth with respect to q. Let X be a vector field on M. Then, for every $T > 0$ there exists a constant L such that, for every interval I of length smaller than T, every extremal trajectory $q : I \to U$, and every corresponding extremal lift $\lambda(\cdot)$, normalized so that $|\lambda(t)| = 1$ at some $t \in I$, the function $t \mapsto \langle \lambda(t), X(q(t))\rangle$ is L-Lipschitz continuous.*

Corollary 1. *Let U be an open, relatively compact subset of M. Consider a family $Y_1, \dots, Y_{n+s}$ of vector fields on M, linearly independent at every point of $\overline{U}$. Let, for every $q \in \overline{U}$ and $\lambda \in T_q^*M$, the norm $|\lambda|$ be given by*

$$|\lambda| = \max\Big\{\big|\langle \lambda, Y_i(q)\rangle\big| \mid i = 1, \dots, n+s\Big\}.$$

Let X be another vector field on M linearly independent of $Y_1, \dots, Y_{n+s-1}$ at every point of $\overline{U}$. Then there exist $\varepsilon_0 \in (0,1)$ and two nonincreasing functions $T, \delta : [0, \varepsilon_0] \to (0, +\infty)$ such that for every $\varepsilon \in [0, \varepsilon_0]$ and every extremal pair $(\lambda(\cdot), q(\cdot))$ defined on a domain I of length smaller than $T(\varepsilon)$, normalized so that $|\lambda(\overline{\tau})| = 1$ at some $\overline{\tau} \in I$, if each of the functions $t \mapsto \big|\langle \lambda(t), Y_i(q(t))\rangle\big|$, $i = 1, \dots, n+s-1$, attains at least one value less than or equal to ε in I, then $\big|\langle \lambda(t), X(q(t))\rangle\big| \geq \delta(\varepsilon)$ for every $t \in I$.

Lemma 1 and Corollary 1 are in the spirit of Lemma 3.3 in [6], to which we refer for the details of the proofs, which follow from a straightforward application of Gronwall inequality to the adjoint equation (3.6).

From now on, fix n vector fields $Y_1, \dots, Y_n$ on M and a relatively compact neighborhood U of q_0 such that, for every $q \in S \cap U$,

$$\mathrm{span}\{Y_1(q), \dots, Y_n(q)\} = T_qS. \tag{5.1}$$

In order to shorten the formulation of local properties which hold near the fixed point $q_0 \in M$, we find it useful to introduce the following agreement: we say that all *short* trajectories of a certain class (e.g., S-extremal trajectories) have a given property $(\mathcal{P})$ if there exist $T > 0$ and a neighborhood U of q_0 such that all trajectories of this class, which are contained in U and have time-length smaller than T, satisfy $(\mathcal{P})$.

6. Cases of small codimension: $s + d \leq 2$

Assume that q_0 is a $(1,0)$-point. By (3.2), we can apply Corollary 1, with $\varepsilon = 0$ and $Y_{n+1} = X = g$. We deduce that the switching function corresponding to a short S-extremal pair has constant sign. Therefore, a short S-extremal trajectory is made of a single bang arc. In particular, it does not contain singular arcs.

The same reasoning as above, applied to the case $(1,1)$, with the choice $Y_{n+1} = X = [f,g]$, implies that for every short S-extremal pair, the derivative of the corresponding switching function does not change sign. In the case $(2,0)$, we fix $Y_{n+1} = g$ and $Y_{n+2} = X = [f,g]$, and the same conclusion holds for every short S-extremal pair such that the corresponding switching function has at least one zero. In both situations $(1,1)$ and $(2,0)$, along any short S-extremal trajectory which is not made of a single bang arc, the switching function is strictly monotone. Therefore, a short S-extremal trajectory does not contain singular arcs and is the concatenation of at most two bang arcs.

7. Cases of higher codimension

If $s \neq 4$ and $s + d = 3, 4$, then V defined as in (2.2) has codimension one in $T_{q_0}M$ and $X_-(q_0) \pitchfork V$. We complete $\{Y_1, \ldots, Y_n\}$ to a full-rank distribution on U. Choose $Y_{n+1}, \ldots, Y_{n+s-1}$ between $g, [f,g]$ so that

$$\operatorname{span}\{Y_1(q_0), \ldots, Y_{n+s-1}(q_0)\} = V \tag{7.1}$$

and set $Y_{n+s} = X_-$. Let U be a relatively compact neighborhood of q_0, such that $\{Y_i\}_{i=1}^{n+s}$ is a moving basis in $\overline{U}$. Further smallness assumptions on U will be made when necessary. Associate with $\{Y_i\}_{i=1}^{n+s}$ the corresponding Euclidean structure $|\cdot|$ on $T^*\overline{U}$, as in the statement of Corollary 1. We can always assume that $T > 0$ is such that, for every admissible control function $u : [0,T] \to [-1,1]$, for every $t \in [0,T]$ and $q \in S \cap \overline{U}$,

$$\left(\overrightarrow{\exp} \int_0^t (f + u(\tau)g)d\tau\right)_* (T_q S) \pitchfork \operatorname{span}\left\{Y_{n+1}(q'), \ldots, Y_{n+s}(q')\right\}, \tag{7.2}$$

where

$$q' = \left(\overrightarrow{\exp} \int_0^t (f + u(\tau)g)d\tau\right)(q).$$

Once a moving basis and its corresponding Euclidean structure are fixed, we say that an extremal pair $(\lambda(\cdot), q(\cdot))$ is a *normalized extremal pair* if, at some time t, $|\lambda(t)| = 1$.

If the switching function has at least two distinct zeros (this is true, e.g., if it contains a singular arc or a compactly contained bang arc), then its derivative annihilates at least once. Let Ξ be the class of short normalized S-extremal pairs whose corresponding φ has at least two zeros. Note that Ξ depends implicitly on the choice of some T and some U. By (3.2), we can apply Corollary 1 (with $\varepsilon = 0$) and obtain that for any vector field X which is transversal to V at q_0, there exists a positive constant δ_X such that for every pair $(\lambda(\cdot), q(\cdot)) \in \Xi$, we have

$$\left|\langle \lambda(t), X(q(t))\rangle\right| \geq \delta_X$$

for all t. A possible choice of X is given by $X = X_-$. Whenever $s + d = 3$, as well as in the case $(2, 2\mathrm{b})$, the role of X can also be played by X_+. In all the other cases studied here, the alternative choice of X_{++} is allowed.

Therefore, we can assume that Ξ satisfies one of the following conditions:

(A) there exists $\delta > 0$ such that $\left|\langle \lambda(t), X_\pm(q(t))\rangle\right| \geq \delta$ for every $(\lambda(\cdot), q(\cdot)) \in \Xi$ and all t;
(B) there exists $\delta > 0$ such that $\left|\langle \lambda(t), X_-(q(t))\rangle\right|$, $\left|\langle \lambda(t), X_{++}(q(t))\rangle\right| \geq \delta$ for every $(\lambda(\cdot), q(\cdot)) \in \Xi$ and all t.

Remark 3. In the case where $\dim S = 0$, where the transversality condition (3.2) has no role in deducing (A) or (B), the assumption can be further strengthened; indeed, we can suppose that (A) or (B) holds for the class of all normalized extremal pairs with at least two zeros of φ. That is, we can neglect the requirement that the initial point of the trajectory lies in S. We will omit to mention at each step this kind of extension, which applies to all regularity properties of S-extremal or quasi-optimal trajectories which are going to be stated. We will come back to the consequences of this fact in Secs. 9 and 10, while formulating Propositions 2 and 3.

The case $(4, 0)$ is reduced to subproblems sharing one of the properties (A) or (B), as follows. Complete $\{Y_1, \dots, Y_n\}$ to a local moving basis by taking $Y_{n+1} = g$, $Y_{n+2} = [f, g]$, $Y_{n+3} = X_+$, and $Y_{n+4} = X_-$. Such basis defines, as above, an Euclidean structure in the cotangent bundle over a compact neighborhood of q_0. Keep on calling Ξ the class of short normalized S-extremal pairs whose switching function annihilates at least twice. By Lemma 1, we can assume that $\left|\langle \lambda(t), Y_i(q(t))\rangle\right|$ is smaller than any prescribed positive constant, for every $i = 1, \dots, n+2$ and every pair $(\lambda(\cdot), q(\cdot))$ in Ξ. Given $\eta \in (0, 1)$, we split Ξ in three subclasses: the class Ξ^1_η of pairs $(\lambda(\cdot), q(\cdot))$ for which

$$\left|\langle \lambda(0), X_+(q(0))\rangle\right| < \eta, \tag{7.3}$$

the class Ξ^2_η defined by

$$\left|\langle \lambda(0), X_-(q(0))\rangle\right| < \eta, \tag{7.4}$$

and the complement Ξ^3_η of $\Xi^1_\eta \cup \Xi^2_\eta$ in Ξ. If η is fixed and sufficiently small, then Lemma 1 and Corollary 1 imply that there exists a common $T = T(\eta)$ such that Ξ^1_η satisfies property (B), Ξ^3_η satisfies property (A) and Ξ^2_η satisfies an analogue of property (B), where the role of $+$ is played by $-$ and vice versa. Since the definition of $(4,0)$-point is symmetric with respect to $+$ and $-$, the regularity properties which can be stated for Ξ^1_η apply to Ξ^2_η as well. Therefore, we will essentially neglect Ξ^2_η, and restrict our attention to Ξ^1_η and Ξ^3_η.

In order to fix η, we impose an additional requirement whose importance will be clear in the next section. Since $[g,[f,g]]$ is transversal to $\text{span}\{Y_1,\dots,Y_{n+s-1}\}$, we can again apply Corollary 1 and assume that $|\langle\lambda(\cdot),[g,[f,g]](q(\cdot))\rangle|$ is separated from zero, uniformly in Ξ^1_η. Moreover, by the monotonicity of the function δ appearing in the statement of Corollary 1, we can choose sufficiently small η so that the sign of

$$\big\langle\lambda(\cdot),X_-(q(\cdot))\big\rangle = \big\langle\lambda(\cdot),X_+(q(\cdot))\big\rangle - 2\big\langle\lambda(\cdot),[g,[f,g]](q(\cdot))\big\rangle$$

is equal to the sign of $-\big\langle\lambda(\cdot),[g,[f,g]](q(\cdot))\big\rangle$ along the trajectory. That is, we choose η such that Ξ^1_η satisfies (B) and

(B′) $\text{sign}(\big\langle\lambda(t),X_-(q(t))\big\rangle) = -\text{sign}(\big\langle\lambda(t),[g,[f,g]](q(t))\big\rangle)$ for every $(\lambda(\cdot),q(\cdot)) \in \Xi^1_\eta$ for all t.

Note that for all (s,d)-points with $s \neq 4$, $s+d = 3,4$, if Ξ does not satisfy (A), then the same reasoning as above shows that Ξ can be assumed to satisfy both (B) and (B′).

The following proposition represents the crucial step toward the proof of Theorem 1.

Proposition 1. *There exists an integer-valued function $k(s,d)$ such that, if q_0 is an (s,d)-point with $s+d \leq 4$, then there exist a neighborhood U of q_0 and a time $T > 0$ for which a trajectory in U of* (2.1)*, of time-length smaller than T, which contains more than $k(s,d)$ concatenated bang arcs, is not quasi-optimal.*

Note that the proposition has already been proved for (s,d)-points such that $s+d \leq 2$. We have shown that a possible choice of k is given by $k(1,0) = 1$, $k(1,1) = k(2,0) = 2$. The values of the function k for which Proposition 1 will be proved are contained in the third column of Table 1.

The next section describes the structure of short quasi-optimal trajectories under the assumption that Proposition 1 holds true. It adapts to the manifold-to-point problem the arguments of [6, Sec. 5].

8. Regularity of non-bang-bang trajectories

Let q_0 be an (s,d)-point, $s+d \leq 4$. Fix a short S-extremal pair $(\lambda(\cdot),q(\cdot))$. We can assume that $(\lambda(\cdot),q(\cdot))$ belongs to a class (Ξ or Ξ^i_η) which satisfies

(A) or (B–B′). In particular, $\lambda(\cdot)$ never annihilates $X_-(q(\cdot))$. The same is true for $X_+(q(\cdot))$ when (A) holds, and for $X_{--}(q(\cdot))$ under the assumption (B).

Lemma 2. *Given a subinterval I of the domain of definition of $q(\cdot)$, if $q|_I$ does not contain bang arcs, then the switching function φ is identically equal to zero on I and $u|_I$ is smooth.*

Proof. Let $t \in I$ be such that $\varphi(t) \neq 0$ and denote by J the maximal neighborhood of t in I on which $u(\cdot)$ is smooth. It cannot be a bang arc by the hypothesis and, therefore, it must be singular. The set

$$\widetilde{J} = \operatorname{int}\{\tau \in J \mid \varphi(\tau) = 0\}$$

is a proper nonempty subset of J. Let $\overline{\tau}$ be in the boundary of $\widetilde{J}$ and in the interior of J. By the definition, $\overline{\tau}$ is both a density point for $\widetilde{J}$, where $\varphi^{(n)} \equiv 0$ for every $n \geq 0$, and for $\{\tau \in J \mid \varphi(\tau) \neq 0\}$, where $|u| = 1$. Since u and φ are smooth on J, it follows that $|u(\overline{\tau})| = 1$ and $\varphi^{(n)}(\overline{\tau}) = 0$ for every $n \geq 0$. As was already noted in Sec. 2, however, $\varphi^{(n)}$ can be computed iterating (3.8). Therefore, $\Big\langle \lambda(\overline{\tau}), \big(\operatorname{ad}^n(f + u(\overline{\tau})g)g\big)(q(\overline{\tau}))\Big\rangle = 0$. We reach a contradiction both with the assumption (A) and with the assumption (B). It follows that $\varphi|_I \equiv 0$.

Therefore, $\dot{\varphi}$ and the further derivatives of φ are also identically equal to zero on I. In particular,

$$\lambda(t) \perp g(q(t)), \quad [f,g](q(t)),$$

and, for almost all $t \in I$,

$$\big\langle \lambda(t), [f,[f,g]](q(t))\big\rangle + u(t)\big\langle \lambda(t), [g,[f,g]](q(t))\big\rangle = 0. \tag{8.1}$$

In both cases (A) and (B), $\big\langle \lambda(t), [g,[f,g]](q(t))\big\rangle \neq 0$ for every t for which (8.1) holds; otherwise, we would have $\big\langle \lambda(t), X_-(q(t))\big\rangle = 0$. If, however,

$$\big\langle \lambda(\bar{t}), [g,[f,g]](q(\bar{t}))\big\rangle = 0$$

for some $\bar{t} \in I$, then the function $t \mapsto \Big|\big\langle \lambda(t), [f,[f,g]](q(t))\big\rangle\Big|$ would be uniformly separated from zero near $\bar{t}$ and, consequently, $|u(t)| > 1$ for some t at which (8.1) holds. Thus, for every $t \in I$, $\big\langle \lambda(t), [g,[f,g]](q(t))\big\rangle \neq 0$ and

$$u(t) = -\frac{\big\langle \lambda(t), [f,[f,g]](q(t))\big\rangle}{\big\langle \lambda(t), [g,[f,g]](q(t))\big\rangle}. \tag{8.2}$$

Substituting (8.2) in (3.3), we obtain that $\lambda|_I$ is a solution of the smooth (autonomous) Hamiltonian system generated by the Hamiltonian

$$h(\lambda) = \langle \lambda, f\rangle - \frac{\big\langle \lambda, [f,[f,g]]\big\rangle}{\big\langle \lambda, [g,[f,g]]\big\rangle}\langle \lambda, g\rangle$$

and, in particular, is smooth. Thus, $q|_I$ is also smooth and, according to (8.2), the same is true for $u|_I$. □

Remark 4. The lemma implies, in particular, that the union of bang and singular arcs is dense in the domain of definition of $q(\cdot)$. A property of this kind turns out to be more general. Indeed, a straightforward generalization of the proof of Proposition 1 in [1] shows that, independently of the dimension of M and of S, if $I(g)$ is the ideal generated by g in the Lie algebra of vector fields generated by f and g, and if

$$\{X(q_0) \mid X \in I(g)\} + T_{q_0}S = T_{q_0}M,$$

then the control function corresponding to a short S-extremal trajectory is smooth on an open and dense subset of its domain of definition.

Lemma 2 implies that along any singular arc, $\varphi \equiv 0$. In particular, if t is such that $\varphi(t) = 0$ and $\dot{\varphi}(t) \neq 0$, then it is the switching time between two concatenated bang arcs. In both cases (A) and (B), the second derivative of φ has constant sign along all "−" arcs. Therefore, $\dot{\varphi}$ is different from zero at the boundary points of each compactly contained "−" arc. It follows that each compactly contained "−" arc is concatenated to two "+" arcs. If (A) holds, a symmetric reasoning for "+" arcs leads to the conclusion that if $q(\cdot)$ has at least one compactly contained bang arc, then it is purely bang-bang, because of Proposition 1. On the other hand, if $q(\cdot)$ does not have compactly contained bang arcs, then it follows from Lemma 2 that it is the concatenation of at most a bang, a singular, and a bang arc.

If (B) holds, the situation is slightly more complicated. Nevertheless, it is still true that a compactly contained "−" arc cannot be concatenated to a singular arc. The condition $\langle \lambda(\cdot), X_{++}(\cdot) \rangle \neq 0$ implies that the third derivative of the switching function along "+" arcs has constant sign. Therefore, $\ddot{\varphi}$ can change sign only once along a "+" arc, and always in the same direction (from negative to positive or the other way round). In particular, $q(\cdot)$ cannot have a bang, a singular, and a bang concatenated compactly contained arcs. In addition, a compactly contained bang arc is always concatenated to at least one more bang arc.

We want to prove that if $q(\cdot)$ has a singular arc, then it cannot have more than one compactly contained bang arc. On the contrary, assume that it has two. Without loss of generality, they are concatenated; indeed, if they are not, they identify a bounded nonempty interval I situated between them. If I contains no bang arc, then, by Lemma 2, we would have detected a BSB compactly contained concatenation, which cannot be the case. Vice versa, a bang arc compactly contained in I is necessarily concatenated to another bang arc. As it will be proved in Lemmas 3 and 4, the existence of two compactly contained concatenated bang arcs implies (at least for small T) the uniqueness of the corresponding covector trajectory. Thus, from the generalized Legendre condition and Theorem 3, it follows that

$$\langle \lambda(t), [g,[f,g]](q(t)) \rangle \leq 0 \tag{8.3}$$

along the singular arcs of the trajectory. Since we assumed that also holds (B′), we have that (8.3) is satisfied for every t and that $\ddot{\varphi}$ is positive along each "−" arc, as φ is. It follows that $q(\cdot)$ cannot have compactly contained "−" arcs. We reached a contradiction, and, therefore, we have proved that, if $q(\cdot)$ is not purely bang-bang, then it admits at most one compactly contained bang arc.

Finally, either a short quasi-optimal trajectory is bang-bang or it is of the type $-+S\pm$ or $\pm S+-$ (allowing some arc to have length zero). In the cases where property (A) holds, we further restricted the possible quasi-optimal concatenations to bang-bang and $\pm S\pm$ trajectories. Theorem 1 is proved, as soon as it is shown that Proposition 1 holds, with $\max\{k(s,d) \mid s+d \leq 4\} \leq 7$.

9. Proof of Proposition 1 for $s < 4$

9.1. **General facts.** Throughout this section, we assume that $s + d = 3, 4$ and $s \neq 4$. Fix $Y_1, \dots, Y_{n+s}$ and the corresponding Euclidean structure on the cotangent bundle over a sufficiently small neighborhood of q_0, as in Sec. 7.

In order to apply Theorems 2 and 3, we need to recover a corank one condition on S-extremal lifts. This is the scope of the following lemma, whose proof introduces many notations and concepts which will be widely used in the sequel.

Lemma 3. *A short S-extremal trajectory which has at least one compactly contained "+" arc and one compactly contained "−" arc admits a unique covector lift, up to multiplication by a positive scalar.*

Proof. Let $(\lambda(\cdot), q(\cdot))$ be an S-extremal pair. Assume that it has at least one compactly contained "+" arc and one compactly contained "−" arc. Denote the compactly contained "+" arc by $(t_0, t_0 + t_1)$. The equations $\varphi(t_0) = 0$ and $\varphi(t_0 + t_1) = 0$ can be written, according to (3.11), as

$$\langle \lambda(t_0), g(q(t_0)) \rangle = 0, \tag{9.1}$$

$$\left\langle \lambda(t_0), e^{t_1 \,\mathrm{ad}(f+g)} g(q(t_0)) \right\rangle = 0. \tag{9.2}$$

Let $\lambda(\cdot)$ be normalized so that

$$|\lambda(t_0)| = 1. \tag{9.3}$$

Define, for every $i = 1, \dots, n+s$,

$$a_i = \langle \lambda(t_0), Y_i(q(t_0)) \rangle . \tag{9.4}$$

By (9.3), we have that

$$\max\left\{|a_i|\,|\,i=1,\dots,n+s\right\}=1.$$

We set

$$\pi_0=\langle\lambda(t_0),[f,g](q(t_0))\rangle \tag{9.5}$$

$$\pi_\star=\langle\lambda(t_0),X_\star(q(t_0))\rangle\,,\quad \star=+,-,++\,. \tag{9.6}$$

Note that $a_{n+s}=\pi_-$, while $a_{n+1},\dots,a_{n+s-1}$ are taken among $\varphi(t_0)=0$ and π_0.

We want to describe the asymptotic behavior, as T goes to zero, of real-valued functions of the trajectory and of the chosen "+" arc. An example of function of this kind is given by t_1, which associates with the trajectory the length of the chosen "+" arc. We say that a function in this class is $O(1)$ if its absolute value can be bounded uniformly on the set of all pairs $q(\cdot)$–(t_0,t_0+t_1) such that $q(\cdot)$ lifts in Ξ. Clearly $t_1=O(1)$. We write that a function is $O(t_1^r)$ or $O(T)$ to express that its quotient with, respectively, t_1^r or the total length of the trajectory is $O(1)$.

From (3.2) we deduce that, for every $i=1,\dots,n$,

$$0=a_i+\left\langle\lambda(t_0),(\mathrm{Ad}\,P^{-1}-\mathrm{Id})Y_i(q(t_0))\right\rangle=a_i+\sum_{j=1}^{n+s}a_jO(T), \tag{9.7}$$

where

$$P=\overrightarrow{\exp}\int_0^{t_0}(f+u(\tau)g)d\tau \tag{9.8}$$

and Id denotes the identity operator on the space of vector fields on M. Similarly, from (9.2) we obtain

$$\pi_0=\sum_{j=1}^{n+s}a_jO(t_1). \tag{9.9}$$

Thus,

$$\max\left\{|a_i|\;|\;i=1,\dots,n+s-1\right\}\le O(T),$$

and, in particular, we can assume that $|a_{n+s}|=1$. Recall now that, since (A) or (B) holds, $\langle\lambda(t),X_-(q(t))\rangle$ does not change sign along the trajectory. On the compactly contained "−" arc of $q(\cdot)$, $\varphi(t)$ is nonnegative and $\ddot\varphi(t)=\langle\lambda(t),X_-(q(t))\rangle$. Thus, $\ddot\varphi$ must be negative and, therefore, $a_{n+s}=\pi_-=-1$.

We can single out a system of $n+s-1$ linear equations for $a_1,\dots,a_{n+s-1}$, associating with any $i=1,\dots,n$ the corresponding equation (9.7) and, eventually, adding some extra equations chosen between (9.1) and (9.9), depending on which vector fields, if any, have been chosen as $Y_{n+1},\dots,Y_{n+s-1}$. The determinant of the $(n+s-1)\times(n+s-1)$-matrix of coefficients of the system is given by $1+O(T)$. The uniqueness of its solution is proved, at least for sufficiently small T. □

Now assume that $(\lambda(\cdot), q(\cdot))$ is a short S-extremal pair and that $q(\cdot)$ contains a bang-bang concatenation of the type $-+-+$. Lemma 3 guarantees that we can apply Theorem 2 to the trajectory $q(\cdot)$.

Let t_0 be the second switching time and denote by t_1 and t_2 the length of, respectively, the second and third bang arcs. The switching times $t_0 - t_1$, t_0, and $t_0 + t_2$ are characterized by the equations

$$\left\langle \lambda(t_0), e^{-t_1 \operatorname{ad}(f+g)} g(q(t_0)) \right\rangle = 0, \tag{9.10}$$

$$\langle \lambda(t_0), g(q(t_0)) \rangle = 0, \tag{9.11}$$

$$\left\langle \lambda(t_0), e^{t_2 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle = 0. \tag{9.12}$$

Renormalize, if necessary, $\lambda(\cdot)$, in order to have $|\lambda(t_0)| = 1$. Let a_i and $\pi_\star$ be defined as in (9.4), (9.5), and (9.6). Lemma 3 states that they can be considered as functions of the trajectory and of the choice of the bang-bang concatenation. Moreover, we can assume that $\pi_- = -1$. Equalities (9.10) and (9.11) imply that

$$\pi_0 = \frac{t_1}{2}\pi_+ - \frac{t_1^2}{6}\pi_{++} + O\left(t_1^3\right). \tag{9.13}$$

Similarly, from (9.12) and (9.13) we deduce that

$$t_2 = 2\pi_0 + O\left(t_2^2\right) = t_1\pi_+ - \frac{t_1^2\pi_{++}}{3} + \pi_+^2 O\left(t_1^2\right) + O\left(t_1^3\right). \tag{9.14}$$

Note that $t_2 = O(t_1)$.

The role of the time $\overline{\tau}$ which appears in the statement of Theorem 2 will be played by t_0. According to (4.1), we have

$$\begin{aligned}
h_0 &= e^{-t_1 \operatorname{ad}(f+g)}(f-g) = f - g + 2t_1[f,g] - t_1^2 X_+ + O\left(t_1^3\right),\\
h_1 &= f + g,\\
h_2 &= f - g,\\
h_3 &= e^{t_2 \operatorname{ad}(f-g)}(f+g) = f + g + 2t_2[f,g] + O\left(t_2^2\right).
\end{aligned}$$

Let

$$\sigma_{ij} = \left\langle \lambda(t_0), [h_i, h_j](q(t_0)) \right\rangle, \quad 0 \le i < j \le 3. \tag{9.15}$$

From the above asymptotic expressions for h_i, $0 \le i \le 3$, we obtain

$$\begin{aligned}
\sigma_{01} &= 2\pi_0 - 2t_1\pi_+ + t_1^2\pi_{++} + O\left(t_1^3\right),\\
\sigma_{02} &= 2t_1 + O\left(t_1^2\right),\\
\sigma_{12} &= -2\pi_0,\\
\sigma_{03} &= \sigma_{01} + \sigma_{23} - 2\pi_0 + O\left(t_1^2 t_2\right),\\
\sigma_{13} &= 2t_2\pi_+ + O\left(t_2^2\right),\\
\sigma_{23} &= 2\pi_0 - 2t_2 + O\left(t_2^2\right).
\end{aligned}$$

A system of coordinates rectifying $P(S)$ can be obtained from the coordinate mapping

$$\begin{aligned}\mathcal{M}(x_1,\dots,x_{n+s}) &= e^{x_{n+s}Y_{n+s}}\circ\cdots\circ e^{x_{n+1}Y_{n+1}}\\ &\circ e^{x_n\,\mathrm{Ad}\,P^{-1}(Y_n)}\circ\cdots\circ e^{x_1\,\mathrm{Ad}\,P^{-1}(Y_1)}(q(t_0)),\end{aligned}\tag{9.16}$$

which is nondegenerate at $(0,\dots,0)$, since we assume that (7.2) holds. Associate with $(x_1,\dots,x_n)$ a horizontal-vertical splitting as in (4.3). From the definition of $\mathcal{M}$ it follows that

$$\partial_{x_j}Y_i^{\mathrm{v}}(q(t_0))=0,\quad Y_i^{\mathrm{h}}(q(t_0))=0\tag{9.17}$$

for every $j=1,\dots,n$ and every $i=n+1,\dots,n+s$. It is important to note that for any fixed vector field X and any $j=1,\dots,n$, the jth component of $X^{\mathrm{h}}(q(t_0))$ is $O(1)$, as well as $\left\langle\lambda(t_0),\partial_{x_j}X^{\mathrm{v}}(q(t_0))\right\rangle$.

In Sec. 9.2, we will treat separately different (s,d) situations. When convenient, we will consider second-order variations of the switching times on a shorter part of the bang-bang piece of $q(\cdot)$, i.e., on the concatenation of three instead of four bang arcs. Denote by K the number of switching times which are involved in the variation.

Let $\overline{Q}(\alpha)$, $R(\alpha)$, and H be defined as in (4.8), (4.10), and (4.5). Recall that H consists of all $(\alpha_0,\dots,\alpha_K)\in\mathbb{R}^{K+1}$ such that

$$\sum_{i=0}^{K}\alpha_i=0,\tag{9.18}$$

$$\sum_{i=0}^{K}\alpha_i h_i(q(t_0))\in\Sigma,\tag{9.19}$$

where

$$\Sigma=P_*\left(T_{q(0)}S\right).\tag{9.20}$$

We find it convenient, in most situations, to replace (9.19) by

$$\sum_{i=0}^{K}\alpha_i(h_i-f)(q(t_0))\in\Sigma,\tag{9.21}$$

as is justified by (9.18).

We claim that the codimension of H in $\mathbb{R}^{K+1}$ is equal to s for sufficiently small T. Indeed, let

$$A:\left\{(\alpha_0,\dots,\alpha_K)\in\mathbb{R}^{K+1}\;\middle|\;\sum_{i=0}^{K}\alpha_i=0\right\}\longrightarrow T_{q(t_0)}M/\Sigma$$

be the linear function which maps $(\alpha_0,\dots,\alpha_K)$ into the class $\sum_{i=0}^{K}\alpha_i h_i(q(0))+\Sigma$. Since $q(\cdot)$ is an S-extremal trajectory, there exists $\lambda\in T^*_{q(t_0))}\setminus\{0\}$ which is orthogonal to Σ and to $(h_i-h_{i-1})(q(t_0))$ for

$i = 1, \ldots, K$. The previous assertion is just a reformulation, obtained through (3.11), of (3.2) and of the fact that φ is equal to zero at the switching times. In the proof of Lemma 3, it has been shown that these orthogonality relations identify λ uniquely, up to multiplication by a scalar. Therefore, the codimension of the image of A in $T_{q(0)}M/\Sigma$ is equal to one. Since H is equal to the kernel of A, its dimension is equal to $K - s + 1$. Finally, as claimed, H has codimension s in $\mathbb{R}^{K+1}$.

9.2. **Case analysis.** In this section, different types of (s, d)-points are considered separately. Each paragraph deals with one or two classes of points, specified by the opening framed declaration.

Cases $(1, 2)$ *and* $(1, 3)$. We compute the second-order variation of $q(\cdot)$ with respect to its $-+-$ concatenation. This means that $K = 2$ and that H is a codimension one subspace of $\mathbb{R}^3$. An explicit expression for H is given by (9.18), as follows:

$$H = \left\{ (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3 \;\middle|\; \alpha_0 = -\alpha_1 - \alpha_2 \right\}.$$

The quadratic form $\overline{Q}$ defined in (4.8) is given by

$$\begin{aligned}\overline{Q}(\alpha_1, \alpha_2) &= \left(2\pi_0 - 2t_1\pi_+ + O\left(t_1^2\right)\right)(-\alpha_1 - \alpha_2)\alpha_1 \\ &\qquad + \left(2t_1 + O\left(t_1^2\right)\right)(-\alpha_1 - \alpha_2)\alpha_2 - 2\pi_0\alpha_1\alpha_2 \\ &= \left(-t_1\pi_+ + O\left(t_1^2\right)\right)\alpha_1^2 + \left(-2t_1 + \pi_+ O(t_1) + O\left(t_1^2\right)\right)\alpha_1\alpha_2 \\ &\qquad - \left(2t_1 + O\left(t_1^2\right)\right)\alpha_2^2.\end{aligned}$$

Let G_j be the jth component of $g(q(t_0))$ and η_j be equal to $\left\langle \lambda(t_0), \partial_{x_j} g^{\mathrm{v}}(q(t_0)) \right\rangle$ for $j = 1, \ldots, n$. Then

$$\begin{aligned}R(\alpha_1, \alpha_2) &= \sum_{j=1}^{n} ((2G_j + O(t_1))\alpha_1 + O(t_1)\alpha_2)\,((2\eta_j + O(t_1))\alpha_1 + O(t_1)\alpha_2) \\ &= O(1)\alpha_1^2 + O(t_1)\alpha_1\alpha_2 + O\left(t_1^2\right)\alpha_2^2.\end{aligned}$$

Whenever t_1 is sufficiently small, the coefficient of α_2^2 of $Q = \overline{Q} + R$ is negative. Note that, even if we computed the variation only on a smaller part of the bang-bang piece, our reasoning relies on the assumption that $\pi_- = -1$, which was justified by the presence of a compactly contained "$-$" arc. It follows from Theorem 2 that a short trajectory which contains a $-+-+$ or a $+-+-$ concatenation cannot be quasi-optimal. Proposition 1 is proved for $(s, d) = (1, 2)$ and $(1, 3)$ with $k(1, 2) = k(1, 3) = 3$.

We have already noted that our results partially overlap those of [11] for the cases $(1, 0)$, $(1, 1)$, and $(1, 2)$. We stress that the restrictions given here, namely, that the maximal possible concatenations for a quasi-optimal trajectory are of the type BBB or BSB, are sharp, since in the classification of time-optimal syntheses given in [11] these kind of concatenations actually appear.

Cases $(2,1)$ *and* $(2,2\mathrm{a})$. Let, as above, $K = 2$. The space H has codimension two and can be described by (9.18) and another independent linear relation deduced from (9.21). Note, e.g., that the component of $\sum_{i=0}^{K} \alpha_i(h_i - f)(q(t_0))$ in the direction $g(q(t_0))$, with respect to the basis

$$P_*(Y_1(q(0))),\ \ldots,\ P_*(Y_n(q(0))),\ g(q(t_0)),\ X_-(q(t_0))$$

is equal to zero. Thus,

$$-(1 + O(t_1))\alpha_0 + \alpha_1 - \alpha_2 = 0. \tag{9.22}$$

From (9.18) and (9.22) we obtain

$$\alpha_1 = O(t_1)\alpha_0, \quad \alpha_2 = -(1 + O(t_1))\alpha_0.$$

The transversality of g and Σ also implies, as follows from (9.17), that $R(\alpha_0) = O(t_1^2)\alpha_0^2$. The quadratic form $\overline{Q}$ is easily computed, and, finally, Q can be written as

$$Q(\alpha_0) = \overline{Q}(\alpha_0) + R(\alpha_0) = -\left(2t_1 + O(t_1^2)\right)\alpha_0^2.$$

Thus, Q is negative definite for small t_1. We conclude, as above, that a short trajectory with four concatenated bang arcs is not quasi-optimal. This proves Proposition 1 in the cases $(2,1)$ and $(2,2\mathrm{a})$ with $k(2,1) = k(2,2\mathrm{a}) = 3$.

Case $(2,2\mathrm{b})$. Let here as in the next cases $K = 3$. Denote by γ the component of $g(q(t_0))$ in the direction $[f,g](q(t_0))$, with respect to the basis

$$P_*(Y_1(q(0))),\ \ldots,\ P_*(Y_n(q(0))),\ [f,g](q(t_0)),\ X_-(q(t_0))."$$

The space H is characterized by (9.18) and by the component of the relation

$$\sum_{i=0}^{3} \alpha_i(h_i - f + g)(q(t_0)) \in \Sigma$$

in the direction $[f,g](q(t_0))$, i.e., by a system of the type

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0, \tag{9.23}$$

$$\left(2t_1 + O\left(t_1^2\right)\right)\alpha_0 + 2\gamma\alpha_1 + \left(2\gamma + 2t_2 + O\left(t_2^2\right)\right)\alpha_3 = 0, \tag{9.24}$$

from which we obtain

$$\begin{aligned} \alpha_0 &= \left(-\frac{\gamma}{t_1} + \gamma O(1)\right)\alpha_1 + \left(-\frac{\gamma + t_2}{t_1} + \gamma O(1) + O(t_1)\right)\alpha_3, \\ \alpha_2 &= -\alpha_0 - \alpha_1 - \alpha_3. \end{aligned} \tag{9.25}$$

Consider the linear change of variables on H given by

$$\beta_1 = \alpha_1 + \alpha_3,$$
$$\beta_2 = \alpha_1 - \alpha_3.$$

The quadratic form $\overline{Q}(\beta_1, \beta_2)$ turns out to have the following expression:

$$\begin{aligned}\overline{Q}(\beta_1, \beta_2) = -\frac{1}{2t_1}\Big[&\big(4\gamma^2 + 2\gamma(\pi_+ - 2)t_1 - \pi_+(\pi_+ + 4)t_1^2 + \gamma^2 O(t_1) \\ &+ \gamma O(t_1^2) + O(t_1^3)\big)\beta_1^2 + 2\big(t_1\pi_+(t_1 - \gamma) + \gamma^2 O(t_1) \\ &+ \gamma O(t_1^2) + O(t_1^3)\big)\beta_1\beta_2 + \big(\pi_+^2 t_1^2 + \gamma O(t_1^2) + O(t_1^3)\big)\beta_2^2\Big].\end{aligned}$$

Let G_j be the jth component of $g(q(t_0))$ and η_j be equal to $\big\langle \lambda(t_0), \partial_{x_j} g^{\mathrm{v}}(q(t_0))\big\rangle$ for $j = 1, \ldots, n$. Taking into account (9.17), one obtains

$$\begin{aligned}R(\beta_1, \beta_2) &= -\sum_{j=1}^{n} \left((2G_j + O(t_1))\beta_1 + O(t_1)\beta_2\right)\left((2\eta_j + O(t_1))\beta_1 + O(t_1)\beta_2\right) \\ &= O(1)\beta_1^2 + O(t_1)\beta_1\beta_2 + O\left(t_1^2\right)\beta_2^2.\end{aligned}$$

Finally, the coefficient of β_2^2 of the quadratic form

$$Q(\beta_1, \beta_2) = \overline{Q}(\beta_1, \beta_2) + R(\beta_1, \beta_2)$$

is given by $\pi_+^2 t_1 + \gamma O(t_1) + O\left(t_1^2\right)$. Since property (A) ensures that π_+^2 is uniformly separated from zero, $q(\cdot)$ cannot be quasi-optimal for sufficiently small T and U (consequently, t_1 and $|\gamma|$).

By the symmetry with respect to "+" and "−," we conclude that Proposition 1 holds in the case (2, 2b) with $k(2, 2\mathrm{b}) = 3$.

Let us focus on the point-to-point problem on a two-dimensional manifold M. As it has been noted in Remark 3, the transversality condition (3.2) gives no restrictions on the extremal lifts of a quasi-optimal trajectory. The above computations actually prove that if $q_0 \in M$ is a $(2, d)$-point, $d = 0, 1, 2a, 2b$, then there exist a neighborhood U of q_0 and a time $T > 0$ such that a bang-bang trajectory contained in U (not necessarily passing through q_0) of time-length smaller than T with four (or more) arcs is not quasi-optimal. Thanks to the additional analysis of non-bang-bang trajectories contained in Sec. 8, standard transversality considerations lead to the following proposition.

Proposition 2. *Let M be a two-dimensional manifold. Then for a generic pair (f, g) of vector fields on M, for every point $q_0 \in M$, there exist a neighborhood U of q_0 and a time $T > 0$ such that a quasi-optimal trajectory of* (1.1) *contained in U and of time-length smaller than T is the concatenation of at most four bang and singular arcs. The only possible maximal concatenations are of the type BBB, $BSBB$, or $BBSB$.*

A quantitative bound of this kind is slightly better than the ones available in the literature (see [9, 12, 16]).

Case $(3,0)$. We describe H, which has codimension three, by (9.18) and the components of (9.21) in the directions $g(q(t_0))$ and $[f,g](q(t_0))$ as follows:

$$\begin{aligned} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 &= 0, \\ -\left(1 + O\left(t_1^2\right)\right)\alpha_0 + \alpha_1 - \alpha_2 + \left(1 + O\left(t_1^2\right)\right)\alpha_3 &= 0, \\ \left(2t_1 + O\left(t_1^2\right)\right)\alpha_0 + \left(2t_2 + O\left(t_1^2\right)\right)\alpha_3 &= 0. \end{aligned}$$

Then we obtain

$$\begin{aligned} \alpha_0 &= -\left(\frac{t_2}{t_1} + O(t_1)\right)\alpha_3, \\ \alpha_1 &= -\left(1 + O\left(t_1^2\right)\right)\alpha_3, \\ \alpha_2 &= \left(\frac{t_2}{t_1} + O(t_1)\right)\alpha_3. \end{aligned}$$

Therefore, after some computations, we obtain

$$\overline{Q}(\alpha_3) = -2\left(t_1\pi_+^2 + O\left(t_1^2\right)\right)\alpha_3^2.$$

Applying (9.17) to $Y_{n+1} = g$, we obtain that $R(\alpha_3) = O\left(t_1^2\right)\alpha_3^2$. Since (A) holds, Q is negative definite, at least for short trajectories. The symmetry with respect to "+" and "−" implies that Proposition 1 holds for the case $(3,0)$ with $k(3,0) = 3$.

Case $(3,1)$. The computations made for the case $(3,0)$ are still valid, but, since the property (A) fails to hold, we cannot conclude as above. We need to take into account higher order terms in the expansion of Q. In particular, it is no more true that $t_1 = O(t_2)$, while we find it convenient to replace, in the estimate of the remainders, the still valid relation $t_2 = O(t_1)$ by a more accurate one $t_2 = \pi_+ O(t_1) + O\left(t_1^2\right)$. We rewrite the parametrization of H obtained for the case $(3,0)$, distinguishing between the roles of t_1 and t_2,

$$\begin{aligned} \alpha_0 &= -\left(\frac{t_2}{t_1} + O(t_2)\right)\alpha_3, \\ \alpha_1 &= -(1 + O(t_1 t_2))\alpha_3, \\ \alpha_2 &= \left(\frac{t_2}{t_1} + O(t_2)\right)\alpha_3. \end{aligned}$$

Recalling that (9.17) holds for $Y_{n+1} = g$ and $Y_{n+2} = [f,g]$, it follows that $R(\alpha_3) = O\left(t_1^2 t_2^2\right)$. On the other hand,

$$\overline{Q}(\alpha_3) = -2t_2\left(\pi_+ + \pi_+ O(t_1) + O(t_1^2)\right)\alpha_3^2.$$

Assume that Q is nonnegative definite. Note that since $\pi_0 = \dot\varphi(t_0) \geq 0$, the following inequality holds:

$$\pi_+ - t_1 \frac{\pi_{++}}{3} + O(t_1^2) \geq 0.$$

From this relation and the sign condition on Q, we deduce that

$$t_1 \pi_{++} + \pi_+ O(t_1) + O(t_1^2) \leq 0.$$

Since (B) holds, π_{++} is uniformly bounded away from 0. A necessary condition for Q to be nonnegative definite is that $\pi_{++} < 0$, provided that T and U are small. In particular, if $X_-(q_0)$ and $X_{++}(q_0)$ point on the opposite side of the hyperplane V, then a short trajectory is not quasi-optimal if it contains a $-+-+$ concatenation of arcs.

We already noted that the time-reversion of a trajectory of (1.1) is admissible for (4.10). If we replace f and g by $-f$ and $-g$, then the roles of $X_\pm$ and $X_{++}(q_0)$ are played, respectively, by $-X_\pm(q_0)$ and $X_{++}(q_0)$. We stress, in particular, that q_0 is a $(3,1)$-point for the time-reversed system as well. In obtaining all the above asymptotic relations, we never used the fact that (3.2) holds *at the starting point* of the trajectory. We used it only to obtain that $a_i = O(T)$ for $i = 1, \ldots, n$. The same relations can be recovered for trajectories attaining S at their final point T, replacing (3.2) by the symmetric transversality condition (4.11). It follows from Remark 2 that if $X_-(q_0)$ and $X_{++}(q_0)$ point on the same side of V, then the quadratic form associated with a short trajectory of the time-reversed system which contains a $-+-+$ concatenation is negative definite and, therefore, a short trajectory of the original system which contains a $+-+-$ concatenation is not quasi-optimal.

Finally, a short quasi-optimal trajectory of the original system has at most four concatenated bang arcs, i.e., we proved Proposition 1 in the case $(3,1)$ with $k(3,1) = 4$.

Remark 5. In the case $n = 0$, we can read the local bounds for $(3,0)$- and $(3,1)$-points from the point of view of generic properties of the point-to-point problem in dimension three, by analogy with what has been done for $s = 2$ in Proposition 2. We recover in this way Theorem 1 in [6]. In particular, it turns out that the presence of general boundary conditions does not weaken the regularity properties which have been proved to hold for the point-to-point problem.

10. Proof of Proposition 1 for $(s,d) = (4,0)$

Throughout this section, we assume that q_0 is a $(4,0)$-point. As for the cases treated in Sec. 9, the preliminary step is to investigate the uniqueness of S-extremal lifts.

Lemma 4. *A short S-extremal trajectory which has at least two concatenated compactly contained bang arcs admits a unique S-extremal lift, up to multiplication by a positive scalar.*

Proof. Let $(\lambda(\cdot), q(\cdot))$ be an S-extremal pair and assume that it has two concatenated compactly contained bang arcs. Denote by t_0 the switching time between them. Let $Y_1, \ldots, Y_{n+4}$ be chosen as in Sec. 7 and define

$$a_i = \langle \lambda(t_0), Y_i(q(t_0)) \rangle$$

for $i = 1, \ldots, n+4$. Normalize $\lambda(\cdot)$ so that

$$\max \left\{ |a_i| \mid i = 1, \ldots, n+4 \right\} = 1.$$

By (3.2), we obtain that

$$a_i = \sum_{j=1}^{n+4} a_j O(T), \quad i = 1, \ldots, n. \tag{10.1}$$

Define $\pi_\star$ as in (9.5) and (9.6), $\star = 0, +, -, ++$. Note that

$$a_{n+1} = \varphi(t_0) = 0, \tag{10.2}$$

$a_{n+2} = \pi_0$, $a_{n+3} = \pi_+$, and $a_{n+4} = \pi_-$. Since $\langle \lambda(t), [f, g](q(t)) \rangle = \dot{\varphi}(t)$ has at least one zero along the trajectory, we can assume that $|a_{n+2}|$ is smaller than one. The presence of compactly contained $+$ and $-$ arcs implies, for small T, that either $a_{n+3} = 1$ or $a_{n+4} = -1$.

Without loss of generality, the control switches at t_0 from $+1$ to -1. Denote by t_1 and t_2 the lengths of the "+" and "−" arc, respectively. From the relation

$$\varphi(t_0 - t_1) = \varphi(t_0) = \varphi(t_0 + t_2) = 0$$

we obtain

$$\pi_0 - \frac{t_1}{2}\pi_+ + \sum_{j=1}^{n+4} a_j O\left(t_1^2\right) = 0, \tag{10.3}$$

$$\pi_0 + \frac{t_2}{2}\pi_- + \sum_{j=1}^{n+4} a_j O\left(t_2^2\right) = 0. \tag{10.4}$$

Collecting (10.1), (10.2), (10.3), and (10.4), we obtain a linear system of $n+3$ homogeneous linear equations satisfied by $a_1, \ldots, a_{n+4}$. The rank of the coefficient matrix of the system is, for small T, equal to $n+3$ and, therefore, the solutions form a one-dimensional linear subspace of $\mathbb{R}^{n+4}$. Its intersection with

$$\left\{ (b_1, \ldots, b_{n+4}) \;\middle|\; |b_i| \le 1 \text{ for every } i = 1, \ldots, n+4;\ b_{n+3} = 1 \text{ or } b_{n+4} = -1 \right\}$$

has cardinality one. ∎

Fix an S-extremal pair $(\lambda(\cdot), q(\cdot))$ and assume that it contains a $+-+-+$ concatenation. Let t_0 be the second switching time of the bang-bang concatenation and denote by t_1, t_2, and t_3 the length of the second, third, and fourth bang arc, respectively. Thus, the following equations are satisfied:

$$\left\langle \lambda(t_0), e^{-t_1 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle = 0, \tag{10.5}$$

$$\langle \lambda(t_0), g(q(t_0)) \rangle = 0, \tag{10.6}$$

$$\left\langle \lambda(t_0), e^{t_2 \operatorname{ad}(f+g)} g(q(t_0)) \right\rangle = 0, \tag{10.7}$$

$$\left\langle \lambda(t_0), e^{t_2 \operatorname{ad}(f+g)} e^{t_3 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle = 0. \tag{10.8}$$

Assume that $(\lambda(\cdot), q(\cdot))$ belongs to $\Xi^1_\eta \cup \Xi^3_\eta$, where Ξ^i_η and η are defined as in Sec. 7. If T is sufficiently small, then, as follows from Lemma 1, we have

$$\big|\langle \lambda(t), X_-(q(t)) \rangle\big| \geq \frac{\eta}{2} \tag{10.9}$$

along $q(\cdot)$. The presence of a compactly contained $-$ arc implies that the sign of $\langle \lambda(t), X_-(q(t)) \rangle$ is negative. Possibly renormalizing $\lambda(\cdot)$, we may assume that

$$\langle \lambda(t_0), X_-(q(t_0)) \rangle = -1. \tag{10.10}$$

Note that applying this renormalization, it is possible that we exit from the class of normalized pairs, as was defined in Sec. 7. Let for $i = 1, 3$, $\widetilde{\Xi}^i_\eta$ be the class of S-extremal pairs containing a $+-+-+$ concatenation which are obtained from a pair in Ξ^i_η by means of renormalization (10.10). Note that since (10.9) holds, the rescaling factor is bounded from below by $2/\eta$. Therefore, for any vector field X, $\langle \lambda(t_0), X(q(t_0)) \rangle = O(1)$, as a function of the pair $(\lambda(\cdot), q(\cdot))$, chosen in $\widetilde{\Xi}^1_\eta \cup \widetilde{\Xi}^3_\eta$, and of the choice of the bang-bang concatenation.

Define $\pi_\star$ as in (9.5) and (9.6), $\star = 0, +, ++$. Note that we can still assume $\widetilde{\Xi}^1_\eta$ to satisfy (B) and $\widetilde{\Xi}^3_\eta$ to satisfy (A).

From equations (10.5)–(10.7) we obtain that

$$\begin{aligned}
\pi_0 &= -\frac{t_2}{2}\pi_+ - \frac{t_2^2}{6}\pi_{++} + O(t_2^3),\\
t_1 &= t_2\pi_+ + \frac{t_2^2}{3}\pi_{++} + \pi_+ O(t_2^2) + O(t_2^3),\\
t_3 &= t_2\pi_+ + \frac{2}{3}t_2^2\pi_{++} + \pi_+ O(t_2^2) + O(t_2^3).
\end{aligned} \tag{10.11}$$

We find it useful to introduce another agreement on how to formulate asymptotic properties. We say that a function is $\Omega(t_2)$ if it can be expressed as a sum of the type $\pi_+ O(1) + O(t_2)$. Briefly,

$$\Omega(t_2) = \pi_+ O(1) + O(t_2). \tag{10.12}$$

For example, $t_1, t_3 = t_2\Omega(t_2)$. In order to recover Q, we compute

$$\begin{aligned}
h_0 &= f + g - 2t_1[f,g] + t_1^2 X_- + O(t_1^3),\\
h_1 &= f - g,\\
h_2 &= f + g,\\
h_3 &= f - g - 2t_2[f,g] - t_2^2 X_+ + O(t_2^3),\\
h_4 &= f + g + 2t_3[f,g] + 2t_2t_3X_+ + t_3^2 X_- + O(t_2^2 t_3).
\end{aligned}$$

The asymptotic expressions of σ_{ij}, $0 \le i < j \le 4$, can be obtained from the above relations. They are omitted for brevity.

By analogy with what was done in Sec. 9, one can derive from Lemma 4 that for sufficiently small T, the space H has codimension four in $\mathbb{R}^5$. A system of equations for H is given by (9.18) and the components of (9.21) in the directions g, $[f,g]$, and X_+ are as follows:

$$\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0,\\
\left(1 + O\left(t_1^2\right)\right)\alpha_0 - \alpha_1 + \alpha_2 - \left(1 + O\left(t_2^3\right)\right)\alpha_3 + (1 + t_2t_3\Omega(t_2))\alpha_4 &= 0,\\
-\left(2t_1 + O\left(t_1^2\right)\right)\alpha_0 - \left(2t_2 + O\left(t_2^3\right)\right)\alpha_3 + (2t_3 + t_2t_3\Omega(t_2))\alpha_4 &= 0,\\
O\left(t_1^2\right)\alpha_0 - \left(t_2^2 + O\left(t_2^3\right)\right)\alpha_3 + (2t_2t_3 + t_2t_3\Omega(t_2))\alpha_4 &= 0.
\end{aligned}$$

From the above relations, we deduce that

$$\begin{aligned}
\alpha_0 &= \frac{t_3}{t_1}\left(1 + \Omega(t_2)\right)\alpha_4,\\
\alpha_1 &= -\alpha_3 + t_2t_3\Omega(t_2)\alpha_4,\\
\alpha_2 &= -\alpha_0 - (1 + t_2t_3\Omega(t_2))\alpha_4,\\
\alpha_3 &= 2\frac{t_3}{t_2}(1 + \Omega(t_2))\alpha_4.
\end{aligned}$$

Finally, one obtains

$$\overline{Q}(\alpha_4) = -\frac{2t_2}{3t_1}\left(t_3\left(6\pi_+^2 + 6t_2\pi_+\pi_{++} + t_2^2\pi_{++}^2\right) + t_2^2\Omega(t_2)\Omega(t_2)\Omega(t_2)\right)\alpha_4^2. \tag{10.13}$$

By (9.17), we have $R(\alpha_4) = O\left(t_2^4 t_3^2\right)$ and, finally, Q has the same asymptotic expression as $\overline{Q}$ in (10.13).

On the class $\widetilde{\Xi}_\eta^3$, the quantity π_+ is uniformly separated from zero, for small T. It follows from Theorem 2 that a short S-extremal pair in $\widetilde{\Xi}_\eta^3$ which contains a $+-+-+$ concatenation is not quasi-optimal. By symmetry with respect "+" and "−," we actually proved that a short S-extremal pair in $\widetilde{\Xi}_\eta^3$ with five concatenated arcs is not quasi-optimal. On the contrary, the case of the class $\widetilde{\Xi}_\eta^1$ deserves a further analysis.

Lemma 5. *If $q(\cdot)$ is short and quasi-optimal, then the pair (π_+, π_{++}) lies in the interior of the second or of the fourth quadrant of $\mathbb{R}^2$.*

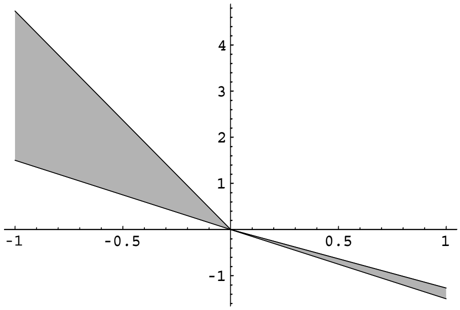

Fig. 1. The set $C \cap D$.

Proof. If $q(\cdot)$ is quasi-optimal, then $Q \geq 0$. Taking into account the asymptotic expression for t_3 given in (10.11), we obtain from (10.13) that

$$\begin{aligned} 0 &\geq (6+O(t_2))\pi_+^3 + (10+O(t_2))\pi_+^2\pi_{++}t_2 \\ &+(5+O(t_2))\pi_+\pi_{++}^2t_2^2 + \left(\frac{2}{3}+O(t_2)\right)\pi_{++}^3t_2^3. \end{aligned} \tag{10.14}$$

Note that the leading term of (10.14) is a homogeneous polynomial inequality in $(\pi_+, t_2\pi_{++})$. Its set of solutions in $\mathbb{R}^2$ is given by the cone

$$\begin{aligned} C &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; 6x^3 + 10x^2y + 5xy^2 + \frac{2}{3}y^3 \leq 0\right\} \\ &= \left\{(r\cos\theta, r\sin\theta) \;\middle|\; r \in [0,+\infty),\ \theta \in \mathcal{S}^1,\ P(\theta) \leq 0\right\}, \end{aligned}$$

where

$$P(\theta) = 6\cos^3\theta + 10\cos^2\theta\sin\theta + 5\cos\theta\sin^2\theta + \frac{2}{3}\sin^3\theta.$$

Fix a cone C' such that $C \subset C'$ and $\partial C \cap \partial C' = \{0\}$. Taking sufficiently small t_2, we have that $(\pi_+, t_2\pi_{++}) \in C'$. Indeed, the trigonometric polynomial P which defines C has six simple zeros on $\mathcal{S}^1$, which are stable under small perturbations of the coefficients.

Let

$$D = \left\{(x,y) \;\middle|\; x + \frac{y}{3} \geq 0\right\}.$$

Consider a cone D' which contains the half-plane D and such that $\partial D \cap \partial D' = \{0\}$. Since the condition $\pi_0 \leq 0$ must also be satisfied, we have $(\pi_+, t_2\pi_{++}) \in D'$ for small t_2. If we choose C' and D' sufficiently close to C and D, then $C' \cap D'$ is contained in the union of the second and fourth quadrant (see Fig. 1) and the lemma is proved. □

Let

$$\tilde{\pi}_+ = \langle \lambda(t_0 - t_1), X_+(q(t_0 - t_1)) \rangle ,$$
$$\tilde{\pi}_{++} = \langle \lambda(t_0 - t_1), X_{++}(q(t_0 - t_1)) \rangle .$$

We have that

$$\tilde{\pi}_+ = \lim_{t \to t_1^-} \ddot{\varphi}(t_0 - t) = \ddot{\varphi}(t_0) - t_1 \varphi^{(3)}(t_0 - \bar{t}),$$

where $\bar{t} \in [0, t_1]$. Since

$$\sup_{t \in [0,t_1]} |\varphi^{(3)}(t_0 - t)| = \sup_{t \in [0,t_1]} |\langle \lambda(t_0 - t), X_{++}(q(t_0 - t)) \rangle| = O(1),$$

we have

$$\tilde{\pi}_+ = \pi_+ + t_2 \Omega(t_2).$$

For the same reason,

$$\tilde{\pi}_{++} = \pi_{++} + t_2 \Omega(t_2).$$

In particular, an inequality in the form (10.14) is still true if we replace π_+ and π_{++} by $\tilde{\pi}_+$ and $\tilde{\pi}_{++}$, respectively. Similarly, the inequality $t_3 > 0$ can be rewritten, in terms of $\tilde{\pi}_+$ and $\tilde{\pi}_{++}$, as

$$\tilde{\pi}_+ + \frac{2}{3} t_2 \tilde{\pi}_{++} + t_2 \Omega(t_2) > 0.$$

Thus, for sufficiently small T, $(\tilde{\pi}_+, t_2 \tilde{\pi}_{++}) \in C' \cap D'$, where C' and D' are chosen as in the proof of Lemma 5. In particular, for short quasi-optimal pairs, $(\tilde{\pi}_+, t_2 \tilde{\pi}_{++})$ lies in the interior of the second or fourth quadrant.

Consider now a short S-extremal pair $(\lambda(\cdot), q(\cdot))$ which contains seven concatenated bang arcs, the first and the last one corresponding to control $+1$. Let $\tau_1 < \cdots < \tau_8$ be the boundary points of such arcs. Note that $q\big|_{[\tau_1,\tau_6]}$ and $q\big|_{[\tau_3,\tau_8]}$ are both $+-+-+$ restrictions of $q(\cdot)$, and that $q(\tau_4)$ is both the first switching point of $q\big|_{[\tau_3,\tau_8]}$ and the second switching point of

$$[\tau_1, \tau_6] \ni t \mapsto q(\tau_1 + \tau_6 - t), \tag{10.15}$$

which is a $+-+-+$ piece of trajectory for the time-reversed system (4.10).

Normalize $\lambda(\cdot)$ according to the choice of the $+-+-+$ concatenation $q\big|_{[\tau_3,\tau_8]}$. Denote by $q'(\cdot)$ the time-reversed trajectory of $q(\cdot)$ and by $\lambda'(\cdot)$ an extremal lift of $q'(\cdot)$ for (4.10), which satisfies (4.11), obtained from $\lambda(\cdot)$ by time-reversion and multiplication by a scalar. Assume that $\lambda'(\cdot)$ is normalized according to the choice of the $+-+-+$ concatenation singled out in (10.15).

Choosing T sufficiently small (depending on η), we can assume either that one between these lifts is in $\widetilde{\Xi}^3_\eta$ or that they both belong to $\widetilde{\Xi}^1_\eta$ or to $\widetilde{\Xi}^2_\eta$. In the first case, we have already proved that the trajectory is not quasi-optimal.

Assume that they both are in $\widetilde{\Xi}^1_\eta$. Let us apply to both of them Lemma 5. The role of $(\tilde\pi_+, \tilde\pi_{++})$ for $(\lambda(\cdot), q(\cdot))$ is played by a pair which is positively proportional to

$$p_1 = \big(\langle \lambda(\tau_4), X_+(q(\tau_4)) \rangle , \langle \lambda(\tau_4), X_{++}(q(\tau_4)) \rangle \big).$$

Similarly, the role of (π_+, π_{++}) for $(\lambda'(\cdot), q'(\cdot))$ is played by a pair which is positively proportional to

$$\begin{aligned} p_2 &= \big(\langle (-\lambda(\tau_4)), -X_+(q(\tau_4)) \rangle , \langle (-\lambda(\tau_4)), X_{++}(q(\tau_4)) \rangle \big) \\ &= \big(\langle \lambda(\tau_4), X_+(q(\tau_4)) \rangle , - \langle \lambda(\tau_4), X_{++}(q(\tau_4)) \rangle \big). \end{aligned}$$

If $q(\cdot)$ were quasi-optimal, then p_1 and p_2 would both lie in the interior of the second or fourth quadrant of $\mathbb{R}^2$, which is, clearly, impossible.

Proposition 1 is proved with $k(4,0) = 7$.

Consider the point-to-point problem corresponding to the case $(4,0)$, i.e., let $\dim S = 0$ and $\dim M = 4$. As was said in Remark 3, the above computations apply to any extremal bang-bang trajectory of sufficiently small time-length in a sufficiently small neighborhood of a $(4,0)$-point q_0 of M. Therefore, the following property holds.

Proposition 3. *Let M be a four-dimensional manifold. Then, for a generic pair (f, g) of vector fields on M, there exists a three-dimensional stratified set $W \subset M$, such that, for every point q_0 in $M \setminus W$, there exist a neighborhood U of q_0 and a time $T > 0$, such that a quasi-optimal trajectory of* (1.1) *contained in U and of time-length smaller than T is the concatenation of at most seven bang and singular arcs. The only possible maximal concatenations including singular arcs are of the type $BSBB$ or $BBSB$.*

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