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Discrete facility location games with different preferences

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Abstract
We study the mechanism design for discrete facility location games with different preferences, where the facilities can only be built at a finite set of candidate locations, and a mechanism maps the agent locations to candidate locations for building facilities. We consider both the *obnoxious preferences*, where the agents want to stay as far away as possible from the facilities, and the *dual preferences*, where each agent may either like or dislike a facility. When the preferences are obnoxious, for two heterogeneous facilities, we present a group strategy-proof mechanism which has an approximation ratio of 2 for both social utility objective and minimum utility objective. Both objectives are proven to have a lower bound of $\frac{3}{2}$. For two homogeneous facilities, we prove there is no deterministic strategy-proof mechanism with bounded approximation ratio. When the preferences are dual, we consider the single facility location games under the social utility objective, and propose a group strategy-proof mechanism with approximation ratio of 4.

Keywords Facility location game · Mechanism design · Strategyproofness · Approximation ratio

Mathematics Subject Classification (2020) 91A68 · 91B03 · 68W25

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1 Introduction

Classical facility location problem is a consideration of where to place one or more facilities to serve the agent and achieve the maximum benefit objective, in other words the trade-off between service and utility. In the facility location game, the location of an agent is private information and will be required to be provided by the individual agent. A mechanism maps the reported locations of the agents to the locations of the facilities. The agents involved in the game are rational, and they will influence the output of the mechanism by misreporting about the information they have, so as to maximize their own benefits. Thus, we are more concerned about how to design a *strategy-proof* mechanism to incentive agents to report their positions truthfully while ensuring a relatively good facility location solution. Therefore, we pay more attention to designing an *strategy-proof* mechanism that, on the one hand, can motivate agents to report their information truthfully and, on the other hand, can achieve a relatively good result compared to the optimal facility placement solution with respect to the system goal. Procaccia and Tennenholtz (2009) first presented this agenda of designing approximate mechanisms without money. This field of study has received a great deal of attention (Chen et al. 2021; Cheng et al. 2013; Zou and Li 2015) in the wake of their work. Chan et al. (2021) have provided a thoroughly researched and comprehensive review.

In the general setting, the focus of scholarly studies has been more on situations where facilities can be placed in any location (e.g., see Anastasiadis and Deligkas 2018; Nehama et al. 2022). But in practice, most facilities can only be located in certain areas or on certain sites. For example, waste treatment plants or landfills are restricted to certain locations. This is due to factors such as geographical location and wind direction. Petrol stations are generally not located in residential centres. Then, in this paper we study the game where facility locations can only be selected from a given set of candidate points, in addition to the fact that multiple facilities cannot be placed in the same location due to space constraints.

We consider different preferences of agents over the facilities, that is, *obnoxious* preferences and *dual* preferences. For obnoxious preferences, the agent wants to avoid the facilities as much as possible, and it models the scenario of building a sewage treatment plant or a garbage station, and the agents want to stay as far away as possible from the facilities. For dual preferences, the agent may enjoy (like) or suffer from (dislike) a facility depending on her personal taste, and it models the scenario of building a wet market, where some people dislike the market due to noise and garbage problems, while other people like the market for fresh cooking ingredients and convenient shopping. That is, an agent may like or dislike a facility.

In this paper, we assume that there are n agents distributed in the interval $[0, 1]$, and the candidate locations are given as a finite set M in the real line. Let $l = \min_{y \in M} y$ and $r = \max_{y \in M} y$ be the leftmost and rightmost candidate location, respectively, and l and r can be any real number. Under obnoxious preferences, for single-facility location game, each agent has a utility equal to the distance from the facility. For two heterogeneous facilities, the agent's utility is the sum of the distances to the facilities (see Xinping et al. 2021 for motivation). For two homogeneous facilities, the agent's utility is the distance to the closer facility. Under dual preferences, we focus on the

Table 1 Main Results in the obnoxious setting under the social utility objective

Facility	Candidate location	Upper bound	Lower bound
Single facility	$\frac{l+r}{2} \le 0$ or $\frac{l+r}{2} \ge 1$	**1** (Thm.1)	/
	$0 < \frac{l+r}{2} < 1$	3	3
Heterogeneous two facilities	$[l, r] \subseteq (-\infty, 0]$ or $[l, r] \subseteq [1, +\infty)$	**1** (Thm.2)	/
	Other cases	**2** (Thm.3)	$\mathbf{\frac{3}{2}}$ (Thm.4)

Table 2 Main results in the dual preferences setting under the social utility objective

Location	Preference	Upper bound	Lower bound
Known	Unknown	**1** (Thm.7)	/
Unknown	Unknown	**4** (Thm.8)	3

The bold numbers indicate the bounds derived in this paper

case when $[l, r]$ is a subinterval contained in $[0, 1]$. We consider a single facility and two different utility settings.

Our results. Under obnoxious preferences, for the single facility location game under the social utility objective, we demonstrate that an optimal group strategy-proof mechanism exists when $\frac{l+r}{2} \le 0$ or $\frac{l+r}{2} \ge 1$, and that a group strategy-proof 3-approximation mechanism exists, which is the best possible. For the heterogeneous two-facility location game where the utility of agent is the total distance to the two facilities, in the scenario where $[l, r]$ does not intersect with $(0, 1)$, we present an optimal group strategy-proof mechanism, and in the scenario where $[l, r]$ intersects with $(0, 1)$, we present a 2-approximation mechanism. The lower bound is proved to be $\frac{3}{2}$. A summary of these results is shown in Table 1, where the bold numbers indicate the bounds derived in this paper.

Further, we study the objective of maximizing the minimum utility. For two heterogeneous facilities, we design a deterministic 2-approximation mechanism and prove a lower bound of $\frac{3}{2}$. For two homogeneous facilities, we prove that there is no deterministic strategy-proof mechanism with bounded approximations.

Under dual preferences, we study the single facility location game, where each agent i have different preferences (like or dislike) for the facility. We further assume that $[l, r] \subseteq [0, 1]$. The preference of agent i is represented by an indicator $p_i \in \{0, 1\}$. $p_i = 0$ indicates that agent i dislikes the facility, and $p_i = 1$ indicates that the agent likes the facility. When the agent's preferences are unknown, the mechanism that outputs the optimal solution is strategy-proof. When both the agent's preferences and locations are unknown, we give deterministic strategy-proof mechanisms with an approximation ratio of 4. A summary of these results is shown in Table 2.

Compared with the preliminary version (Gai et al. 2022), several new results are added in this full version. First, we consider the objective of maximizing the minimum utility for two heterogeneous facilities and two homogeneous facilities with obnoxious preferences. Upper and lower bounds on the approximation ratios are derived (see

Theorem 5 and Theorem 6). Second, we additionally consider the facility location game with dual preferences (see Sects. 5).

Related work. Below we briefly review the results of facility location games with candidate locations. If agents is interested in the facility, and agents on a continuous graph and facility location only selected in candidates, Thang et al. (2010) assumed that an agent could control multiple locations and designed a randomised strategy-proof mechanism with 3 approximations and a deterministic group strategy-proof mechanism with an approximation ratio of $2n + 1$ for the social cost objective. And For the social cost objective and the maximum cost objective, Feldman et al. (2016), Tang et al. (2020) considered single-facility and two-facility location games. Similarly, agents on a continuous graph, but the facility is an arbitrary optional point within a specific interval of a line, Walsh (2021) considered six different objectives when limiting facilities to particular locations makes the problem more difficult to approximate. On the other hand, when agents are located on a discrete graph, Kanellopoulos et al. (2023) studied the heterogeneous facility location games on the basis of Serafino and Ventre (2016) which gave deterministic and randomized mechanisms, and bounded approximation ratio of mechanisms for the social objective. Kanellopoulos et al. (2023) devised deterministic mechanisms with better bounds on both the social and the maximum cost. A different version of the facility location games was explored by Kanellopoulos et al. (2022). Their assumption was that the positions of the agents are common knowledge, whereas each agent has a private preference with respect to the candidate facilities. Thus, rather than accessing the nearest facility, each agent's cost is defined as the distance to the set of facilities he is interested in. Dokow et al. (2012) analysed the discrete unweighted graph location game, restricting agents and facilities to vertices. They have given a full characterization of the strategy-proof mechanisms on lines and on cycles of sufficient size. A characterization of strategy-proof mechanisms on discrete trees was introduced by Filimonov and Meir (2021).

2 Preliminaries

We will define $N = \{1, 2, ..., n\}$ to be the set of all agents. They are located in an interval $[0, 1]$ and their set of locations is named S. Let M be the set of candidate locations for the facility, $M \subseteq [l, r]$. That is, the candidate for the facility on the leftmost is point l and the candidate for the facility on the rightmost is point r, $l, r \in \Re$. The distance between any two points $s, f \in \Re$ is $d(s, f) = |s - f|$. We denote the agents reported location profile as $\mathbf{s} = (s_1, s_2, ..., s_n)$. A deterministic mechanism f uses the agents' location profile $\mathbf{s}$ as input and it outputs k facility locations $\mathbf{f} = (f_1, ..., f_k) \in M^k$.

Obnoxious preferences. For the obnoxious preference single facility location games $(k = 1)$, assume that the facility location is $\mathbf{f} = f(\mathbf{s}) = f$. The utility of agent i is his distance to the facility f, that is,

$$u(s_i, \mathbf{f}) = u(s_i, f) = d(s_i, f).$$

For two-facility location games ($k = 2$), suppose the locations output by the mechanism are $\mathbf{f} = f(\mathbf{s}) = \{f_1, f_2\}$. Then for the *heterogeneous* facilities case, the utility is defined to be the sum of distances to both facilities (Xinping et al. 2021)

$$u(s_i, \mathbf{f}) = d(s_i, f_1) + d(s_i, f_2).$$

For the *homogeneous* facilities case, the utility is defined to be the distances to the closer facility

$$u(s_i, \mathbf{f}) = \min\{d(s_i, f_1), d(s_i, f_2)\}.$$

Dual preferences. For the dual preference single facility location games ($k = 1$), each agent $i \in N$ may like or dislike the facility. The preference is represented by an indicator p_i: $p_i = 0$ if agent i dislikes the facility, and $p_i = 1$ if he/she likes. We assume that $[l, r] \subseteq [0, 1]$, given the facility location f, when $p_i = 0$, the utility is defined as $u_i(f) = d(s_i, f)$. When $p_i = 1$, the utility is defined according to the agent location:

$$u_i(f) = \begin{cases} r - d(s_i, f) & \text{if } s_i \in [0, l] \\ r - l - d(s_i, f) & \text{if } s_i \in (l, r) \\ 1 - l - d(s_i, f) & \text{if } s_i \in [r, 1] \end{cases} \tag{1}$$

We study two objectives, maximizing the social utility, and maximizing the minimum utility. The social utility with respect to agent locations $\mathbf{s}$ and facility locations $\mathbf{f}$ is the sum of utilities of n agents,

$$SU(\mathbf{s}, \mathbf{f}) = \sum_{i \in N} u(s_i, \mathbf{f}).$$

The minimum utility with respect to $\mathbf{s}$ and $\mathbf{f}$ is the minimum utility of the agents,

$$MU(\mathbf{s}, \mathbf{f}) = \min_{i \in N} u(s_i, \mathbf{f}).$$

A mechanism f is *strategy-proof* if no agent can acquire more utility from misreporting. Specifically, assume that an agent $i \in N$ misreports its location profile s_i as s_i', then

$$u(s_i, f(s_i', \mathbf{s}_{-i})) \leq u(s_i, f(s_i, \mathbf{s}_{-i}))$$

A mechanism f is *group strategy-proof* if there exists at least one agent in group T who cannot benefit from misreporting. That is, there exists an agent $i \in T \subseteq N$ such that

$$u(s_i, f(\mathbf{s}_T', \mathbf{s}_{-T})) \leq u(s_i, f(\mathbf{s}_T, \mathbf{s}_{-T}))$$

Given an instance $\mathbf{c}$, let $OPT(\mathbf{c})$ be the value of the social utility corresponding to the optimal facility location solution, and $f(\mathbf{c})$ be the value of the social utility corresponding to the facility location solution output by the mechanism f. We say mechanism f has an approximate ratio β if for any instance $\mathbf{c}$ there exists a number β such that $\frac{OPT(\mathbf{c})}{f(\mathbf{c})} \leq \beta$.

3 Single facility location game with obnoxious preference

In this section, we study the single facility location game with obnoxious preferences. The objective is to maximize social utility. In particular, given the set $S \subseteq [0, 1]$ of locations of agents, the placement locations of facilities are selected from the set M of candidate points, and M is within the interval $[l, r]$, $l, r \in \Re$. The utility of an agent is the distance between itself and the facility output by the mechanism, and the social utility is the sum of the utilities of all agents.

It can clearly be seen that finding the best place to locate a facility does not guarantee that the strategy-proofness. For instance, there are an agent location profile $\mathbf{s} = (s_1, s_2) = (\frac{1}{3}, \frac{3}{5})$ and the two facility candidates at point 0, point 1. Since $\sum_{i=1}^{2} d(s_i, 1) > \sum_{i=1}^{2} d(s_i, 0)$, place the facility at point 1 is the optimal facility location solution, that is, $f^* = 1$. However, if agent 2 misreports its location to be 1, the mechanism returns the facility location $f^* = 0$ and agent 2 can benefit from misreporting. In the next analysis, we present a mechanism for group strategy-proofness. The proof of the approximation ratio of the mechanism is not only based on the location of the interval $[l, r]$, but is also related to the intersection relation between it and the set S where the agent is located.

Mechanism 1 *Let n_1 be the number of agents with $s_i \leq \frac{l+r}{2}$ and n_2 be the number of agents with $s_i > \frac{l+r}{2}$. If $n_1 \geq n_2$, return the rightmost facility candidate point r; otherwise, return the leftmost facility candidate point l.*

Theorem 1 *For the single facility location game with obnoxious preferences, Mechanism 1 is group strategy-proof. For the social utility objective, it is optimal if $[l, r] \cap S = \emptyset$, or $[l, r] \cap S \neq \emptyset$, $\frac{l+r}{2} \leq 0$ or $[l, r] \cap S \neq \emptyset$, $\frac{l+r}{2} \geq 1$; and is 3-approximate if $[l, r] \cap S \neq \emptyset$ and $0 < \frac{l+r}{2} < 1$.*

Proof As shown in Fig. 1, there is no intersection between $[l, r]$ and S. As $n_1 < n_2$ in cases (i)(ii)(iii), l (point a in Fig. 1) will be chosen as the location where the facility is placed as the output of the mechanism. And for all the agents, point l is the farthest of all the facility candidates. This means that it is optimal, and no agent is motivated to misreport where they actually are located. Cases (v)(vi)(vii) are analysed similarly.

For case (iv) with $l > 0, r < 1, 0 < \frac{l+r}{2} < 1$. Suppose that N_1 is the set of agents whose location $s_i \in [0, \frac{l+r}{2}]$ and N_2 is the set of agents whose location $s_i \in (\frac{l+r}{2}, 1]$, and $i \in N$. Assuming $n_1 \geq n_2$, then Mechanism 1 selects the rightmost position r as the facility location, i.e., $f = r$. For any $l < f' < f$, we have,

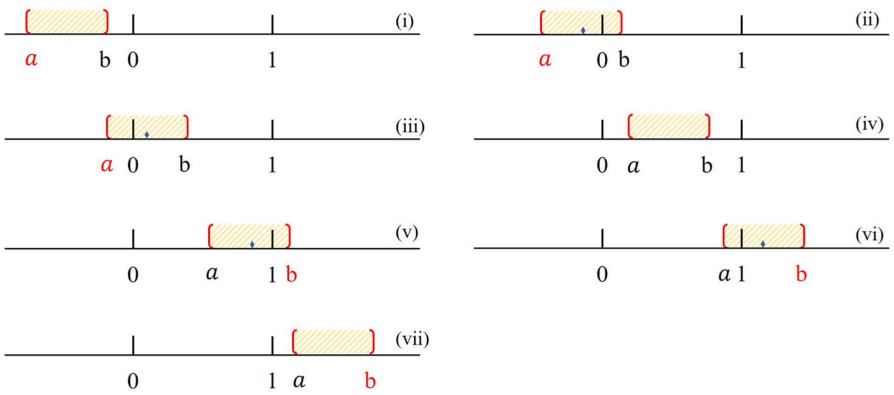

Fig. 1 Seven possible positions of interval $[l, r]$ comparing to $[0, 1]$, $[l, r] \cap S = \emptyset$

$$\begin{aligned}\sum_{i=1}^{n} d(s_i, f) &= \sum_{i \in N_1} d(s_i, f') + \sum_{i \in N_1} d(f', f) + \sum_{i \in N_2} d(s_i, f') - \sum_{i \in N_2} d(f', f) \\ &= \sum_{i=1}^{n} d(s_i, f') + (n_1 - n_2) \cdot d(f', f) \\ &\geq \sum_{i=1}^{n} d(s_i, f'),\end{aligned}$$

where the last inequality holds because $n_1 \geq n_2$, which implies the optimality of f.

Next, we analyse the group strategy-proofness of Mechanism 1. Let $T \subseteq N$ be a group. Our purpose is to prove that there exists at least one agent in T who cannot profit by misreporting about its location information. It can be assumed without loss of generality that $n_1 \geq n_2$, then the facility location solution corresponding to the mechanism is $\mathbf{f} = r$. If the output of the mechanism is to be changed, i.e., the facility is located on point l, at least one agent with $s_i \leq \frac{l+r}{2}$ misreports his/her location to $s_i' > \frac{l+r}{2}$. However, $u(s_i, f(\mathbf{s}_T', \mathbf{s}_{-T})) = d(s_i, l) \leq u(s_i, f(\mathbf{s}_T, \mathbf{s}_{-T})) = d(s_i, r)$, implies no utility increase for agent i.

When there are agents with positions between $[l, r]$, while the midpoint $\frac{l+r}{2}$ is outside of $[0, 1]$, Mechanism 1 is still group strategy-proof and optimal. Suppose $\frac{l+r}{2} \leq 0$, then Mechanism 1 returns $f = l$, which is the farthest facility candidate location for all agents. Therefore, Mechanism 1 is optimal.

When $[l, r] \cap S \neq \emptyset, 0 < \frac{l+r}{2} < 1$, Tang et al. (2022) proved that this mechanism is 3-approximation and group strategy-proof. They further proved that no strategyproof mechanism have an approximation ratio better than 3 for the utility, and no strategyproof mechanism can have a bounded approximation ratio for the minimum utility. $\square$

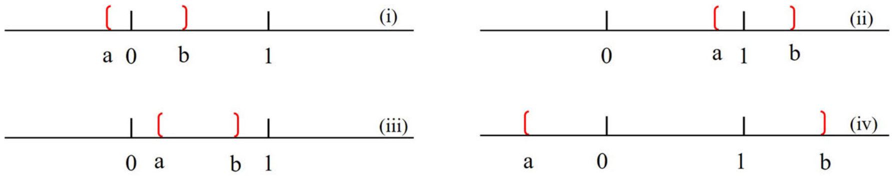

Fig. 2 Four possible positions of interval $[l, r]$ comparing to $[0, 1]$

4 Heterogeneous two-facility location game with obnoxious preferences

In this section, we will select two positions in the facility candidate location set M as the placement points for two heterogeneous facilities, and $M \subseteq [l, r], l, r \in \Re$. Agents remain situated in the interval $[0, 1]$. They are eager to increase the total distance to two facilities, that is, for agent i, his utility is $d(s_i, f_1) + d(s_i, f_2)$. Let the optimal location solution be $\mathbf{f}^* = (f_1^*, f_2^*)$ and the output of our mechanism as (f_1, f_2). Without loss of generality, suppose $f_1^* < f_2^*$ and $f_1 < f_2$.

Mechanism 2 *Two facilities are placed at the two candidate locations which are farthest from point* 0.

Theorem 2 *For the heterogeneous two-facility location game with obnoxious preferences, Mechanism 2 is a group strategy-proof optimal mechanism if* $[l, r] \subseteq (-\infty, 0]$ *or* $[l, r] \subseteq [1, +\infty)$.

Proof The facility placement strategy of Mechanism 2 is independent of the reported information of the agent, the group strategy-proofness of Mechanism 2 can be easily proven. The conclusion of optimal is trivial. Mechanism 2 guarantees that the locations of the output facilities are the farthest away for each agent compared to the other candidate locations. □

Considering other possible situations in the interval $[l, r]$, we proceed with our study.

Mechanism 3 (Endpoints Mechanism) *Select two endpoints of the facility candidate locations, i.e., point l and point r.*

Theorem 3 *Mechanism 3 is group strategy-proof for the heterogeneous two-facility location game with obnoxious preferences under the social utility objective. When* $[l, r] \subseteq [0, 1]$, *or* $l < 0$ *and* $r > 1$,*or* $l < 0$ *and* $0 < r \leq 1$, *or* $0 \leq l < 1$ *and* $r > 1$, *the approximation ratio is 2.*

Proof The output of the mechanism is not affected by misreporting by any agent, i.e. the output of the mechanism is fixed. So, Mechanism 3 is group strategy-proof.

Four cases are shown in Fig. 2. In either case, the approximate ratio of Mechanism 3 is 2. For each agent, it is either farthest from l or r. For one agent, suppose it's farther from point l than it is from point r. Its utility obtained from any mechanism is at most $2|s_i - l|$. Since our mechanism outputs two endpoints, it must output the point l. Then, the utility of the agent under our mechanism is at least $|s_i - l|$. Thus, when all agents are considered, the social utility of Mechanism 3 is at least half of the optimal social utility. Therefore, the approximate ratio of Mechanism 3 is 2. □

As for the lower bound of Mechanism 3, consider the following example. Assume that all agents are located at point ε. The facility candidate locations set $M = \{0, 1-\varepsilon, 1\}$. Mechanism 3 outputs two points, point $l = 0$ and point $r = 1$, and the social utility is $n(1-0)$. The optimal solution will give the two locations, point $1-\varepsilon$ and point 1. The maximum social utility is $n(2-3\varepsilon)$. So we have $\frac{n(2-3\varepsilon)}{n(1-0)} \to 2$, when $\varepsilon \to 0$. Therefore, the analysis of the approximation ratio is tight.

Theorem 4 *For the heterogeneous two-facility location game with obnoxious preferences, no deterministic strategy-proof mechanism f can have an approximation ratio better than $\frac{3}{2}$ if $[l, r] \subseteq [0, 1]$, or $l < 0$ and $r > 1$, or $l < 0$ and $0 < r \leq 1$, or $0 \leq l < 1$ and $r > 1$, under the social utility objective.*

Proof Suppose mechanism f is a deterministic strategy-proof mechanism with an approximation ratio of less than $\frac{3}{2}$. Consider an example where two agents are located at points $\frac{1}{2}-\varepsilon$ and $\frac{1}{2}+\varepsilon$, and the set of facility candidate locations $M = \{0, \varepsilon, 1-\varepsilon, 1\}$. First of all, suppose that f places at least one facility in the position of point $1-\varepsilon$ or point 1. In this instance, agent 2's utility is at most $\frac{1}{2}+\varepsilon+1-(\frac{1}{2}+\varepsilon) = 1$. However, if agent 2 misreports its location to point 1, and in order to achieve the approximate ratio, f must output two locations, points 0 and ε, as facilities placements. And thus the utility of agent 2 increases to $\frac{1}{2}+\varepsilon+\frac{1}{2} = 1+\varepsilon$, giving a contradiction to the strategy-proofness. Secondly, the proof is similar if mechanism f places at least one facility in the position of point 0 or point ε. □

Theorem 5 *Mechanism 3 is 2-approximation group strategy-proof for the heterogeneous two-facility location game with obnoxious preferences under the minimum utility objective.*

Proof The discussion is divided into two cases. First, there are agents with positions between $[l, r]$; Second, there are no agent with position between $[l, r]$. In the first case, the agent with the minimum utility of Mechanism 3 exists between the interval $[l, r]$ and its utility is $r-l$. For the agent, the maximum utility that the optimal solution can achieve is $2(r-l)$, so the approximate ratio is 2. In the second case, the agent with the minimum utility of Mechanism 3 exists in the set of agents closest to the endpoints l,r. Suppose there exist two agents s_i and s_j on each side of l, r, and $|s_i - r| \geq |s_j - l|$. Then, the minimum utility at this point $u_j \geq |s_j - l|$. And for the agent, the maximum utility that the optimal solution can achieve is $2|s_j - l|$, so the approximate ratio is 2. In summary, Mechanism 3 achieves an approximate ratio of 2 even with the objective of maximizing the minimum utility.

Regarding the lower bound of the mechanism, consider the following example. Suppose there exist three facility candidates: $l, l+\varepsilon$ and r. There is an agent at point r, and the mechanism outputs points l and r with minimum utility of $r-l$. The optimal facility solution is points l and $l+\varepsilon$ with minimum utility of $r-l+r-l-\varepsilon$. When $\varepsilon \to 0$, the ratio is 2. □

Theorem 6 *For the heterogeneous two-facility location game with obnoxious preference, no deterministic strategy-proof mechanism f can have an approximation ratio better than $\frac{3}{2}$ under the minimum utility objective.*

Proof Suppose f is a strategy-proof mechanism with approximation ratio better than $\frac{3}{2}$ for heterogeneous two-facility location. Consider an instance with 2 agents: the location profile is $(0, 1)$. There are 3 facility candidates, points 0, $\frac{2}{5}$ and $\frac{3}{5}$. The optimal minimum utility is 1, attained by solutions $(\frac{2}{5}, \frac{3}{5})$. It is not hard to see that any other solution has a utility of at most $\frac{3}{5}$. By the approximation ratio, f must output one of solutions $(\frac{2}{5}, \frac{3}{5})$.

We consider the case when f outputs $(\frac{2}{5}, \frac{3}{5})$. Assume that agent 2 misreports her location as $s_2' = \frac{3}{5}$. In this new instance, the optimal solution is $(0, \frac{3}{5})$, and the optimal minimum utility is $\frac{3}{5}$. Any other solution has a minimum utility of at most $\frac{2}{5}$. By the approximation ratio, f must output the optimal solution $(0, \frac{3}{5})$. Then the utility of agent 2 after the misreporting is $\frac{7}{5}$, while the utility when reporting truthfully is 1, which gives a contradiction to the strategy-proofness. □

Remark 1 We remark that, for the obnoxious *homogeneous* two-facility location game, there is no deterministic strategy-proof mechanism with bounded approximation ratio under the minimum utility objective. Given the agent's location profile $(\varepsilon, 1)$ with a sufficiently small ε. The set of facility candidate locations is $\{0, \frac{1}{2}, \frac{1}{2}+\varepsilon\}$. Suppose the approximation ratio of a deterministic strategy-proof mechanism f is c. To achieve an approximate ratio of c, the mechanism f will place the facilities at points $\frac{1}{2}$ and $\frac{1}{2} + \varepsilon$. In this case, agent 2 misreports its location to point $\frac{1}{2} + \varepsilon$, then mechanism f outputs points 0 and $\frac{1}{2}$. The utility of agent 2 increases from the original $\frac{1}{2} - \varepsilon$ to $\frac{1}{2}$, which contradicts the strategy-proofness of the mechanism.

5 Dual preference single facility location games

In this section, we consider the dual preferences of agents, that is, agents may have different preferences ($p_i = 0$ or $p_i = 1$) for the same facility. When the agents can only misreport locations, there are two special cases where the agent's preferences are all 1 and all 0, corresponding to the classical facility location game and the facility location game with obnoxious preferences, respectively. Thus, we study the cases when the agents can only misreport preferences, and when the agents can misreport both locations and preferences. We assume that $[l, r] \subseteq [0, 1]$.

5.1 Private preferences

First, we assume that the location information of each agent is known, and agents are required to report preference information by themselves. Thus, we begin with an analysis of agents misreport only on preferences.

Mechanism 4 *Locate the facility at the candidate location f such that* $\arg\max_{f \in M} \sum_{i=1}^{n} u(s_i, f)$.

Theorem 7 *For the dual preference single facility location game, Mechanism 4 is strategy-proof and optimal under the social utility objective, when agents only misreport on preference.*

Proof Suppose the mechanism originally output $\mathbf{f} = f$, and there is an agent s_i misreports profile $c_i = (s_i, p_i) = (s_i, 0)$ to $c_i' = (s_i, p_i') = (s_i, 1)$ that causes the mechanism to output $\mathbf{f}' = f'$.

Assume $s_i \in [0, l]$, then $u(c_i, f) = d(s_i, f)$ and $u(c_i', f) = r - d(s_i, f)$. By definition, we get

$$\begin{aligned} SU(\mathbf{c}, \mathbf{f}) = \sum_{i=1}^{n} u(c_i, \mathbf{f}) &= u(c_i, \mathbf{f}) + \sum_{j \in [1,n] \& j \neq i} u(c_j, \mathbf{f}) \\ &= d(s_i, f) + \sum_{j \in [1,n] \& j \neq i} u(c_j, \mathbf{f}), \end{aligned}$$

and

$$\begin{aligned} SU(\mathbf{c}', \mathbf{f}) = \sum_{i=1}^{n} u(c_i', \mathbf{f}) &= u(c_i', \mathbf{f}) + \sum_{j \in [1,n] \& j \neq i} u(c_j, \mathbf{f}) \\ &= r - d(s_i, f) + \sum_{j \in [1,n] \& j \neq i} u(c_j, \mathbf{f}). \end{aligned}$$

We assume that $df(f) = SU(\mathbf{c}', \mathbf{f}) - SU(\mathbf{c}, \mathbf{f}) = r - 2 \cdot d(s_i, f)$, because of except for agent i, no other agents' preference have changed. Similarly, $df(f') = SU(\mathbf{c}', \mathbf{f}') - SU(\mathbf{c}, \mathbf{f}') = r - 2 \cdot d(s_i, f')$. That's given by Mechanism 4, $SU(\mathbf{c}, \mathbf{f}) \geq SU(\mathbf{c}, \mathbf{f}')$, $SU(\mathbf{c}', \mathbf{f}') \geq SU(\mathbf{c}', \mathbf{f})$. Thus, $SU(\mathbf{c}, \mathbf{f}') + df(f') \geq SU(\mathbf{c}, \mathbf{f}) + df(f)$, $df(f') - df(f) \geq SU(\mathbf{c}, \mathbf{f}) - SU(\mathbf{c}, \mathbf{f}') \geq 0$, i.e., $r - 2 \cdot d(s_i, f') - [r - 2 \cdot d(s_i, f)] \geq 0$, that is, $d(s_i, f) \geq d(s_i, f')$.

The analysis is similar when the agent is located in other position intervals. The situation is similar if $p_i = 1$. □

5.2 Private preferences and locations

In this section, we continue with the analysis of situations where the preferences and location of an agent can be misreported at the same time. Let's start by defining some of the required symbols.

Let $H = \{i | p_i = 0\}$, $L = \{i | p_i = 1\}$,
$H_1 = \{i | s_i \in [0, \frac{l+r}{2}], i \in H\}$, $H_2 = \{i | s_i \in (\frac{l+r}{2}, 1], i \in H\}$,
$L_1 = \{i | s_i \in [0, \frac{l+r}{2}], i \in L\}$, $L_2 = \{i | s_i \in (\frac{l+r}{2}, 1], i \in L\}$,
$L_{11} = \{i | s_i \in [0, l), i \in L\}$, $L_{12} = \{i | s_i \in [l, \frac{l+r}{2}], i \in L\}$,
$L_{21} = \{i | s_i \in (\frac{l+r}{2}, r), i \in L\}$, $L_{22} = \{i | s_i \in [r, 1], i \in L\}$,
$h = |H|$, $h_1 = |H_1|$, $h_2 = |H_2|$,
$l = |L|$, $l_1 = |L_1|$, $l_2 = |L_2|$,

$l_{11} = |L_{11}|, l_{12} = |L_{12}|, l_{21} = |L_{21}|, l_{22} = |L_{22}|,$
$n_1 = h_1 - l_1, n_2 = h_2 - l_2.$

According to the previous definition, when $i \in H$, $u_i = d(s_i, f)$; when $i \in L_{11}$, $u_i = r - d(s_i, f)$; when $i \in L_{12}$, $u_i = r - l - d(s_i, f)$; when $i \in L_{21}$, $u_i = r - l - d(s_i, f)$; when $i \in L_{22}$, $u_i = 1 - l - d(s_i, f)$.

Mechanism 5 *If $n_1 \geq n_2$, return the rightmost facility candidate point r; otherwise, return the leftmost facility candidate point l.*

Theorem 8 *For the dual preference single facility location game, Mechanism 5 is group strategy-proof. It is 4-approximate under the social utility objective when agents both misreport on preference and location.*

Proof First, we analyse the group strategy-proofness of Mechanism 5. Let $T \in N$ be a union. Our objective is to demonstrate that there exists at least one agent in T who does not profit from lying. Without loss of generality, we assume that $n_1 \geq n_2$, and the mechanism outputs the facility location profile $\mathbf{f} = r$. In order to alter the output of the mechanism, i.e., relocating the facility to point l,, it is necessary for at least one agent member to intentionally misreport information in two ways. (1) One agent with $s_i \in [0, \frac{l+r}{2}]$ and $p_i = 0$ misreports his/her location to $s_i' \in (\frac{l+r}{2}, 1]$ or preference to $p_i' = 1$. However, $d(s_i, r) \geq d(s_i, l)$, implies no utility increase for agent i. (2) One agent with $s_i \in (\frac{l+r}{2}, 1]$ and $p_i = 1$ misreports his/her location to $s_i' \in [0, \frac{l+r}{2}]$ or preference to $p_i' = 0$. However, if $s_i \in (\frac{l+r}{2}, r)$, $r - l - d(s_i, r) > r - l - d(s_i, l)$; if $s_i \in [r, 1]$, $1 - l - d(s_i, r) > 1 - l - d(s_i, l)$, implies no utility increase for agent i. In summary, at least one of the agent members does not benefit from misreporting, i.e., the group strategy-proofness of the mechanism is proved.

Suppose $n_1 \geq n_2$, the optimal location is $\mathbf{f}^* = f^*$ and Mechanism 5 returns $f = r$. Then,

$$\begin{aligned}
SU(\mathbf{s}, \mathbf{f}) &= \sum_{i \in H_1} u(s_i, \mathbf{f}) + \sum_{i \in H_2} u(s_i, \mathbf{f}) + \sum_{i \in L_1} u(s_i, \mathbf{f}) + \sum_{i \in L_2} u(s_i, \mathbf{f}) \\
&= \sum_{i \in H_1} d(s_i, r) + \sum_{i \in H_2} d(s_i, r) + \sum_{i \in L_{11}} [r - d(s_i, r)] + \sum_{i \in L_{12}} [r - l - d(s_i, r)] \\
&\quad + \sum_{i \in L_{21}} [r - l - d(s_i, r)] + \sum_{i \in L_{22}} [1 - l - d(s_i, r)] \\
&\geq h_1 \cdot \frac{r-l}{2} + \sum_{i \in H_2} d(s_i, r) + \sum_{i \in L_{11}} s_i + \sum_{i \in L_{12}} (s_i - l) \\
&\quad + l_{21} \cdot \frac{r-l}{2} + l_{22} \cdot 2 \cdot \frac{r-l}{2} \\
&\geq h_1 \cdot \frac{r-l}{2} + l_2 \cdot \frac{r-l}{2}.
\end{aligned}$$

Let $D_1 = SU(\mathbf{s}, \mathbf{f}) - (h_1 \cdot \frac{r-l}{2} + l_2 \cdot \frac{r-l}{2})$, we have

$$D_1 = \sum_{i \in H_1} d(s_i, r) + \sum_{i \in H_2} d(s_i, r) + \sum_{i \in L_{11}} s_i + \sum_{i \in L_{12}} (s_i - l)$$

$$
\begin{aligned}
&+ \sum_{i \in L_{21}} (s_i - l) + \sum_{i \in L_{22}} [(r-l) + (1-s_i)] - (h_1 \cdot \frac{r-l}{2} + l_2 \cdot \frac{r-l}{2}) \\
&= \sum_{i \in H_1} d(\frac{l+r}{2}, s_i) + \sum_{i \in H_2} d(s_i, r) + \sum_{i \in L_{11}} s_i + \sum_{i \in L_{12}} (s_i - l) \\
&+ \sum_{i \in L_{21}} d(\frac{l+r}{2}, s_i) + \sum_{i \in L_{22}} [\frac{r-l}{2} + (1-s_i)].
\end{aligned}
$$

Consider a new location profile $\mathbf{s}'$, in which there are h_1 agents at point $\frac{l+r}{2}$, h_2 agents are at point r, l_{11} agents at point 0 , l_{12} agents are at point l, l_{21} agents at point $\frac{l+r}{2}$, l_{22} agents are at point 1. Next, we show that

$$
SU(\mathbf{s}', \mathbf{f}^*) \leq 4(h_1 + l_2) \cdot \frac{r-l}{2}.
$$

We discuss two cases, $f^* \in [0, \frac{l+r}{2}]$ and $f^* \in (\frac{l+r}{2}, 1]$. When the optimal solution is $f^* \in [0, \frac{l+r}{2}]$, for agents whose preference is 0, the total utility of h_1 agents located at $\frac{l+r}{2}$ is $h_1 \cdot d(\frac{l+r}{2}, f^*)$, and the total utility of h_2 agents located at r is $h_2 \cdot d(r, f^*)$. For agents whose preference is 1, first, the total utility of the l_{11} agents located at 0 is $l_{11} \cdot [r - d(0, f^*)] = l_{11} \cdot d(r, f^*)$, and that of the l_{12} agents at point l is $l_{12} \cdot [r - l - d(l, f^*)] = l_{12} \cdot d(r, f^*)$. Further, the total utility of the l_{21} agents at point $\frac{l+r}{2}$ is $l_{21} \cdot [r - l - d(\frac{l+r}{2}, f^*)]$, and that of the l_{22} agents at point 1 is $l_{22} \cdot [1 - l - d(1, f^*)]$. Therefore, we have

$$
\begin{aligned}
SU(\mathbf{s}', \mathbf{f}^*) &= h_1 \cdot d(\frac{l+r}{2}, f^*) + h_2 \cdot d(r, f^*) + l_{11} \cdot d(r, f^*) + l_{12} \cdot d(r, f^*) \\
&+ l_{21} \cdot [r - l - d(\frac{l+r}{2}, f^*)] + l_{22} \cdot [1 - l - d(1, f^*)] \\
&\leq h_1 \cdot d(\frac{l+r}{2}, f^*) + h_1 \cdot d(r, f^*) + l_2 \cdot d(r, f^*) \\
&+ l_{21} \cdot [r - l - d(\frac{l+r}{2}, f^*)] + l_{22} \cdot [1 - l - d(1, f^*)] \\
&\leq h_1 \cdot \frac{r-l}{2} + h_1 \cdot 2 \cdot \frac{r-l}{2} + l_2 \cdot 2 \cdot \frac{r-l}{2} \\
&+ l_{21} \cdot 2 \cdot \frac{r-l}{2} + l_{22} \cdot \frac{r-l}{2} \\
&< 3h_1 \cdot \frac{r-l}{2} + 4l_2 \cdot \frac{r-l}{2} < 4(h_1 + l_2) \cdot \frac{r-l}{2}.
\end{aligned}
$$

When $f^* \in (\frac{l+r}{2}, 1]$, since $d(r, f^*) \leq \frac{r-l}{2}$ and $[1 - l - d(1, f^*)] \leq 2 \cdot \frac{r-l}{2}$, similarly, we have

$$
SU(\mathbf{s}', \mathbf{f}^*) \leq h_1 \cdot d(\frac{l+r}{2}, f^*) + h_1 \cdot d(r, f^*) + l_2 \cdot d(r, f^*)
$$

$$
\begin{aligned}
&+ l_{21} \cdot [r - l - d(\frac{l+r}{2}, f^*)] + l_{22} \cdot [1 - l - d(1, f^*)] \\
&\leq h_1 \cdot \frac{r-l}{2} + h_1 \cdot \frac{r-l}{2} + l_2 \cdot \frac{r-l}{2} \\
&+ l_{21} \cdot 2 \cdot \frac{r-l}{2} + l_{22} \cdot 2 \cdot \frac{r-l}{2} \\
&= 2h_1 \cdot \frac{r-l}{2} + 3l_2 \cdot \frac{r-l}{2} < 4(h_1 + l_2) \cdot \frac{r-l}{2}.
\end{aligned}
$$

Let D_2 be the difference of $SU(\mathbf{s}, \mathbf{f}^*)$ and $SU(\mathbf{s}', \mathbf{f}^*)$. Then,

$$
\begin{aligned}
D_2 =& SU(\mathbf{s}, \mathbf{f}^*) - SU(\mathbf{s}', \mathbf{f}^*) \\
=& \sum_{i \in H_1} [d(s_i, f^*) - d(\frac{l+r}{2}, f^*)] + \sum_{i \in H_2} [d(s_i, f^*) - d(r, f^*)] \\
&+ \sum_{i \in L_{11}} [r - d(s_i, f^*) - r + d(0, f^*)] \\
&+ \sum_{i \in L_{12}} [r - l - d(s_i, f^*) - r + l + d(l, f^*)] \\
&+ \sum_{i \in L_{21}} [r - l - d(s_i, f^*) - r + l + d(\frac{l+r}{2}, f^*)] \\
&+ \sum_{i \in L_{22}} [1 - l - d(s_i, f^*) - 1 + l + d(1, f^*)].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
D_1 - D_2 = & \sum_{i \in H_1} [d(\frac{l+r}{2}, s_i) + d(\frac{l+r}{2}, f^*) - d(s_i, f^*)] \\
&+ \sum_{i \in H_2} [d(r, s_i) + d(r, f^*) - d(s_i, f^*)] \\
&+ \sum_{i \in L_{11}} [d(s_i, 0) + d(s_i, f^*) - d(0, f^*)] \\
&+ \sum_{i \in L_{12}} [d(s_i, l) + d(s_i, f^*) - d(l, f^*)] \\
&+ \sum_{i \in L_{21}} [d(s_i, \frac{l+r}{2}) + d(s_i, f^*) - d(\frac{l+r}{2}, f^*)] \\
&+ \sum_{i \in L_{22}} [\frac{r-l}{2} + d(s_i, 1) + d(s_i, f^*) - d(1, f^*)].
\end{aligned}
$$

By the triangle inequality, it is easy to get that $D_1 - D_2 \geq 0$. In summary,

$$\begin{aligned}\frac{OPT(\mathbf{s})}{SU(\mathbf{s},\mathbf{f})} &= \frac{SU(\mathbf{s},\mathbf{f}^*)}{SU(\mathbf{s},\mathbf{f})} \leq \frac{SU(\mathbf{s},\mathbf{f}^*) - D_1}{SU(\mathbf{s},\mathbf{f}) - D_1} \leq \frac{SU(\mathbf{s},\mathbf{f}^*) - D_2}{SU(\mathbf{s},\mathbf{f}) - D_1} \\ &= \frac{SU(\mathbf{s}',\mathbf{f}^*)}{h_1 \cdot \frac{r-l}{2} + l_2 \cdot \frac{r-l}{2}} < \frac{4(h_1 + l_2) \cdot \frac{r-l}{2}}{h_1 \cdot \frac{r-l}{2} + l_2 \cdot \frac{r-l}{2}} = 4.\end{aligned}$$

We established the proof. □

6 Conclusion

This work studies the discrete facility location games with obnoxious preferences and dual preferences.

We propose deterministic strategy-proof mechanisms and evaluate how well they approximate the optimal solution. For facility location games with obnoxious preferences, we consider both the single facility location game and the two-facility location game. In the single facility game, the agent's utility is defined as the distance between the agent and that facility; in the two-facility game, the agent's utility is defined as the sum of the distances to the two facilities in the heterogeneous case; and in the homogeneous case, it is defined as the minimum distance between an agent and any facility.

Under obnoxious preferences, we propose a group strategy-proof mechanism with a 3-approximation ratio for the single facility location game, and prove that this ratio is tight. We provide a 2-approximation mechanism and an optimal group strategy-proof mechanism for the heterogeneous two-facility game under social utility and minimum utility objectives in various situations of $[l, r]$. We have also demonstrated that no deterministic strategy-proof mechanism can achieve an approximation ratio of less than $\frac{3}{2}$ for the social utility, and no such mechanism has bounded ratio for the minimum utility. Under dual preferences, we have developed a group strategy-proof mechanism with an approximation ratio of 4 for single facility location games under the social utility objective.

For future directions, it is interesting to consider the scenarios where agents are distributed on a tree structure, and extend the results to general graphs.

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

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