



On optimal due date assignment without restriction and resource allocation in group technology scheduling

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Accepted: 18 January 2023 / Published online: 2 February 2023

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Abstract

A single machine group scheduling problem with due date assignment and resource allocation is investigated. Based on production similarities, jobs are classified into groups and it is required that jobs within the same group are processed contiguously, in order to achieve high-volume production efficiency. Jobs in the same group are allowed to have different due dates. The job processing times are resource dependent, and both convex and bounded linear resource consumption functions are considered. The aim is minimizing an aggregate cost which takes into account earliness, tardiness, due date assignment and resource allocation costs, by finding a group schedule, due date assignment and resource allocation for all jobs. For both resource consumption functions, we present properties of the optimal solutions, and for the special case where the size of every group is the same and the minimum of the due date assignment cost and the tardiness cost for each job is identical, we present an algorithm to optimally solve the problem in $O(n^3)$ time, where n is the total number of jobs.

A preliminary version of this paper appears in Combinatorial Optimization and Applications, COCOA 2021. Lecture Notes in Computer Science, vol 13135, Springer, Cham.

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Keywords Single machine scheduling · Due date assignment · Group technology · Resource allocation

1 Introduction

We study a single machine group scheduling problem with due date assignment and resource allocation. By grouping various products with similar designs and/or production processes, one could increase the efficiency for high-volume production. The utilization of group technology has many advantages, as reported in, e.g., Liu et al. (2014), Keshavarz et al. (2015), Qin et al. (2016) and Wang et al. (2021). Many studies view due date assignment as part of the scheduling decision, since due dates are often determined during sales negotiations. Setting further due dates may defer customers from placing orders, which often results in lost sales, hence a cost is associated with the due date assignment. In addition, products completed processing before their due dates usually incur earliness costs such as storage cost or insurance fees. On the other hand, products completed after their due dates usually incur tardiness penalties such as compensation for customers. In practice, usually it is unlikely to complete all jobs exactly on their due dates, thus for many manufacturing systems all the above mentioned costs are relevant and should be considered (Seidmann et al. 1981; Panwalkar et al. 1982; Shabtay 2016).

Seidmann et al. (1981) and Panwalkar et al. (1982) initiated the study of scheduling problems with due date assignment. Seidmann et al. (1981) studied a single machine scheduling problem where jobs are allowed to be assigned different due dates, and presented an $O(n \log n)$ time algorithm to solve the problem. Panwalkar et al. (1982) analyzed a scheduling problem where all jobs are required to be assigned a common due date, and an $O(n \log n)$ time algorithm is presented. Li et al. (2011) studied a single machine group scheduling problem considering three different due date assignment methods. The objective is to minimize an aggregate cost including earliness, tardiness, due date assignment and flow time costs. Li and Zhao (2015) and Ji et al. (2016) studied single machine group scheduling problems with multiple due windows assignment. Bajwa et al. (2019) studied a single machine group scheduling problem to minimize the number of tardy jobs, in which jobs are allowed to be assigned different due dates.

For many modern industrial processes the job processing times are affected by the allocation of certain resource such as fuel or manpower. Shabtay et al. (2010) investigated a group scheduling problem with due date assignment, and extended the study to the case where job processing times are resource dependent. Lv et al. (2021) studied a single machine common flow allowance group scheduling problem with learning effect, and two resource allocation functions are considered. Yin et al. (2014) studied single machine group scheduling problems with learning effect, deteriorating jobs and resource allocation.

To the best of our knowledge, there is no research on group scheduling with resource allocation and due date assignment, in which jobs in the same group could have different due dates assigned and different unit costs (including due date, earliness, tardiness, and resource allocation costs). We will fill the research gap in this paper. An application of our considered scheduling problem is presented as follows. In modern manufac-

turing, products exported to overseas destinations from a multinational company are often adapted to local regulations, or modified to meet different electricity or other infrastructure standards. In order to improve production efficiency and reduce cost, companies usually join orders with the same processing standards to be processed together. These orders can be considered as a group of jobs and each job of the same group can be assigned a different due date, since the orders of the same group may come from different customers or areas, and they may own different unit costs. Moreover, the actual processing time of an order can often be compressed if additional resources (e.g., money, fuel, or manpower) are allocated.

The remainder of this paper is structured in the following manner. In Sect. 2, we give a formal description of the scheduling problem that will be studied in this paper. Several preliminary results are presented in Sect. 3. In Sect. 4, properties of the optimal solutions are presented, and for the special case where the size of every group is the same and the minimum of the due date assignment cost and the tardiness cost for each job is identical, for both convex and bounded linear resource consumption functions, an $O(n^3)$ time algorithm is presented, where n is the total number of jobs. Numerical examples are presented in Sect. 5, and we conclude our paper in Sect. 6.

2 Problem formulation

There are n jobs J_1, J_2, \dots, J_n to be processed on a single machine, and the jobs are partitioned into m groups in advance. Let the i th group be $G_i = \{J_{i1}, J_{i2}, \dots, J_{in_i}\}$, where n_i is the number of jobs in G_i , for $i = 1, 2, \dots, m$. Hence, $\sum_{i=1}^m n_i = n$. The jobs are non-preemptive, and it is required that jobs in the same group are processed contiguously. For each group G_i , prior to the processing of the first job in G_i , a machine setup time s_i is required, which does not depend on the job sequence in the group. It is permissible to assign different due dates to jobs in the same group. Let d_{ij} and C_{ij} be the due date and completion time of job J_{ij} , respectively. Then, the earliness of job J_{ij} is $E_{ij} = \max\{d_{ij} - C_{ij}, 0\}$, and the tardiness of job J_{ij} is $T_{ij} = \max\{C_{ij} - d_{ij}, 0\}$.

In practical manufacturing systems, the actual processing times of jobs are often controllable by allocating a continuous resource, and this relationship is characterized by the resource consumption function. Two different types of resource consumption functions have been widely used in literature (see, e.g., Shabtay et al. 2010). One is a convex function $p_{ij}(u_{ij}) = (\frac{w_{ij}}{u_{ij}})^r$, where u_{ij} is the amount of resource allocated to J_{ij} , w_{ij} denotes the workload of J_{ij} , and $r > 0$ is a constant. The second resource consumption function is a bounded linear function $p_{ij}(u_{ij}) = \overline{p}_{ij} - a_{ij}u_{ij}$, where \overline{p}_{ij} is the non-compressed processing time of job J_{ij} , u_{ij} is the amount of resource allocated to J_{ij} , and a_{ij} is the positive compression rate. The value of u_{ij} is within the range $0 \leq u_{ij} \leq \overline{u}_{ij} \leq \frac{\overline{p}_{ij}}{a_{ij}}$, where \overline{u}_{ij} is the upper bound on the amount of resource which can be allocated to the job.

Our objective is finding a schedule π for the group sequence and job sequence in every group, a due date assignment

$$d(\pi) = (d_{11}(\pi), \dots, d_{1n_1}(\pi), \dots, d_{m1}(\pi), \dots, d_{mn_m}(\pi))$$

Table 1 Notations

Symbol	Definition
$G_{[i]}$	The group scheduled in the i th position
$n_{[i]}$	The number of jobs in $G_{[i]}$
$J_{[i][j]}$	The job scheduled in the j th position in $G_{[i]}$
$p_{[i][j]}$	The processing time of $J_{[i][j]}$
$s_{[i]}$	The setup time for $G_{[i]}$
$d_{[i][j]}$	The due date assigned to $J_{[i][j]}$
$\psi_{[i][j]}$	$\min(\alpha_{[i][j]}, \gamma_{[i][j]})$
$P_{[i]} = \sum_{j=1}^{n_{[i]}} p_{[i][j]}$	The total processing time of jobs in $G_{[i]}$
$\Psi_{[i]} = \sum_{j=1}^{n_{[i]}} \psi_{[i][j]}$	The sum of $\psi_{[i][j]}$ for all jobs in $G_{[i]}$
$J_{[i]j}$	The j th job in group $G_{[i]}$
$J_{i[j]}$	The job scheduled in the j th position in group G_i

for all jobs, and a resource allocation vector

$$u(\pi) = (u_{11}(\pi), \dots, u_{1n_1}(\pi), \dots, u_{m1}(\pi), \dots, u_{mn_m}(\pi))$$

specifying the amount of resource allocated to each job, to minimize the following aggregate cost function

$$Z(\pi, d(\pi), u(\pi)) = \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}),$$

which includes earliness, tardiness, due date assignment and resource allocation costs. Here α_{ij} , β_{ij} , γ_{ij} and v_{ij} are positive parameters representing the unit due date, earliness, tardiness and resource allocation costs for J_{ij} , respectively. Parameters n_i , s_i , α_{ij} , β_{ij} , γ_{ij} and v_{ij} are all given.

Using the three-field notation of Graham et al. (1979), we denote our problem with the convex and linear resource consumption functions by

$$1|GT, DIF, CONV| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$$

and

$$1|GT, DIF, LIN| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}),$$

respectively, where GT represents group technology, DIF indicates that jobs within the same group are allowed to be assigned different due dates, CONV indicates that the

resource consumption function adopted is convex, and LIN indicates that the resource consumption function adopted is bounded linear. The notations used in this paper are listed in Table 1.

3 Preliminary analysis

First, we present two lemmas that will be useful for later analysis.

Lemma 3.1 *For problem*

$$1|GT, DIF, X| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}), \quad X \in \{CONV, LIN\},$$

given a schedule π , the optimal due date assignment $d^*(\pi)$ can be calculated by the following rule: For $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n_i$, if $\alpha_{[i][j]} < \gamma_{[i][j]}$ then $d_{[i][j]}^*(\pi) = C_{[i][j]}(\pi)$, if $\alpha_{[i][j]} > \gamma_{[i][j]}$ then $d_{[i][j]}^*(\pi) = 0$, and if $\alpha_{[i][j]} = \gamma_{[i][j]}$ then $d_{[i][j]}^*(\pi)$ can be any value in interval $[0, C_{[i][j]}(\pi)]$.

Proof Since the due date assignment vector only affects the first three terms in the objective function, and by the definition of the earliness and tardiness of a job given in Sect. 2, that is, $E_{ij} = \max\{d_{ij} - C_{ij}, 0\}$ and $T_{ij} = \max\{C_{ij} - d_{ij}, 0\}$, the lemma follows immediately. □

Lemma 3.2 *There exists an optimal schedule with zero machine idle times for problem*

$$1|GT, DIF, X| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}), \quad X \in \{CONV, LIN\}$$

Proof We omit the proof here because it is similar with the proof to Lemma 3.1 in Li et al. (2011). □

For a schedule π and its corresponding optimal due date assignment $d^*(\pi)$ determined in Lemma 3.1, let $Z_{[i][j]}(\pi, d^*(\pi), u(\pi))$ denote the contribution to the objective function from $J_{[i][j]}$, and let $\psi_{[i][j]} = \min(\alpha_{[i][j]}, \gamma_{[i][j]})$, for $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n_i$. By Lemma 3.1 and the same analysis as in Sect. 3 in Chen and Cheng (2021), we can write $Z_{[i][j]}(\pi, d^*(\pi), u(\pi))$ in the following unified way:

$$Z_{[i][j]}(\pi, d^*(\pi), u(\pi)) = \psi_{[i][j]}C_{[i][j]} + v_{[i][j]}u_{[i][j]}.$$

Hence, we can write the objective function of our problem, $Z(\pi, d^*(\pi), u(\pi))$, as follows

$$Z(\pi, d^*(\pi), u(\pi)) = \sum_{i=1}^m \sum_{j=1}^{n_i} (\psi_{[i][j]}C_{[i][j]} + v_{[i][j]}u_{[i][j]}). \tag{1}$$

4 The optimal schedule

We first analyze the optimal schedule for

$$1|GT, DIF, X| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}), \quad X \in \{CONV, LIN\}.$$

According to Lemma 3.2, we only consider schedules with no idle time. Thus, for any job $J_{[i][j]}$, we have the following formula for its completion time:

$$\begin{aligned} C_{[i][j]} &= \sum_{k=1}^{i-1} P_{[k]} + \sum_{k=1}^i s_{[k]} + \sum_{\ell=1}^j P_{[i][\ell]} \\ &= \sum_{k=1}^{i-1} \sum_{\ell=1}^{n_{[k]}} P_{[k][\ell]} + \sum_{k=1}^i s_{[k]} + \sum_{\ell=1}^j P_{[i][\ell]}. \end{aligned}$$

Next, we calculate the objective function based on Eq. (1) and the above equalities.

$$\begin{aligned} Z(\pi, d^*(\pi), u(\pi)) &= \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} (\psi_{[i][j]} C_{[i][j]} + v_{[i][j]} u_{[i][j]}) \\ &= \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \left(\sum_{k=1}^{i-1} \sum_{\ell=1}^{n_{[k]}} P_{[k][\ell]} + \sum_{k=1}^i s_{[k]} + \sum_{\ell=1}^j P_{[i][\ell]} \right) + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} v_{[i][j]} u_{[i][j]} \\ &= \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \left(\sum_{k=1}^i s_{[k]} \right) + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} v_{[i][j]} u_{[i][j]} + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \left(\sum_{k=1}^{i-1} \sum_{\ell=1}^{n_{[k]}} P_{[k][\ell]} \right) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \left(\sum_{\ell=1}^j P_{[i][\ell]} \right). \end{aligned}$$

We rewrite the first, the third and the fourth term in the above formula. For the first term,

$$\sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \left(\sum_{k=1}^i s_{[k]} \right) = \sum_{i=1}^m \left(\sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \right) \left(\sum_{k=1}^i s_{[k]} \right) = \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i s_{[k]} \right),$$

for the third term,

$$\sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \left(\sum_{k=1}^{i-1} \sum_{\ell=1}^{n_{[k]}} P_{[k][\ell]} \right) = \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^{i-1} \sum_{\ell=1}^{n_{[k]}} P_{[k][\ell]} \right)$$

$$\begin{aligned}
 &= \sum_{i=1}^m \sum_{k=1}^{i-1} \left(\Psi_{[i]} \sum_{\ell=1}^{n_{[k]}} P_{[k][\ell]} \right) \\
 &= \sum_{i=1}^m \left(\sum_{k=i+1}^m \Psi_{[k]} \right) \left(\sum_{\ell=1}^{n_{[i]}} P_{[i][\ell]} \right) \\
 &= \sum_{i=1}^m \left(\sum_{k=i+1}^m \Psi_{[k]} \right) \left(\sum_{j=1}^{n_{[i]}} P_{[i][j]} \right) \\
 &= \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(\sum_{k=i+1}^m \Psi_{[k]} \right) P_{[i][j]},
 \end{aligned}$$

and for the fourth term,

$$\sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \psi_{[i][j]} \left(\sum_{\ell=1}^j P_{[i][\ell]} \right) = \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(\sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right) P_{[i][j]}.$$

By combining the above, it follows that

$$\begin{aligned}
 &Z(\pi, d^*(\pi), u(\pi)) \\
 &= \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i S_{[k]} \right) + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} v_{[i][j]} u_{[i][j]} + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(\sum_{k=i+1}^m \Psi_{[k]} \right) P_{[i][j]} \\
 &\quad + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(\sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right) P_{[i][j]} \\
 &= \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i S_{[k]} \right) + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} v_{[i][j]} u_{[i][j]} + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right) P_{[i][j]}.
 \end{aligned} \tag{2}$$

Next, we analyze the optimal schedule for convex and bounded linear resource consumption functions, respectively.

4.1 Optimal schedule for problem

$$1|GT, DIF, CONV| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij} d_{ij} + \beta_{ij} E_{ij} + \gamma_{ij} T_{ij} + v_{ij} u_{ij})$$

By $p_{ij}(u_{ij}) = \left(\frac{w_{ij}}{u_{ij}} \right)^r$ and Eq. (2), we have

$$Z(\pi, d^*(\pi), u(\pi)) = \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i S_{[k]} \right) + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right)$$

$$\times \left(\frac{w_{[i][j]}}{u_{[i][j]}} \right)^r + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} v_{[i][j]} u_{[i][j]}. \tag{3}$$

Lemma 4.1 For problem

$$1|GT, DIF, CONV| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij} d_{ij} + \beta_{ij} E_{ij} + \gamma_{ij} T_{ij} + v_{ij} u_{ij}),$$

given a schedule π for the group sequence and job sequence within each group, assume that the due date assignment vector $d^*(\pi)$ is adopted, then the following formula gives the optimal resource allocation with respect to π

$$u_{[i][j]}^*(\pi) = w_{[i][j]}^{r/(r+1)} \times \left(\frac{r \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right)}{v_{[i][j]}} \right)^{1/(r+1)}. \tag{4}$$

Proof The objective function in Eq. (3) can be viewed as a function in $u_{[i][j]}$'s, and each variable $u_{[i][j]}$ is only in the following summation consisting of two terms

$$v_{[i][j]} u_{[i][j]} + \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right) \left(\frac{w_{[i][j]}}{u_{[i][j]}} \right)^r.$$

Hence, taking derivative of the above formula with respect to $u_{[i][j]}$ and equating it to zero give the formula in Eq. (4). Note that the above formula is convex in $u_{[i][j]}$, it follows that Eq. (4) indeed gives the optimal value for $u_{[i][j]}$. \square

Let $\theta_{ij} = (w_{ij} v_{ij})^{r/(r+1)}$, for $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n_i$. Plugging Eq. (4) into Eq. (3) gives the following formula for the objective value when the resource allocation and the due date assignment are both optimal, under a given schedule π for the group sequence and job sequence within each group.

$$\begin{aligned} Z(\pi, d^*(\pi), u^*(\pi)) &= \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i s_{[k]} \right) \\ &+ (r^{1/(r+1)} + r^{-r/(r+1)}) \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \theta_{[i][j]} \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right)^{1/(r+1)}. \end{aligned} \tag{5}$$

In order to determine the optimal job sequence within each group, the following lemmas will be used.

Lemma 4.2 (Hardy et al. 1967) *The sum of products of two sequences x_i and y_i , $\sum_i x_i y_i$, attains its maximum (minimum) value if sequences x_i and y_i are monotonic in the same (opposite) sense.*

Lemma 4.3 *For each $i = 1, 2, \dots, m$, the optimal job sequence in G_i is in non-decreasing order of θ_{ij} , where $\theta_{ij} = (w_{ij} v_{ij})^{r/(r+1)}$.*

Proof In Eq. (5), $\sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i s_{[k]} \right)$ and $\sum_{k=i+1}^m \Psi_{[k]}$ do not depend on job sequences in groups. Moreover, $r^{1/(r+1)} + r^{-r/(r+1)}$ is a constant as r is a positive constant parameter. Since $\sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]}$ is monotonically decreasing in j , by Lemma 4.2, the lemma follows. \square

Lemma 4.3 gives the optimal job sequence in every group. To solve the original problem, it remains to calculate the optimal group sequence that minimizes Eq. (5). Next, we consider a special case where the size of each group is the same, and we use \bar{n} to denote this size. In addition, $\psi_{ij} = \bar{\psi}$ is identical for every job.

Define $x_{i\ell} = 1$ if group G_i is assigned to position ℓ in π , and $x_{i\ell} = 0$ otherwise. According to Lemma 6 in Chen and Cheng (2021), for this special case, the optimal group sequence can be calculated in $O(m \times \max(n, m^2))$ time by solving the following linear assignment problem in $x_{i\ell}$.

$$\begin{aligned}
 (P1) \quad & \min \sum_{i=1}^m \sum_{\ell=1}^m t_{i\ell} \times x_{i\ell} \\
 \text{s.t.} \quad & \sum_{\ell=1}^m x_{i\ell} = 1, \quad \text{for } i = 1, 2, \dots, m. \\
 & \sum_{i=1}^m x_{i\ell} = 1, \quad \text{for } \ell = 1, 2, \dots, m. \\
 & x_{i\ell} \in \{0, 1\}, \quad \text{for } i, \ell = 1, 2, \dots, m.
 \end{aligned}$$

In the above,

$$\begin{aligned}
 t_{i\ell} = & \bar{\Psi}(m - \ell + 1)s_i + \left(r^{1/(r+1)} + r^{-r/(r+1)} \right) \\
 & \times \sum_{j=1}^{\bar{n}} \theta_{i[j]} \left((m - \ell)\bar{\Psi} + (\bar{n} - j + 1)\bar{\psi} \right)^{1/(r+1)}, \tag{6}
 \end{aligned}$$

where $\bar{\Psi} = \bar{n}\bar{\psi}$.

The following algorithm, Algorithm 1, optimally solves this special case, i.e., problem

$$1|GT, DIF, CONV, n_i = \bar{n}, \psi_{ij} = \bar{\psi} | \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}).$$

Theorem 4.4 Algorithm 1 optimally solves $1|GT, DIF, CONV, n_i = \bar{n}, \psi_{ij} = \bar{\psi} | \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$ in $O(n^3)$ time.

Proof The correctness of Algorithm 1 is established by Lemmas 3.1, 3.2, and 4.1, together with Lemma 6 in Chen and Cheng (2021). Next we analyze its time complexity. Since $\sum_{i=1}^m n_i = n$, calculating all the optimal job sequences within the m groups

Algorithm 1

- 1: Calculate the optimal job sequences in all the m groups according to Lemma 4.3.
- 2: Solve problem P1 to obtain the optimal group sequence. Let the resulting schedule be π .
- 3: Use Equ. (4) to calculate the optimal resource allocation $u^*(\pi)$.
- 4: Calculate for all jobs their optimal due dates by Lemma 3.1.

in Step 1 requires $O(\sum_{i=1}^m n_i \log n_i) = O(n \log n)$ time. By Lemma 6 in Chen and Cheng (2021), Step 2 requires $O(m \times \max(n, m^2)) = O(n^3)$ time, where the equality holds because $m = O(n)$. It requires $O(n)$ time to calculate the optimal resource allocation in Step 3, and requires $O(n)$ time to calculate the optimal due dates in Step 4. Therefore, Algorithm 1 runs in $O(n^3)$ time. \square

4.2 Optimal schedule for problem

$$1|GT, DIF, LIN| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$$

First, the following lemma to determine the optimal resource allocation with respect to a given schedule π (for the group sequence and job sequence within each group) is presented.

Lemma 4.5 *For problem*

$$1|GT, DIF, LIN| \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}),$$

given a schedule π for the group sequence and job sequence within each group, assume that the due date assignment vector $d^*(\pi)$ is adopted, then the optimal resource allocation with respect to π can be determined by

$$u^*_{[i][j]}(\pi) = \begin{cases} 0, & \text{if } \eta_{[i][j]} < 0 \\ \text{any value in } [0, \frac{u_{[i][j]}}{a_{[i][j]}}], & \text{if } \eta_{[i][j]} = 0 \\ \frac{u_{[i][j]}}{a_{[i][j]}}, & \text{if } \eta_{[i][j]} > 0 \end{cases} \tag{7}$$

where $\eta_{[i][j]} = \sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} - \frac{v_{[i][j]}}{a_{[i][j]}}$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n_i$.

Proof By $p_{ij}(u_{ij}) = \overline{p_{ij}} - a_{ij}u_{ij}$, we rewrite Eq. (2) as follows

$$\begin{aligned} Z(\pi, d^*(\pi), u(\pi)) &= \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i s_{[k]} \right) + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right) \overline{p_{[i][j]}} \\ &+ \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \left(v_{[i][j]} - a_{ij} \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right) \right) u_{[i][j]}. \end{aligned}$$

From the above formula for the objective value, for a given schedule π , the first two terms are constants, and the objective value can be viewed as a linear function in

$u_{[i][j]}$'s. Moreover, each variable $u_{[i][j]}$ only appears in the following term

$$\left(v_{[i][j]} - a_{ij} \left(\sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} \right) \right) u_{[i][j]}.$$

Since all $a_{[i][j]}$'s are given positive parameters, if $\eta_{[i][j]} < 0$ then no resource should be allocated to job $J_{[i][j]}$; if $\eta_{[i][j]} > 0$ then job $J_{[i][j]}$ should be allocated the maximum amount of resource allowed; and if $\eta_{[i][j]} = 0$, any feasible amount should be optimal. The lemma is established. \square

To determine the optimal job sequence within each group, we consider a special case where $\psi_{ij} = \psi_i$ is identical for all jobs in the same group G_i , for $i = 1, 2, \dots, m$. We will show that, for any given group sequence, in this case the optimal job sequence within each group can be obtained via solving a linear assignment problem, as described in the following lemma.

Lemma 4.6 *Given a group sequence, for the case where $\psi_{ij} = \psi_i$ for all jobs J_{ij} in group G_i , for each $i = 1, 2, \dots, m$, the optimal job sequence within group G_i can be determined in $O(n_i^3)$ time, and all optimal job sequences for the m groups can be determined in $O(n^3)$ time.*

Proof Rewrite $p_{ij}(u_{ij}) = \overline{p_{ij}} - a_{ij}u_{ij}$ as $u_{ij} = \frac{\overline{p_{ij}} - p_{ij}}{a_{ij}}$, and plug it into Eq. (2) for the objective function, we obtain

$$Z(\pi, d^*(\pi), u(\pi)) = \sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i s_{[k]} \right) + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \frac{\overline{p_{[i][j]}}}{a_{[i][j]}} v_{[i][j]} + \sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \eta_{[i][j]} p_{[i][j]}, \tag{8}$$

where $\eta_{[i][j]} = \sum_{k=i+1}^m \Psi_{[k]} + \sum_{\ell=j}^{n_{[i]}} \psi_{[i][\ell]} - \frac{v_{[i][j]}}{a_{[i][j]}}$.

When the group sequence is given, both the first term $\sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i s_{[k]} \right)$ and the second term $\sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \frac{\overline{p_{[i][j]}}}{a_{[i][j]}} v_{[i][j]}$ in Eq. (8) are constants. Hence, to determine the optimal job sequence within each group, we only need to consider the last term $\sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \eta_{[i][j]} p_{[i][j]}$ in Eq. (8).

Since for each i , $\psi_{ij} = \psi_i$ for $j = 1, 2, \dots, n_i$, it follows that $\eta_{[i][j]}$ in Eq. (8) can be rewritten as $\eta_{[i][j]} = \sum_{k=i+1}^m \Psi_{[k]} + (n_{[i]} - j + 1)\psi_{[i]} - \frac{v_{[i][j]}}{a_{[i][j]}}$. Define

$$\eta_{[i]rs} = \sum_{k=i+1}^m \Psi_{[k]} + (n_{[i]} - s + 1)\psi_{[i]} - \frac{v_{[i]r}}{a_{[i]r}}, \tag{9}$$

for $i = 1, 2, \dots, m$ and $r, s = 1, 2, \dots, n_{[i]}$, which is the penalty if job $J_{[i]r}$ is assigned to position s in group $G_{[i]}$.

According to the optimal resource allocation in Eq. (7), and by $p_{ij}(u_{ij}) = \overline{p_{ij}} - a_{ij}u_{ij}$, we obtain the optimal processing time of job $J_{[i]r}$ if it is scheduled in the s th

position in group $G_{[i]}$, as follows:

$$p_{[i]rs} = \begin{cases} \overline{p_{[i]r}}, & \text{if } \eta_{[i]rs} < 0 \\ \text{Any value in } [\overline{p_{[i]r}} - a_{[i]r} \times \overline{u_{[i]r}}, \overline{p_{[i]r}}], & \text{if } \eta_{[i]rs} = 0 \\ \overline{p_{[i]r}} - a_{[i]r} \times \overline{u_{[i]r}}, & \text{if } \eta_{[i]rs} > 0 \end{cases} \quad (10)$$

Let $w_{[i]rs}$ be the minimum cost incurred from assigning job $J_{[i]r}$ to position s in group $G_{[i]}$. Then $w_{[i]rs}$ can be expressed as follows:

$$w_{[i]rs} = \eta_{[i]rs} \times p_{[i]rs} = \begin{cases} \eta_{[i]rs} \times \overline{p_{[i]r}}, & \text{if } \eta_{[i]rs} < 0 \\ 0, & \text{if } \eta_{[i]rs} = 0 \\ \eta_{[i]rs} \times (\overline{p_{[i]r}} - a_{[i]r} \times \overline{u_{[i]r}}), & \text{if } \eta_{[i]rs} > 0 \end{cases} \quad (11)$$

Similar to problem $P1$, let us define $x_{[i]rs} = 1$ if job $J_{[i]r}$ is assigned to position s in group $G_{[i]}$ and $x_{[i]rs} = 0$ otherwise. For a given group sequence, the problem of determining optimal positions for the $n_{[i]}$ jobs in group $G_{[i]}$ can be formulated as the following linear assignment problem:

$$(P2) \quad \min \sum_{r=1}^{n_{[i]}} \sum_{s=1}^{n_{[i]}} w_{[i]rs} \times x_{[i]rs}$$

$$s.t. \quad \sum_{s=1}^{n_{[i]}} x_{[i]rs} = 1, \quad \text{for } r = 1, 2, \dots, n_{[i]}.$$

$$\sum_{r=1}^{n_{[i]}} x_{[i]rs} = 1, \quad \text{for } s = 1, 2, \dots, n_{[i]}.$$

$$x_{[i]rs} \in \{0, 1\}, \quad \text{for } r, s = 1, 2, \dots, n_{[i]}.$$

Since the linear assignment problem can be solved in cubic time, the optimal job sequence within each group G_i can be determined in $O(n_i^3)$ time, and by $\sum_{i=1}^m n_i = n$, all optimal job sequences for the m groups can be determined in $O(n_1^3) + O(n_2^3) + \dots + O(n_m^3) = O(n^3)$ time. \square

To obtain the optimal group sequence, we further consider a special case which is the same as that in Sect. 4.1, that is the number of jobs within each group is the same, i.e., $n_i = \bar{n} = n/m$ for $i = 1, 2, \dots, m$, and $\psi_{ij} = \bar{\psi}$ is identical for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, \bar{n}$.

Lemma 4.7 *When $n_i = n/m = \bar{n}$ for $i = 1, 2, \dots, m$, and $\psi_{ij} = \bar{\psi}$ is identical for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, \bar{n}$, the optimal group sequence can be determined in $O(n^3)$ time.*

Proof Since $n_1 = n_2 = \dots = n_m = n/m = \bar{n}$, and $\psi_{ij} = \bar{\psi}$ is identical for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, \bar{n}$, recall that $\Psi_i = \sum_{j=1}^{n_i} \psi_{ij}$, it follows that the value of Ψ_i for each group is identical, i.e., $\Psi_1 = \Psi_2 = \dots = \Psi_m = \bar{n}\bar{\psi} = \bar{\Psi}$.

Let $t'_{i\ell}$ be the minimal cost contributed to the objective function from assigning group G_i to position ℓ . We will show how to determine the three additive terms of $t'_{i\ell}$ based on the three terms in the objective function as in Eq. (8).

Since $\Psi_1 = \Psi_2 = \dots = \Psi_m = \bar{\Psi}$, the first term in Eq. (8) can be rewritten as

$$\sum_{i=1}^m \Psi_{[i]} \left(\sum_{k=1}^i s_{[k]} \right) = \bar{\Psi} \sum_{i=1}^m \sum_{k=1}^i s_{[k]} = \bar{\Psi} \sum_{\ell=1}^m \sum_{k=1}^{\ell} s_{[k]} = \bar{\Psi} \sum_{\ell=1}^m (m - \ell + 1) s_{[\ell]},$$

which indicates that the first additive term of $t'_{i\ell}$ is $\bar{\Psi}(m - \ell + 1)s_i$.

The second term in Eq. (8) can be rewritten as

$$\sum_{i=1}^m \sum_{j=1}^{n_{[i]}} \frac{\overline{P_{[i][j]}}}{a_{[i][j]}} v_{[i][j]} = \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{\overline{P_{ij}}}{a_{ij}} v_{ij} = \sum_{i=1}^m \sum_{j=1}^{\bar{n}} \frac{\overline{P_{ij}}}{a_{ij}} v_{ij},$$

which is a constant, that is, does not depend on schedule π . Hence, the second additive term of $t'_{i\ell}$ is $\sum_{j=1}^{\bar{n}} \frac{\overline{P_{ij}}}{a_{ij}} v_{ij}$.

By Eq. (9), the positional penalty of assigning job $J_{[\ell]r}$ to position s in group $G_{[\ell]}$ can be rewritten as

$$\eta_{[\ell]rs} = (m - \ell)\bar{\Psi} + (\bar{n} - s + 1)\bar{\psi} - \frac{v_{[\ell]r}}{a_{[\ell]r}}, \tag{12}$$

for $\ell = 1, 2, \dots, m$ and $r, s = 1, 2, \dots, \bar{n}$. From the above formula, if group G_i is assigned to position ℓ in π , then this penalty is $\eta_{[\ell]rs} = (m - \ell)\bar{\Psi} + (\bar{n} - s + 1)\bar{\psi} - \frac{v_{ir}}{a_{ir}}$, which depends on ℓ , the position that group G_i is assigned to in schedule π , and s , the position that job J_{ir} is assigned to within group G_i in π . That is, this penalty is independent of the positions of other groups, and the job sequences within other groups. Similarly, by Eq. (10), the optimal processing time of job $J_{[\ell]r}$ (i.e., J_{ir}) if it is scheduled in the s th position within group $G_{[\ell]}$ (i.e., G_i) only depends on $\eta_{[\ell]rs}$ given above, and given parameters of job J_{ir} (to be specific, $\overline{p_{ir}}$, a_{ir} , and $\overline{u_{ir}}$). Hence, if G_i is assigned to position ℓ , by applying Lemma 4.6, we can determine in $O(\bar{n}^3)$ time an optimal job sequence within group G_i and the corresponding minimum cost $Z_{i\ell}^*$, which is the optimal value of the corresponding linear assignment problem P2.

From the above, for each $i, \ell = 1, 2, \dots, m$, $t'_{i\ell} = \bar{\Psi}(m - \ell + 1)s_i + \sum_{j=1}^{\bar{n}} \frac{\overline{P_{ij}}}{a_{ij}} v_{ij} + Z_{i\ell}^*$ can be determined in $O(\bar{n}^3)$ time, hence the overall complexity of determining all values $t'_{i\ell}$ for $i, \ell = 1, 2, \dots, m$ is $m^2 \times O(\bar{n}^3)$, which is $O(n^3/m) = O(n^3)$ since $\bar{n} = n/m$.

Let us now define $x_{i\ell} = 1$ if group G_i is assigned to position ℓ in π and $x_{i\ell} = 0$ otherwise. The problem of determining the optimal positions for the m groups in π can be formulated as the following linear assignment problem:

$$(P3) \quad \min \sum_{i=1}^m \sum_{\ell=1}^m t'_{i\ell} \times x_{i\ell}$$

$$\begin{aligned}
 \text{s.t. } & \sum_{\ell=1}^m x_{i\ell} = 1, \quad \text{for } i = 1, 2, \dots, m. \\
 & \sum_{i=1}^m x_{i\ell} = 1, \quad \text{for } \ell = 1, 2, \dots, m. \\
 & x_{i\ell} \in \{0, 1\}, \quad \text{for } i, \ell = 1, 2, \dots, m.
 \end{aligned}$$

Since the linear assignment problem can be solved in $O(m^3)$ time, the overall complexity of determining the optimal group sequence for this special case is $O(n^3) + O(m^3) = O(n^3)$, which establishes the lemma. \square

Now we are ready to present Algorithm 2, which optimally solves problem $1|GT, DIF, LIN, n_i = \bar{n}, \psi_{ij} = \bar{\psi} | \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$.

Algorithm 2

- 1: Calculate the optimal group sequence according to Lemma 4.7.
 - 2: For the above obtained group sequence, determine the optimal job sequences within all the m groups according to Lemma 4.6. Let the resulting schedule be π .
 - 3: Use Equ. (7) to calculate the optimal resource allocation $u^*(\pi)$.
 - 4: Calculate for all jobs their optimal due dates by Lemma 3.1.
-

Theorem 4.8 *Algorithm 2 optimally solves $1|GT, DIF, LIN, n_i = \bar{n}, \psi_{ij} = \bar{\psi} | \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$ in $O(n^3)$ time.*

Proof The correctness of Algorithm 2 is established by Lemmas 3.1, 3.2, and 4.5–4.7. Step 1 requires $O(n^3)$ time by Lemma 4.7, Step 2 requires $O(n^3)$ time by Lemma 4.6, and calculating the optimal resource allocation and due date assignment vector in Steps 3 and 4 both require $O(n)$ time. Therefore, Algorithm 2 runs in $O(n^3)$ time. \square

5 Numerical examples

In this section, we illustrate applying Algorithms 1 and 2 to problems

$$1|GT, DIF, CONV, n_i = \bar{n}, \psi_{ij} = \bar{\psi} | \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$$

and

$$1|GT, DIF, LIN, n_i = \bar{n}, \psi_{ij} = \bar{\psi} | \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij}),$$

Table 2 Parameters for the numerical examples

Group	J_{ij}	α_{ij}	β_{ij}	γ_{ij}	v_{ij}	w_{ij}	a_{ij}	$\overline{p_{ij}}$	$\overline{u_{ij}}$	s_i
G_1	J_{11}	2	4	5	2	5	0.2	8	10	4
	J_{12}	2	5	2	2	3	0.1	5	20	
	J_{13}	2	2	4	4	2	0.4	6	10	
G_2	J_{21}	5	3	2	4	2	0.5	12	10	3
	J_{22}	2	4	6	5	3	0.2	8	10	
	J_{23}	2	2	3	3	4	0.3	7	20	
G_3	J_{31}	2	3	4	5	3	0.5	5	8	5
	J_{32}	2	5	6	2	4	0.4	8	15	
	J_{33}	2	6	2	3	3	0.15	8	20	
G_4	J_{41}	4	5	2	3	2	0.3	6	15	6
	J_{42}	2	4	5	3	5	0.1	5	30	
	J_{43}	2	2	3	6	2	0.6	10	5	

respectively, with parameters $n = 12, m = 4, n_i = \bar{n} = 3, r = 1$, and those presented in Table 2.

From Table 2, we can see that $\psi_{ij} = \min(\alpha_{ij}, \gamma_{ij}) = \bar{\psi} = 2$, and $\Psi_i = \bar{n}\bar{\psi} = \bar{\Psi} = 6$, for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$.

5.1 Convex resource consumption function

We use Algorithm 1 to solve the above given instance of problem 1|GT, DIF, CONV, $n_i = \bar{n}, \psi_{ij} = \bar{\psi} | \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$.

Step 1. Compute θ_{ij} 's by $\theta_{ij} = (w_{ij}v_{ij})^{r/(r+1)}$.

	i, j	1	2	3
$\theta_{ij} =$	1	$\sqrt{10}$	$\sqrt{6}$	$\sqrt{8}$
	2	$\sqrt{8}$	$\sqrt{15}$	$\sqrt{12}$
	3	$\sqrt{15}$	$\sqrt{8}$	$\sqrt{9}$
	4	$\sqrt{6}$	$\sqrt{15}$	$\sqrt{12}$

By Lemma 4.3, the optimal job sequences in the four groups can be determined as follows.

$$G_1 : \{J_{12}, J_{13}, J_{11}\}; \quad G_2 : \{J_{21}, J_{23}, J_{22}\}; \quad G_3 : \{J_{32}, J_{33}, J_{31}\}; \quad G_4 : \{J_{41}, J_{43}, J_{42}\}.$$

Step 2. Compute $t_{i\ell}$'s by Eq. (6).

	i, ℓ	1	2	3	4
$t_{i\ell} =$	1	174.817	139.076	100.748	56.258
	2	166.850	134.696	99.414	56.667
	3	210.496	166.983	120.478	66.811
	4	235.137	185.480	132.788	72.811

Compute the optimal group sequence by solving problem $P1$, the result is $\{G_2, G_1, G_3, G_4\}$. Hence, the resulting schedule is $\pi = \{J_{21}, J_{23}, J_{22}, J_{12}, J_{13}, J_{11}, J_{32}, J_{33}, J_{31}, J_{41}, J_{43}, J_{42}\}$.

Step 3. Use Eq. (4) to compute the optimal resource allocation $u^*(\pi)$.

	i, j	1	2	3
$u_{[i][j]}^*(\pi) =$	1	3.464	5.416	3.464
	2	5.196	2.828	5.916
	3	4.899	3.162	2.191
	4	2	1.155	1.826

Step 4. By $p_{ij}(u_{ij}) = (\frac{w_{ij}}{u_{ij}})^r$, the actual processing times of all jobs can be calculated as follows.

	i, j	1	2	3
$p_{[i][j]}(u^*) =$	1	0.577	0.739	0.866
	2	0.577	0.707	0.845
	3	0.816	0.949	1.369
	4	1	1.732	2.739

Based on the above matrix of actual processing times of all jobs and by Lemma 3.1, assign to jobs the following optimal due dates.

	i, j	1	2	3
$d_{[i][j]}^*(\pi) =$	1	0	4.316	5.182
	2	9.759	10.466	11.311
	3	17.127	18.076	19.445
	4	0	28.177	30.916

5.2 Linear resource consumption function

We use Algorithm 2 to solve the above given instance of problem 1|GT, DIF, LIN, $n_i = \bar{n}$, $\psi_{ij} = \bar{\psi}$ | $\sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_{ij}d_{ij} + \beta_{ij}E_{ij} + \gamma_{ij}T_{ij} + v_{ij}u_{ij})$.

Step 1. Compute $t'_{i\ell}$'s by Lemma 4.7.

i, ℓ	1	2	3	4	
$t'_{i\ell} =$	1	430	332	226	94
	2	510	396	282	116
	3	418	322	226	108
	4	505	388	265	110

Compute the optimal group sequence by solving problem P3, the result is $\{G_3, G_1, G_4, G_2\}$.

Step 2. For the above obtained group sequence, for each position i , calculate the values of $w_{[i]rs}$ in Eq. (11) for $r, s = 1, 2, 3$. We illustrate the results for $i = 1$ as follows.

r, s	1	2	3	
$w_{[1]rs} =$	1	14	12	10
	2	38	34	30
	3	20	10	0

Compute the optimal job sequences in four groups by solving problem P2 as in Lemma 4.6. The optimal schedule π is determined as $\pi = \{J_{31}, J_{32}, J_{33}, J_{13}, J_{12}, J_{11}, J_{41}, J_{42}, J_{43}, J_{23}, J_{22}, J_{21}\}$.

Step 3. Use Eq. (7) to compute the optimal resource allocation $u^*(\pi)$.

i, j	1	2	3	
$u^*_{[i][j]}(\pi) =$	1	8	15	20
	2	10	0	10
	3	15	0	0
	4	0	0	0

Step 4. By $p_{ij}(u_{ij}) = \bar{p}_{ij} - a_{ij}u_{ij}$, the actual processing times of all jobs can be calculated as follows.

Based on the above matrix of actual processing times of all jobs and by Lemma 3.1, assign to jobs the following optimal due dates.

	i, j	1	2	3
$P_{[i][j]}(u^*)=$	1	1	2	5
	2	2	5	6
	3	1.5	5	10
	4	7	8	12
<hr/>				
	i, j	1	2	3
$d_{[i][j]}^*(\pi)=$	1	6	8	13
	2	19	24	30
	3	0	42.5	52.5
	4	62.5	70.5	0

6 Conclusion

A single machine group scheduling problem with due date assignment and resource allocation is investigated. Jobs in the same group are allowed to have different due dates. The job processing times are resource dependent, and both convex and bounded linear resource consumption functions are considered. The aim is minimizing an aggregate cost which takes into account earliness, tardiness, due date assignment and resource allocation costs, by finding a group schedule, due date assignment and resource allocation for all jobs. For both resource consumption functions, we present properties of the optimal solutions, and for the special case where the size of every group is the same and the minimum of the due date assignment cost and the tardiness cost for each job is identical, we present an algorithm to optimally solve the problem in $O(n^3)$ time.

As the actual processing time of jobs may be shortened due to learning over time, one possible future research direction is to take into account the learning effect. Also, it is worthwhile to consider the group scheduling problem with resource allocation in the flow shop setting.

Acknowledgements The authors are grateful to the two anonymous referees for their helpful comments and suggestions.

Funding This work was partially supported by the Major Program of National Natural Science Foundation of China under Grant Nos. 72192830 and 72192834, and the Key Project of National Natural Science Foundation of China under Grant No. 71732006.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

References

- Bajwa N, Melouk S, Bryant P (2019) A hybrid heuristic approach to minimize number of tardy jobs in group technology systems. *Int Trans Oper Res* 26(5):1847–1867
- Chen Y, Cheng Y (2021) Optimal due date assignment without restriction and convex resource allocation in group technology scheduling. In: Du DZ, Du D, Wu C, Xu D (eds) *Combinatorial optimization and applications, COCOA 2021. Lecture notes in computer science*, vol 13135, Springer, Cham
- Graham RL, Lawler EL, Lenstra JK et al (1979) Optimization and approximation in deterministic sequencing and scheduling: a survey. *Ann Discrete Math* 5:287–326
- Hardy G, Littlewood JE, Polya G (1967) *Inequalities*. Cambridge University Press, London
- Ji M, Zhang X, Tang X et al (2016) Group scheduling with group-dependent multiple due windows assignment. *Int J Prod Res* 54(4):1244–1256
- Keshavarz T, Savelsbergh M, Salmasi N (2015) A branch-and-bound algorithm for the single machine sequence-dependent group scheduling problem with earliness and tardiness penalties. *Appl Math Model* 39(20):6410–6424
- Li W, Zhao C (2015) Single machine scheduling problem with multiple due windows assignment in a group technology. *J Appl Math Comput* 44:477–494
- Li S, Ng CT, Yuan J (2011) Group scheduling and due date assignment on a single machine. *Int J Prod Econ* 130(2):230–235
- Liu L, Xu Y, Yin N et al (2014) Single machine group scheduling problem with deteriorating jobs and release dates. *Appl Mech Mater* 513–517:2145–2148
- Lv D, Luo S, Xue J et al (2021) A note on single machine common flow allowance group scheduling with learning effect and resource allocation. *Comput Ind Eng* 151:106941
- Panwalkar SS, Smith ML, Seidmann A (1982) Common due date assignment to minimize total penalty for the one machine scheduling problem. *Oper Res* 30(2):391–399
- Qin H, Zhang Z, Bai D (2016) Permutation flowshop group scheduling with position-based learning effect. *Comput Ind Eng* 92:1–15
- Seidmann A, Panwalkar SS, Smith ML (1981) Optimal assignment of due-dates for a single processor scheduling problem. *Int J Prod Res* 19(4):393–399
- Shabtay D (2016) Optimal restricted due date assignment in scheduling. *Eur J Oper Res* 252(1):79–89
- Shabtay D, Itskovich Y, Yedidsion L et al (2010) Optimal due date assignment and resource allocation in a group technology scheduling environment. *Comput Oper Res* 37(12):2218–2228
- Wang L, Liu M, Wang J et al (2021) Optimization for due-date assignment single-machine scheduling under group technology. *Complexity* 6656261:1–9
- Yin N, Kang L, Wang X (2014) Single machine group scheduling with processing times dependent on position, starting time and allotted resource. *Appl Math Model* 38(19–20):4602–4613

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