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Due date assignment single-machine scheduling with delivery times, position-dependent weights and deteriorating jobs

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Abstract
This article studies the single-machine scheduling problem with due date assignments, deteriorating jobs, and past-sequence-dependent delivery times. Under three assignments (i.e., common, slack, and different due dates), the goal is to determine a feasible sequence and due dates of all jobs in order to minimize the weighted sum of earliness, tardiness, and due date costs of all jobs, where the weight is not related to the job but to the position in which some job is scheduled. Through a series of optimal properties, efficient and fast polynomial time algorithms are designed for solving the studied scheduling problem with three due date assignments.

Keywords Scheduling · Combinatorial optimization · Production · Delivery time · Due date assignment

1 Introduction

In classical production scheduling problems, it is typically assumed that each job's processing time on a specific machine is a given constant. But under many practical cases, the processing time may be related to the start time of each job; that is, the processing time of the job increases with the increase of its start time. More specifically, the later the job starts to process, the longer the processing times on machines and vice versa. This problem is called a deteriorating job (time-dependent) scheduling (Li et al. 2019; Gawiejnowicz 2020). For example, Cheng et al. (2020) proved that the

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due date assignment scheduling problem with deteriorating jobs and minimal the total completion time is NP-hard, and thus proposed a pseudo-polynomial time algorithm for the considered problem, in which the processing time of a job on a specific machine is a step function of its start time.

In practical problems, it is also necessary to consider the extra time for delivering the items to their customers, which is known as the past-sequence-dependent delivery time in the literature (Koulamas and Kyparisis 2010). For example, Zhao and Tang (2014) studied the scheduling problem with the minimization of the makespan, the total completion time, and the absolute difference between the completion time and the past-sequence-dependent delivery time. Their results verified that two types of scheduling problems, namely including and excluding the due date, can be solved in polynomial time. Ji et al. (2015) investigated the single-machine scheduling problem with simultaneous considering the slack due window assignment, the past-sequence-dependent delivery time and the controllable processing time. Note that in their work, each job's processing time was supposed to be a linear or convex function of learning effect and resource allocation, and the authors proposed an efficient polynomial algorithm with the time complexity of $O(\bar{N}^3)$, where $\bar{N}$ denotes the number of jobs. Recently, Qian and Han (2022) investigated a problem considering the due date assignment, the delivery time, and deterioration effects at the same time, and proposed a polynomial algorithm, which runs in $O(\bar{N}\log\bar{N})$ time.

Similarly, as an important research direction, due date assignment problems can be divided into three cases (see Gordon et al. 2002a; Gordon et al. 2002b; Yin et al. 2014; Yin et al. 2018). Among them, the first case is common due date, abbreviated as CON (see Yin et al. 2016; Xiong et al. 2018; T'kindt V, Shang L, Croce FD, 2020; Lv and Wang 2021; Cheng et al. 2020). That is, each job has the same due date, which can be represented by d_{opt}. Toksari and Güner Toksari (2010) simultaneously worked on the common due date earliness/tardiness scheduling problem with time-dependent learning effects and linear or non-linear deterioration effects under parallel machines, and verified that the optimal schedule follows the V-shaped sequence. Koulamas (2017) proposed a dynamic programming (DP) algorithm for the CON problem with general earliness/tardiness penalties under the single-machine scenario and proposed a faster algorithm within $O(\bar{N})$ for the special situation without taking into account the earliness/tardiness of jobs. The second case of the due date assignment problem is slack due date, abbreviated as SLK (see Yin et al. 2016; Chen et al. 2023); that is, there is the same common flow allowance for each job, which is usually represented by q_{opt}, and the due date of job J_i can be denoted as $d_i = p_i + q_{opt}$, in which p_i is the processing time of J_i. Liu et al. (2017) studied the earliness-tardiness scheduling problem with only considering the single-machine and proposed effective polynomial time algorithms for general processing time and position-dependent processing time functions, respectively, with corresponding time complexities of $O(\bar{N}\log\bar{N})$ and $O(\bar{N}^3)$. Recently, Liu et al. (2020) scrutinized the single-machine resource allocation problem with deterioration effects and position-dependent weights. Under the CON and SLK cases, they gave a corresponding optimal algorithm and optimal resource allocation. For the third case of due date assignments, that is, an unrestricted due date assignment scheduling problem, abbreviated as DIF (see Yang et al. 2022), namely the due date of each job J_i can be expressed as d_i, where $i = 1, 2, \ldots, \bar{N}$. Very recently, Yin et al.

(2021) examined the serial-batch delivery scheduling problem with only considering the single-machine and two competing agents. Under the CON and DIF due date models, they proved that the total cost (comprising the earliness, tardiness (weighted number of tardy jobs), job holding, due date assignment, and batch delivery costs) minimization is NP-hard and presented a pseudo-polynomial DP algorithm for each of the explored problems.

Furthermore, Qian and Han (2022) researched the single-machine scheduling problem with simple linear deterioration and past-sequence-dependent delivery times. Under three due date assignment methods, the objective was to simultaneously minimize the weighted sum of earliness, tardiness, and the due date of all jobs. They have proved that the optimal schedule to the minimization problem can be obtained in $O(\bar{N} \log \bar{N})$ time. However, the scheduling problems with position-dependent weights exist in many practical production services environments, such as in Didi taxi dispatching, orders placed in the morning offer a higher bonus to the driver, which can effectively improve customer satisfaction in these locations by better meeting the needs of customers going to work in the morning (see Sun et al. 2020). Hence, in this article, we mainly investigate the three due date assignment problems with position-dependent weights (see Wang et al. 2020; Wang et al. 2021), deteriorating jobs, and past-sequence-dependent delivery times. The research objective is to minimize the weighted sum of earliness, tardiness and due date of all jobs, where the weights depend on the position in which a job is scheduled. The contributions of this study are given as follows: (1) we examine the single-machine due date assignment scheduling problem along with deteriorating jobs and past-sequence-dependent delivery times; (2) under CON, SLK and DIF, our goal is to minimize the weighted sum of earliness, tardiness, and due date assignment cost, where the weights are position-dependent weights (i.e., the weight is not related to the job but to the position in which some job is scheduled); (3) we present the structural properties of the optimal solutions and demonstrate that the problem is polynomially solvable. The structure of the paper is organized as follows: Sect. 2 describes the studied problem. Sections 3, 4 and 5 give the solution algorithms for the researched problem under CON, SLK and DIF, respectively. Section 6 gives the conclusion.

2 Problem description

Consider that there exist $\bar{N}$ jobs in total to be processed continuously on a single-machine, and all jobs are processed at time instant s_0, where $s_0 > 0$. Under the simple linear deterioration, the processing time of job J_i can be defined as $p_i = \chi_i s_i$, in which s_i and χ_i represent the start time and the deterioration rate of job J_i, respectively. The past-sequence-dependent delivery time (denoted by $psddt$) of job J_i can be defined as $q_i = \upsilon s_i$, where υ is the delivery rate. For the case of common due date assignment, i.e., CON, the earliness and tardiness of job $J_{[i]}$ at the i-th position in the sequence can be expressed as $\widehat{E}_{[i]} = \max\{0, d_{opt} - C_{[i]}\}$ and $\widehat{T}_{[i]} = \max\{0, C_{[i]} - d_{opt}\}$, respectively, where d_{opt} is the optimal common due date of all jobs, and $C_{[i]}$ is the completion time of job $J_{[i]}$. For the slack due date assignment problem, i.e., SLK, the earliness and tardiness of job $J_{[i]}$ can be written as $\widehat{E}_{[i]} = \max\{0, p_{[i]} + q_{opt} - C_{[i]}\}$

and $\widehat{T}_{[i]} = \max\{0, C_{[i]} - p_{[i]} - q_{opt}\}$, in which q_{opt} denotes the optimal common flow allowance of all jobs. For the different (unrestricted) due dates assignment, i.e., DIF, the earliness and tardiness of job $J_{[i]}$ are $\widehat{E}_{[i]} = \max\{0, d_{[i]} - C_{[i]}\}$ and $\widehat{T}_{[i]} = \max\{0, C_{[i]} - d_{[i]}\}$ respectively, where $d_{[i]}$ is the due date of job $J_{[i]}$ $(i = 1, 2, \ldots, n)$.

Brucker (2001) has demonstrated that the problem $1|CON|\sum_{i=1}^{\bar{N}}(\widehat{\xi}_i|\widehat{L}_{[i]}|+\widehat{\gamma}d_{opt})$ is polynomially solvable, where $\widehat{L}_{[i]} = C_{[i]} - d_{opt}$ is the lateness of job $J_{[i]}$, $\widehat{\xi}_i$ is the position-dependent weight for lateness of ith position and $\widehat{\gamma}$ is the weight for common due date (obviously, $|\widehat{L}_{[i]}| = \widehat{E}_{[i]} + \widehat{T}_{[i]}$). Liu et al. (2017) have showed that the problem $1|SLK|\sum_{i=1}^{\bar{N}}(\widehat{\xi}_i|\widehat{L}_{[i]}| + \widehat{\gamma}q_{opt})$ is can be solved efficiently. Qian and Han (2022) have proved that the problems $1|p_i = \chi_i s_i, CON, psddt|\sum_{i=1}^{\bar{N}}(\widehat{\alpha}\widehat{E}_{[i]} + \widehat{\beta}\widehat{T}_{[i]} + \widehat{\gamma}d_{opt})$, $1|p_i = \chi_i s_i, SLK, psddt|\sum_{i=1}^{\bar{N}}\left(\widehat{\alpha}\widehat{E}_{[i]} + \widehat{\beta}\widehat{T}_{[i]} + \widehat{\gamma}q_{opt}\right)$ and $1|p_i = \chi_i s_i, DIF, psddt|\sum_{i=1}^{\bar{N}}\left(\widehat{\alpha}\widehat{E}_{[i]} + \widehat{\beta}\widehat{T}_{[i]} + \widehat{\gamma}d_{[i]}\right)$ can be solved in polynomial time, respectively, where $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\gamma}$ are the weights for earliness, tardiness, and due date, respectively. In this paper, we consider the problem of minimizing the weighted sum of earliness, tardiness and due date of all jobs, in which the weights we considered refer to the position-dependent weights, i.e., the weight is related to position of the job and not simply the job itself (Jiang et al. 2020). Under three due date assignment cases, the problems can be expressed as follows:

$$1|p_i = \chi_i s_i, CON, psddt|\sum_{i=1}^{\bar{N}}\left(\alpha_i\widehat{E}_{[i]} + \beta_i\widehat{T}_{[i]} + \gamma_i d_{opt}\right);$$

$$1|p_i = \chi_i s_i, SLK, psddt|\sum_{i=1}^{\bar{N}}\left(\alpha_i\widehat{E}_{[i]} + \beta_i\widehat{T}_{[i]} + \gamma_i q_{opt}\right);$$

$$1|p_i = \chi_i s_i, DIF, psddt|\sum_{i=1}^{\bar{N}}\left(\alpha_i\widehat{E}_{[i]} + \beta_i\widehat{T}_{[i]} + \gamma_i d_{[i]}\right),$$

where 1 denotes the single-machine, the middle (resp. third) field denotes the job characteristics (resp. the objective function), α_i, β_i and γ_i are the position-dependent weights for earliness, tardiness, and due date, respectively, in which a position-dependent weight does not change with a change in some job.

Obviously, there exists an optimal schedule or sequence in which all jobs are processed consecutively without any idle time from the start time s_0. For a given job schedule or sequence, by the mathematical induction, we have $C_{[i]} = s_0\prod_{h=1}^{i}(1 + v + \chi_{[h]})$. From this equation, if the jobs at any two positions are exchanged with the others positions remaining unchanged, then the completion time and start time of the jobs before and after the two jobs will essentially remain unchanged.

3 CON case

In this section, we focus on the problem $1|p_i = \chi_i s_i, CON, psddt|\sum_{i=1}^{\bar{N}}\left(\alpha_i\widehat{E}_{[i]} + \beta_i\widehat{T}_{[i]} + \gamma_i d_{opt}\right)$. First, we present some properties that will be useful for later analysis.

Property 3.1 *For a given job schedule or sequence, suppose that there exists an optimal common due date, then we can know that d_{opt} equals either the completion of a job (i.e., $d_{opt} = C_{[l]}$) or s_0 (i.e., $d_{opt} = s_0$).*

Proof First, suppose that the d_{opt} does not represent the job's the completion time, namely $C_{[l]} < d_{opt} < C_{[l+1]}$, where $1 \leq l < \bar{N}$. Then, based on the above assumption, the objective function could be written as the following:

$$
\begin{aligned}
f &= \sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{opt}\right) \\
&= \sum_{i=1}^{l} \alpha_i (d_{opt} - C_{[i]}) + \sum_{i=l+1}^{\bar{N}} \beta_i (C_{[i]} - d_{opt}) + \sum_{i=1}^{\bar{N}} \gamma_i d_{opt}.
\end{aligned}
$$

Next, move the common due date d_{opt} to the left to $C_{[l]}$ and set $x = d_{opt} - C_{[l]}$; that is, $d_{opt} = x + C_{[l]}$, and $x > 0$. Then,

$$
f_1 = \sum_{i=1}^{l} \alpha_i (C_{[l]} - C_{[i]}) + \sum_{i=l+1}^{\bar{N}} \beta_i (C_{[i]} - C_{[l]}) + \sum_{i=1}^{\bar{N}} \gamma_i C_{[l]},
$$

and

$$
f - f_1 = \left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) x.
$$

In the next, shift the common due date d_{opt} to the right to $C_{[l+1]}$, and let $y = C_{[l+1]} - d_{opt}$, i.e., $d_{opt} = C_{[l+1]} - y$, and $y > 0$. Then,

$$
f_2 = \sum_{i=1}^{l} \alpha_i (C_{[l+1]} - C_{[i]}) + \sum_{i=l+1}^{\bar{N}} \beta_i (C_{[i]} - C_{[l+1]}) + \sum_{i=1}^{\bar{N}} \gamma_i C_{[l+1]},
$$

and

$$
f - f_2 = -\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) y.
$$

When $\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \geq 0$, then we can easily know that $f \geq f_1$ holds; If $\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) < 0$, then $f > f_2$ holds. That is, either $d_o pt$ equals the completion of a job or s_0. $\square$

Property 3.2 *For a given job schedule or sequence, its common due date d_{opt} is equal to the completion time of the l-th position, i.e., $d_{opt} = C_{[l]}$, where l satisfies* $\left(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \leq 0$ *and* $\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \geq 0.$

Proof The conclusion can be easily proved by the classical small perturbation technique. On the basis of Property 3.1, assuming that $d_{opt} = C_{[l]}$ and the corresponding objective function f can be obtained as follows:

$$f = \sum_{i=1}^{l-1} \alpha_i (C_{[l]} - C_{[i]}) + \sum_{i=l}^{\bar{N}} \beta_i (C_{[i]} - C_{[l]}) + \sum_{i=1}^{\bar{N}} \gamma_i C_{[l]}.$$

Moving the common due date d_{opt} to the left by u units, it follows that

$$f_1 = \sum_{i=1}^{l-1} \alpha_i (C_{[l]} - u - C_{[i]}) + \sum_{i=l}^{\bar{N}} \beta_i (C_{[i]} - C_{[l]} + u) + \sum_{i=1}^{\bar{N}} \gamma_i (C_{[l]} - u),$$

then

$$f_1 = f - \left(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) u.$$

When f is optimal, then $\left(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \leq 0$ is satisfied.

Shifting the common due date d_{opt} to the right by u units yields

$$f_2 = \sum_{i=1}^{l} \alpha_i (C_{[l]} + u - C_{[i]}) + \sum_{i=l+1}^{\bar{N}} \beta_i (C_{[i]} - C_{[l]} - u) + \sum_{i=1}^{\bar{N}} \gamma_i (C_{[l]} + u),$$

then

$$f_2 = f + \left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) u.$$

Similarly, $\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \geq 0$ can be satisfied.

Hence, the optimal common due date is $d_{opt} = C_{[l]}$, where l satisfies the following inequalities: $\left(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \leq 0$ and $\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \geq 0.$ □

For convenience, let the sets be $S = \{J_i \in \sigma | C_i \leq d_{opt}\}$ and $R = \{J_i \in \sigma | C_i > d_{opt}\}$, where σ is a given sequence.

Property 3.3 *In the optimal schedule or sequence, all jobs in set S are sorted according to the non-increasing order of χ_i, i.e., the LDR (Largest Deterioration Rate first) order.*

Proof Suppose that there exist two successive jobs J_j and J_k, where J_j is at the u-th position in a sequence A_1, in which $A_1 = \{J_1, \ldots, J_j, J_k, \ldots, J_{\bar{N}}\}$, the start time of job $J_{[1]}$ is s_0, and $d_{opt} = C_{[l]}$, where $1 \le u < l \le \bar{N}$. Now, by exchanging the positions of jobs J_j and J_k, the sequence $A_2 = \{J_1, \ldots, J_k, J_j, \ldots, J_{\bar{N}}\}$ can be obtained. For the objective function f_1 and f_2 of sequences A_1 and A_2, it follows that

$$f_1 - f_2 = -\alpha_u s_0(\chi_j - \chi_k)\prod_{i=1}^{u-1}(1 + \chi_{[i]} + v).$$

If $\chi_j \ge \chi_k$, we have $f_1 \le f_2$, that is, the jobs in set S are sorted in non-increasing order of χ_i. □

Property 3.4 *In the optimal schedule or sequence, all jobs in set R are sorted according to the non-decreasing order of χ_i, i.e., the SDR (Smallest Deterioration Rate first) order.*

Proof As in Property 3.3, suppose that there exist two successive jobs J_j and J_k in set R, and J_j is at the u-th position in sequence A_1, in which $A_1 = \{J_1, \ldots, J_j, J_k, \ldots, J_{\bar{N}}\}$. Similarly, $C_{[0]} = s_0$ and $d_{opt} = C_{[l]}$, where $l+1 \le u < \bar{N}$. Now the sequence $A_2 = \{J_1, \ldots, J_k, J_j, \ldots, J_{\bar{N}}\}$ can be obtained by exchanging jobs J_j and J_k, then, for the objective function f_1 and f_2 of sequences A_1 and A_2, it follows that

$$f_1 - f_2 = \beta_u s_0(\chi_j - \chi_k)\prod_{i=1}^{u-1}(1 + \chi_{[i]} + v).$$

If $\chi_j \le \chi_k$, we have $f_1 \le f_2$, that is, the jobs in set R are sorted in non-decreasing order of χ_i. □

Now, the following notation is defined:

$$Q_{x,y} = \begin{bmatrix} -\sum_{i=x}^{l-1}\alpha_i\prod_{g=x+1}^{i}(1+\chi_{[g]}+v) + \sum_{i=l+1}^{y-1}\beta_i\prod_{g=x+1}^{i}(1+\chi_{[g]}+v) \\ +\left(\sum_{i=1}^{l-1}\alpha_i - \sum_{i=l+1}^{\bar{N}}\beta_i + \sum_{i=1}^{\bar{N}}\gamma_i\right)\prod_{i=x+1}^{l}(1+\chi_{[i]}+v) \end{bmatrix}$$

where $1 \le x \le l$ and $l+1 \le y \le \bar{N}$.

Property 3.5 *Supposing there are two different jobs J_j and J_k in which job J_j is supposed to be at the x-th position and job J_k is supposed to be at the y-th position ($1 \le x \le l$ and $l+1 \le y \le \bar{N}$), consider the following cases:*

1. *If $Q_{x,y} \ge 0$, then $\chi_j < \chi_k$;*
2. *If $Q_{x,y} < 0$, then $\chi_j \ge \chi_k$.*

Proof First, we assume that there are two different jobs J_j and J_k in an optimal sequence $A_1 = \{J_1, \ldots, J_j, \ldots, J_k, \ldots, J_{\bar{N}}\}$ and the former is supposed to be processed at time s_0, in which J_j is at the x-th position and J_k is supposed to be at the y-th position, $d_{opt} = C_{[l]}$, $1 \leq x \leq l$, and $l+1 \leq y \leq \bar{N}$. Based on the above assumptions, the sequence $A_2 = \{J_1, \ldots, J_k, \ldots, J_j, \ldots, J_{\bar{N}}\}$ can be obtained by swapping jobs J_j and J_k. As a result, it follows that

$$f_1 - f_2 = s_0(\chi_j - \chi_k)\prod_{i=1}^{x-1}(1 + \chi_{[i]} + v) \times \left[\begin{array}{l} -\sum_{i=x}^{l-1}\alpha_i\prod_{g=x+1}^{i}(1+\chi_{[g]}+v)+\sum_{i=l+1}^{y-1}\beta_i\prod_{g=x+1}^{i}(1+\chi_{[g]}+v) \\ +\left(\sum_{i=1}^{l-1}\alpha_i-\sum_{i=l+1}^{\bar{N}}\beta_i+\sum_{i=1}^{\bar{N}}\gamma_i\right)\prod_{i=x+1}^{l}(1+\chi_{[i]}+v) \end{array}\right]$$

Because sequence A_1 is an optimal sequence, it follows that $f_1 - f_2 \leq 0$. Therefore, if $Q_{x,y} \geq 0$, then $\chi_j < \chi_k$ definitely holds; if $Q_{x,y} < 0$, then $\chi_j \geq \chi_k$ must hold. □

Based on the above analysis, the following algorithm holds:

Algorithm 3.1
Input: $\bar{N}$, v, $\chi_i, \alpha_i, \beta_i, \gamma_i$ $(1 \leq i \leq \bar{N})$.
Output: An optimal sequence A, and an optimal d_{opt}.
Step 1. Arrange jobs in non-decreasing order of χ_i, i.e., $\chi_{[1]} \leq \chi_{[2]} \leq \ldots \leq \chi_{[\bar{N}]}$;
Step 2. Calculate the value of l based on Property 3.2;
Step 3. Determine the optimal sequence based on Property 3.5;
Step 4. Calculate $d_{opt} = C_{[l]}$.

Theorem 3.1 *The scheduling problem* $1|p_i = \chi_i s_i, CON, psddt|\sum_{i=1}^{\bar{N}}\left(\alpha_i\widehat{E}_{[i]} + \beta_i\widehat{T}_{[i]} + \gamma_i d_{opt}\right)$ *can be solved by Algorithm 3.1 in* $O(\bar{N}\log\bar{N})$ *time.*

Proof First, it is not hard to find that the time required for Step 1 is $O(\bar{N}\log\bar{N})$. Besides, it is easy to know that Step 2 can be solved in a constant time. Steps 3 and 4 require $O(\bar{N})$ time. In summary, the total time of Algorithm 3.1 is $O(\bar{N}\log\bar{N})$. □

4 SLK case

In this section, we focus on the problem $1|p_i = \chi_i s_i, SLK, psddt|\sum_{i=1}^{\bar{N}}\left(\alpha_i\widehat{E}_{[i]} + \beta_i\widehat{T}_{[i]} + \gamma_i q_{opt}\right)$. First, the following properties to obtain the optimal solution are presented.

Property 4.1 *For a given sequence, the optimal common flow allowance* q_{opt} *is equal to* $(1 + v)$ *times the completion time of a job or* $(1 + v)s_0$.

Proof Proof by contradiction. That is, assuming that $(1+v)C_{[l-1]} < q_{opt} < (1+v)C_{[l]}$, $1 \le l \le \bar{N}$, and the start time of the job J_1 is s_0, then

$$
\begin{aligned}
f &= \sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i q\right) \\
&= \sum_{i=1}^{l} \alpha_i (p_{[i]} + q_{opt} - C_{[i]})s + \sum_{i=l+1}^{\bar{N}} \beta_i (C_{[i]} - p_{[i]} - q_{opt}) + \sum_{i=1}^{\bar{N}} \gamma_i q_{opt} \\
&= -\sum_{i=1}^{l} \alpha_i (1+v)C_{[i-1]} + \sum_{i=l+1}^{\bar{N}} \beta_i (1+v)C_{[i-1]} \\
&\quad + \left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) q_{opt}.
\end{aligned}
$$

First, move the q_{opt} to the left side of the $(1+v)C_{[l-1]}$ and set $q_{opt} - (1+v)C_{[l-1]} = s$, where $s > 0$;, then,

$$
\begin{aligned}
f_1 &= \sum_{i=1}^{l-1} \alpha_i (p_{[i]} + q_{opt} - C_{[i]}) + \sum_{i=l+1}^{\bar{N}} \beta_i (C_{[i]} - p_{[i]} - q_{opt}) + \sum_{i=1}^{\bar{N}} \gamma_i q_{opt} \\
&= -\sum_{i=1}^{l-1} \alpha_i (1+v)C_{[i-1]} + \sum_{i=l+1}^{\bar{N}} \beta_i (1+v)C_{[i-1]} \\
&\quad + \left(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) (1+v)C_{[l-1]}.
\end{aligned}
$$

Then, shift the q_{opt} to the right side of the $(1+v)C_{[l]}$ and set $(1+v)C_{[l]} - q_{opt} = t$, where $t > 0$; it holds that

$$
\begin{aligned}
f_2 &= \sum_{i=1}^{l} \alpha_i (p_{[i]} + q_{opt} - C_{[i]}) + \sum_{i=l+1}^{\bar{N}} \beta_i (C_{[i]} - p_{[i]} - q_{opt}) + \sum_{i=1}^{\bar{N}} \gamma_i q_{opt} \\
&= -\sum_{i=1}^{l} \alpha_i (1+v)C_{[i-1]} + \sum_{i=l+2}^{\bar{N}} \beta_i (1+v)C_{[i-1]} \\
&\quad + \left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+2}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) (1+v)C_{[l]}.
\end{aligned}
$$

Hence

$$
f - f_1 = \left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) s
$$

and

$$f - f_2 = -\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) t.$$

When $\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \geq 0$, it is easy to find that $f \geq f_1$ holds, otherwise $f > f_2$ must hold. That is to say, q_{opt} is equal to $(1+v)$ times the completion of a job or $(1+v)s_0$. □

Property 4.2 *For a given sequence,* $q_{opt} = (1+v)C_{[l-1]}$*, where* l *satisfies*

$$\left(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \leq 0$$

and

$$\left(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i\right) \geq 0.$$

Proof It is similar to the proof of Property 3.3. □

Now we define two sets: $W = \{J_j \in \sigma | C_j \leq q_{opt}\}$ and $U = \{J_j \in \sigma | C_j > q_{opt}\}$, where $q_{opt} = (1+v)C_{[l-1]}$, and σ is the given sequence.

Property 4.3 *In an optimal schedule or sequence, all jobs in the set* W *are sorted by the LDR order of* χ_i*.*

Proof First, suppose there are two consecutive jobs, J_j and J_k in W, where J_j is at the u-th position in sequence $A_1 = \{J_1, \ldots, J_j, J_k, \ldots, J_{\bar{N}}\}$. Also, suppose that $C_{[0]} = s_0$, and $q_{opt} = (1+v)C_{[l-1]}$, where $1 \leq u \leq l-2$. Then the sequence $A_2 = \{J_1, \ldots, J_k, J_j, \ldots, J_{\bar{N}}\}$ can be obtained by exchanging the two jobs, then the objective function f_1 of A_1 is subtracted from f_2 of A_2:

$$f_1 - f_2 = -\alpha_{u+1} s_0 (1+v)(\chi_j - \chi_k) \prod_{i=1}^{u-1} (1 + \chi_{[i]} + v).$$

If $f_1 \leq f_2$, $\chi_j \geq \chi_k$, that means the jobs in W are sorted in descending order of χ_i. □

Property 4.4 *In an optimal schedule or sequence, all jobs in the set* U *are sorted by the SDR order of* χ_i*.*

Proof Also, suppose that there are two successive jobs, J_j and J_k, in set U, where J_j is at the u-th position in sequence $A_1 = \{J_1, \ldots, J_j, J_k, \ldots, J_{\bar{N}}\}$. Assume that $q_{opt} = (1+v)C_{[l-1]}$, $l \leq u < \bar{N}$. Now exchange the job J_j and J_k, and the sequence

$A_2 = \{J_1, \ldots, J_k, J_j, \ldots, J_{\bar{N}}\}$ can be obtained. Then, the objective function f_1 of A_1 is subtracted from f_2 of A_2:

$$f_1 - f_2 = \beta_{u+1} s_0 (1+v)(\chi_j - \chi_k) \prod_{i=1}^{u-1} (1 + \chi_{[i]} + v).$$

If $f_1 \le f_2$, then $\chi_j \le \chi_k$ holds, which indicates that all jobs in U are sorted in ascending order of χ_i. □

Similarly, we can define:

$$Q_{x,y} = \begin{bmatrix} -\sum_{i=x+1}^{l-1} \alpha_i \prod_{g=x+1}^{i-1} (1 + \chi_{[g]} + v) + \sum_{i=l+1}^{y} \beta_i \prod_{g=x+1}^{i} (1 + \chi_{[g]} + v) \\ + \left(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i \right) \prod_{i=x+1}^{l} (1 + \chi_{[i]} + v) \end{bmatrix}$$

If $Q_{xy} \ge 0$, it is known that the job J_j should be at the x-th position; otherwise, the job J_j should be at y-th position.

Property 4.5 *Assuming there are two different jobs, J_j and J_k, in which J_j is at the x-th position and J_k is at the y-th position ($1 \le x \le l-1$ and $l \le y \le \bar{N}$), consider the following two cases:*

1. *If $Q_{x,y} \ge 0$, then $\chi_j < \chi_k$;*
2. *If $Q_{x,y} < 0$, then $\chi_j \ge \chi_k$.*

Proof Similar to the proofing process of Property 3.6. □

In summary, the following scheduling algorithm and theorem can be obtained:

Algorithm 4.1
Input: $\bar{N}$, v, $\chi_i, \alpha_i, \beta_i, \gamma_i$ ($1 \le i \le \bar{N}$).
Output: An optimal sequence A, and an optimal q_{opt}.
Step 1. First, arrange jobs in non-decreasing order of χ_i, i.e., $\chi_{[1]} \le \chi_{[2]} \le \cdots \le \chi_{[\bar{N}]}$;
Step 2. Then, calculate the value of l based on Property 4.2;
Step 3. Next, determine the optimal sequence based on Property 4.5;
Step 4. Finally, calculate $q_{opt} = (1+v)C_{[l-1]}$.

Theorem 4.1 *Algorithm 4.1 solves $1|p_i = \chi_i s_i, SLK, psddt| \sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i q_{opt} \right)$ in $O(n \log n)$ time.*

5 DIF case

In this section, we first analyze the optimal properties for the problem $1|p_{[i]} = \chi_{[i]} s_i, DIF, psddt| \sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{[i]} \right)$.

Property 5.1 *For a given schedule or sequence, when $\gamma_i \ge \beta_i$, the due date of job J_i is equal to 0; that is, $\min\{\beta_i, \gamma_i\} = \beta_i$, $d_{[i]} = 0$. Otherwise, its due date time is equal to the completion time of job J_i; that is, $\min\{\beta_i, \gamma_i\} = \gamma_i$, $d_{[i]} = C_{[i]}$.*

Proof For job J_i, it follows that

$$\begin{aligned} f_i &= \alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{[i]} \\ &= \alpha_i \max\{0, d_{[i]} - C_{[i]}\} + \beta_i \max\{0, C_{[i]} - d_{[i]}\} + \gamma_i d_{[i]}. \end{aligned}$$

Then, it has the following two cases:
Case I When J_i is a tardy job, that is, $d_{[i]} \leq C_{[i]}$, the corresponding objective function can be rewritten as the following:

$$f_i = \beta_i C_{[i]} + (\gamma_i - \beta_i) d_{[i]}.$$

If $\gamma_i \geq \beta_i$, in order to minimize the objective function, there exists $d_{[i]} = 0$, and the objective function is $f_i = \beta_i C_{[i]}$; If $\gamma_i < \beta_i$, then it follows that $d_{[i]} = C_{[i]}$, and $f_i = \gamma_i C_{[i]}$.
Case II If J_i is an early job, that is, $d_{[i]} \geq C_{[i]}$, then the objective function of job J_i can be rewritten as follows:

$$f_i = (\alpha_i + \gamma_i) d_{[i]} - \alpha_i C_{[i]}.$$

Then, if there exists $d_{[i]} = C_{[i]}$, it follows that $f_i = \gamma_i C_{[i]}$. □

From Lemma 5.1, it is easily known that, if $\min\{\beta_i, \gamma_i\} = \beta_i$, then $d_{[i]} = 0$ and $f = \sum_{i=1}^{n} \beta_i C_{[i]}$ hold; If $\min\{\beta_i, \gamma_i\} = \gamma_i$, then $d_{[i]} = C_{[i]}$, and $f = \sum_{i=1}^{n} \gamma_i C_{[i]}$ must hold. Based on the above works, it follows that

$$f = \sum_{i=1}^{n} \min\{\beta_i, \gamma_i\} C_{[i]}.$$

Property 5.2 *If an optimal sequence exists, then it can be obtained by the SDR order of χ_i, i.e., $\chi_{[1]} \leq \chi_{[2]} \leq \cdots \leq \chi_{[\bar{N}]}$.*

Proof For sequence $A_1 = \{J_1, \ldots, J_j, J_k, \ldots, J_{\bar{N}}\}$, J_j and J_k are two consecutive jobs in the sequence, job J_j is supposed to be at the x-th position, and job J_k is supposed to be at the $(x+1)$-th position. Now swap the two jobs to obtain sequence $A_2 = \{J_1, \ldots, J_k, J_j, \ldots, J_{\bar{N}}\}$. Subsequently, the objective function f_1 of A_1 is subtracted from the objective function f_2 of A_2:

$$f_1 - f_2 = \min\{\beta_x, \gamma_x\}(\chi_j - \chi_k) s_0 \prod_{i=1}^{x-1} (1 + \chi_{[i]} + v),$$

and the optimal sequence is that the jobs are sorted in the increasing order of χ_i.

In summary, the following scheduling algorithm and theorem can be obtained: □

Algorithm 5.1
Input: $\bar{N}, v, \chi_i, \alpha_i, \beta_i, \gamma_i$ $(1 \leq i \leq \bar{N})$.

Output: An optimal sequence A, and an optimal d_i.
Step 1. Determine the optimal sequence by using the SDR order of χ_i, i.e., $\chi_{[1]} \leq \chi_{[2]} \leq \cdots \leq \chi_{[\bar{N}]}$;
Step 2. If $\gamma_i \geq \beta_i$, the optimal due date is $d_{[i]} = 0$. Else, the optimal due date is $d_{[i]} = C_{[i]}$.

Theorem 5.1 *Algorithm 5.1 can solve the* $1|p_{[i]} = \chi_{[i]}s_i, DIF, psddt|\sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{[i]}\right)$ *problem within* $O(\bar{N} \log \bar{N})$ *time.*

6 Example

Example 6.1 There are four jobs, that is, $\bar{N} = 4$, in which $s_0 = 2$, $v = 0.3$, $\{\chi_1, \chi_2, \chi_3, \chi_4\} = \{0.2, 0.5, 0.4, 0.3\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{1, 1, 2, 3\}$, $\{\beta_1, \beta_2, \beta_3, \beta_4\} = \{1, 1, 3, 5\}$, and $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \{1, 2, 1, 3\}$.

For the problem $1|p_i = \chi_i s_i, CON, psddt|\sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{opt}\right)$, according to Algorithm 3.1, the solution steps are given as follows:
Step 1. $J_1 \to J_4 \to J_3 \to J_2$ can be obtained by $b_1 < b_4 < b_3 < b_2$.
Step 2. Calculate the value of l according to Property 3.2:

When $l = 1$, $(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = -10 + 7 = -3 < 0$, and $(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 1 - 9 + 7 = -1 < 0$;

When $l = 2$, $(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = -1 < 0$, and $(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 2 - 8 + 7 = 1 > 0$;

When $l = 3$, $(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 1 > 0$, and $(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 4 - 5 + 7 = 6 > 0$;

When $l = 4$, $(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 6 > 0$, and $(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 7 + 7 = 14 > 0$;

Then $l = 2$ can be obtained.
Step 3. From *Step 2*, $l = 2$, and the solution steps are given as follows:

(1) $x = 2$, $y = 3$, and $Q_{2,3} = 0$; then, job J_1 is at the 2-th position;
(2) $x = 1$, $y = 3$, and $Q_{1,3} = -1 < 0$; then, job J_4 is at the 3-th position;
(3) $x = 2$, $y = 4$, and $Q_{2,4} = 4.8 > 0$; then, job J_2 is at the 4-th position;
(4) $x = 1$, $y = 4$, and $Q_{1,4} = 6.2 > 0$; then, job J_3 is at the 1-th position.

Based on the above works, the optimal sequence is $J_3 \to J_1 \to J_4 \to J_2$, and the processing times, delivery times, and completion times can be seen as follows (see Table 1):

It can be seen that $d_{opt} = C_{[2]} = 5.1000$, and we can easily know that the corresponding optimal value is $f = \sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{opt}\right) = 94.5200$.

For the problem $1|p_i = \chi_i s_i, SLK, psddt|\sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i q_{opt}\right)$, according to Algorithm 4.1, the solution steps are given as follows:

Table 1 Results for CON

J_i	1	2	3	4
$p_{[i]}$	0.8	0.6800	1.5300	4.0800
$q_{[i]}$	0.6	1.0200	1.5300	2.4480
$C_{[i]}$	3.4	5.1000	8.1600	14.6880

Table 2 Results for SLK

J_i	1	2	3	4
$p_{[i]}$	0.4000	0.9000	1.9200	4.0800
$q_{[i]}$	0.6000	0.9000	1.4400	2.4480
$C_{[i]}$	3.0000	4.8000	8.1600	14.6880
$d_{[i]}$	6.6400	7.1400	8.1600	10.3200

Table 3 Results for DIF

J_i	1	2	3	4
$p_{[i]}$	0.4	0.9000	1.9200	4.0800
$q_{[i]}$	0.6	0.9000	1.4400	2.4480
$C_{[i]}$	3.0	4.8000	8.1600	14.6880
$d_{[i]}$	0	0	8.1600	14.6880

Step 1. $J_1 \to J_4 \to J_3 \to J_2$ can be obtained by $\chi_1 < \chi_4 < \chi_3 < \chi_2$.
Step 2. Calculate the value of l according to Property 4.2:

When $l = 2$, $(\sum_{i=1}^{l-1} \alpha_i - \sum_{i=l}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 1 - 9 + 7 = -1 < 0$, and $(\sum_{i=1}^{l} \alpha_i - \sum_{i=l+1}^{\bar{N}} \beta_i + \sum_{i=1}^{\bar{N}} \gamma_i) = 2 - 8 + 7 = 1 > 0$, that is $l = 2$.
Step 3. From *Step 2*, $l = 2$, and the solution steps are given as follows:

(1) $x = 2$, $y = 3$, and $Q_{2,3} = 5.1 > 0$; then, job J_4 is at the 2-th position;
(2) $x = 1$, $y = 3$, and $Q_{1,3} = 8.16 > 0$; then, job J_1 is at the 1-th position;
(3) $x = 2$, $y = 4$, and $Q_{2,4} = 20.4 > 0$; then, job J_2 is at the 4-th position;
(4) $x = 1$, $y = 4$, and $Q_{1,4} = 32.64 > 0$; then, job J_3 is at the 3-th position.

Based on the above works, the optimal sequence is $J_1 \to J_4 \to J_3 \to J_2$, and the processing times, delivery times, completion times and due dates can be seen in Table 2.

It can be known that $q_{opt} = (1 + v)C_{[2]} = 6.2400$, and the corresponding objective function value is $f = \sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i q_{opt}\right) = 71.2000$.

For the problem $1|p_{[i]} = \chi_{[i]} s_i, DIF, psddt| \sum_{i=1}^{\bar{N}} \left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{[i]}\right)$, according to Algorithm 5.1, the solution steps are given as follows:

Step 1. The optimal sequence: $J_1 \to J_4 \to J_3 \to J_2$ can be obtained by $\chi_1 < \chi_4 < \chi_3 < \chi_2$.

Step 2. The processing times, delivery times, completion times and due dates (see Property 5.1) are shown in Table 3.

The corresponding objective function value is $\sum_{i=1}^{\bar{N}}\left(\alpha_i \widehat{E}_{[i]} + \beta_i \widehat{T}_{[i]} + \gamma_i d_{[i]}\right) = 60.0240$.

7 Conclusion

We extended the results of scheduling with position-dependent weights and due date assignment to a setting of scheduling with psddt and deterioration effects. Under three kinds of due date assignments, our objective is to minimize weighted sum of earliness, tardiness, and due date cost, where the weights are position-dependent weights. Through a series of optimal properties, it can be obtained that the CON, SLK, and DIF assignment scheduling problems can be solved in $O(\bar{N} \log \bar{N})$ time. Future research could delve into a general linear deterioration, such as $p_i = a_i + \chi_i s_i$ (where a_i is the normal processing time of job J_i), consider the model under a flow shop setting, or study the model with job-dependent weights (i.e., the objective function would be to minimize $\sum_{i=1}^{\bar{N}}\left(\alpha_i \widehat{E}_i + \beta_i \widehat{T}_i + \gamma_i d_i\right)$, where α_i, β_i and γ_i are the weights of job J_i for earliness, tardiness and due date).

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References

Brucker P (2001) Scheduling algorithms, 3rd edn. Springer, Berlin

Chen Y, Ma X, Zhang G, Cheng Y (2023) On optimal due date assignment without restriction and resource allocation in group technology scheduling. J Comb Optim 45:64

Cheng TCE, Kravchenko SA, Lin BMT (2020) Scheduling step-deteriorating jobs to minimize the total completion time. Comput Ind Eng 144:106329

Gawiejnowicz S (2020) Models and algorithms of time-dependent scheduling. Springer, Berlin

Gordon VS, Proth JM, Chu CB (2002) A survey of the state of-the-art of common due date assignment and scheduling research. Eur J Oper Res 139:1–25

Gordon VS, Proth JM, Chu CB (2002) Due date assignment and scheduling: SLK, TWK and other due date assignment models. Prod Plan Control 13:117–132

Ji M, Ge Yao DL, JJ, Cheng TCE (2015) Single-machine slack due-window assignment and scheduling with past-sequence-dependent delivery times and controllable job processing times. Eur J Ind Eng 9:794–818

Jiang C, Zou D, Bai D, Wang J-B (2020) Proportionate flowshop scheduling with position-dependent weights. Eng Optim 52(1):37–52

Koulamas C (2017) Common due date assignment with generalized earliness/tardiness penalties. Comput Ind Eng 109:79–83

Koulamas C, Kyparisis GJ (2010) Single-machine scheduling problems with past-sequence-dependent delivery times. Int J Prod Econ 126:264–266
Li S-S, Chen R-X, Feng Q, Jiao C-W (2019) Parallel-machine scheduling with job-dependent cumulative deterioration effect and rejection. J Comb Optim 38:957–971
Liu W, Hu X, Wang X (2017) Single machine scheduling with slack due dates assignment. Eng Optim 49:709–717
Liu W-W, Yao Y, Jiang C (2020) Single-machine resource allocation scheduling with due-date assignment, deterioration effect and position-dependent weights. Eng Optim 52:701–714
Lv D-Y, Wang J-B (2021) Study on proportionate flowshop scheduling with due-date assignment and position-dependent weights. Optim Lett 15(6):2311–2319
Qian J, Han H (2022) The due date assignment scheduling problem with the deteriorating jobs and delivery time. J Appl Math Comput 68:2173–2186
Sun X, Geng X-N, Liu T (2020) Due-window assignment scheduling in the proportionate flow shop setting. Ann Oper Res 292:113–131
T'kindt V, Shang L, Croce FD (2020) Exponential time algorithms for just-in-time scheduling problems with common due date and symmetric weights. J Comb Optim 39:764–775
Toksari, MD, Guner, E (2010) The common due-date early/tardy scheduling problem on a parallel machine under the effects of time-dependent learning and linear and nonlinear deterioration. Expert Syst Appl 37:92–112
Wang J-B, Zhang B, Li L, Bai D-Y, Feng Y-B (2020) Due-window assignment scheduling problems with position-dependent weights on a single machine. Eng Optim 52(2):185–193
Wang J-B, Cui B, Ji P, Liu W-W (2021) Research on single-machine scheduling with position-dependent weights and past-sequence-dependent delivery times. J Comb Optim 41:290–303
Xiong X, Wang D, Cheng TCE, Wu C-C, Yin Y (2018) Single-machine scheduling and common due date assignment with potential machine disruption. Int J Prod Res 56(3):1345–1360
Yang Y, Yin G, Wang C, Yin Y (2022) Due date assignment and two-agent scheduling under multitasking environment. J Comb Optim 44(4):2207–2223
Yin Y, Wu W-H, Cheng TCE, Wu C-C (2014) Due date assignment and single-machine scheduling with generalized positional deteriorating jobs and deteriorating multi-maintenance activities. Int J Prod Res 52:2311–2326
Yin Y, Wang D, Wu C-C, Cheng TCE (2016) CON/SLK due date assignment and scheduling on a single machine with two agents. Nav Res Logist 63:416–429
Yin Y, Yang Y, Wang D, Cheng TCE, Wu C-C (2018) Integrated production, inventory, and batch delivery scheduling with due date assignment and two competing agents. Nav Res Logist 65:393–409
Yin Y, Li D, Wang D, Cheng TCE (2021) Single-machine serial-batch delivery scheduling with two competing agents and due date assignment. Ann Oper Res 298:497–523
Zhao CL, Tang HY (2014) Single machine scheduling problems with general position-dependent processing times and past-sequence-dependent delivery times. J Appl Math Comput 45:259–274