



Derivations of large classes of facet defining inequalities of the weak order polytope using ranking structures

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Abstract

Ordering polytopes have been instrumental to the study of combinatorial optimization problems arising in a variety of fields including comparative probability, computational social choice, and group decision-making. The weak order polytope is defined as the convex hull of the characteristic vectors of all binary orders on n alternatives that are reflexive, transitive, and total. By and large, facet defining inequalities (FDIs) of this polytope have been obtained through simple enumeration and through connections with other combinatorial polytopes. This paper derives five new large classes of FDIs by utilizing the equivalent representations of a weak order as a ranking of n alternatives that allows ties; this connection simplifies the construction of valid inequalities, and it enables groupings of characteristic vectors into useful structures. We demonstrate that a number of FDIs previously obtained through enumeration are actually special cases of the large classes. This work also introduces novel construction procedures for generating affinely independent members of the identified ranking structures. Additionally, it states two conjectures on how to derive many more large classes of FDIs using the featured techniques.

Keywords Order polyhedra · Weak orders · Rankings

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1 Introduction

Ordering polytopes have been instrumental to the study of problems from a variety of fields including comparative probability (Anscombe et al. 1963; Chevryev et al. 2013), computational social choice (Barthelemy and Monjardet 1981; Marley and Regenwetter 2016) and group decision-making (Marti and Reinelt 2011; Yoo and Escobedo 2021; Yoo et al. 2020). The derivation of facet defining inequalities (FDIs) for these polytopes is a focal point owing primarily to their usefulness in tackling challenging combinatorial optimization problems. For example, their implementation via branch-and-cut approaches has been cited as an effective solution methodology (e.g., see Coll et al. 2002; Grötschel and Wakabayashi 1990; Marti and Reinelt 2011). Such polyhedral studies have historically centered on the linear ordering polytope (e.g., see Bolotashvili et al. 1999; Fishburn 1992; Grötschel et al. 1985). In recent years, different classes of FDIs have been introduced for various other ordering polytopes (e.g., see Doignon and Fiorini 2001; Doignon and Rexhep 2016; Müller 1996; Oswald and Reinelt 2003), driven by a fundamental need to consider a broader range of ordinal relationships. For example, the incorporation of non-strict orderings (i.e., possibly containing ties) is regarded as a fundamental form of preference expression in various group decision-making applications (Yoo and Escobedo 2021).

There are many situations in which it is necessary to determine an ordering of all alternatives of a set $N := \{1, \dots, n\}$ or, equivalently a *complete ranking* of n alternatives that best achieves a given objective. For instance, the objective of the consensus ranking problem is to find a ranking $\mathbf{r} \in N^n$ that minimizes the cumulative distance to a set of input rankings $\{\mathbf{a}^k\}_{k=1}^m$, the latter of which typically represents a collection of evaluations of the alternatives provided by m sources. It is often prudent, if not necessary, to allow the consensus ranking (as well as the input rankings (Kendall 1945)) to contain ties for a myriad of reasons—e.g., contradictory information in the given evaluations, high cost of obtaining a strict linear ordering when more than a handful of alternatives are involved, etc. However, the universe of complete non-strict rankings is exceedingly large. This underscores the ongoing need to better characterize the weak order polytope, P_{WO}^n , whose vertices are equivalent to the individual members of this set. Specifically, a *weak order* on N is defined as a binary relation that is reflexive, transitive, and total.

For the most part, FDIs of P_{WO}^n have been either obtained through enumeration approaches with specialized software or derived from known FDIs of other existing polytopes. The complete sets of FDIs for P_{WO}^4 and P_{WO}^5 listed in Fiorini and Fishburn (2004) and Regenwetter and Davis-Stober (2012) respectively, were generated using the `Porta` program (Christof et al. 1997). Furthermore, in Doignon and Fiorini (2001) certain members of the classes of *2-partition*, *2-chorded cycle*, *2-chorded path*, and *2-chorded wheel* FDIs (Grötschel and Wakabayashi 1990) of the partial order polytope P_{PA}^n (also known as the clique partitioning polytope) were lifted into FDIs of P_{WO}^n . This work introduces a new approach for deriving large classes of FDIs for P_{WO}^n from structural insights that are rooted in the connection between weak orders and complete non-strict rankings.

The main contributions of this work are as follows. We derive six large classes of valid inequalities (VIs) of P_{WO}^n , for $n \geq 4$. Each VI class is obtained by considering a

relatively small number of ordinal relationship combinations, specifically the orderings of only up to two alternatives with one another and with an unspecified number of other alternatives, which are treated as a group. We prove that five of the six new large classes of VIs are FDIs. This is done by characterizing the weak orders that satisfy a VI at equality into a small number of *ranking structures* and then devising construction procedures that generate expedient sequences of affinely independent points from them. It is important to remark that each fixed value of n provides a VI expression that is not applicable in a dimension lower than n ; in other words, the statement about the applicability of these FDI classes to any $n \geq 4$ is not due to the Lifting Lemma (Fiorini and Fishburn 2004). In fact, we show that a number of FDIs previously obtained through enumeration are special cases of these large classes of FDIs. Lastly, we conjecture how the above contributions may be extended to derive numerous other large classes of VIs, many of which we expect to be FDIs.

The rest of this paper is organized as follows: Sect 2 introduces basic notation and definitions used throughout this work. Section 3 introduces six classes of valid inequalities of the weak order polytope, along with a description of the ranking structures whose members satisfy each inequality at equality. Section 4 introduces novel characteristic vector construction procedures, which are then utilized to demonstrate that five of the six aforementioned large classes of valid inequalities are facet defining for any $n \geq 4$. Section 5 concludes with additional insights and two conjectures on how the fundamental insights herein presented may induce additional large classes of FDIs.

2 Notation, definitions, and other background concepts

Let $N := [n]$ be a set of alternatives or objects, where $[n]$ is shorthand for the set $\{1, \dots, n\}$, and let $\mathcal{A}^N := \{(i, j) : i, j \in N, i \neq j\}$ be the set of all possible ordered pairings on N . Let \mathcal{W} be the family of all weak orders on N . In certain related works, the elements of a weak order $W \in \mathcal{W}$ are expressed as $i \leq j$ (i is not preferred over j). Without loss of generality, we describe the elements of W as $i \geq j$ (alternative i is preferred over or tied with j) and express this preference relation succinctly as the ordered pair (i, j) . We do this primarily to help make the description of ranking structures more intuitive. Next, we introduce three different representations of W to be used throughout this paper.

Definition 1 The characteristic vector representation of $W \in \mathcal{W}$ is typified by a vector $x^W \in \{0, 1\}^{\mathcal{A}^N}$ whose entry $(i, j) \in \mathcal{A}^N$ is given by:

$$x_{(i,j)}^W = \begin{cases} 1 & \text{if } (i, j) \in W \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2 The (unique) *ranking representation* of $W \in \mathcal{W}$ is typified by a vector $r^W \in N^n$ whose entry $i \in N$ is given by:

$$r_i^W = n - \sum_{j \in N \setminus \{i\}} \mathbb{I}(i \succeq j \in W) = n - \sum_{(i,j) \in \mathcal{A}^N} x_{(i,j)}^W.$$

Definition 3 The *alternative-ordering representation* of $W \in \mathcal{W}$ can be typified by a collection of nonempty subsets $S_1^W, \dots, S_p^W \subseteq N$ which forms a preference partition of N of size p , written as $\{\{S_k^W\}_{k=1}^p\}_{\succeq}$, if:

- (a) $S_k^W \cap S_{k'}^W = \emptyset$ for $k \neq k'$;
- (b) $S_1^W \cup S_2^W \cup \dots \cup S_p^W = N$;
- (c) $i \approx j \forall i, j \in S_k^W$; $i \succ j$ for $i \in S_k^W, j \in S_{k'}^W$ where $1 \leq k < k' \leq p$;

where, for a pair of elements $i, j \in N$, the expression “ $i \approx j$ ” indicates that alternative i is tied with alternative j and “ $i \succ j$ ” indicates that i is strictly preferred over j . Here, the k th subset of the partition (S_k^W), contains all alternatives in the k th *bucket* or equivalence class of W .

Stated otherwise, $\{\{S_k^W\}_{k=1}^p\}_{\succeq}$ is an ordered set of sets. Its k th element, S_k^W , contains alternatives that are all tied amongst themselves. An example application of the presented terminology is as follows.

Example 1 Let $W = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (4, 3)\}$. The corresponding ranking is $r^W = (1, 1, 4, 3)$; the rank of alternative 3 is $r_3^W = 4$. The corresponding alternative-ordering is $\{\{S_k^W\}_{k=1}^3\}_{\succeq} = \{\{1, 2\}, \{4\}, \{3\}\}$; and the contents of bucket 2 are $S_2^W = \{4\}$.

For notational convenience, when working with a collection of m weak orders, each $W^w \in \mathcal{W}$ may be represented in abbreviated fashion as x^w, r^w , or $\{S^w\}$ for $w = 1, \dots, m$, when there is no ambiguity; the number of buckets is evident from the given context or is written as $|\{S^w\}|$. Additionally, individual elements in the first two representations may be written as $x_{(i,j)}^w$ and r_i^w , respectively, for $i, j \in N$, and individual subsets in the third may be written as S_k^w for $k \leq |\{S^w\}|$.

Next, we state a few relevant polyhedral theory concepts.

Definition 4 A polyhedron P is a subset of \mathbb{R}^n that can be described by a finite set of linear constraints as $P = \{x : \pi x \leq \pi_0\} \subseteq \mathbb{R}^n$.

Definition 5 The ordered pair (π, π_0) is a valid inequality (VI) for a polyhedron P if $\pi x \leq \pi_0$ holds $\forall x \in P$ or, equivalently, $\max\{\pi x : x \in P\} \leq \pi_0$.

Definition 6 The valid inequality (π, π_0) defines a facet of a polyhedron P if $\emptyset \neq P \cap \{x : \pi x = \pi_0\} \neq P$ and if there exists $\dim(P)$ affinely independent points in $P \cap \{x : \pi x = \pi_0\}$, i.e., the dimension of the VI is $\dim(P) - 1$.

Since P_{WO}^n has dimension $n(n - 1)$ (i.e., the polyhedron is full-dimensional), its facets have dimension $n(n - 1) - 1$ (Gurgel 1992). The FDIs of P_{WO}^n for $n \geq 3$ that follow directly from the characteristic vector representation and the completeness and transitivity properties of a weak order are (Fiorini and Fishburn 2004):

$$x_{ij} \leq 1 \quad i, j = 1, \dots, n; i \neq j \tag{1a}$$

$$x_{ij} + x_{ji} \geq 1 \quad i, j = 1, \dots, n; i \neq j \tag{1b}$$

$$x_{ij} - x_{ik} - x_{kj} \geq -1 \quad i, j, k = 1, \dots, n; i \neq j \neq k \neq i \tag{1c}$$

where x_{ij} is used in place of $x_{(i,j)}^W$ for visual simplification.

Later in this work, we refer to the FDIs of P_{WO}^4 , as categorized into nine classes WO_1-WO_9 (Regenwetter and Davis-Stober 2008) in Table 1. For visual clarity, only nonzero coefficients of WO_i are displayed and the coordinates $(j_k, j_{k'})$ of each VI coefficient-vector $\pi \in \mathbb{R}^{4 \times 3}$ are abbreviated as $j_k j_{k'}$, for $(k, k') \in \mathcal{A}^4$.

The cardinality of WO_i in P_{WO}^4 , written in the rightmost table column as $|WO_i^4|$, is obtained by counting the number of ways that the labels $j_1, j_2, j_3, j_4 \in [4]$ can be permuted within each expression. We remark that WO_1-WO_9 can be reduced into just seven classes (Fiorini 2001; Fiorini and Fishburn 2004) by leveraging symmetries between WO_6 and WO_7 and between WO_8 and WO_9 . In the list, WO_1-WO_3 match Inequalities (1a)–(1c) and are known as the *axiomatic* inequalities, which comprise the full set of FDIs for P_{WO}^3 . They are also FDIs for $n \geq 4$ due to the Lifting Lemma of Fiorini and Fishburn (2004) restated below for future reference.

Lemma 1 (Lifting Lemma (Fiorini and Fishburn 2004)) *Let (π, π_0) be an FDI of P_{WO}^n , and let $\bar{\pi} \in \mathbb{R}^{(n+1) \times n}$ be defined by:*

$$\bar{\pi}_{jj'} = \begin{cases} \pi_{jj'} & \text{if } j, j' \in [n] : j \neq j', \\ 0 & \text{if } (j = n + 1, j' \in [n]) \text{ or } (j \in [n], j' = n + 1). \end{cases}$$

Then, $(\bar{\pi}, \pi_0)$ is an FDI of P_{WO}^{n+1} .

From the Lifting Lemma, WO_4-WO_9 are FDIs for any $n \geq 4$. Note that the total number of FDIs that can be generated from WO_4-WO_9 in higher dimensions is greater than in P_{WO}^4 . Expressly, the number of these FDIs that can be generated in P_{WO}^n is given by:

$$|WO_i^n| = \binom{n}{4} |WO_i^4| = \frac{n(n-1)(n-2)(n-3)}{24} |WO_i^4|, \tag{2}$$

where $4 \leq i \leq 9$ and $n \geq 4$. Stated otherwise, $|WO_i^4|$ distinct FDIs from class WO_i^4 can be generated for every combination of distinct indices $j_1, j_2, j_3, j_4 \in [n]$, each of which can be lifted into any higher dimension.

3 Constructing valid inequalities

This section introduces six new classes of VIs. For fixed dimension $\hat{n} \geq 4$, each class defines a set of VIs specific to $P_{WO}^{\hat{n}}$, that is, no individual member of the specific VIs for dimension \hat{n} is applicable in dimension $n < \hat{n}$. While the VIs defined for \hat{n} are also valid for $n > \hat{n}$ owing to the Lifting Lemma, the applicability of the classes to any dimension $n \geq 4$ is independent of lifting. Hence, the cardinality of each VI class

Table 1 Facet defining inequalities (π, π_0) of P_{WO}^4 and their cardinalities

i	$\pi_{j_1j_2}$	$\pi_{j_2j_1}$	$\pi_{j_1j_3}$	$\pi_{j_3j_1}$	$\pi_{j_1j_4}$	$\pi_{j_4j_1}$	$\pi_{j_2j_3}$	$\pi_{j_3j_2}$	$\pi_{j_2j_4}$	$\pi_{j_4j_2}$	$\pi_{j_3j_4}$	$\pi_{j_4j_3}$	π_0	$ WO_i^4 $
1	1												1	12
2	-1	-1											-1	6
3	1		-1				1						-1	24
4	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	4
5	-1	1	1	1	1	1	1	1		1	-1	-1	2	12
6	1		1	1	1	1	-1	-1		-1	-1	-1	2	12
7		1	1	1	1	1	-1	-1	-1		-1	-1	2	12
8	-1		1	1	1	1	1	1		1	-1	-1	3	12
9		-1	1	1	1	1	1	1	1		-1	-1	3	12

is much larger than those of VIs WO_4-WO_9 , which will be elaborated in Sect. 5; therein, it will also be explained that the latter existing FDIs are in fact special cases of the featured VI classes.

Surprisingly, each of the featured large classes of VIs can be derived by considering only a small number of ordinal relationship combinations. Fig. 1 helps illustrate this insight with digraphs $G = (N, \mathcal{A})$ that represent the left-hand side coefficients π of each of the six classes of VIs, where $\mathcal{A} \subseteq \mathcal{A}^N$. Each node $j \in N$ represents an alternative and the format of the arc between nodes j, j' represents the coefficient of $x_{jj'}$; expressly, $\pi_{jj'} = 0, 1$, or -1 if there is no arc, a solid arc, or a dashed arc, respectively, from j to j' . When $\pi_{jj'} = \pi_{j'j} \neq 0$, a bidirectional arc of the appropriate kind is drawn between j and j' to represent arcs (j, j') and (j', j) . Furthermore, a light gray node represents a “fixed” alternative $i_1 \in N$; a dark gray node represents a second fixed alternative $i_2 \in N$ (when applicable), with $i_2 \neq i_1$, and blank nodes represent the remaining “unfixed” alternatives $N \setminus \{i_1\}$ or $N \setminus \{i_1, i_2\}$, as applicable. Using this characterization, we categorize the featured VI classes as being of Type 1 or Type 2 (T1 or T2 for short) indicating the number of fixed alternatives in each expression; multiple classes of the same type are differentiated accordingly. Henceforth, the fixed alternative set is denoted as \hat{N} , and the unfixed alternative set as $\hat{N}^c := N \setminus \hat{N}$ (the complement of \hat{N} in N). Additionally, i -indices are reserved for elements in \hat{N} while j -indices are used for elements of either \hat{N}^c or N , depending on the context. As an important note, although the shaded nodes in each digraph represent the actual number (one or two) of fixed alternatives, the six blank nodes represent a variable number of unfixed alternatives that grows with n —more specifically equal to $n - |\hat{N}|$, for $n \geq 4$.

Within each digraph depicted in Fig. 1, all arcs belonging to one of the following three sets have a uniform format (all are solid or all are dashed) and a uniform orientation (all point in the same direction or all are bidirectional): $\mathcal{A}^{(i_1, j)} := \{(i_1, j), (j, i_1) : j \in \hat{N}^c\}$, $\mathcal{A}^{(i_2, j)} := \{(i_2, j), (j, i_2) : j \in \hat{N}^c\}$, and $\mathcal{A}^{(j, j')} := \{(j, j') : j \in \hat{N}^c, j \neq j'\}$. In fact, the characteristics of the arcs belonging to the third set are the same across the six digraphs. In other words, even though each of the digraphs (i.e., VI classes) induces a distinctive mathematical expression for any $n \geq 4$, their general forms are obtained from a small number of possible uniform-format and uniform-orientation options for $\mathcal{A}^{(i_1, j)}$ and $\mathcal{A}^{(i_2, j)}$, combined with the different arc choices between i_1 and i_2 . For example, in Fig. 1c all arcs in $\mathcal{A}^{(i_1, j)}$ are solid and bidirectional, all arcs in $\mathcal{A}^{(i_2, j)}$ are dashed and directed from $j \in \hat{N}^c$ to i_2 , and arc (i_1, i_2) is solid (with no arc from i_2 to i_1).

The mathematical expressions and derivations of the T1 and T2 VI classes are given in the ensuing paragraphs. In each VI, all non-zero variable coefficients $\pi_{jj'}$ are equal to either 1 or -1 . Hence, the left-hand side of the VI is reexpressed as the difference between the *selected positive arcs* and the *selected negative arcs* from G , that is:

$$\| \mathbf{x}_+^W \| - \| \mathbf{x}_-^W \| := \sum_{(j, j') \in \mathcal{A}_+} x_{jj'} - \sum_{(j, j') \in \mathcal{A}_-} x_{jj'}$$

where $\mathcal{A}_+ := \{(j, j') \in \mathcal{A} : \pi_{jj'} = 1\}$, $\mathcal{A}_- := \{(j, j') \in \mathcal{A} : \pi_{jj'} = -1\}$, $\mathbf{x}_+^W \in \{0, 1\}^{|\mathcal{A}_+|}$, $\mathbf{x}_-^W \in \{0, 1\}^{|\mathcal{A}_-|}$, and $\| \cdot \|$ is the $L1$ -norm. The derivation of each VI

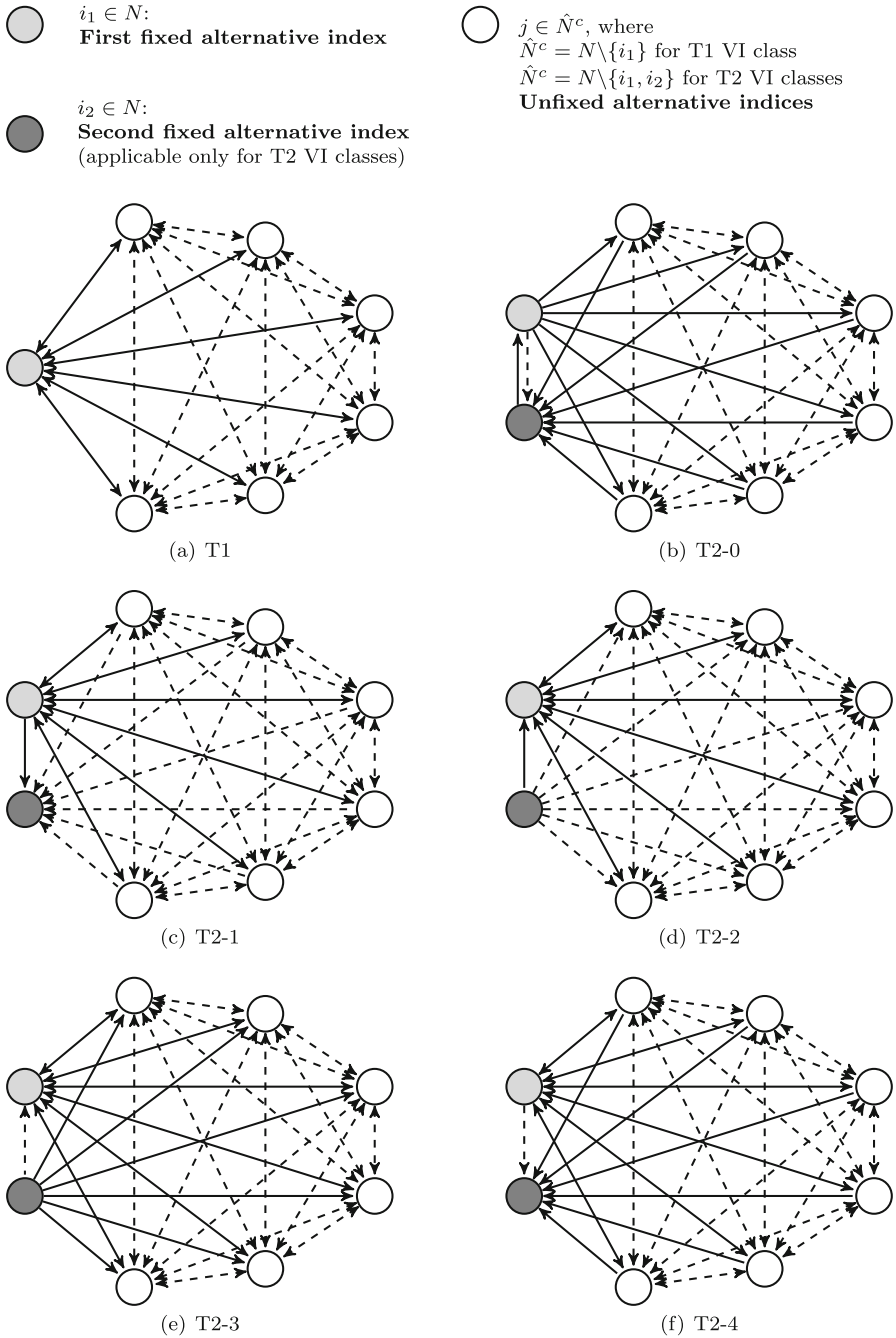


Fig. 1 Digraphs representing coefficients π of the six featured classes of valid inequalities

centers on determining the value of $\max\{\|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\|\}$ in the left-hand side of the given expression, where \mathbf{x}_+^W and \mathbf{x}_-^W together must induce a weak ordering on N . The weak orders that achieve this maximum value are described and conveniently grouped into (non-strict) *ranking structures*. For completeness and clarity, the ranking structures are first described in mathematical notation in tables and then in words within the proof narratives. Each proof begins with an arbitrary weak order W , where the fixed alternatives (i_1 in Theorem 1, and i_1 and i_2 in Theorems 2–6) are tied with a fixed number, $t \geq 0$, of unfixed alternatives. For each possible value of t , the proof describes modifications to W that lead to one or more of ranking structures listed in the respective ranking structure table. Each set of modifications leads to some weak order W' , i.e., corresponding to the enumerated ranking structures. The proofs then present constructive arguments as to why the value of $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|$ is higher than that achieved by any other weak order. To facilitate understanding, readers are advised to refer to the corresponding digraphs in Fig. 1 while following the proofs.

Due to space limitations, some proofs are located in the ‘‘Appendix’’. Accompanying examples of the weak order constructions described in the VI proofs are given in the Supplementary Materials.

Theorem 1 (T1 VI) *Let $\hat{N} = \{i_1\}$ be the fixed index set, where $i_1 \in N$, and let $\hat{N}^c = N \setminus \hat{N}$. The following is a valid inequality for \mathbf{P}_{WO}^n , for any $n \geq 4$:*

$$\sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1}) - \sum_{j, j' \in \hat{N}^c: j \neq j'} x_{j j'} \leq 2 - \frac{(n - 2)(n - 3)}{2}. \tag{3}$$

Proof The VI is satisfied at equality by the ranking structures listed in Table 2, which are justified as follows. The arc sets \mathcal{A}_+ and \mathcal{A}_- are given by:

$$\begin{aligned} \mathcal{A}_+ &= \{(i_1, j) : j \in \hat{N}^c\} \cup \{(j, i_1) : j \in \hat{N}^c\}, \\ \mathcal{A}_- &= \{(j, j') : j, j' \in \hat{N}^c, j \neq j'\}. \end{aligned}$$

Let W be an arbitrary weak order defined over the set of alternatives N , where alternative i_1 is tied with exactly $t \geq 0$ unfixed alternatives, $[j_1, j_t] \subseteq \hat{N}^c$, where $[j_1, j_t]$ is used as shorthand for the index set $\{j_1, j_2, \dots, j_t\}$. The structure of the weak orders generated by different values of t can be encapsulated by the following four cases:

Case 1 ($t = 1$) There are precisely n selected positive arcs, $\|\mathbf{x}_+^W\|$ (i.e., from \mathcal{A}_+): 2 from the tie of i_1 with an unfixed alternative, say j_1 , and $(n - 2)$ from the strict ordering of i_1 with each $j \in \hat{N}^c \setminus \{j_1\}$. Now, break the ties (if any) between $j, j' \in \hat{N}^c \setminus \{j_1\}$, where $j \neq j'$, by removing either negative arc (j, j') or negative arc (j', j) from W ; denote the generated weak order as W' . This gives a total of $(n - 1)(n - 2)/2$ selected negative arcs (i.e., from \mathcal{A}_-). Since this removal does not increase the value of $\|\mathbf{x}_-^W\|$ and keeps the value of $\|\mathbf{x}_+^W\|$ unchanged, we have that $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\| \leq \|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\|$. For any such weak order, equivalently characterized by ranking structure #1, the value of $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|$ equals the right-hand side of inequality (3). Note that, since there must be at least one arc between every pair of alternatives to induce a total order, further reduction in negative arcs is not possible.

Case 2 ($t = 2$) This case is addressed using the same arguments as in Case 1 by replacing $\hat{N}^c \setminus \{j_1\}$ with $\hat{N}^c \setminus \{j_1, j_2\}$; the generated weak order is defined by ranking structure #2.

Case 3 ($t \geq 3$) First, break the tie between \hat{j} and j_t by removing either arc (\hat{j}, j_t) or arc (j_t, \hat{j}) from W , for each $\hat{j} \in [j_1, j_t - 1] \cup \{i_1\}$; denote the generated weak order as W_{t-1} . Since this removal decreases the value of $\|\mathbf{x}_-^W\|$ and $\|\mathbf{x}_+^W\|$ by $(t - 1) \geq 2$ and 1, respectively, we have that $\|\mathbf{x}_+^{W_{t-1}}\| - \|\mathbf{x}_-^{W_{t-1}}\| < \|\mathbf{x}_+^{W_{t-2}}\| - \|\mathbf{x}_-^{W_{t-2}}\|$. Next, break the ties between each of $\hat{j} \in [j_1, j_t - 2] \cup \{i_1\}$ with j_{t-1} in W_{t-1} in a similar fashion to obtain a modified weak order, denoted as W_{t-2} . Repeat this process until arriving at a weak order, denoted as W_2 , in which i_1 is tied with exactly two unfixed alternatives j_1 and j_2 . At this point, apply the results of Case 2 by setting $W := W_2$.

Case 4 ($t = 0$) Suppose that the rank of alternative i_1 is fixed as $r_{i_1} = k > 0$. Let $[j_1, j_l] \subseteq \hat{N}^c$ denote those alternatives with rank r equal to $k - 1$ when $k \geq 2$, or with rank r equal to $k + 1 = 2$ when $k = 1$. Add to W arc (i_1, \hat{j}) in the former, and add to W arcs (\hat{j}, i_1) in the latter, for each $\hat{j} \in [j_1, j_l]$; denote the generated weak order in either case as W' . Since these additions keep the value of $\|\mathbf{x}_-^W\|$ unchanged and increase the value of $\|\mathbf{x}_+^W\|$, we have that $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\| < \|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\|$. At this point, apply the results of Cases 1–3 by setting $W := W'$. \square

Table 2 List of ranking structures that satisfy T1 VI at equality

Ranking structure description	$\ \mathbf{x}_+^{W'}\ $	$\ \mathbf{x}_-^{W'}\ $
#1 $r_{i_1} = r_{j_1} = k$, where $k \in [n], j_1 \in \hat{N}^c$; $r_j, r_{j'} \in [n] \setminus \{k, k + 1\}, r_j \neq r_{j'},$ for $j, j' \in \hat{N}^c \setminus \{j_1\}$	n	$\frac{(n-1)(n-2)}{2}$
#2 $r_{i_1} = r_{j_1} = r_{j_2} = k$, where $k \in [n], j_1, j_2 \in \hat{N}^c : j_1 \neq j_2$; $r_j, r_{j'} \in [n] \setminus \{k, k + 1, k + 2\}, r_j \neq r_{j'},$ for $j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$	$n + 1$	$\frac{(n-1)(n-2)}{2} + 1$

Table 3 List of ranking structures that satisfy T2-0 VI at equality

Ranking structure description	$\ x_+^{W'}\ $	$\ x_-^{W'}\ $
#1 $r_{i_1} = 1, r_{i_2} = n;$ $r_j, r_{j'} \in [n] \setminus \{1, n\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c$	$2(n-2)$	$\frac{(n-2)(n-3)}{2} + 1$
#2 $r_{i_1} = r_{j_1} = 1, r_{i_2} = n, \text{ where } j_1 \in \hat{N}^c;$ $r_j, r_{j'} \in [n] \setminus \{1, 2, n\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c \setminus \{j_1\}$	$2(n-2)$	$\frac{(n-2)(n-3)}{2} + 1$
#3 $r_{i_1} = 1, r_{i_2} = r_{j_2} = n-1, \text{ where } j_2 \in \hat{N}^c;$ $r_j, r_{j'} \in [n] \setminus \{1, n-1, n\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c \setminus \{j_2\}$	$2(n-2)$	$\frac{(n-2)(n-3)}{2} + 1$
#4 $r_{i_1} = r_{j_1} = 1, r_{i_2} = r_{j_2} = n-1, \text{ where } j_1, j_2 \in \hat{N}^c : j_1 \neq j_2;$ $r_j, r_{j'} \in [n] \setminus \{1, 2, n-1, n\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$	$2(n-2)$	$\frac{(n-2)(n-3)}{2} + 1$

Theorem 2 (T2-0 VI) Let $\hat{N} = \{i_1, i_2\}$ be the fixed index set, where $i_1, i_2 \in N$ s.t. $i_1 \neq i_2$, and let $\hat{N}^c = N \setminus \hat{N}$. The following is a valid inequality of \mathbf{P}_{WO}^n , for any $n \geq 4$:

$$x_{i_2 i_1} + \sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_2}) - x_{i_1 i_2} - \sum_{j, j' \in \hat{N}^c: j \neq j'} x_{j j'} \leq 2 - \frac{(n-4)(n-5)}{2}. \quad (4)$$

Proof The VI is satisfied at equality by the ranking structures listed in Table 3. Because the VI is only an FDI for $n = 4$ (see Sect. 5), the remainder of the proof is relegated to ‘‘Appendix A.1’’. \square

Theorem 3 (T2-1 VI) Let $\hat{N} = \{i_1, i_2\}$ be the fixed index set, where $i_1, i_2 \in N$ s.t. $i_1 \neq i_2$, and let $\hat{N}^c = N \setminus \hat{N}$. The following is a valid inequality of \mathbf{P}_{WO}^n , for any $n \geq 4$:

$$x_{i_1 i_2} + \sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1}) - \sum_{j, j' \in \hat{N}^c: j \neq j'} (x_{j j'} + x_{j i_2}) \leq 2 - \frac{(n-3)(n-4)}{2}. \quad (5)$$

Proof The VI is satisfied at equality by the ranking structures listed in Table 4, which are justified as follows. The arc sets \mathcal{A}_+ and \mathcal{A}_- are given by:

$$\begin{aligned} \mathcal{A}_+ &= \{(i_1, i_2)\} \cup \{(i_1, j) : j \in \hat{N}^c\} \cup \{(j, i_1) : j \in \hat{N}^c\}, \\ \mathcal{A}_- &= \{(j, i_2) : j \in \hat{N}^c\} \cup \{(j, j') : j, j' \in \hat{N}^c, j \neq j'\}. \end{aligned}$$

Let W be an arbitrary weak order defined over the set of alternatives N , where the fixed alternatives i_1 and i_2 are tied with exactly $t_1 \geq 0$ and $t_2 \geq 0$ unfixed alternatives, respectively, such that $t_1 + t_2 \leq n - 2$. Now suppose that the ranks of these alternatives are fixed as $r_{i_1} = k_1 > 0$ and $r_{i_2} = k_2 > 0$. Additionally, let $\hat{N}_{<k_1}^c \subseteq \hat{N}^c$ denote the set of unfixed alternatives with rank $r < r_{i_1}$, and $\hat{N}_{<k_2}^c \subseteq \hat{N}^c$ denote the set of

unfixed alternatives with rank $r' < r_{i_2}$. Note that, $\hat{N}_{<k_1}^c \cap \hat{N}_{<k_2}^c = \emptyset$ only when either $\hat{N}_{<k_1}^c = \emptyset$ or $\hat{N}_{<k_2}^c = \emptyset$ or both. The structure of the weak orders generated by the different values of t_1 and t_2 can be encapsulated by the following cases:

Case 1 ($t_1 = 0, t_2 = 0$) For each $\hat{j} \in \hat{N}_{<k_1}^c$, replace positive arc (\hat{j}, i_1) with positive arc (i_1, \hat{j}) and for each $\hat{j}' \in \hat{N}_{<k_2}^c$, replace negative arc (\hat{j}', i_2) with zero arc (i_2, \hat{j}') to obtain a weak order where alternatives i_1 and i_2 are placed in front of all $j \in \hat{N}^c$. Additionally, if $k_1 > k_2$, replace zero arc (i_2, i_1) with positive arc (i_1, i_2) to place alternative i_1 in front of i_2 . This gives precisely $n - 1$ selected positive arcs: 1 from the relative position of i_1 with i_2 and $n - 2$ from the strict ordering of i_1 with each unfixed alternative. Next, break the ties (if any) between $j, j' \in \hat{N}^c$, where $j \neq j'$, by removing either negative arc (j, j') or negative arc (j', j) from W ; denote the generated weak order as W' . This gives a total of $(n - 3)(n - 2)/2$ selected negative arcs. Since all preceding operations neither increase the value of $\|\mathbf{x}_+^W\|$ nor decrease the value of $\|\mathbf{x}_-^W\|$, we have that $\|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\| \leq \|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|$. For any such weak order, equivalently characterized by either ranking structure #1, if $k_1 \neq k_2$, or ranking structure #2, if $k_1 = k_2$, the value of $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|$ equals the right-hand side of inequality (5).

Case 2 ($t_1 = 1, t_2 = 0$) Let $j_1 \in \hat{N}^c$ denote the unfixed alternative tied with i_1 in rank position k_1 . Based on the relative values of k_1 and k_2 , the generated weak order can be further divided into the following two sub-cases:

Case 2a ($k_1 > k_2$) For each $\hat{j}' \in \hat{N}_{<k_2}^c$, replace negative arc (\hat{j}', i_2) with zero arc (i_2, \hat{j}') to obtain a weak order where alternative i_2 is uniquely in first place. Next, break the ties (if any) between $j, j' \in \hat{N}^c \setminus \{j_1\}$, where $j \neq j'$, by removing either negative arc (j, j') or negative arc (j', j) from W ; denote the generated weak order as W' . Since these removals do not increase the value of $\|\mathbf{x}_-^W\|$ and keep the value of $\|\mathbf{x}_+^W\|$ unchanged, we have that $\|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\| \leq \|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|$. Here, W' is a member of ranking structure #5.

Case 2b ($k_1 \leq k_2$) For each $\hat{j} \in \hat{N}_{<k_1}^c$, replace positive arc (\hat{j}, i_1) with positive arc (i_1, \hat{j}) and negative arc (\hat{j}, j_1) with negative arc (j_1, \hat{j}) in W . Additionally, for each $\hat{j}' \in \hat{N}_{<k_2}^c \setminus \{j_1\}$, replace negative arc (\hat{j}', i_2) with zero arc (i_2, \hat{j}') to obtain a weak order where alternatives i_1, j_1 , and i_2 are placed in front of each $j \in \hat{N}^c \setminus \{j_1\}$. Next, break the ties (if any) between $j, j' \in \hat{N}^c \setminus \{j_1\}$, where $j \neq j'$, by removing either negative arc (j, j') or negative arc (j', j) from W ; denote the generated weak order as W' . Since all preceding operations neither increase the value of $\|\mathbf{x}_-^W\|$ nor decrease the value of $\|\mathbf{x}_+^W\|$, we have that $\|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\| \leq \|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|$. Here, W' is a member of ranking structure #3, if $k_1 < k_2$, or #4, if $k_1 = k_2$.

Case 3 ($t_1 \geq 2, t_2 = 0$) Let $[j_1, j_{t_1}] \subseteq \hat{N}^c$ denote the set of unfixed alternatives tied with i_1 in rank position k_1 . First, break the tie between \hat{j} and j_{t_1} by removing arc (j_{t_1}, \hat{j}) from W , for each $\hat{j} \in \{i_1\} \cup [j_1, j_{t_1} - 1]$ (if $k_1 = k_2, \hat{j} \in \{i_1, i_2\} \cup [j_1, j_{t_1} - 1]$); denote the generated weak order as $W_{t_1-1,0}$. Since this removal decreases the value

Table 4 List of ranking structures that satisfy T2-1 VI at equality

Ranking structure description	$\ \mathbf{x}_+^{W'}\ $	$\ \mathbf{x}_-^{W'}\ $
#1 $r_{i_1} = 1, r_{i_2} = 2;$ $r_j, r_{j'} \in [n] \setminus \{1, 2\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c$	$n - 1$	$\frac{(n-2)(n-3)}{2}$
#2 $r_{i_1} = r_{i_2} = 1;$ $r_j, r_{j'} \in [n] \setminus \{1, 2\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c$	$n - 1$	$\frac{(n-2)(n-3)}{2}$
#3 $r_{i_1} = r_{j_1} = 1, r_{i_2} = 3, \text{ where } j_1 \in \hat{N}^c;$ $r_j, r_{j'} \in [n] \setminus \{1, 2, 3\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c \setminus \{j_1\}$	n	$\frac{(n-2)(n-3)}{2} + 1$
#4 $r_{i_1} = r_{i_2} = r_{j_1} = 1, \text{ where } j_1 \in \hat{N}^c;$ $r_j, r_{j'} \in [n] \setminus \{1, 2, 3\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c \setminus \{j_1\}$	n	$\frac{(n-2)(n-3)}{2} + 1$
#5 $r_{i_2} = 1; r_{i_1} = r_{j_1} = k, \text{ where } k \in [n] \setminus \{1\} \text{ and } j_1 \in \hat{N}^c;$ $r_j, r_{j'} \in [n] \setminus \{1, k, k + 1\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c \setminus \{j_1\}$	$n - 1$	$\frac{(n-2)(n-3)}{2}$
#6 $r_{i_2} = 1, r_{i_1} = r_{j_1} = r_{j_2} = k, \text{ where } k \in [n] \setminus \{1\}, j_1, j_2 \in \hat{N}^c : j_1 \neq j_2;$ $r_j, r_{j'} \in [n] \setminus \{1, k, k + 1, k + 2\}, r_j \neq r_{j'}, \text{ for } j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$	n	$\frac{(n-2)(n-3)}{2} + 1$

of $\|\mathbf{x}_-^W\|$ and $\|\mathbf{x}_+^W\|$ by at least $(t_1 - 1) \geq 1$ and at most 1, respectively, we have that $\|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\| \leq \|\mathbf{x}_+^{W_{t_1-1,0}}\| - \|\mathbf{x}_-^{W_{t_1-1,0}}\|$. Next, break the ties between each of $\hat{j} \in \{i_1\} \cup [j_1, j_{t_1-2}]$ (if $k_1 = k_2, \hat{j} \in \{i_1, i_2\} \cup [j_1, j_{t_1-2}]$) with j_{t_1-1} in $W_{t_1-1,0}$ in a similar fashion to obtain a modified weak order, denoted as $W_{t_1-2,0}$. Repeat this process until arriving at a weak order, denoted as $W_{2,0}$, in which i_1 is tied with exactly two unfixed alternatives j_1 and j_2 . Afterwards, we consider the following two sub-cases:

Case 3a ($k_1 > k_2$) This case is addressed using the same arguments as in Case 2a by replacing $\hat{N}^c \setminus \{j_1\}$ with $\hat{N}^c \setminus \{j_1, j_2\}$; the generated weak order is a member of ranking structure #6.

Case 3b ($k_1 \leq k_2$) The ties between each of i_1 and j_1 (if $k_1 = k_2$, between each of i_1, i_2 , and j_1) with j_2 in $W_{2,0}$ are further broken to obtain a modified weak order, denoted as $W_{1,0}$, in which i_1 is tied with exactly one unfixed alternative j_1 (if $k_1 = k_2, i_1$ and i_2 are tied with j_1). At this point, apply the results of Case 2b by setting $W := W_{1,0}$.

Case 4 ($t_2 \geq 1$) Let $[j_1, j_{t_2}] \subseteq \hat{N}^c$ denote the unfixed alternatives tied with i_2 in rank position k_2 . This case is addressed using the same arguments as in Case 3 by replacing $\{i_1\} \cup [j_1, j_{t'-1}]$ with $\{i_2\} \cup [j_1, j_{t'-1}]$ in each step, where $t' = t_2, t_2 - 1, \dots, 0$. In the generated weak order, denoted as $W_{t_1,0}$, i_2 is not tied with any unfixed alternatives. At this point, apply the results of Cases 1–3 by setting $W := W_{t_1,0}$.

□

Theorem 4 (T2-2 VI) Let $\hat{N} = \{i_1, i_2\}$ be the fixed index set, where $i_1, i_2 \in N$ s.t. $i_1 \neq i_2$, and let $\hat{N}^c = N \setminus \hat{N}$. The following is a valid inequality of $\mathbf{P}_{W,0}^n$, for any $n \geq 4$:

Table 5 List of ranking structures that satisfy T2-3 VI at equality

Ranking structure description	$\ \mathbf{x}_+^{W'}\ $	$\ \mathbf{x}_-^{W'}\ $
#1 T2-1 #1	$2(n-2)$	$\frac{(n-2)(n-3)}{2}$
#2 $r_{i_1} = 1, r_{i_2} = r_{j_1} = 2$, where $j_1 \in \hat{N}^c$ $r_j, r_{j'} \in [n] \setminus \{1, 2, 3\}, r_j \neq r_{j'},$ for $j, j' \in \hat{N}^c \setminus \{j_1\}$	$2(n-2)$	$\frac{(n-2)(n-3)}{2}$
#3 T2-1 #3	$2(n-2)$	$\frac{(n-2)(n-3)}{2}$
#4 $r_{i_1} = r_{j_1} = 1; r_{i_2} = r_{j_2} = 3$, where $j_1, j_2 \in \hat{N}^c : j_1 \neq j_2$ $r_j, r_{j'} \in [n] \setminus \{1, 2, 3, 4\}, r_j \neq r_{j'},$ for $j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$	$2(n-2)$	$\frac{(n-2)(n-3)}{2}$
#5 T2-1 #4	$2(n-2) + 1$	$\frac{(n-2)(n-3)}{2} + 1$
#6 $r_{i_1} = r_{i_2} = r_{j_1} = r_{j_2} = 1$, where $j_1, j_2 \in \hat{N}^c : j_1 \neq j_2$ $r_j, r_{j'} \in [n] \setminus \{1, 2, 3, 4\}, r_j \neq r_{j'},$ for $j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$	$2(n-1)$	$\frac{(n-2)(n-3)}{2} + 2$
#7 T2-1 #5	$2(n-2) + 1$	$\frac{(n-2)(n-3)}{2} + 1$
#8 $r_{i_2} = r_{j_2} = 1; r_{i_1} = r_{j_1} = k$, where $k \in [n] \setminus \{1, 2\}, j_1, j_2 \in \hat{N}^c : j_1 \neq j_2$ $r_j, r_{j'} \in [n] \setminus \{1, 2, k, k+1\}, r_j \neq r_{j'},$ for $j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$	$2(n-2) + 1$	$\frac{(n-2)(n-3)}{2} + 1$
#9 T2-1 #6	$2(n-1)$	$\frac{(n-2)(n-3)}{2} + 2$
#10 $r_{i_2} = r_{j_3} = 1; r_{i_1} = r_{j_1} = r_{j_2} = k$, where $k \in [n] \setminus \{1, 2\}, j_1, j_2, j_3 \in \hat{N}^c : j_1 \neq j_2 \neq j_3$ $r_j, r_{j'} \in [n] \setminus \{1, 2, k, k+1, k+2\}, r_j \neq r_{j'},$ for $j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$	$2(n-1)$	$\frac{(n-2)(n-3)}{2} + 2$

$$x_{i_2 i_1} + \sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1}) - \sum_{j, j' \in \hat{N}^c : j \neq j'} (x_{j j'} + x_{i_2 j}) \leq 2 - \frac{(n-3)(n-4)}{2}. \tag{6}$$

Proof The above inequality is satisfied at equality by six ranking structures that are symmetric images of those listed in Table 4. That is, those alternatives set to the best and second-best available ranking positions in the T2-1 structures are set to the last and second-to-last available positions, respectively, in the T2-2 structures. Alternatives occupying the remaining inferior positions in the T2-1 structures occupy the remaining superior positions in the T2-2 structures. Hence, a string of arguments paralleling the proof of Theorem 3 establishes that these ranking structures achieve a value of $\max\{\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|\}$ equal to the right-hand side of inequality (6) and all others achieve a lower value. \square

Theorem 5 (T2-3 VI) Let $\hat{N} = \{i_1, i_2\}$ be the fixed index set, where $i_1, i_2 \in N$ s.t. $i_1 \neq i_2$, and let $\hat{N}^c = N \setminus \hat{N}$. The following is a valid inequality of \mathbf{P}_{WO}^n , for any $n \geq 4$:

$$\sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1} + x_{i_2 j}) - x_{i_2 i_1} - \sum_{j, j' \in \hat{N}^c : j \neq j'} x_{j j'} \leq 3 - \frac{(n-4)(n-5)}{2}. \tag{7}$$

Proof The VI is satisfied at equality by the ranking structures listed in Table 5. The remainder of the proof can be found in “Appendix A.2”. □

Theorem 6 (T2-4 VI) *Let $\hat{N} = \{i_1, i_2\}$ be the fixed index set, where $i_1, i_2 \in N$ s.t. $i_1 \neq i_2$, and let $\hat{N}^c = N \setminus \hat{N}$. The following is a valid inequality of P_{WO}^n for any $n \geq 4$:*

$$\sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1} + x_{j i_2}) - x_{i_1 i_2} - \sum_{j, j' \in \hat{N}^c: j \neq j'} x_{j j'} \leq 3 - \frac{(n - 4)(n - 5)}{2}. \tag{8}$$

Proof The result can be obtained using a nearly identical line of reasoning as the proof to Theorem 4, with the difference that the T2-4 VI ranking structures are the symmetric images of the T2-3 VI ranking structures (see Table 5). □

To conclude this section, it is worth restating that Fig. 1 depicts only a small number of the possible format and orientation choices for $\mathcal{A}^{i_1, j}$, $\mathcal{A}^{i_2, j}$, and the arcs that join the fixed alternatives. Fig. 1a is only 1 of 7 possible digraphs for the case with $|\hat{N}| = 1$ and Fig. 1b–e are only 5 of 343 possible digraphs for the case with $|\hat{N}| = 2$. While additional T1 and T2 VIs could be constructed for the omitted combinations using similar constructive arguments, these would likely be equivalent to or dominated by one of the featured VIs. Indeed, as Sect. 5 demonstrates, T2-0 is an FDI only for $n = 4$, while T1 and T2-1 to T2-4 are FDIs for all $n \geq 4$. It remains an open question whether some of the omitted digraphs represent FDIs for some $n > 4$ even though they are not FDIs for $n = 4$. Moreover, it remains an open question whether expanding the techniques from this section to cases with $|\hat{N}| \geq 3$ (i.e., T3 VIs, T4 VIs, etc.) can produce FDIs of P_{WO}^n for some or all $n \geq 5$. These inquiries are formalized into conjectures in Sect. 5.

4 Constructing facet defining inequalities

To obtain the dimensionality of the faces induced by the T1 and T2 VIs, we first devise systematic processes for selecting members of their respective ranking structures such that simple patterns of linearly independent characteristic vectors are formed. The key idea is to select these so that consecutive pairs of vectors differ minimally, thereby simplifying the respective proofs.

4.1 Building block procedures

Assume that alternatives $i \in I$ belong to bucket k of alternative-ordering $\{S^w\}$, that is, $I \subseteq S_k^w$, where $1 \leq k \leq p = |\{S^w\}|$. Additionally, define a *step parameter* $q \in \mathbb{Q}$, where $-k < q < p - k + 1$, $q = t/2$, and $t \in \mathbb{Z}$.

Definition 7 A move of q steps of I in $\{S^w\}$ is an operation that yields an alternative-ordering in which all $i \in I$ are removed from their current bucket k and either merged with the alternatives in bucket $k + q$, when $q \in \mathbb{Z}$, or separated into a new bucket

inserted immediately after (before, resp.) bucket $\lfloor k + q \rfloor$ ($\lceil k + q \rceil$, resp.), when $q \notin \mathbb{Z}$ is positive (negative, resp.). The operation is abbreviated as the triple $\langle I, q, \{S^w\} \rangle$.

Example 2 Consider five different move operations and their outputs:

1. $\langle \{2\}, 1, \{\{1, 2\}, \{4\}, \{3\}\} \rangle = \{\{1\}, \{2, 4\}, \{3\}\}$
2. $\langle \{2, 4\}, -1, \{\{1\}, \{2, 4\}, \{3\}\} \rangle = \{\{1, 2, 4\}, \{3\}\}$
3. $\langle \{3\}, -2, \{\{1, 2\}, \{4\}, \{3\}\} \rangle = \{\{1, 2, 3\}, \{4\}\}$
4. $\langle \{1, 3\}, \frac{3}{2}, \{\{1, 2, 3\}, \{4\}\} \rangle = \{\{2\}, \{4\}, \{1, 3\}\}$
5. $\langle \{3\}, \frac{-5}{2}, \{\{1, 2\}, \{4\}, \{3\}\} \rangle = \{\{3\}, \{1, 2\}, \{4\}\}$.

As the example shows, move operations can be used to change not only the contents of a bucket for a given alternative-ordering, but also the ordering and total number of buckets. For instance, the second move operation merges the entire contents of buckets 1 and 2. Additionally, the output alternative-ordering in the third move operation has one fewer bucket than the input alternative-ordering, since the bucket where alternative 3 resides contains only one alternative and $q \in \mathbb{Z}$; the reverse holds for the fourth move operation since the bucket where alternatives 1 and 3 reside contains three alternatives and $q \notin \mathbb{Z}$.

The Merge and Reverse Construction Procedure (M&R), whose pseudocode is given in Algorithm 1, is at the core of the characteristic vector constructions. It begins with an alternative-ordering $\{S^0\}$ with p buckets (associated with a weak ordering $W^0 \in \mathcal{W}$), and proceeds to iteratively merge and then reverse adjacent buckets in $\{S^0\}$, generating potential characteristic vectors from each related move operation. To be more precise, the pseudocode displays a shell of the M&R procedure, which can be customized by incorporating optional steps that allow certain alternatives to move more freely between buckets.

Algorithm 1 Merge and Reverse Construction Procedure (M&R) (Shell)

```

1: procedure M&R( $\{S^0\}, I^0, \hat{p}$ )
2:    $p = |\{S^0\}|$                                  $\triangleright$  number of buckets in input alternative-ordering
3:    $I^1 = \emptyset$                                  $\triangleright$  initiate working alternative subset
4:    $\{S^1\} = \{S^0\}$                                 $\triangleright$  initiate working alternative-ordering
5:    $X = []$                                         $\triangleright$  initiate characteristic vector matrix
6:   if  $\hat{p} \leq p$  then
7:     for  $j = 1, \dots, \hat{p}$  do
8:        $I^1 \leftarrow S^1_j$                         $\triangleright$  set  $I^1$  to first bucket of  $\{S^1\}$ 
9:       for  $k = 1, \dots, p - j$  do
10:         $\{S^1\} \leftarrow \langle I^1, 1, \{S^1\} \rangle$     $\triangleright$  merge
11:         $X.append(toBinary(\{S^1\}))$ 
12:         $\{S^1\} \leftarrow \langle I^1, \frac{1}{2}, \{S^1\} \rangle$     $\triangleright$  reverse
13:         $X.append(toBinary(\{S^1\}))$ 
14:      end for
15:    perform optional outer steps (possibly involving  $I^0$ )
16:  end for
17: end if
18: return  $(X, \{S^1\})$ 
19: end procedure

```

Example 3 Let $\{S^0\} = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$, $I^0 = \{1\}$ and $\hat{p} = 4$; here, $S_1^0 = \{1, 2\}$, $S_2^0 = \{3\}$, $S_3^0 = \{4\}$, $S_4^0 = \{5\}$, and $p = 4$. Additionally, define the j th optional outer step (pseudocode line 15) for the M&R as:

$$\{S^1\} \leftarrow \left\langle I^0, j - p, \{S^1\} \right\rangle \tag{9}$$

Put simply, the optional outer step moves alternative 1 to the first bucket of $\{S^1\}$ after each execution of the inner for-loop. Performing $M\&R(\{S^0\}, I^0, \hat{p})$ with this optional outer step produces the following sequence of alternative-orderings:

j	k	Merge	Reverse	j th Outer Step
1	1	$\{\{1,2, 3\}, \{4\}, \{5\}\}$	$\{\{3\}\{1,2\}, \{4\}, \{5\}\}$	–
1	2	$\{\{3\}\{1,2, 4\}, \{5\}\}$	$\{\{3\}, \{4\}, \{1,2\}, \{5\}\}$	–
1	3	$\{\{3\}, \{4\}, \{1,2, 5\}\}$	$\{\{3\}, \{4\}, \{5\}, \{1,2\}\}$	$\{\{1, 3\}, \{4\}, \{5\}, \{2\}\}$
2	1	$\{\{1,3, 4\}, \{5\}, \{2\}\}$	$\{\{4\}, \{1,3\}, \{5\}, \{2\}\}$	–
2	2	$\{\{4\}, \{1,3, 5\}, \{2\}\}$	$\{\{4\}, \{5\}, \{1,3\}, \{2\}\}$	$\{\{1, 4\}, \{5\}, \{3\}, \{2\}\}$
3	1	$\{\{1,4, 5\}, \{3\}, \{2\}\}$	$\{\{5\}, \{1,4\}, \{3\}, \{2\}\}$	$\{\{1, 5\}, \{4\}, \{3\}, \{2\}\}$

where the numbers in bold are the members of I^1 at iteration j . Here, M&R directly returns 12 characteristic vectors for the weak orders under the “Merge” and “Reverse” columns, which are then appended to matrix X . As a point of emphasis, the optional outer steps serve primarily an auxiliary purpose of setting up $\{S^1\}$ between iterations; characteristic vectors may or may not be stored from their respective weak orders.

Theorem 7 Let $\mathbf{x}^1, \dots, \mathbf{x}^{\hat{p}(\hat{p}-1)} \in \{0, 1\}^{n(n-1)}$ denote the $\hat{p}(\hat{p}-1)$ characteristic vectors generated by the non-optional steps of $M\&R(\{S^0\}, I^0, \hat{p})$, where $p = |\{S^0\}|$ and $\hat{p} \leq p \leq n$; let $\mathbf{x}^0 \in \{0, 1\}^{n(n-1)}$ denote the corresponding characteristic vector for $\{S^0\}$; and assume that these vectors occupy rows $1, \dots, \hat{p}(\hat{p}-1)$ and $\hat{p}(\hat{p}-1) + 1$, respectively, of $X \in \{0, 1\}^{m \times n(n-1)}$, with $m \geq \hat{p}(\hat{p}-1) + 1$. Additionally, let $\{j_k\}_{k=1}^p$ represent a set of alternative indices, one from each bucket in $\{S^0\}$, which are not permitted to move during the optional outer steps of M&R; that is, $j_k \in S_k^0$ with $j_k \notin I^0$, for $k \in [p]$. Independent of the optional outer steps performed subject to this restriction, M&R yields at least $\hat{p}(\hat{p}-1) + 1$ affinely independent characteristic vectors.

Proof Assume that $\hat{p} = p$ in M&R. The proof focuses on the elements $(j, j') \in \mathcal{A}^N$ such that $j, j' \in \{j_k\}_{k=1}^p$. Restricted to these elements, each pair of consecutively generated alternative-orderings differs only in that two alternatives that are in the same bucket in one ordering are in separate adjacent buckets in the other ordering. Stated otherwise, in each successive move operation, $(p-2)$ of the alternatives from $\{j_k\}_{k=1}^p$ retain their ordinal relationships. After subtracting \mathbf{x}^{i-1} from \mathbf{x}^i , for $i = p(p-1), \dots, 1$, a submatrix $\tilde{X}_{M\&R} \in \{0, -1, 1\}^{[p(p-1)+1] \times p(p-1)}$ can be

extracted from X having the following entry pattern:

$$\bar{X}_{M\&R} = \begin{matrix} \text{Row}\backslash\text{Col} & (j_1, j_2) & (j_2, j_1) & (j_1, j_3) & (j_3, j_1) & \dots & (j_{p-1}, j_p) & (j_p, j_{p-1}) \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p(p-1)-1 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ p(p-1) & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 1 & 0 \end{matrix} ; \quad (10)$$

where the 0-row (corresponding to \mathbf{x}^0) is placed at the bottom to highlight the convenient structure of this submatrix—expressly, the first $p(p-1)$ rows contain all the possible elementary vectors of size $p(p-1)$, times ± 1 . Next, add the rows with nonzero even indices to row 0 to yield the all-zeros vector, $\mathbf{0}$, of size $p(p-1)$, and let $\bar{X}'_{M\&R}$ be the resulting submatrix. Now, since the rows of the augmented matrix $[\bar{X}'_{M\&R} \mathbf{1}]$ are linearly independent, where $\mathbf{1}$ is the all-ones column vector of size $p(p-1) + 1$, the characteristic vectors $\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^{p(p-1)}$ are affinely independent. Lastly, when $\hat{p} < p$, the resulting matrix $\bar{X}'_{M\&R}$ that is extracted from the characteristic vectors generated by subroutine $M\&R(\{S^0\}, I^0, \hat{p})$ is contained within the larger matrix obtained with subroutine $M\&R(\{S^0\}, I^0, p)$ and, thus, the $\hat{p}(\hat{p}-1) + 1$ characteristic vectors produced must also be affinely independent. \square

4.2 Facet constructions and proofs

The presented construction procedures iteratively generate individual characteristic vectors so that they fall into the respective ranking structures of a VI class and differ minimally from preceding vectors (or from other specific reference vectors). These procedures are simplified with the incorporation of M&R subroutines, which are used to generate a large portion of the needed $n(n-1)$ affinely independent vectors; the remaining vectors are generated to yield other convenient patterns. The respective M&R subroutines differ in their definition of the optional outer step of the j th inner loop; the shorthand statement used to represent a specific M&R variant in the pseudocode is:

$$M\&R(S, I, \hat{p}) \mid j\text{th optional outer step defined by}(\#)$$

where $(\#)$ denotes a set of algorithmic expressions. The characteristic vectors are stored in a matrix $X \in \mathbb{R}^{n(n-1) \times n(n-1)}$. Each row of X is obtained by converting an alternative-ordering $\{S\}$ into its characteristic vector, which is represented in the pseudocode by the operation $X.append(toBinary(\{S\}))$.

Each FDI proof begins with a *difference matrix* $\bar{X} \in \mathbb{R}^{n(n-1) \times n(n-1)}$ that reflects the dissimilarities between (mostly) consecutively generated vectors in X ; the precise entries of each difference matrix are described in the ‘‘Appendix’’. Row operations are applied to show that \bar{X} is non-singular or, equivalently, that the characteristic vectors are linearly (and affinely) independent. A salient feature of these proofs is

that, by leveraging the characteristic vector patterns devised within each construction procedure, proving the non-singularity of \bar{X} reduces to showing the non-singularity of a symbolic 4×4 matrix. For the reader’s convenience, numerical examples of the step-by-step matrix operations described within each proof are included in the Supplementary Materials.

The remainder of this subsection will introduce different construction procedures and demonstrate that five of the six featured VI classes are FDIs for any $n \geq 4$. It will also explain why the remaining class is an FDI only for $n = 4$. Only one of the FDI proofs is shown within the body of the paper. The others are similar in structure and are located in the “Appendix” due to length considerations. Next, the construction procedure used to show that T1 VI is an FDI is presented in Algorithm 2; an example of this procedure is also included.

Example 4 Perform CPT1(5, 1) via Algorithm 2, with $j_1 = 2, j_2 = 3, j_3 = 4, j_4 = 5$ (and $i_1 = 1$). With these values, line 2 of Algorithm 2 initializes $\{S^0\}$ to the starting weak order from Example 3, and line 4 yields the 12 characteristic vectors for the weak orders under the “Merge” and “Reverse” columns therein (i.e., these are the direct outputs of the M&R subroutine). Line 5 yields the weak order from the last M&R outer step of Example 3—assigned to $\{S^1\}$ in line 4—as the 13th characteristic vector. Through a sequence of move operations that start from $\{S^1\}$, Lines 6–9 yield the next three characteristic vectors and Lines 10–13 three more after that; finally, line 14 yields the initial weak order as the 20th characteristic vector. These last seven weak orders are as follows:

$$\begin{aligned} & \{\{4\}, \{1, 5\}, \{3\}, \{2\}\}, \\ & \{\{4\}, \{3\}, \{1, 5\}, \{2\}\}, \\ & \{\{4\}, \{3\}, \{2\}, \{1, 5\}\}, \\ & \{\{4\}, \{3\}, \{1, 2\}, \{5\}\}, \\ & \{\{4\}, \{1, 3\}, \{2\}, \{5\}\}, \\ & \{\{1, 4\}, \{3\}, \{2\}, \{5\}\}, \\ & \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}. \end{aligned}$$

Theorem 8 (T1 FDI) *T1 VI is an FDI of \mathbf{P}_{WO}^n , for any $n \geq 4$.*

Proof It is straightforward to verify that each row of X output by CPT1 belongs to the ranking structures listed in Table 2.

For ease of exposition, fix $i_1 = 1$ and $j_k = k + 1$, for $k = 1, \dots, n - 1$ (or assume a corresponding relabeling of the alternatives is performed a priori). To begin, set \bar{X} as the matrix obtained after iteratively subtracting row $i - 1$ from row i , for $i = n(n - 1) - 1, \dots, 2$, and also subtracting row $i = n(n - 1)$ from row 1 of X ; see “Appendix 2” for a full characterization of \bar{X} . To proceed with row operations, define $A^0 \in \mathbb{R}^{n(n-1) \times n(n-1)}$ and set this matrix with the elements of \bar{X} , such that, all comparisons between $j, j' \in \hat{N}^c = N \setminus \{1\}$ appear in the first $(n - 1)(n - 2)$ columns, and the elements involving the comparison of alternative 1 with $j \in \hat{N}^c$ show up in the last $2(n - 1)$ columns. The columns of \bar{X} and A^0 are further organized as in Eq. (10); that is, odd columns follow a lexicographical ordering of the respective arcs and

Algorithm 2 Construction Procedure for Type 1 VI (CPT1)

```

1: procedure CPT1( $n, i_1$ )
2:    $\{S^0\} = \{\{i_1, j_1\}, \{j_2\}, \{j_3\}, \dots, \{j_{n-1}\}\}$ , where  $\{j_k\}_{k=1}^{n-1} = [n] \setminus \{i_1\}$ 
3:    $p = |\{S^0\}| = n - 1$ 
4:    $(X, \{S^1\}) \leftarrow [M\&R(\{S^0\}, i_1, p) \mid j\text{th optional outer step defined by (9)}]$ 
5:    $X.append(toBinary(\{S^1\}))$ 
6:   for  $j = 1, \dots, p - 1$  do
7:      $\{S^1\} \leftarrow \left\langle S^1_j, \frac{3}{2}, \{S^1\} \right\rangle$ 
8:      $X.append(toBinary(\{S^1\}))$ 
9:   end for
10:  for  $j = 1, \dots, p - 1$  do
11:     $\{S^1\} \leftarrow \left\langle i_1, -1, \{S^1\} \right\rangle$ 
12:     $X.append(toBinary(\{S^1\}))$ 
13:  end for
14:   $X.append(toBinary(\{S^0\}))$ 
15:  return  $X$ 
16: end procedure

```

even columns have the reverse indexing of the preceding odd columns. Next, add the first $(n - 1)(n - 2)/2$ even-index rows to row $n(n - 1)$ (i.e., the initial weak order) and partition the resulting matrix, A^1 , as follows:

$$A^1 = \begin{bmatrix} B^1 & D^1 \\ C^1 & E^1 \end{bmatrix} \tag{11}$$

where $B^1 \in \mathbb{Z}^{(n-1)(n-2) \times (n-1)(n-2)}$, $C^1 \in \mathbb{Z}^{(2n-2) \times (n-1)(n-2)}$, $D^1 \in \mathbb{Z}^{(n-1)(n-2) \times (2n-2)}$, and $E^1 \in \mathbb{Z}^{(2n-2) \times (2n-2)}$. The pertinent entries of the four submatrices are as follows. First, B^1 consists of all but the last row of the $\tilde{X}_{M\&R}$ submatrix [Eq. (10) with $p = n - 1$], which implies that $|\det(B^1)| = 1$. C^1 is mostly a zero matrix, with the exception of row i whose values under columns $(n - i + 1, n)$ and $(n, n - i + 1)$ are 1 and -1 , respectively, for $i = 2, \dots, n - 1$. Although D^1 has a more intricate structure than B^1 and C^1 , it is only necessary to know the contents of a subset of rows aligned with those rows in B^1 that will be used to turn C^1 into a zero matrix. Expressly, each nonzero row i of C^1 is eliminated to yield an all-zero matrix C^2 by adding to it the two consecutive elementary vectors from B^1 with the opposite signs under columns $(n - i + 1, n)$ and $(n, n - i + 1)$; call them $k(i)$ and $k(i) + 1$. Rows $k(i)$ and $k(i) + 1$ of D^1 have a 1 under column $(n, 1)$ and a -1 under column $(1, n)$ respectively with no other nonzero entries, for $i = 3, \dots, n - 1$. For, $i = 2$, row $k(i)$ of D^1 has $-1, 1,$ and 1 under columns $(n - 2, 1), (1, n - 1),$ and $(1, n)$, respectively, and no other nonzero entries; row $k(i) + 1$ of D^1 has -1 under column $(1, n)$ and no other nonzero entries. Finally, E^1 is comprised primarily of a ‘wraparound staircase’ structure of nonzeros, illustrated as follows:

$$\mathbf{E}^1 = \begin{matrix} & (1,2) & (2,1) & (1,3) & (3,1) & (1,4) & (4,1) & (1,5) & \dots & (1,n-2) & (n-2,1) & (1,n-1) & (n-1,1) & (1,n) & (n,1) \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ n-3 \\ n-2 \\ n-1 \\ n \\ n+1 \\ n+2 \\ n+3 \\ \vdots \\ 2n-3 \\ 2n-2 \end{matrix} & \left[\begin{array}{cccccccccccccc} & & & & & & & & & & & & & -1 & 1 \\ & & & & & & & & & & & & -1 & 1 & \\ & & & & & & & & & & & & & & & -1 & 1 \\ & & & & & & & & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & & & & & & & & & & \ddots & \ddots \\ & \ddots & \ddots \\ & \ddots & \ddots \\ & -1 \\ & -1 \end{array} \right] ; \end{matrix}$$

E^1 's rows align with rows $(n - 1)(n - 2) + 1, \dots, n(n - 1)$ of \bar{X} , which correspond to the characteristic vectors generated following the M&R subroutine. We remark that in addition to having a 1 under each of its first two columns, row $2n - 2$ of E^1 has a decreasing sequence of consecutive negative integers under columns $(1, j)$, for $j = 4, \dots, n$. Based on the above explanations, after eliminating the nonzero elements of C^1 using the aforementioned rows from B^1 (and D^1), E^1 changes into E^2 , given by:

$$\mathbf{E}^2 = \begin{matrix} & (1,2) & (2,1) & (1,3) & (3,1) & (1,4) & (4,1) & (1,5) & \dots & (1,n-2) & (n-2,1) & (1,n-1) & (n-1,1) & (1,n) & (n,1) \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ n-3 \\ n-2 \\ n-1 \\ n \\ n+1 \\ n+2 \\ n+3 \\ \vdots \\ 2n-3 \\ 2n-2 \end{matrix} & \left[\begin{array}{cccccccccccccc} & & & & & & & & & & & & & -1 & 1 \\ & & & & & & & & & & & & & -1 & 1 \\ & & & & & & & & & & & & & & & -1 & 1 \\ & & & & & & & & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & & & & & & & & & & \ddots & \ddots \\ & \ddots & \ddots \\ & \ddots & \ddots \\ & -1 \\ & -1 \end{array} \right] ; \end{matrix}$$

Now, since C^2 is a zero matrix, $|\det(A^1)| = |\det(B^1) \det(E^2)| = |\det(E^2)|$ and, therefore, we can operate exclusively on E^2 from this point. First, eliminate the nonzero entries along row $2n - 2$, one by one, from column $(1, 2)$ to column $(n - 2, 1)$ by adding to row $2n - 2$ a multiple of some row i , where $3 \leq i \leq 2n - 3$. Specifically, beginning with row $i = n$, alternate between a row with index $i \leq n$ and a row with index $i > n$ to select each succeeding pivot row. Note that in each such elimination step, there is only one pivot element available from the designated row-index subset to eliminate the next nonzero entry from row $2n - 2$, whose value may have been modified by the previous elimination steps. For $4 \leq j < n$, the sequence value under column $(1, j)$ remains intact until the nonzero under column $(j - 1, 1)$ (the column immediately to the left of $(1, j)$) is eliminated; the latter has a value equal to $1 - \sum_{k=1}^{j-3} k$, that is, the sum of the preceding negative integers in the sequence, plus one. Second, subtract row $2n - 3$ (the penultimate row) from row 2 and add rows 3 to $2n - 4$ to row $2n - 3$. Upon completion of these row operations, the resulting matrix E^3 possesses the following form:

shown to be facet-defining in Fiorini and Fishburn (2004) using the `Porta` program (Christof et al. 1997). Therefore T1 VI is an FDI for any $n \geq 4$. \square

Next, we describe why T2-0 VI is an FDI for $n = 4$ but not $n \geq 5$. Since P_{WO}^n is full dimensional, the rank of the characteristic vectors generated by any construction procedure must be at least $n(n - 1) - 1$. Because the maximum number of affinely independent weak orders on N that do not contain ties is approximately half of this value, this means that a significant fraction of the generated vectors must contain ties. It can be seen in Table 3, however, that the placement of ties is limited to rankings with a two-way tie with alternative i_1 for first position (#2 & #4) and rankings with a two way-tie with alternative i_2 for last position (#3 & #4). When $n = 4$, enough affinely independent vectors can be drawn from the four T2-0 VI ranking structures to form a facet—in fact, it is straightforward to verify that substituting this value of n in T2-0 VI expression yields WO_5 . However, as n becomes larger, they do not add up to a sufficiently large fraction. Through enumeration of the weak orders that satisfy the given ranking structures, we verified for many n that the rank of the resulting matrix is far lower than $n(n - 1)$; for example, for $n = 6$, the rank is 15 (i.e., half the required number). Therefore, T2-0 VI is not an FDI for $n \geq 5$.

The construction procedure used to generate characteristic vectors for T2-1 VI is given in Algorithm 3, where the j th optional outer step for the respective M&R subroutine is defined as:

$$\{S^1\} \leftarrow \langle i_2, \frac{3}{2}, \{S^1\} \rangle \tag{12a}$$

$$\{S^1\} \leftarrow \langle i_1, j - p, \{S^1\} \rangle \tag{12b}$$

$$X.append(toBinary(\{S^1\})). \tag{12c}$$

Theorem 9 (T2-1 FDI) *T2-1 VI is an FDI of P_{WO}^n , for any $n \geq 4$.*

Proof It is straightforward to verify that all points output by CPT2-1 correspond to the characteristic vectors associated with the six ranking structures that satisfy inequality (5) at equality. The remainder of the proof can be found in “Appendix B.2”. \square

Theorem 10 (T2-2 FDI) *T2-2 VI is an FDI of P_{WO}^n , for any $n \geq 4$.*

Proof It is evident from the proof of Theorem 4 that the applicable characteristic vectors associated with the ranking structures that satisfy inequality (6) at equality are symmetric images of those output by CPT2-1. Therefore, an almost identical set of arguments used in the proof of Theorem 9 establishes that inequality (6) is an FDI for $n \geq 4$. \square

A comparison of Tables 4 and 5 reveals that the only ranking structure from Table 4 that does not satisfy valid inequality T2-3 is #2, in which two fixed alternatives are tied for the first position and the unfixed alternatives are strictly ordered to occupy positions 2 to $n - 1$. The characteristic vector that represents ranking structure #2 from Table 4 is generated by line 7 of the pseudocode(CPT2-1), after which a number of move operations are performed to generate a vector representing structure #3 from

the same table. Hence, certain modifications to lines 7–11 are needed to make the construction procedure applicable to the ranking structures associated with T2-3 VI. These modifications are as follows. First, after Line 6, move the second fixed alternative, i_2 , one step to the right to generate a ranking representative of structure #2 in Table 5. Second, break the tie between alternatives i_2 and j_1 created by the previous step by first moving i_2 a half step to the right and then shifting i_1 one step to the right to tie it with j_1 ; this generates a ranking representative of structure #3 in Table 5. The above changes can be encapsulated by replacing lines 7–11 of Algorithm 3 with the following algorithmic expressions:

- Line 7' : $\{S^0\} \leftarrow \langle i_2, 1, \{S^0\} \rangle$
- Line 8' : $\mathbf{X.append}(toBinary(\{S^0\}))$
- Line 9' : $\{S^0\} \leftarrow \langle i_2, \frac{1}{2}, \{S^0\} \rangle$
- Line 10' : $\{S^0\} \leftarrow \langle i_1, 1, \{S^0\} \rangle$
- Line 11' : $\mathbf{X.append}(toBinary(\{S^0\}))$

Algorithm 3 Construction Procedure for Type 2-1 VI (CPT2-1)

- 1: **procedure** CPT2- 1(n, i_1, i_2)
 - 2: $\{S^0\} = \{i_1, i_2, j_1, \dots, j_{n-4}, j_{n-2}, j_{n-3}\}$, where $\{j_k\}_{k=1}^{n-2} = [n] \setminus \{i_1, i_2\}$
 - 3: $X = []$
 - 4: $X.append(toBinary(\{S^0\}))$
 - 5: $\{S^0\} \leftarrow \langle S_n^0, -\frac{3}{2}, \{S^0\} \rangle$
 - 6: $X.append(toBinary(\{S^0\}))$
 - 7: $\{S^0\} \leftarrow \langle i_1, 1, \{S^0\} \rangle$
 - 8: $X.append(toBinary(\{S^0\}))$
 - 9: $\{S^0\} \leftarrow \langle S_1^0, 1, \{S^0\} \rangle$
 - 10: $\{S^0\} \leftarrow \langle i_2, \frac{1}{2}, \{S^0\} \rangle$
 - 11: $X.append(toBinary(\{S^0\}))$
 - 12: $p = |\{S^0\}| = n - 1$
 - 13: $(X, \{S^1\}) \leftarrow [M\&R(\{S^0\}, i_1, p - 2) \mid j\text{th optional outer step defined by (12)}]$
 - 14: $\{S^1\} \leftarrow \langle i_2, -1, \{S^1\} \rangle$
 - 15: $X.append(toBinary(\{S^1\}))$
 - 16: $\{S^1\} \leftarrow \langle i_2, \frac{1}{2}, \{S^1\} \rangle$
 - 17: **for** $j = 1, \dots, p - 1$ **do**
 - 18: $\{S^1\} \leftarrow \langle S_j^1, \frac{3}{2}, \{S^1\} \rangle$
 - 19: $X.append(toBinary(\{S^1\}))$
 - 20: **end for**
 - 21: **return** X
 - 22: **end procedure**
-

Theorem 11 (T2-3 FDI) *T2-3 VI is an FDI of \mathbf{P}_{WO}^n , for $n \geq 4$.*

Proof The proof is similar to that of Theorem 9 and can be found in ‘‘Appendix B.3’’. □

Theorem 12 (T2-4 FDI) *T2-4 VI is an FDI of \mathbf{P}_{WO}^n , for any $n \geq 4$.*

Proof This theorem is proved by using a string of arguments similar to the proof of Theorem 10, with the difference that the applicable characteristic vectors are symmetric images of those generated by the algorithm used to establish that inequality (7) is facet defining for $n \geq 4$. □

5 Additional insights and conjectures

We begin this section by discussing how the featured VIs yield FDIs WO_4 – WO_9 (see Sect. 2) as a special case. By fixing $n = 4$ in each expression, it is straightforward to verify that VI T1 yields FDI WO_4 , while VIs T2-0, T2-1, and T2-2 correspond to FDIs WO_5 , WO_6 , and WO_7 , respectively; finally VI T2-3 induces FDI WO_9 and VI T2-4 induces FDI WO_8 . According to this correspondence, $O(n^4)$ FDIs can be generated from this single variant of each featured VI in a higher dimension n [see Eq. (2)]. On that note, it is important to elaborate on how the featured VI classes provide a distinctive expression for each setting of $n \geq 4$. To see this, we present the expressions for T1 VI for $n = 4, 5$, with $i_1 = 1$ as the fixed alternative:

$$x_{12} + x_{21} + x_{13} + x_{31} + x_{14} + x_{41} - x_{23} - x_{32} - x_{24} - x_{42} - x_{34} - x_{43} \leq 1. \tag{13a}$$

$$x_{12} + x_{21} + x_{13} + x_{31} + x_{14} + x_{41} - x_{23} - x_{32} - x_{24} - x_{42} - x_{34} - x_{43} + x_{15} + x_{51} - x_{25} - x_{52} - x_{35} - x_{53} - x_{45} - x_{54} \leq -1. \tag{13b}$$

Although the first 12 terms of the left-hand side of Eq. (13b) mirror the left-hand side of Eq. (13a), neither of the two inequalities dominates the other. Specifically, for a VI (π, π_0) to be dominated by a VI (π', π'_0) , the relationships $\pi' \geq \mu\pi$ and $\pi'_0 \leq \mu\pi_0$ must hold, for some scalar $\mu > 0$, with at least one of the two inequalities holding in the strict sense. For the two above inequalities, we have that $\pi'_0 = -1 < 1 = \pi_0$ —choosing $\mu = 1$ since the coefficients of the variables that do not involve alternative index $j = 5$ already match. However, the left-hand side terms related to $j = 5$ are both $+1$ and -1 in Eq. (13b), while they are all zero-valued in Eq. (13a), implying that $\pi' \not\geq \mu\pi$. In general, as n increases, the right-hand side will get smaller. Conversely, on the left-hand side, the coefficients of some newly added variables will become smaller than 0, and the coefficient of other newly added variables will become larger than 0 (hence, the incomparability will hold for any $\mu > 0$). Similar arguments can be applied to all of the T2 VIs. Therefore, for each of these large VI classes, each fixed value $\hat{n} \geq 4$ yields a distinctive VI that does not dominate or is dominated by another member of the same class in a lower dimension.

Since a distinctive set of FDIs can be generated from VIs T1, T2-1, T2-2, T2-3, and T2-4 for any fixed dimension $\hat{n} \geq 4$, the number of FDIs that can be generated

from these new classes of FDIs is very large. Specifically, the total number of FDIs that can be generated from VI class of Type i in dimension n is given by:

$$|WO_{Ti}^n| = \sum_{\hat{n}=4}^n |WO_{Ti}^{\hat{n}}| = \sum_{\hat{n}=4}^n \binom{n}{\hat{n}} \binom{\hat{n}}{i} i! = n! \sum_{\hat{n}=4}^n \frac{1}{(n - \hat{n})!(\hat{n} - i)!}. \tag{14}$$

That is, each of the five VI classes generates $O(n!)$ FDIs in dimension n .

We conclude the paper by stating two conjectures related to how the insights herein presented may induce many more large classes of FDIs.

Conjecture 13 *Let $\hat{N} = \{i_1, i_2, i_3\}$ be the fixed index set, where $i_1, i_2, i_3 \in N$ s.t. $i_1 \neq i_2 \neq i_3$ and let $\hat{N}^c = N \setminus \hat{N}$. The following are T3 VIs and FDIs of \mathbf{P}_{WO}^n , for any $n \geq 5$:*

$$x_{i_1 i_2} + x_{i_3 i_2} + \sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1} + x_{i_3 j} + x_{j i_3}) - x_{i_1 i_3} - x_{i_3 i_1} - \sum_{j, j' \in \hat{N}^c: j \neq j'} (x_{j j'} + x_{j i_2}) \leq 4 - \frac{(n-5)(n-6)}{2} \tag{15a}$$

$$x_{i_2 i_1} + x_{i_2 i_3} + \sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1} + x_{i_3 j} + x_{j i_3}) - x_{i_1 i_3} - x_{i_3 i_1} - \sum_{j, j' \in \hat{N}^c: j \neq j'} (x_{j j'} + x_{i_2 j}) \leq 4 - \frac{(n-5)(n-6)}{2} \tag{15b}$$

$$x_{i_1 i_3} + x_{i_2 i_3} + \sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1} + x_{i_2 j} + x_{j i_2}) - x_{i_1 i_2} - x_{i_2 i_1} - \sum_{j, j' \in \hat{N}^c: j \neq j'} (x_{j j'} + x_{j i_3}) \leq 4 - \frac{(n-5)(n-6)}{2} \tag{15c}$$

$$x_{i_3 i_1} + x_{i_3 i_2} + \sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1} + x_{i_2 j} + x_{j i_2}) - x_{i_1 i_2} - x_{i_2 i_1} - \sum_{j, j' \in \hat{N}^c: j \neq j'} (x_{j j'} + x_{i_3 j}) \leq 4 - \frac{(n-5)(n-6)}{2} \tag{15d}$$

$$\sum_{j \in \hat{N}^c} (x_{i_1 j} + x_{j i_1} + x_{j i_2} + x_{i_3 j} + x_{j i_3}) - x_{i_1 i_2} - x_{i_1 i_3} - x_{i_3 i_1} - x_{i_3 i_2} - \sum_{j, j' \in \hat{N}^c: j \neq j'} x_{j j'} \leq 5 - \frac{(n-6)(n-7)}{2} \tag{15e}$$

$$\sum_{j \in \hat{N}^c} (x_{i_1j} + x_{j i_1} + x_{i_2j} + x_{i_3j} + x_{j i_3}) - x_{i_2i_1} - x_{i_1i_3} - x_{i_3i_1} - x_{i_2i_3} - \sum_{j, j' \in \hat{N}^c: j \neq j'} x_{j j'} \leq 5 - \frac{(n-6)(n-7)}{2} \tag{15f}$$

$$\sum_{j \in \hat{N}^c} (x_{i_1j} + x_{j i_1} + x_{i_2j} + x_{j i_2} + x_{i_3j}) - x_{i_1i_2} - x_{i_2i_1} - x_{i_1i_3} - x_{i_2i_3} - \sum_{j, j' \in \hat{N}^c: j \neq j'} x_{j j'} \leq 5 - \frac{(n-6)(n-7)}{2} \tag{15g}$$

$$\sum_{j \in \hat{N}^c} (x_{i_1j} + x_{j i_1} + x_{i_2j} + x_{j i_2} + x_{i_3j}) - x_{i_1i_2} - x_{i_2i_1} - x_{i_3i_1} - x_{i_3i_2} - \sum_{j, j' \in \hat{N}^c: j \neq j'} x_{j j'} \leq 5 - \frac{(n-6)(n-7)}{2} \tag{15h}$$

This conjecture was conceived as a two-step process. In the first step, the following binary programming problem was solved for all possible format and orientation choices of arcs that join only the fixed alternatives (i.e., $\{i_1, i_2, i_3\} = \hat{N}$) combined with the possible format and orientation choices of arc sets $\mathcal{A}^{(i_1, j)}$, $\mathcal{A}^{(i_2, j)}$, $\mathcal{A}^{(i_3, j)}$ (where $j \in \hat{N}^c$ is an unfixed alternative index):

$$\begin{aligned} \max \quad & \sum_{(i, j) \in \mathcal{A}_+} x_{ij} - \sum_{(i, j) \in \mathcal{A}_-} x_{ij} \\ \text{st.} \quad & \text{Constraints (1a)–(1c)} \\ & x_{ij} \in \{0, 1\} \qquad i, j \in N; i \neq j. \end{aligned}$$

These format and orientation choices determine the contents of the sets \mathcal{A}_+ and \mathcal{A}_- ; we remark that the characteristics of the arcs that join only the unfixed alternatives, namely $\mathcal{A}^{(j, j')} := \{(j, j') : j \in \hat{N}^c, j \neq j'\}$, were set to the constant format and orientation as the six digraphs given in Fig. 1 (all dashed and bidirectional). In general, the above binary program can be defined for a specific number of fixed alternatives and solved for various values of n so as to deduce a generalizable expression. The objective function provides an algebraic expression for the left-hand side of a VI, while the sequence of numerical objective function values obtained from solving for consecutive values of n are analyzed to derive an algebraic expression for its right-hand side.

The second step to conceive the above conjecture involved verifying that the VIs generated in the first step are FDIs for at least a few values of n . This was done by enumerating all characteristic vectors that satisfy each inequality in (15) at equality and verifying that these vectors induce a matrix of rank $n(n-1)$, for $5 \leq n \leq 10$. Hence, the above conjecture is true for at least these values. It remains to show that the result will hold for any $n \geq 11$. Proving this using the techniques introduced in this work would require introducing new construction procedures for each VI, followed

by additional formal proofs (which would likely be even more protracted). Hence, it is left for future work.

Conjecture 14 *Let, $\hat{N} = \{i_1, i_2 \dots i_k\}$ be a subset of $N = [n]$, where $k \leq n - 2$ and $i_1, i_2 \dots i_k \in N$ s.t. $i_1 \neq i_2 \dots \neq i_k$. By fixing these $|\hat{N}|$ alternatives, new large classes of FDIs that incorporate at least one variable from each pair of alternatives in N can be obtained using the insights presented in this work.*

Verification of Conjecture 14 could lead to the development of new techniques for deriving numerous new large classes of FDIs. Each such class would define a set of FDIs specific to $P_{WO}^{\hat{n}}$, that is, no individual member of the FDIs specific to dimension \hat{n} would be defined in dimension $q < \hat{n}$. As such, the FDIs would require the involvement of all alternative pairs on \hat{N} . Conversely, when using the Lifting Lemma to generate FDIs of $P_{WO}^{\hat{n}}$ from FDIs of P_{WO}^q , only the alternative pairs from a smaller set are utilized—for example, from $\hat{N} \setminus \{q + 1, q + 2, \dots, \hat{n}\}$.

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Availability of data and material The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

Code Availability The computer code used to generate the characteristic vectors and the matrices used in the proofs of the T1, T2-1, and T2-3 FDIs is available at: <https://github.com/ryasmin/Ranking-VI-proofs.git>.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Valid inequality proofs

A.1 T2-0 VI proof

Theorem 2 (T2-0 VI) *Inequality (4) is a VI of \mathbf{P}_{WO}^n , for any $n \geq 4$.*

Proof Inequality (4) is satisfied at equality by the characteristic vectors corresponding to the ranking structures listed in Table 3, which are justified as follows. Here, the positive and negative arc subsets are given by:

$$\begin{aligned} \mathcal{A}_+ &= \{(i_2, i_1)\} \cup \{(i_1, j) : j \in \hat{N}^c\} \cup \{(j, i_2) : j \in \hat{N}^c\}, \\ \mathcal{A}_- &= \{(i_1, i_2)\} \cup \{(j, j') : j, j' \in \hat{N}^c, j \neq j'\}. \end{aligned}$$

Let W be an arbitrary weak order defined over the set of alternatives N , where the fixed alternatives i_1 and i_2 are tied with exactly $t_1 \geq 0$ and $t_2 \geq 0$ unfixed alternatives, respectively, such that $t_1 + t_2 \leq n - 2$. The structure of the weak orders generated by the different values of t_1 and t_2 can be encapsulated by the following cases:

Case 1 ($t_1 \leq 1, t_2 \leq 1$) Suppose that the ranks of alternatives i_1 and i_2 are fixed as $r_{i_1} = k_1 > 0$ and $r_{i_2} = k_2 > 0$, respectively. Furthermore, let $\hat{N}_{<k_1}^c \subseteq \hat{N}^c$ denote the set of unfixed alternatives with rank $r < k_1$, and $\hat{N}_{>k_2}^c \subseteq \hat{N}^c$ denote the set of unfixed alternatives with rank $r' > k_2$. Note that $\hat{N}_{<k_1}^c \cap \hat{N}_{>k_2}^c = \emptyset$ only when either $k_1 \leq k_2$ or $k_1 = k_2 + t_2$. Based on the relative values of k_1 and k_2 , the weak orders can be further divided into the following two sub-cases:

Case 1a ($k_1 < k_2$) For each $\hat{j} \in \hat{N}_{<k_1}^c$ and $\hat{j}' \in \hat{N}_{>k_2}^c$ add positive arcs (i_1, \hat{j}) and (\hat{j}', i_2) , respectively, to W to obtain a weak order where alternative i_1 is in first place and alternative i_2 is in last place. This gives precisely $2(n - 2)$ selected positive arcs: $n - 2$ each from the ordering of the two fixed alternatives i_1 and i_2 with each unfixed alternative $j \in \hat{N}^c$. In the event of ties between i_1 and an unfixed alternative, say j_1 (i.e., $t_1 = 1$), or/and between i_2 and an unfixed alternative, say j_2 (i.e., $t_2 = 1$), it may be necessary to remove the corresponding arcs, specifically arcs (j_1, i_1) and (i_2, j_2) , respectively, to maintain transitivity. Since these are zero arcs (i.e., $\pi_{j_1, i_1} = \pi_{i_2, j_2} = 0$), their removal does not affect the values of $\|\mathbf{x}_-^W\|$ or $\|\mathbf{x}_+^W\|$. Next, break the ties (if any) between $j, j' \in \hat{N}^c$, where $j \neq j'$, by removing either negative arc (j, j') or negative arc (j', j) from W ; denote the generated weak order as W' . This gives a total of $(n - 2)(n - 3)/2 + 1$ selected negative arcs. Since all preceding operations do not increase the value of $\|\mathbf{x}_-^W\|$ or decrease the value of $\|\mathbf{x}_+^W\|$, we have that $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\| \leq \|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\|$. For any member of the presented ranking structure, the value of $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\|$ equals the right-hand side of inequality (4). Note that, since there must be at least one arc between every pair of alternatives to induce a total order, further reduction in negative arcs is not possible. Now, depending on the initial position of i_1 and i_2 and the value of t_1 and t_2 , W' attains one of four ranking structures: alternative i_1 is uniquely in first place and alternative i_2 is uniquely in last place (#1), an alternative $j_1 \in \hat{N}^c$ is tied with i_1 for first place (#2), an alternative $j_2 \in \hat{N}^c$ is tied with i_2 for the last available position, $n - 1$ in this case (#3), or both of these ties occur, with $j_1 \neq j_2$ (#4).

Case 1b ($k_1 \geq k_2$) This case is addressed using the same arguments as in Case 1a with the addition that, to maintain transitivity, it is necessary either to replace positive arc (i_2, i_1) with negative arc (i_1, i_2) (when $k_1 > k_2$) or to remove positive arc (i_2, i_1) (when $k_1 = k_2$) to place i_1 strictly in front of i_2 ; denote the generated weak order as W' . In this case, the relevant operations increase the value of $\|\mathbf{x}_-^W\|$ by at most 1 and of $\|\mathbf{x}_+^W\|$ by at least $n - 3 \geq 1$. As a result, we have that $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\| \leq \|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\|$. This also leads to the same ranking structures as in Case 1a.

Case 2 ($t_1 \geq 2, t_2 \leq 1$) Let $[j_1, j_{t_1}] \subseteq \hat{N}^c$ denote those unfixed alternatives tied with i_1 in rank position k_1 . First, break the ties between \hat{j} and j_{t_1} by removing arc (j_{t_1}, \hat{j}) from W , for each $\hat{j} \in \{i_1\} \cup [j_1, j_{t_1 - 1}]$; denote the generated weak order as $W_{t_1 - 1, t_2}$. Since this removal decreases the value of $\|\mathbf{x}_-^W\|$ by $(t_1 - 1) \geq 1$ and keeps the value of $\|\mathbf{x}_+^W\|$ unchanged, we have that $\|\mathbf{x}_+^{W'}\| - \|\mathbf{x}_-^{W'}\| < \|\mathbf{x}_+^{W_{t_1 - 1, t_2}}\| - \|\mathbf{x}_-^{W_{t_1 - 1, t_2}}\|$. It is worth noting that, as (j_{t_1}, i_1) is a zero arc, its removal does not affect the values of

$\|\mathbf{x}_+^W\|$ or $\|\mathbf{x}_+^W\|$. Next, break the ties between each of $\hat{j} \in \{i_1\} \cup [j_1, j_{t_1} - 2]$ with $j_{t_1} - 1$ in $W_{t_1 - 1, t_2}$ in a similar fashion to obtain a modified weak order, denoted as $W_{t_1 - 2, t_2}$. Repeat this process until arriving at a weak order, denoted as W_{1, t_2} , in which i_1 is tied with exactly one unfixed alternative j_1 . At this point, apply the results of Case 1 by setting $W := W_{1, t_2}$.

Case 3 ($t_2 \geq 2$) Let $[j_1, j_{t_2}] \subseteq \hat{N}^c$ denote those unfixed alternatives tied with alternative i_2 in rank position k_2 . This case is addressed using the same arguments as in Case 2 by replacing $\{i_1\} \cup [j_1, j_{t'} - 1]$ and $(j_{t'}, \hat{j})$ with $\{i_2\} \cup [j_1, j_{t'} - 1]$ and $(\hat{j}, j_{t'})$, respectively, in each step, where $t' = t_2, t_2 - 1, \dots, 2$. In the generated weak order, denoted as $W_{t_1, 1}$, i_2 is tied with exactly one unfixed alternative j_1 . At this point, apply the results of Case 1 or 2 by setting $W := W_{t_1, 1}$. □

A.2 T2-3 VI proof

Theorem 5 (T2-3 VI) *Inequality (7) is a VI of $\mathbf{P}_{W,O}^n$, for any $n \geq 4$.*

Proof Inequality (7) is satisfied at equality by the characteristic vectors corresponding to the ranking structures listed in Table 5. Here, the positive and negative arc subsets are given by:

$$\begin{aligned} \mathcal{A}_+ &= \{(i_2, j) : j \in \hat{N}^c\} \cup \{(i_1, j) : j \in \hat{N}^c\} \cup \{(j, i_1) : j \in \hat{N}^c\}, \\ \mathcal{A}_- &= \{(i_2, i_1)\} \cup \{(j, j') : j, j' \in \hat{N}^c, j \neq j'\}. \end{aligned}$$

Let W be an arbitrary weak order defined over the set of alternatives N , where the fixed alternatives i_1 and i_2 are tied with exactly $t_1 \geq 0$ and $t_2 \geq 0$ unfixed alternatives, respectively, such that $t_1 + t_2 \leq n - 2$. Now suppose that the ranks of these alternatives are fixed as $r_{i_1} = k_1 > 0$ and $r_{i_2} = k_2 > 0$. Additionally, let $\hat{N}_{<k_1}^c \subseteq \hat{N}^c$ denote the set of unfixed alternatives with rank $r < r_{i_1}$, and $\hat{N}_{<k_2}^c \subseteq \hat{N}^c$ denote the set of unfixed alternatives with rank $r' < r_{i_2}$. Note that, $\hat{N}_{<k_1}^c \cap \hat{N}_{<k_2}^c = \emptyset$ only when either $\hat{N}_{<k_1}^c = \emptyset$ or $\hat{N}_{<k_2}^c = \emptyset$ or both. The structure of the weak orders generated by the different values of t_1 and t_2 can be encapsulated by the following cases:

Case 1 ($t_1 = 0, t_2 \leq 1$) For each $\hat{j} \in \hat{N}_{<k_1}^c$, replace positive arc (\hat{j}, i_1) with positive arc (i_1, \hat{j}) and for each $\hat{j}' \in \hat{N}_{<k_2}^c$, add positive arc (i_2, \hat{j}') to W to obtain a weak order where alternatives i_1 and i_2 are placed in front of all $j \in \hat{N}^c$. This gives precisely $2(n - 2)$ selected positive arcs: $n - 2$ each from the strict ordering of the two fixed alternatives i_1 and i_2 with each unfixed alternative $j \in \hat{N}^c$. Next, break the ties (if any) between $j, j' \in \hat{N}^c$, where $j \neq j'$, by removing either negative arc (j, j') or negative arc (j', j) from W . Additionally, when $k_1 \geq k_2$, replace negative arc (i_2, i_1) with zero arc (i_1, i_2) to obtain a weak order, denoted as W' , in which i_1 is uniquely in first place and i_2 is uniquely in second place. This gives a total of $(n - 3)(n - 2)/2$ selected negative arcs. In the event of ties between i_2 and an unfixed alternative, say $j_1 \in \hat{N}^c$ (i.e., $t_2 = 1$), it is also necessary to replace negative arc (\hat{j}', j_1) with negative arc (j_1, \hat{j}') for each $\hat{j}' \in \hat{N}_{<k_2}^c$ to maintain the tie between i_2 and j_1 . Furthermore,

when $k_1 = k_2$ remove positive arc (j_1, i_1) , whereas when $k_1 > k_2$ replace positive arc (j_1, i_1) with positive arc (i_1, j_1) to place i_1 strictly in front of j_1 . Since all preceding operations neither increase the value of $\|x_-^W\|$ nor decrease the value of $\|x_+^W\|$, we have that $\|x_+^W\| - \|x_-^W\| \leq \|x_+^{W'}\| - \|x_-^{W'}\|$. For any such weak order, equivalently characterized by either ranking structure #1, if $t_2 = 0$, or #2, if $t_2 = 1$, the value of $\|x_+^{W'}\| - \|x_-^{W'}\|$ equals the right-hand side of inequality (7).

Case 2 ($t_1 = 1, t_2 \leq 1$) Let $j_1 \in \hat{N}^c$ denote the unfixed alternative tied with i_1 in rank position k_1 . For each $\hat{j}' \in \hat{N}_{<k_2}^c \setminus \{j_1\}$, add positive arc (i_2, \hat{j}') to W to place i_2 in front of all $j \in \hat{N}^c \setminus \{j_1\}$. Additionally, when $k_1 \leq k_2$, for each $\hat{j} \in \hat{N}_{<k_1}^c$, replace positive arc (\hat{j}, i_1) with positive arc (i_1, \hat{j}) and negative arc (\hat{j}, j_1) with negative arc (j_1, \hat{j}) , to obtain a weak order where alternatives i_1, j_1 , and i_2 are placed in front of all $j \in \hat{N}^c \setminus \{j_1\}$. In the event of tie between i_2 and an unfixed alternative, say $j_2 \in \hat{N}^c \setminus \{j_1\}$ (i.e., $t_2 = 1$), it is also necessary to replace negative arc (\hat{j}', j_2) with negative arc (j_2, \hat{j}') for each $\hat{j}' \in \hat{N}_{<k_2}^c \setminus \{j_1\}$ to maintain the tie between i_2 and j_2 . Next the ties (if any) between $j, j' \in \hat{N}^c \setminus \{j_1\}$, where $j \neq j'$, are broken by removing either negative arc (j, j') or negative arc (j', j) from W ; denote the generated weak order as W' . Since the preceding operations neither increase the value of $\|x_-^W\|$ nor decrease the value of $\|x_+^W\|$, we have that $\|x_+^W\| - \|x_-^W\| \leq \|x_+^{W'}\| - \|x_-^{W'}\|$. Now, depending on the initial position of i_1 and i_2 and the value of t_2 , W' attains one of six ranking structures: i_1 is tied with j_1 for first place (#3); i_1 is tied with j_1 for first place and i_2 is tied with j_2 for third place (#4); i_1, i_2 , and j_1 all remain tied for first place (#5); i_1, i_2, j_1 , and j_2 all remain tied for first place (#6); i_1 and j_1 remain tied for any position but first, which is occupied by i_2 (#7); i_1 and j_1 remain tied for any position but first, which is occupied by i_2 jointly with j_2 (#8).

Case 3 ($t_1 = 2, t_2 \leq 1$) Let $j_1, j_2 \in \hat{N}^c$ denote the unfixed alternatives tied with i_1 in rank position k_1 . For each $\hat{j}' \in \hat{N}_{<k_2}^c$, add positive arc (i_2, \hat{j}') to W to place i_2 in front of all alternatives $j \in N$. Additionally, when $k_1 < k_2$, replace zero arc (i_1, i_2) with negative arc (i_2, i_1) , whereas when $k_1 = k_2$ remove zero arc (i_1, i_2) to place i_2 strictly in front of i_1 . In the event of tie between i_2 and an alternative, say $j_3 \in \hat{N}^c \setminus \{j_1, j_2\}$ (i.e., $t_2 = 1$), it is also necessary to replace negative arc (\hat{j}', j_3) with negative arc (j_3, \hat{j}') for each $\hat{j}' \in \hat{N}_{<k_2}^c$ to maintain the tie between i_2 and j_3 . Furthermore, when $k_1 < k_2$ replace arc (\hat{j}, j_3) with arc (j_3, \hat{j}) , whereas when $k_1 = k_2$ remove arc (\hat{j}, j_3) for each $\hat{j} \in \{i_1, j_1, j_2\}$. Next, the ties (if any) between $j, j' \in \hat{N}^c \setminus \{j_1, j_2\}$, where $j \neq j'$, are broken by removing either negative arc (j, j') or negative arc (j', j) from W ; denote the generated weak order as W' . Since the preceding operations neither increase the value of $\|x_-^W\|$ nor decrease the value of $\|x_+^W\|$, we have that $\|x_+^W\| - \|x_-^W\| \leq \|x_+^{W'}\| - \|x_-^{W'}\|$. Here, W' is a member of ranking structure #9, if $t_2 = 0$, or #10, if $t_2 = 1$.

Case 4 ($t_1 \geq 3, t_2 \leq 1$) Let $[j_1, j_{t_1}] \subseteq \hat{N}^c$ denote the unfixed alternatives tied with i_1 in rank position k_1 . First, break the tie between \hat{j} and j_{t_1} by removing arc (j_{t_1}, \hat{j}) from W , for each $\hat{j} \in \{i_1\} \cup [j_1, j_{t_1} - 1]$ (when $k_1 = k_2, \hat{j} \in \{i_1, i_2\} \cup [j_1, j_{t_1} - 1]$); denote the generated weak order as W_{t_1-1, t_2} . Since this removal decreases the value

of $\|\mathbf{x}_-^W\|$ and $\|\mathbf{x}_+^W\|$ by at least $(t_1 - 1) \geq 1$ and at most 1, respectively, we have that $\|\mathbf{x}_+^W\| - \|\mathbf{x}_-^W\| \leq \|\mathbf{x}_+^{W_{t_1-1,t_2}}\| - \|\mathbf{x}_-^{W_{t_1-1,t_2}}\|$. Next, break the ties between each of $\hat{j} \in \{i_1\} \cup [j_1, j_{t_1-2}]$ (when $k_1 = k_2$, $\hat{j} \in \{i_1, i_2\} \cup [j_1, j_{t_1-2}]$) with j_{t_1-1} in W_{t_1-1,t_2} in a similar fashion to obtain a modified weak order, denoted as W_{t_1-2,t_2} . Repeat this process until arriving at a weak order, denoted as W_{2,t_2} , in which i_1 is tied with exactly two unfixed alternatives, say j_1 and j_2 (when $k_1 = k_2$, both i_1 and i_2 are tied with j_1 and j_2). At this point, apply the results of Case 3 by setting $W := W_{2,t_2}$.

Case 5 ($t_2 \geq 2$) Let $[j_1, j_{t_2}] \subseteq \hat{N}^c$ denote the unfixed alternatives tied with i_2 in rank position k_2 . This case is addressed using the same arguments as in Case 4 by replacing $\{i_1\} \cup [j_1, j_{t'-1}]$ with $\{i_2\} \cup [j_1, j_{t'-1}]$ in each step, where $t' = t_2, t_2 - 1, \dots, 1$. In the generated weak order, denoted as $W_{t_1,1}$, i_2 is tied with exactly one unfixed alternative, say j_1 . At this point, apply the results of Cases 1-4 by setting $W := W_{t_1,1}$. \square

B Facet defining inequality proofs

B.1 T1 FDI differences matrix

To introduce the differences matrix we fixed $i_1 = 1$ and $j_k = k + 1$, for $k = 1, \dots, n - 1$; note that the same relabeling was used in the proof. The difference matrix, \bar{X} is generated after iteratively subtracting several rows of X as described in the T1 FDI proof (see Theorem 8). For ease of visualization, \bar{X} is partitioned into two matrices as,

$$\bar{X} = \begin{bmatrix} \bar{X}^1 & \bar{X}^2 \end{bmatrix} \tag{16}$$

where $\bar{X}_1 \in \mathbb{Z}^{n(n-1) \times 2(n-1)}$ and $\bar{X}_2 \in \mathbb{Z}^{n(n-1) \times (n-1)(n-2)}$ such that, the first $2n - 2$ columns i.e., elements involving the comparison of alternative 1 with $j \in \hat{N}^c$ show up in \bar{X}_1 and, the last $(n - 1)(n - 2)$ columns i.e., all comparisons between $j, j' \in \hat{N}^c = N \setminus \{1\}$ appear in \bar{X}_2 . Both of these matrices are illustrated as follows:

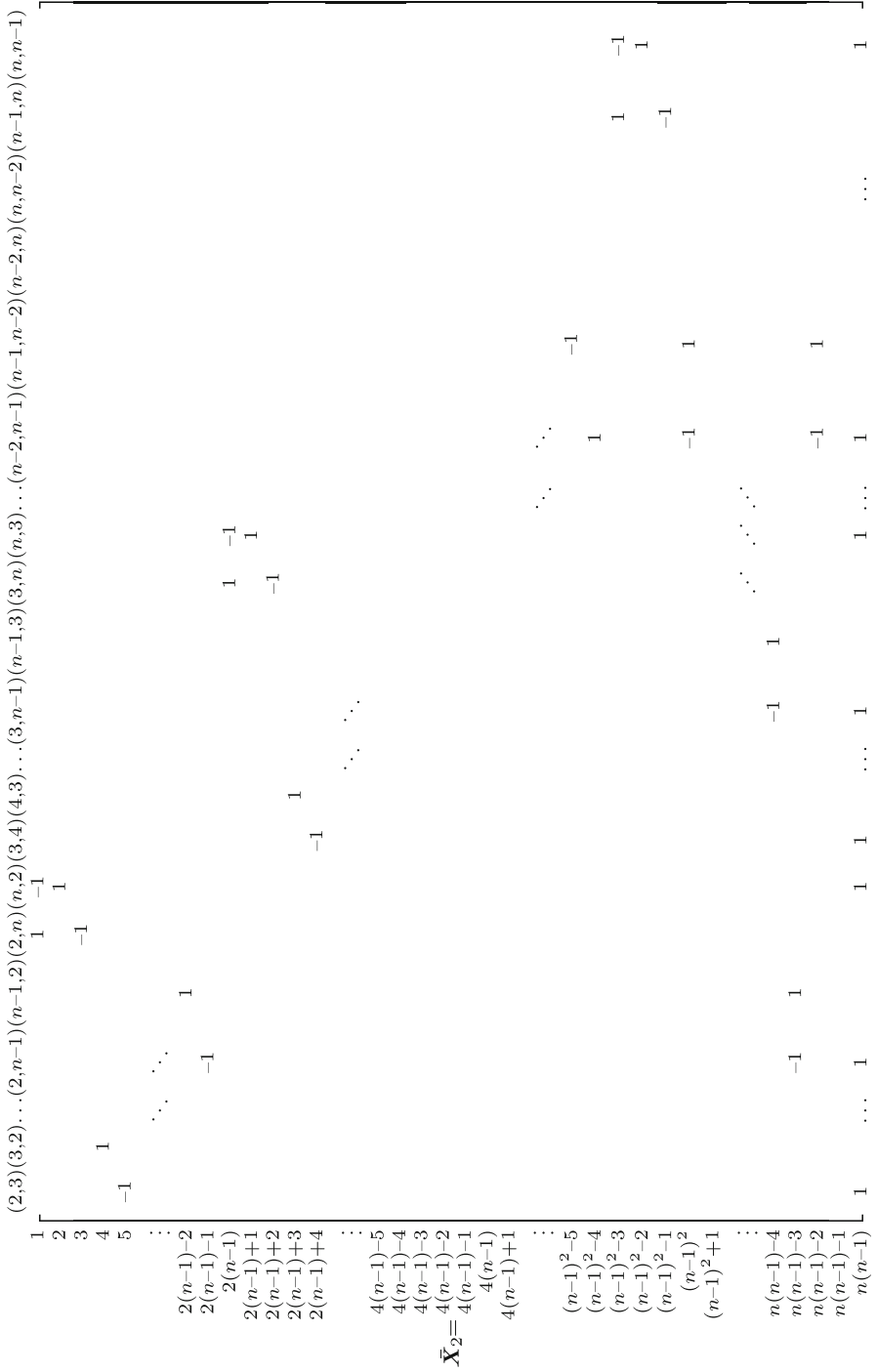
B.2 T2-1 FDI differences matrix and proof

To introduce the differences matrix for the T2-1 FDI proof we fixed $i_1 = 1$, $i_2 = n$ and $j_k = k + 1$, for $k = 1, \dots, n - 2$. The difference matrix, \bar{X} is generated after iteratively subtracting several rows of X as described in the following proof. For ease of visualization, \bar{X} is partitioned into two matrices as,

$$\bar{X} = \begin{bmatrix} \bar{X}^1 & \bar{X}^2 \end{bmatrix} \quad (17)$$

where $\bar{X}_1 \in \mathbb{Z}^{n(n-1) \times 2(n-1)}$ and $\bar{X}_2 \in \mathbb{Z}^{n(n-1) \times (n-1)(n-2)}$ such that, the first $2n - 2$ columns i.e., elements involving the comparison of alternative 1 with $j \in N \setminus \{1\}$ show up in \bar{X}_1 and, the last $(n - 1)(n - 2)$ columns i.e., all comparisons between $j, j' \in N \setminus \{1\}$ appear in \bar{X}_2 . Both of these matrices are illustrated as follows:

$$\mathbf{X}_1 = \begin{matrix} & \begin{matrix} (1,2) & (2,1) & (1,3) & (3,1) & (1,4) & (4,1) & (1,5) & (5,1) & \dots & (1,n-3) & (n-3,1) & (1,n-2) & (n-2,1) & (1,n-1) & (n-1,1) & (1,n) & (n,1) \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \\ 2(n-1)-2 \\ 2(n-1)-1 \\ 2(n-1) \\ 2(n-1)+1 \\ 2(n-1)+2 \\ 2(n-1)+3 \\ 2(n-1)+4 \\ \vdots \\ 4(n-1)-5 \\ 4(n-1)-4 \\ 4(n-1)-3 \\ 4(n-1)-2 \\ 4(n-1)-1 \\ 4(n-1) \\ 4(n-1)+1 \\ \vdots \\ (n-1)^2-5 \\ (n-1)^2-4 \\ (n-1)^2-3 \\ (n-1)^2-2 \\ (n-1)^2-1 \\ (n-1)^2 \\ (n-1)^2+1 \\ \vdots \\ n(n-1)-4 \\ n(n-1)-3 \\ n(n-1)-2 \\ n(n-1)-1 \\ n(n-1) \end{matrix} & \begin{bmatrix} 1 & & & & & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & & & & & & \\ & & -1 & & & & & & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & & & & & & \\ & & & & \dots & & & & & & & & & & & & & & \\ & & & & & \dots & & & & & & & & & & & & & \\ & & & & & & \dots & & & & & & & & & & & & \\ & & & & & & & \dots & & & & & & & & & & & \\ & & & & & & & & \dots & & & & & & & & & & \\ & & & & & & & & & \dots & & & & & & & & & \\ & & & & & & & & & & \dots & & & & & & & & \\ & & & & & & & & & & & \dots & & & & & & & \\ & & & & & & & & & & & & \dots & & & & & & \\ & & & & & & & & & & & & & \dots & & & & & \\ & & & & & & & & & & & & & & \dots & & & & \\ & & & & & & & & & & & & & & & \dots & & & \\ & & & & & & & & & & & & & & & & \dots & & \\ & & & & & & & & & & & & & & & & & \dots & \\ & & & & & & & & & & & & & & & & & & \dots \end{bmatrix} \end{matrix}$$



Theorem 9 (T2-1 FDI) T2-1 VI is an FDI of \mathbf{P}_{WO}^n , for any $n \geq 4$.

Proof In the T2-1 valid inequality expression, fix $i_1 = 1$ and $i_2 = n$ for ease of exposition—or assume a corresponding relabeling of the alternatives is performed a priori. It is straightforward to verify that all points output by CPT-2-1 belong to the six ranking structures given in Table 4 that satisfy the inequality at equality.

To obtain the difference matrix, \bar{X} , first shift rows 1, 2 and 3 to the bottom of the matrix such that they become rows $n(n-1)-2$, $n(n-1)-1$ and $n(n-1)$ respectively; all other rows are shifted upward. Second, iteratively subtract row $i-1$ from row i , for $i = n(n-1)-3, \dots, 2$. Third, subtract row $n(n-1)$ from row 1, row $n(n-1)-1$ from row $n(n-1)-2$ and row $n(n-1)-2$ from row $n(n-1)-1$. To proceed, set $\mathbf{A}^0 \in \mathbb{R}^{n(n-1) \times n(n-1)}$ with a rearranged column ordering of \bar{X} such that, all comparisons between the alternatives $j, j' \in \hat{N}^c = N \setminus \{1, n\}$ appear in the first $(n-2)(n-3)$ columns, the next $2(n-2)$ columns involve the comparisons between $i_2 = n$ and $j \in \hat{N}^c$ and the finally elements involving the comparison of $i_1 = 1$ with $j \in \hat{N}^c$ and $i_2 = n$ shows up in the last $2(n-1)$ columns. The first thing to remark about the structure of \mathbf{A}^0 is that for the submatrix involving the first $(n-1)(n-2)$ columns and the first $(n-1)^2-1$ rows, nearly all rows have either a 1 or a -1 as the only nonzero element and the nonzero occurs under a unique column. The only $n-2$ rows that do not fit this pattern are rows $2n(i-1)-i^2+2$, for $i = 1, \dots, n-2$ which have a 1 and a -1 under columns $(i+1, n)$ and $(n, i+1)$, respectively. The two consecutive vectors after each of these $n-2$ rows have a nonzero element of the opposite sign under the same columns. In particular, row $2n(i-1)-i^2+3$ has a 1 under column $(n, i+1)$ and row $2n(i-1)-i^2+4$ has a -1 under column $(i+1, n)$, where $1 \leq i \leq n-2$. Another thing to note about \mathbf{A}^0 is that, the binary values of its final row corresponds to the alternative-ordering in which items $\{1, n\}$ are tied for the first position and the remaining alternatives are in a lexicographical linear ordering occupying positions 2 to $(n-1)$. To eliminate the nonzero elements of the first $(n-1)(n-2)$ columns of this row add to it rows $2n(i-1)-i^2+2j$, where, $1 \leq i \leq n-2$ and $1 \leq j \leq n-i$. As the next step, eliminate the first two nonzero entries of row $2n(i-1)-i^2+2$ by adding to it rows $2n(i-1)-i^2+3$ and $2n(i-1)-i^2+4$, for $i = 1, \dots, n-2$. Then shift these $n-2$ rows to the bottom of the matrix and denote the resulting matrix as \mathbf{A}^1 . More explicitly, row $(n-1)^2+i+1$ of \mathbf{A}^1 receives row $2n(i-1)-i^2+2$ from \mathbf{A}^0 , for $1 \leq i \leq n-2$; all other rows are shifted upwards. Afterwards, the structure of \mathbf{A}^1 can be described via a partition with the same number of submatrices and related dimensions as defined by Eq. (11).

Similar to the proof of Theorem 8, it is only necessary to know a part of the contents of these submatrices to proceed. \mathbf{B}^1 is comprised entirely of positive or negative unit vectors and, thus we have, $|\det(\mathbf{B}^1)| = 1$. \mathbf{C}^1 is mostly a zero matrix, with the exception of row i whose values under columns $(n-i-1, n-1)$ and $(n-1, n-i-1)$ are 1 and -1 , respectively, for $i = 1, \dots, n-3$ and row $n-2$ which has a -1 under column $(n-2, n-1)$ and a 1 under column $(n-1, n-2)$. To turn \mathbf{C}^1 into a zero matrix first add to row i , where $1 \leq i \leq n-3$, the two consecutive elementary vectors from \mathbf{B}^1 that have nonzeros of the opposite sign under columns $(n-i-1, n-1)$ and $(n-1, n-i-1)$. Next, to eliminate the entries of row $n-2$ subtract from it rows $(n-1)(n-2)-4$ and $(n-1)(n-2)-3$ of \mathbf{B}^1 . Similar to

The determinant of this matrix is $-\beta - 1$, which equals 0 when,

$$\begin{aligned}\beta + 1 = 0 &\Leftrightarrow \frac{(n-1) - (n-3)(n-5)}{2} + 1 \\ &= -\frac{1}{2}(n^2 - 9n + 14) = -\frac{1}{2}(n-2)(n-7) = 0 \Leftrightarrow n = 7 \text{ or } 2.\end{aligned}$$

Therefore, for all $n \geq 4$, except $n = 7$, the $n(n-1)$ characteristic vectors generated by CPT2-3 that satisfy inequality (7) at equality are linearly independent and, thus, affinely independent. To prove that the result holds for $n = 7$ as well, subtract row $n(n-1)$ of matrix X from row i , where $i = 1, 2, \dots, n(n-1) - 1$, to yield a new matrix \hat{X} . It is straightforward to verify that the first 41 vectors of \hat{X} are linearly independent (i.e., the row rank is 41) and, therefore, the 42 characteristic vectors are indeed affinely independent. Hence, the result also holds for $n = 7$, and T2-3 VI is facet defining for any $n \geq 4$. \square

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