Journal of Combinatorial Optimization (2023) 45:30
https://doi.org/10.1007/s10878-022-00962-y

On the complexity of finding well-balanced orientations with upper bounds on the out-degrees

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Accepted: 30 November 2022 / Published online: 12 December 2022

Abstract
We show that the problem of deciding whether a given graph G has a well-balanced orientation $\vec{G}$ such that $d^+_{\vec{G}}(v) \leq \ell(v)$ for all $v \in V(G)$ for a given function $\ell : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ is NP-complete. We also prove a similar result for best-balanced orientations. This improves a result of Bernáth, Iwata, Király, Király and Szigeti and answers a question of Frank.

Keywords Graph orientation · Well-balanced · Complexity

1 Introduction

This article contains a negative result concerning the possibility of deciding whether a given graph has a well-balanced or best-balanced orientation with a certain extra property. Any undefined notions can be found in Sect. 2.

During the history of graph orientations, the problem of characterizing graphs admitting orientations with certain connectivity properties has played a decisive role. The first important theorem due to Robbins (1939) states that a graph has a strongly connected orientation if and only if it is 2-edge-connected. Nash-Williams (1960) proved several theorems generalizing the result of Robbins. The first one is the following natural generalization of the result of Robbins to higher global arc-connectivity.

Theorem 1 *Let G be a graph and k a positive integer. Then G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

While Theorem 1 resolves the problem of finding graph orientations of high global arc-connectivity, Nash-Williams also considered orientations satisfying local

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arc-connectivity conditions. We say that an orientation $\vec{G}$ of a graph G is well-balanced if $\lambda_{\vec{G}}(u,v) \geq \lfloor \frac{\lambda_G(u,v)}{2} \rfloor$ for all $(u,v) \in V(G) \times V(G)$. If additionally $d^+_{\vec{G}}(v) \in \{\lfloor \frac{d_G(v)}{2} \rfloor, \lceil \frac{d_G(v)}{2} \rceil\}$ holds for all $v \in V(G)$, then $\vec{G}$ is called best-balanced. Nash-Williams (1960) proved the following result.

Theorem 2 *Every graph has a best-balanced orientation.*

Observe that Theorem 2 implies Theorem 1. In the last decades, numerous attempts have been made to develop theory surrounding Theorems 1 and 2. These attempts turned out to be much more successful when concerning Theorem 1 than when concerning Theorem 2. For example, while a relatively simple proof of Theorem 1 relying on a splitting off theorem of Lovász has been found by Frank (2011), no simple proof of Theorem 2 is known. Even though since the original, very complicated proof of Nash-Williams new proofs have been found by Mader (1978) and Frank (1993), all of them are pretty involved.

Another branch of research in the theory surrounding Theorems 1 and 2 consists in characterizing graphs which admit orientations satisfying some extra properties in addition to the connectivity conditions. These problems turn out to be much more tractable when trying to generalize Theorem 1 than when trying to generalize Theorem 2.

For generalizing Theorem 1, polymatroid theory has proven to be a valuable tool. It allowed Frank (2011) to solve the problem of deciding whether a mixed graph has a k-arc-connected orientation for some given positive integer k and to solve the more general problem of finding a minimum cost k-arc-connected orientation of a given graph where a cost is given for both possible orientations of each edge.

Bernáth et al. (2008) attempted to obtain similar generalizations for Theorem 2 which yielded several negative results, see also (Bernáth 2006). For example, the problems of finding well-balanced and best-balanced orientations minimizing a given weight function were proven to be NP-complete in Bernáth et al. (2008). The problem of deciding whether a mixed graph has a best-balanced orientation has also been proven to be NP-complete in Bernáth et al. (2008). A proof that the problem of deciding whether a mixed graph has a well-balanced orientation is NP-complete has been found by Bernáth and Joret (2008).

Another extra property which can be imposed on the orientation is degree constraints. Here a generalization of Theorem 1 has been obtained by Frank (1980) using comparatively elementary methods. As its proof is constructive, he obtained the following result.

Theorem 3 *There is a polynomial-time algorithm which, given a graph G, a positive integer k and two functions $\ell_1, \ell_2 : V(G) \to \mathbb{Z}_{\geq 0}$, decides whether there is a k-arc-connected orientation $\vec{G}$ of G such that $\ell_1(v) \leq d^+_{\vec{G}}(v) \leq \ell_2(v)$ for all $v \in V(G)$.*

Yet again, a similar generalization of Theorem 2 was proven to be out of reach in Bernáth et al. (2008).

Theorem 4 *The problem of deciding whether, given a graph G and two functions $\ell_1, \ell_2 : V(G) \to \mathbb{Z}_{\geq 0}$, there is a well-balanced orientation $\vec{G}$ of G such that $\ell_1(v) \leq d^+_{\vec{G}}(v) \leq \ell_2(v)$ for all $v \in V(G)$, is NP-complete.*

A similar result for best-balanced orientations is also proven in Bernáth et al. (2008).

In this article, we deal with the question whether a version of the above problem with milder restrictions on the vertex degrees is better tractable. We are interested in the case when instead of imposing an upper and a lower bound on the out-degree of every vertex only an upper bound is imposed.

More concretely, we consider the following two problems:

Upper-bounded well-balanced orientation (UBWBO):
Input: A graph G, a function $\ell : V(G) \to \mathbb{Z}_{\geq 0}$.
Question: Is there a well-balanced orientation $\vec{G}$ of G such that $d^+_{\vec{G}}(v) \leq \ell(v)$ for all $v \in V(G)$?

Upper-bounded best-balanced orientation (UBBBO):
Input: A graph G, a function $\ell : V(G) \to \mathbb{Z}_{\geq 0}$.
Question: Is there a best-balanced orientation $\vec{G}$ of G such that $d^+_{\vec{G}}(v) \leq \ell(v)$ for all $v \in V(G)$?

Observe that any orientation obtained from a well-balanced (best-balanced) orientation by reversing the orientation of all arcs is again well-balanced (best-balanced). Hence imposing lower bounds instead of upper bounds on the out-degrees would lead to equivalent problems. Similarly, the bounds could be imposed on the in-degrees instead of the out-degrees.

The question of the complexity of UBBBO can be found in various sources. It is mentioned by Frank (2011), by Bernáth et al. (2008) and there is an online posting on it in the open problem collection of the Egerváry Research group. The contribution of this article is to prove that even these problems involving milder restrictions remain hard. We prove the following two results:

Theorem 5 *UBWBO is NP-complete.*

Theorem 6 *UBBBO is NP-complete.*

Observe that Theorem 5 implies Theorem 4. Theorems 5 and 6 can be considered yet another indication of the isolated position that Theorem 2 has in the theory of graph orientations.

After a collection of formal definitions and preliminary results in Sect. 2, we prove Theorems 5 and 6 in Sect. 3 using a reduction from Cubic Vertex Cover. While our reduction is inspired by the one used in Bernáth et al. (2008) to prove Theorem 4, it is more involved.

2 Preliminaries

This section is dedicated to providing the background for the proof of the main results in Sect. 3. We first define all important terms in Sect. 2.1 and then give some preliminary results in Sect. 2.2.

2.1 Definitions

We first give some basic notions of graph theory. A *mixed graph* F consists of a vertex set $V(F)$, an edge set $E(F)$, and an arc set $A(F)$. We also say that F *contains* $V(F)$, $E(F)$, and $A(F)$. An edge $e = uv \in E(F)$ is a set containing the vertices u and v. We say that e *links* u and v and e is *incident* to u and v. More generally, we say that e *links* two disjoint sets $X, Y \subseteq V(F)$ if $u \in X$ and $v \in Y$. If e links X and $V(F)-X$, we say that e *enters* X. An arc $a = uv \in A(F)$ is an ordered tuple of the vertices $u, v \in V(F)$ where u is called the *tail* of a and v is called the *head* of a. For some $X \subseteq V(F)$ with $u \in X$ and $v \in V(F) - X$, we say that e *enters* $V(F) - X$ and *leaves* X. For some $e = uv \in E(F) \cup A(F)$, we say that u and v are the *endvertices* of e. A *mixed subgraph* F' of F is a mixed graph F' with $V(F') \subseteq V(F)$, $E(F') \subseteq E(F)$, and $A(F') \subseteq A(F)$. For some $X \subseteq V(F)$, we let $F[X]$ denote the mixed subgraph of F whose vertex set is X and that contains all the edges in $E(F)$ and all the arcs in $A(F)$ whose both endvertices are in X.

A mixed graph G without arcs is called a *graph*. For a graph G and some $X \subseteq V(G)$, we let $d_G(X)$ denote the number of edges in $E(G)$ that have exactly one endvertex in X and we let $i_G(X)$ denote the number of edges in $E(G)$ that have both endvertices in X. For a single vertex $v \in V(G)$, we abbreviate $d_G(\{v\})$ to $d_G(v)$ and call this number the *degree* of v in G. If $d_G(v) = 3$ for all $v \in V(G)$, we say that G is *cubic*. For two vertices $u, v \in V(G)$, we use $\lambda_G(u, v)$ for $\min_{u \in X \subseteq V(G)-v} d_G(X)$. Observe that $\lambda_G(u, v) = \lambda_G(v, u)$. For some positive integer k, we say that G is *k-edge-connected* if $\lambda_G(u, v) \geq k$ for all $u, v \in V(G)$. A 1-edge-connected graph which contains two vertices u, v of degree 1 and in which all other vertices are of degree 2 is called a *uv-path*. We also say that u and v are the *endvertices* of the path. Two graphs whose edge sets are disjoint are called *edge-disjoint*. For two paths T_1, T_2 with $V(T_1) \cap V(T_2) = x$ for a vertex x that is an endvertex of both T_1 and T_2, we denote by T_1T_2 the path with $V(T_1T_2) = V(T_1) \cup V(T_2)$ and $E(T_1T_2) = E(T_1) \cup E(T_2)$.

A mixed graph D without edges is called a *digraph*. For a digraph D and some $X \subseteq V(D)$, we let $d_D^+(X)$ denote the number of arcs whose tail is in X and whose head is in $V(D) - X$. We use $d_D^-(X)$ for $d_D^+(V(D) - X)$. For a single vertex $v \in V(D)$, we abbreviate $d_D^+(\{v\})(d_D^-(\{v\}))$ to $d_D^+(v)(d_D^-(v))$ and call this number the *out-degree (in-degree)* of v in D. If $d_D^+(v) = d_D^-(v)$ for all $v \in V(D)$, we say that D is *eulerian*. Given a function $\ell : V(D) \to \mathbb{Z}_{\geq 0}$, we say that D is *ℓ-bounded* if $d_D^+(v) \leq \ell(v)$ for all $v \in V(D)$. For two vertices $u, v \in V(D)$, we use $\lambda_D(u, v)$ for $\min_{v \in X \subseteq V(D)-u} d_D^-(X)$. For some positive integer k, we say that D is *k-arc-connected* if $\lambda_D(u, v) \geq k$ for all $(u, v) \in V(D) \times V(D)$. We abbreviate 1-arc-connected to *strongly connected.* The operation of exchanging the head and the tail of an arc is called *reversing* the arc. Two digraphs whose arc sets are disjoint are called *arc-disjoint*.

A mixed graph F' is called a *partial orientation* of another mixed graph F if F' can be obtained from F by replacing some of the edges in $E(F)$ by an arc with the same two endvertices. This operation is called *orienting* the edge. If F' is a digraph, then F' is called an *orientation* of F. The unique graph G such that F is an orientation of G is called the *underlying graph* of F. A strongly connected orientation of a graph all of whose vertices are of degree 2 is called a *circuit*. An orientation T of a uv-

path with $\lambda_T(u, v) = 1$ is called a *directed uv-path*. For a directed uv-path T_1 and a directed vw-path T_2 for some vertices u, v and w, we denote by T_1T_2 the uw-path with $V(T_1T_2) = V(T_1) \cup V(T_2)$ and $A(T_1T_2) = A(T_1) \cup A(T_2)$. We say that an orientation $\vec{G}$ of a graph G is *well-balanced* if $\lambda_{\vec{G}}(u, v) \geq \lfloor \frac{\lambda_G(u,v)}{2} \rfloor$ for all $(u, v) \in V(G) \times V(G)$. If additionally $d^+_{\vec{G}}(v) \in \{\lfloor \frac{d_G(v)}{2} \rfloor, \lceil \frac{d_G(v)}{2} \rceil\}$ holds for all $v \in V(G)$, then $\vec{G}$ is called *best-balanced*. We also say that a digraph is *well-balanced (best-balanced)* if it is a well-balanced (best-balanced) orientation of its underlying graph.

For basic notions of complexity theory, see Garey and Johnson (1979). Given a graph H, a *vertex cover* of H is a subset U of $V(H)$ such that every $e \in E(H)$ is incident to at least one vertex in U. We consider the following algorithmic problem:

Cubic Vertex Cover (CVC):

Input: A cubic graph H, a positive integer k. Question: Is there a vertex cover of H of size at most k?

2.2 Preliminary results

For proving the correctness of our reduction, we need a few preliminaries.

The following classic results are due to Menger (1927) and fundamental to graph connectivity.

Theorem 7 *Let G be a graph and $s_1, s_2 \in V(G)$. Then the maximum number of pairwise edge-disjoint s_1s_2-paths in G is $\lambda_G(s_1, s_2)$.*

The second result is the directed analogue of Theorem 7.

Theorem 8 *Let D be a digraph and $s_1, s_2 \in V(D)$. Then the maximum number of pairwise arc-disjoint directed s_1s_2-paths in D is $\lambda_D(s_1, s_2)$.*

The next result is helpful when proving that a given orientation is well-balanced.

Proposition 1 *Let G be a graph and $a \in V(G)$. Let $\vec{G}$ be an orientation of G such that $\lambda_{\vec{G}}(a, s) \geq \lfloor \frac{d_G(s)}{2} \rfloor$ and $\lambda_{\vec{G}}(s, a) \geq \lfloor \frac{d_G(s)}{2} \rfloor$ hold for all $s \in V(G) - a$. Then $\vec{G}$ is well-balanced.*

Proof Let $s_1, s_2 \in V(G)$ and $R \subseteq V(G) - s_1$ with $s_2 \in R$. If $a \in R$, we have $d^-_{\vec{G}}(R) \geq \lambda_{\vec{G}}(s_1, a) \geq \lfloor \frac{d_G(s_1)}{2} \rfloor \geq \lfloor \frac{\lambda_G(s_1,s_2)}{2} \rfloor$. If $a \in V(G) - R$, we have $d^-_{\vec{G}}(R) \geq \lambda_{\vec{G}}(a, s_2) \geq \lfloor \frac{d_G(s_2)}{2} \rfloor \geq \lfloor \frac{\lambda_G(s_1,s_2)}{2} \rfloor$. In either case, we obtain $d^-_{\vec{G}}(R) \geq \lfloor \frac{\lambda_G(s_1,s_2)}{2} \rfloor$, so $\lambda_{\vec{G}}(s_1, s_2) \geq \lfloor \frac{\lambda_G(s_1,s_2)}{2} \rfloor$. Hence $\vec{G}$ is well-balanced. □

The next simple result allows to modify orientations maintaining important properties.

Proposition 2 *Let G be a graph, $\ell : V(G) \to \mathbb{Z}_{\geq 0}$ a function, $\vec{G}_0$ an ℓ-bounded, well-balanced orientation of G, D an eulerian directed subgraph of $\vec{G}_0$ and $\vec{G}_1$ the orientation of G which is obtained by reversing all the arcs of D. Then $\vec{G}_1$ is ℓ-bounded and well-balanced.*

Proof Since D is eulerian, we have $d^+_{\vec{G}_1}(s) = d^+_{\vec{G}_0}(s)$ for all $s \in V(G)$. Hence, as $\vec{G}_0$ is ℓ-bounded, so is $\vec{G}_1$. Similarly, we have $d^-_{\vec{G}_1}(s) = d^-_{\vec{G}_0}(s)$ for all $s \in V(G)$. We hence have $d^-_{\vec{G}_1}(R) = \sum_{s\in R} d^-_{\vec{G}_1}(s) - i_G(R) = \sum_{s\in R} d^-_{\vec{G}_0}(s) - i_G(R) = d^-_{\vec{G}_0}(R)$ for all $R \subseteq V$. Hence $\lambda_{\vec{G}_1}(s_1, s_2) = \min_{s_2\in R\subseteq V-s_1} d^-_{\vec{G}_1}(R) = \min_{s_2\in R\subseteq V-s_1} d^-_{\vec{G}_0}(R) = \lambda_{\vec{G}_0}(s_1, s_2)$ for all $(s_1, s_2) \in V(G) \times V(G)$. Thus, as $\vec{G}_0$ is well-balanced, so is $\vec{G}_1$. □

Finally, we need the following result to justify the usefulness of our reduction. It can be found in Garey and Johnson (1979).

Theorem 9 *Cubic Vertex Cover is NP-complete.*

3 The reduction

In this section, we give the reduction we need to prove Theorems 5 and 6. We first give a reduction for Theorem 5 and then show how to adapt it to prove Theorem 6. In Sect. 3.1, we describe the instance (G, ℓ) of UBWBO we create from a given instance (H, k) of CVC. In the remaining part of the paper (H, k) and (G, ℓ) are fixed. In Sect. 3.2, we describe a particular kind of orientations, called convenient orientations that play a crucial role in the proof of the reduction. In Sect. 3.3, we give the first direction of the reduction showing how to obtain an ℓ-bounded, well-balanced orientation of G from a vertex cover of H. The other direction is divided into two parts. First, we show in Sect. 3.4 how an ℓ-bounded, well-balanced orientation of G can be turned into one that additionally has the property of being convenient. After, in Sect. 3.5, we show how an orientation with this extra property yields a vertex cover of H. In Sect. 3.6, we show how to adapt our construction for the proof of Theorem 6. Finally, in Sect. 3.7, we conclude our proof.

3.1 The construction

We here show how to create an instance of UBWBO from an instance of CVC. Let (H, k) be an instance of CVC. Since H is cubic, we have $|V(H)| = 2n$ and $|E(H)| = 3n$ for some integer $n \geq 2$.

We first describe, for every $v \in V(H)$, a vertex gadget G^v that contains 6 vertices: $p^v_0, p^v_1, p^v_2, q^v_0, q^v_1, q^v_2$ and 5 edges: $p^v_0p^v_1, p^v_1p^v_2, q^v_0q^v_1, q^v_1q^v_2, p^v_0q^v_0$. We next describe, for every $e \in E(H)$, an edge gadget G^e that contains 6 vertices $x^e, y^e, z^e_1, z^e_2, z^e_3, z^e_4$ and 5 edges: $x^ey^e, x^ez^e_1, y^ez^e_2, y^ez^e_3, y^ez^e_4$. An illustration of these gadgets can be found in Fig. 1.

We are now ready to describe G. For every $v \in V(H)$, we let G contain a vertex gadget G^v and for every $e \in E(H)$, we let G contain an edge gadget G^e. Let $P = \bigcup_{v\in V(H)} V(G^v)$, $X = \bigcup_{e\in E(H)}\{x^e, y^e\}$, and $Z = \bigcup_{e\in E(H)}\{z^e_1, z^e_2, z^e_3, z^e_4\}$. We let $V(G)$ contain two more vertices a and b. We now finish the description of G by linking these components by some additional edges. For every $z \in Z$, we let $E(G)$ contain an edge az and an edge bz. Further, for every $v \in V(H)$, let e_1, e_2, e_3 be an arbitrary ordering of the edges in $E(H)$ which are incident to v in H. We add

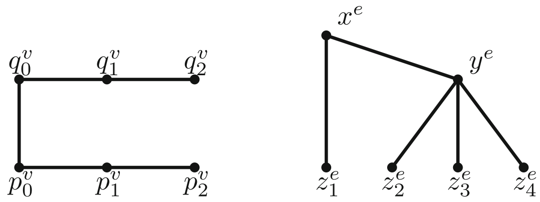

Fig. 1 A vertex gadget for a vertex v and an edge gadget for an edge e

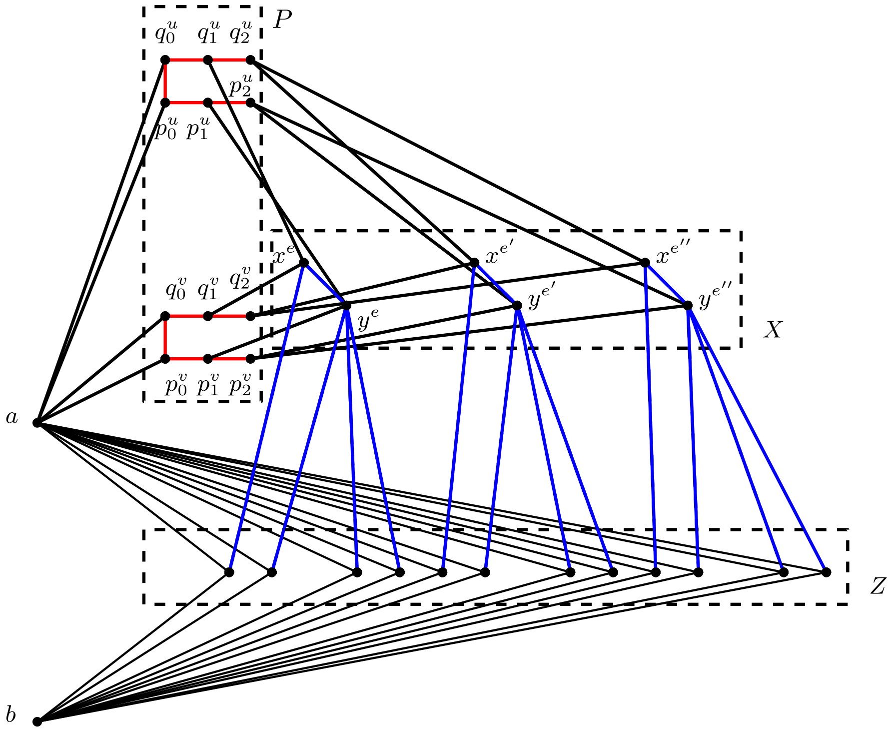

Fig. 2 An example for the graph G created from a graph H where $V(H)$ contains two vertices u and v and $E(H)$ contains three parallel edges e, e', and e'' linking u and v. All the edges belonging to a vertex gadget are marked in red while all the edges belonging to an edge gadget are marked in blue. The names of the vertices in Z have been omitted due to space restrictions. They are from left to right: $z_1^e, z_2^e, z_3^e, z_4^e, z_1^{e'}, z_2^{e'}, z_3^{e'}, z_4^{e'}, z_1^{e''}, z_2^{e''}, z_3^{e''}, z_4^{e''}$. See also Fig. 1 (Color figure online)

the edges $ap_0^v, aq_0^v, p_1^v y^{e_1}, p_2^v y^{e_2}, p_2^v y^{e_3}, q_1^v x^{e_1}, q_2^v x^{e_2}$, and $q_2^v x^{e_3}$. This finishes the construction of G.

Observe that $d_G(a) = 4|E(H)| + 2|V(H)| = 16n, d_G(b) = 4|E(H)| = 12n, d_G(s) = 3$ for all $s \in P \cup Z$, and $d_G(x^e) = 4$ and $d_G(y^e) = 6$ for all $e \in E(H)$. An illustration can be found in Fig. 2.

We now define ℓ. We set $\ell(a) = 8n + k$ and $\ell(z) = 1$ for all $z \in Z$. For all $s \in V(G) - (Z \cup a)$, we set the trivial bound $\ell(s) = d_G(s)$.

We now give an important result on the connectivity properties of G.

Proposition 3 $\lambda_G(s,a) = d_G(s)$ *for all* $s \in V(G) - a$.

Proof By definition, $\lambda_G(s,a) \leq d_G(s)$ for all $s \in V(G) - a$. First observe that for every $z \in Z$, G contains the ba-path bza. By Theorem 7, this yields $\lambda_G(b,a) \geq |Z| = 12n = d_G(b)$. Now consider some $e = uv \in E(H)$. By construction, there are some $i, j \in \{1,2\}$ such that G contains the edges $q_i^u x^e$, $p_i^u y^e$, $q_j^v x^e$ and $p_j^v y^e$. Due to the pairwise edge-disjoint $x^e a$-paths $T_1 = x^e z_1^e a$, $T_2 = x^e y^e z_2^e a$, $T_3 = x^e q_i^u \dots q_0^u a$ and $T_4 = x^e q_j^v \dots q_0^v a$ and Theorem 7, we obtain that $\lambda_G(x^e, a) \geq 4 = d_G(x^e)$. Due to the pairwise edge-disjoint $y^e a$-paths $T_1 = y^e x^e z_1^e a$, $T_2 = y^e z_2^e a$, $T_3 = y^e z_3^e a$, $T_4 = y^e z_4^e a$, $T_5 = y^e p_i^u \dots p_0^u a$ and $T_6 = y^e p_j^v \dots p_0^v a$ and Theorem 7, we obtain that $\lambda_G(y^e, a) \geq 6 = d_G(y^e)$. Finally, suppose for the sake of a contradiction that for some $t \in P \cup Z$, $\lambda_G(t,a) < d_G(t) = 3$. Let $a \in R \subseteq V(G) - t$ with $d_G(R) = \lambda_G(t,a)$. As $\lambda_G(s,a) \geq 3$ for all $s \in X \cup b$, we obtain $X \cup b \subseteq R$. Hence, since every $z \in Z$ is adjacent to three vertices in R, we obtain $Z \subseteq R$, so $t \in P$. As t is adjacent to three distinct vertices in G and every vertex in P is linked to R, we obtain $d_G(R) \geq 3$, a contradiction. □

By Propositions 1 and 3, we have the following characterization of well-balanced orientations of G.

Corollary 1 *Let* $\vec{G}$ *be an orientation of* G.

(a) $\vec{G}$ *is well-balanced if and only if* $\lambda_{\vec{G}}(a,s) \geq \lfloor \frac{d_G(s)}{2} \rfloor$ *and* $\lambda_{\vec{G}}(s,a) \geq \lfloor \frac{d_G(s)}{2} \rfloor$ *for all* $s \in V(G) - a$.
(b) If $\vec{G}$ *is well-balanced, then* $d^+_{\vec{G}}(s) = d^-_{\vec{G}}(s) = \frac{d_G(s)}{2} = \lambda_{\vec{G}}(a,s) = \lambda_{\vec{G}}(s,a)$ *for all* $s \in X \cup b$.

Proof (a) The sufficiency is Proposition 1. The necessity is an immediate consequence of the definition of well-balanced orientations and Proposition 3.
(b) Suppose that $\vec{G}$ is well-balanced and let $s \in X \cup b$. As $d_G(s)$ is even and by (a), we have $d_G(s) = 2\lfloor \frac{d_G(s)}{2} \rfloor \leq \lambda_{\vec{G}}(s,a) + \lambda_{\vec{G}}(a,s) \leq d^+_{\vec{G}}(s) + d^-_{\vec{G}}(s) = d_G(s)$, hence equality holds throughout.

□

3.2 Convenient orientations

In order to prove that the reduction works indeed, we wish to consider a certain restricted class of orientations. We now define a mixed graph F which is obtained as a partial orientation of G.

First for every $e \in E(H)$ and $i \in \{1,2\}$, let the edge az_i^e be oriented from a to z_i^e and the edge bz_i^e be oriented from z_i^e to b. For every $e \in E(H)$ and $i \in \{3,4\}$, let the edge az_i^e be oriented from z_i^e to a and the edge bz_i^e be oriented from b to z_i^e. Let all the edges linking X and Z be oriented from X to Z. For every $e \in E(H)$, let the edge $x^e y^e$ be oriented from x^e to y^e. Next, let all the edges linking P and X be oriented from P to X. For every $v \in V(H)$ and $i \in \{0,1\}$, let the edge $p_i^v p_{i+1}^v$ be oriented from p_i^v to p_{i+1}^v and let the edge $q_i^v q_{i+1}^v$ be oriented from q_i^v to q_{i+1}^v. We denote the obtained partial orientation of G by F.

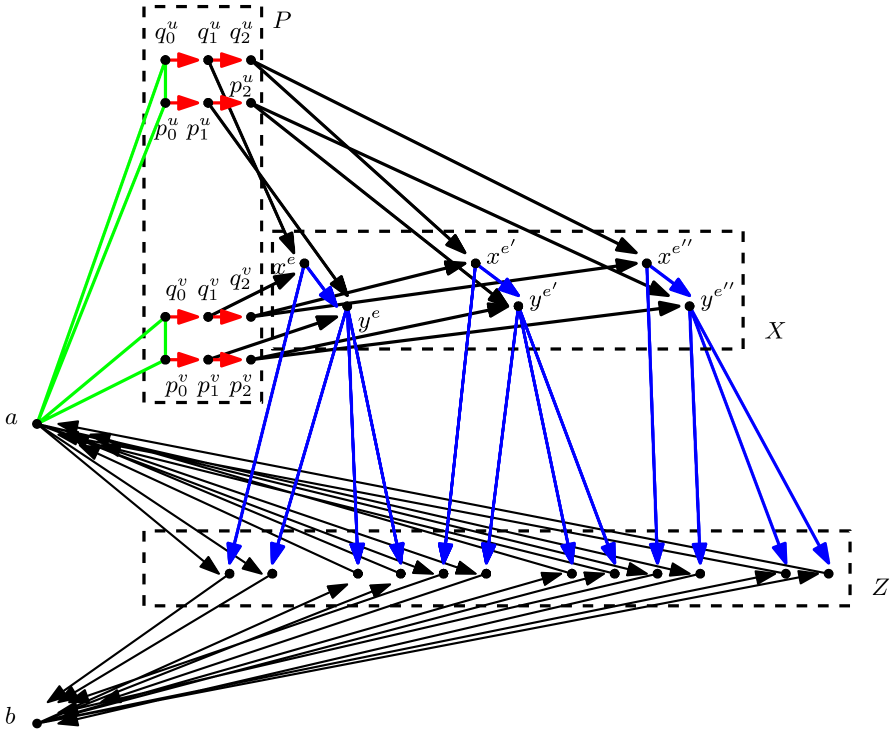

Fig. 3 An example for the mixed graph F created from the same graph H as considered in Fig. 2. The edges of F are marked in green (Color figure online)

Observe that the edge set of F consists of the 3 edges $aq_0^v, ap_0^v, p_0^v q_0^v$ for every $v \in V(H)$. An illustration of F can be found in Fig. 3.

We now say that an orientation $\vec{G}$ of G is *convenient* if $\vec{G}$ is also an orientation of F. The following lemma contains a characterization of convenient, well-balanced orientations of G which is a crucial ingredient for proving the correctness of our reduction.

Lemma 1 *A convenient orientation $\vec{G}$ of G is well-balanced if and only if for every $uv \in E(H)$,*

(i) either the edges from a to $\{p_0^u, q_0^u\}$ are oriented from a to $\{p_0^u, q_0^u\}$ or the edges from a to $\{p_0^v, q_0^v\}$ are oriented from a to $\{p_0^v, q_0^v\}$,

(ii) in $\vec{G}[\{a, p_0^u, q_0^u, p_0^v, q_0^v\}]$, there is a directed as-path for all $s \in \{p_0^u, q_0^u, p_0^v, q_0^v\}$.

Proof First suppose that $\vec{G}$ is well-balanced and let $e = uv \in E(H)$. Consider the set $R = V(G^u) \cup V(G^v) \cup \{x^e, y^e\}$. By $y^e \in R \subseteq V - a$ and Corollary 1(b), we have $d^-_{\vec{G}}(R) \geq \lambda_{\vec{G}}(a, y^e) = \frac{d_G(y^e)}{2} = 3$. As $\vec{G}$ is convenient, it follows that the only arcs entering R in $\vec{G}$ have the tail a. Since the set of edges linking a and R consists of the four edges ap_0^u, aq_0^u, ap_0^v, and aq_0^v, we get that either the arcs ap_0^u and aq_0^u exist in $\vec{G}$ or the arcs ap_0^v and aq_0^v exist in $\vec{G}$, that is, (i) holds.

Consider a vertex s in $\{p_0^u, q_0^u, p_0^v, q_0^v\}$. Since $\vec{G}$ is well-balanced, by Corollary 1(a), $d_G(s) = 3$ and Theorem 8, there exists a directed as-path in $\vec{G}$. Since $\vec{G}$ is convenient, this path also exists in $\vec{G}[\{a, p_0^u, q_0^u, p_0^v, q_0^v\}]$, that is, (ii) holds.

For the other direction, we will use that $\vec{G}$ is convenient several times without explicit mention. By Corollary 1(a), it suffices to prove that $\lambda_{\vec{G}}(a, s) \geq \lfloor \frac{d_G(s)}{2} \rfloor$ and $\lambda_{\vec{G}}(s, a) \geq \lfloor \frac{d_G(s)}{2} \rfloor$ hold for all $s \in V(G) - a$.

First we consider b. For every $e \in E(H)$, $i \in \{1, 2\}$ and $j \in \{3, 4\}$, $az_i^e b$ and $bz_j^e a$ is a directed ab-path and ba-path, respectively, in $\vec{G}$. We obtain, by Theorem 8, that $\min\{\lambda_{\vec{G}}(a, b), \lambda_{\vec{G}}(b, a)\} \geq 2|E(H)| = \lfloor \frac{d_G(b)}{2} \rfloor$.

We next consider the vertices in X. Let $e = uv \in E(H)$. By construction, there are indices $i, j \in \{1, 2\}$ such that $\vec{G}$ contains the arcs $p_i^u y^e$, $q_i^u x^e$, $p_j^v y^e$, and $q_j^v x^e$. By (ii), there is a directed as-path T_s in $\vec{G}[\{a, p_0^u, q_0^u, p_0^v, q_0^v\}]$ for every $s \in \{p_0^u, q_0^u, p_0^v, q_0^v\}$.

Since $T_1 = T_{q_0^u} q_0^u q_1^u \dots q_i^u x^e$ and $T_2 = T_{q_0^v} q_0^v q_1^v \dots q_i^v x^e$ are two arc-disjoint directed ax^e-paths, by Theorem 8, we obtain $\lambda_{\vec{G}}(a, x^e) \geq 2 = \lfloor \frac{d_G(x^e)}{2} \rfloor$. Since $T_1 = x^e z_1^e b z_3^e a$ and $T_2 = x^e y^e z_4^e a$ are two arc-disjoint directed $x^e a$-paths, by Theorem 8, we obtain $\lambda_{\vec{G}}(x^e, a) \geq 2 = \lfloor \frac{d_G(x^e)}{2} \rfloor$.

Next consider $T_1 = y^e z_2^e b z_3^{e'} a$, $T_2 = y^e z_3^e a$, and $T_3 = y^e z_4^e a$, where $e' \in E(H) - e$ is chosen arbitrarily. These are three arc-disjoint directed $y^e a$-paths, so by Theorem 8, we have $\lambda_{\vec{G}}(y^e, a) \geq 3 = \lfloor \frac{d_G(y^e)}{2} \rfloor$. For the next part, by (i) and symmetry, we may suppose that the arcs ap_0^u, aq_0^u exist in $\vec{G}$. Since $T_1 = ap_0^u \dots p_i^u y^e$, $T_2 = aq_0^u \dots q_i^u y^e$, and $T_3 = T_{p_0^v} p_0^v p_1^v \dots p_j^v y^e$ are three arc-disjoint directed ay^e-paths, by Theorem 8, we obtain $\lambda_{\vec{G}}(a, y^e) \geq 3 = \lfloor \frac{d_G(y^e)}{2} \rfloor$.

Let R be the vertex set of the strongly connected component of $\vec{G}$ containing a. By the above, we have $X \cup b \subseteq R$. Next, every $z \in Z$ is incident to an arc entering $R - z$ and an arc leaving $R - z$, so $Z \subseteq R$. Now let $v \in V(H)$. For every $p \in V(G^v)$, by (ii), a directed ap-path is contained in $\vec{G}[V(G^v) \cup a]$. Further, $\vec{G}$ contains a directed path from p to X. We hence obtain that $P \subseteq R$, so $\vec{G}$ is strongly connected. This yields $\lambda_{\vec{G}}(a, s) \geq 1 = \lfloor \frac{d_G(s)}{2} \rfloor$ and $\lambda_{\vec{G}}(s, a) \geq 1 = \lfloor \frac{d_G(s)}{2} \rfloor$ for all $s \in P \cup Z$. □

3.3 From vertex cover to orientation

In this section, we give the first direction of the reduction. More formally, we prove the following result.

Lemma 2 *If there exists a vertex cover of size at most k of H, then there exists an ℓ-bounded, well-balanced orientation of G.*

Proof Let U be a vertex cover of size at most k of H. Let $\vec{G}$ be the unique convenient orientation of G in which for every $v \in V(H)$, the edges ap_0^v, $p_0^v q_0^v$ are oriented to a directed path $ap_0^v q_0^v$; further the edge aq_0^v is oriented from a to q_0^v if and only if $v \in U$. By Lemma 1 and as U is a vertex cover, we obtain that $\vec{G}$ is well-balanced. By construction, we have $d^+_{\vec{G}}(s) \leq \ell(s)$ for all $s \in V(G) - a$. Finally, $\vec{G}$ contains $2|E(H)|$ arcs from a to Z, one arc from a to p_0^v for all $v \in V(H)$ and one arc from a to q_0^v for all $v \in U$. This yields $d^+_{\vec{G}}(a) = 2|E(H)| + |V(H)| + |U| \leq 6n + 2n + k = \ell(a)$, so $\vec{G}$ is ℓ-bounded. □

3.4 Making a well-balanced orientation convenient

In this section, we give a slightly technical lemma that shows that if an ℓ-bounded, well-balanced orientation of G exists, we can also find one which is additionally convenient.

Lemma 3 *If there exists a well-balanced, ℓ-bounded orientation of G, then there also exists a convenient, well-balanced, ℓ-bounded orientation of G.*

Proof Let $\vec{G}_0$ be a well-balanced, ℓ-bounded orientation of G.

Let Z_0^+ be the set of all $z \in Z$ such that $\vec{G}_0$ contains the arc bz and let $Z_0^- = Z - Z_0^+$. As $\vec{G}_0$ is well-balanced and by Corollary 1 (b), we have $|Z_0^+| = |Z_0^-| = 6n$. Further, let Z_0^* be the set of all $z \in Z$ such that $\vec{G}_0$ contains an arc from z to X. Observe that $Z_0^* \subseteq Z_0^+$ because $\vec{G}_0$ is ℓ-bounded. □

Claim 1 *There is a set of pairwise arc-disjoint circuits $\{C_z : z \in Z_0^*\}$ such that $V(C_z) \cap Z = z$ for all $z \in Z_0^*$.*

Proof By Corollary 1(b), we have $\lambda_{\vec{G}_0}(b, a) = d^+_{\vec{G}_0}(b) = \frac{d_G(b)}{2} = 6n$. By Theorem 8, there is a set $\mathcal{T}$ of $6n$ pairwise arc-disjoint directed ba-paths in $\vec{G}_0$. For all $z \in Z_0^-$, as $\vec{G}_0$ is ℓ-bounded and contains the arc zb, we obtain that z is not contained in a directed ba-path of $\mathcal{T}$.

Clearly, every $T \in \mathcal{T}$ contains a vertex in Z_0^+. Further, as $\vec{G}_0$ is ℓ-bounded and the directed ba-paths in $\mathcal{T}$ are pairwise arc-disjoint, no vertex in Z_0^+ can be contained in two distinct ba-paths of $\mathcal{T}$. As $|Z_0^+| = 6n = |\mathcal{T}|$, we obtain that every $z \in Z_0^+$ is contained in exactly one directed ba-path T_z of $\mathcal{T}$ and T_z satisfies $V(T_z) \cap Z = z$. For every $z \in Z_0^*$, as $\vec{G}_0$ is ℓ-bounded, the arc az is contained in $\vec{G}_0$. Now let C_z be obtained from T_z by deleting the arc bz and adding the arc az. Then C_z is a circuit. Since the directed ba-paths in $\mathcal{T}$ are arc-disjoint, $\{C_z : z \in Z_0^*\}$ has the desired properties. □

Let $\vec{G}_1$ be obtained from $\vec{G}_0$ by reversing all the arcs of $\cup_{z \in Z_0^*} A(C_z)$. Observe that in $\vec{G}_1$ all the edges linking Z and X are oriented from X to Z. Further observe that for all $z \in Z_0^+$, $\vec{G}_1$ contains the directed ba-path bza and for all $z \in Z_0^-$, $\vec{G}_1$ contains the directed ab-path azb. Now let $Z^{1,2} = \bigcup_{e \in E(H)} \{z_1^e, z_2^e\}$ and $Z^{3,4} = Z - Z^{1,2}$. Let D be the spanning directed subgraph of $\vec{G}_1$ whose arc set is $\bigcup_{z \in Z_0^+ \cap Z^{1,2}} \{bz, za\} \cup \bigcup_{z \in Z_0^- \cap Z^{3,4}} \{az, zb\}$.

Claim 2 *D is eulerian.*

Proof Clearly, we have $d_D^+(s) = d_D^-(s)$ for all $s \in V(G) - \{a, b\}$. Further, we have $d_D^+(b) = |Z_0^+ \cap Z^{1,2}| = |Z^{1,2}| - |Z_0^- \cap Z^{1,2}| = 6n - |Z_0^- \cap Z^{1,2}| = |Z_0^-| - |Z_0^- \cap Z^{1,2}| = |Z_0^- \cap Z^{3,4}| = d_D^-(b)$ and similarly $d_D^+(a) = d_D^-(a)$. □

Let $\vec{G}_2$ be obtained from $\vec{G}_1$ by reversing the orientation of every arc of D.

Observe that all the edges in G incident to a vertex of Z have the same orientation in $\vec{G}_2$ and F. Applying Proposition 2 twice, we obtain that $\vec{G}_2$ is well-balanced and ℓ-bounded. In order to complete the proof of Lemma 3, we show in the following that $\vec{G}_2$ is convenient.

Claim 3 *All the edges in G incident to at least one vertex in X have the same orientation in $\vec{G}_2$ and F.*

Proof Let $e \in E(H)$. As observed above, all the edges linking $\{x^e, y^e\}$ and Z are oriented from $\{x^e, y^e\}$ to Z in $\vec{G}_2$. By Corollary 1(b), we obtain $d^+_{\vec{G}_2}(y^e) = d^-_{\vec{G}_2}(y^e) = \frac{d_G(y^e)}{2} = 3$ and $d^+_{\vec{G}_2}(x^e) = d^-_{\vec{G}_2}(x^e) = \frac{d_G(x^e)}{2} = 2$. As $\vec{G}_2$ contains 3 arcs from y^e to Z, we obtain that the edges linking P and y^e are oriented from P to y^e in $\vec{G}_2$ and that the edge $x^e y^e$ is oriented from x^e to y^e in $\vec{G}_2$. As $\vec{G}_2$ contains two arcs from x^e to $Z \cup y^e$, we obtain that the edges linking P and x^e are oriented from P to x^e in $\vec{G}_2$. □

Claim 4 *For every $v \in V(H)$, the edges in $E(G^v) - \{p_0^v q_0^v\}$ have the same orientation in $\vec{G}_2$ and F.*

Proof For every $v \in V(H)$, as $\vec{G}_2$ is well-balanced and by Corollary 1(a), we have $\lambda_{\vec{G}_2}(a, p_2^v) \geq \lfloor \frac{d_G(p_2^v)}{2} \rfloor = 1$. Hence, by construction and Claim 3, we obtain that there is a directed $p_0^v p_2^v$-path in $\vec{G}_2$, namely $p_0^v p_1^v p_2^v$. Similarly, $q_0^v q_1^v q_2^v$ is a directed $q_0^v q_2^v$-path in $\vec{G}_2$. □

Claims 3 and 4 finish the proof of the fact that $\vec{G}_2$ is convenient. □

3.5 From convenient orientation to vertex cover

We now give the last step of the other direction of our reduction. More formally, we prove the following result.

Lemma 4 *If there is a convenient, well-balanced, ℓ-bounded orientation of G, then there is a vertex cover of size at most k of H.*

Proof Let $\vec{G}$ be a convenient, well-balanced, ℓ-bounded orientation of G. Let $U \subseteq V(H)$ be the set of vertices v for which the arcs ap_0^v and aq_0^v exist in $\vec{G}$. By Lemma 1, we get that U is a vertex cover of H and for every $v \in V(H)$, at least one arc exists in $\vec{G}$ from a to $V(G^v)$. Next note that there are exactly $2|E(H)| = 6n$ arcs leaving a in F. As $\vec{G}$ is a convenient, ℓ-bounded orientation of G, we have $|U| = d^+_{\vec{G}}(a) - d^+_F(a) - |V(H)| \leq \ell(a) - 6n - 2n = (8n + k) - 8n = k$. □

3.6 Best-balanced orientations

We now show how to extend our reduction to best-balanced orientations. We create an instance (G', ℓ') of UBBBO by altering the instance (G, ℓ) of UBWBO created in

Sect. 3.1. Let G' be obtained from G by adding a set W of $2k$ new vertices and an edge wa for all $w \in W$. Observe that $d_{G'}(a) = d_G(a) + |W| = 16n + 2k$ and $d_{G'}(w) = 1$ for all $w \in W$. Further, we set $\ell'(z) = 1$ for all $z \in Z$ and we set the trivial bound $\ell'(s) = d_{G'}(s)$ for all $s \in V(G') - Z$.

Lemma 5 *There exists an ℓ'-bounded, best-balanced orientation of G' if and only if there exists an ℓ-bounded, well-balanced orientation of G.*

Proof First suppose that there exists an ℓ'-bounded, best-balanced orientation $\vec{G}'$ of G'. Let $\vec{G} = \vec{G}'[V(G)]$. Observe that $\vec{G}$ is an orientation of G. Further, as $\vec{G}'$ is well-balanced, for any $(s_1, s_2) \in V(G) \times V(G)$, we have $\lambda_{\vec{G}}(s_1, s_2) = \lambda_{\vec{G}'}(s_1, s_2) \geq \lfloor\frac{\lambda_{G'}(s_1,s_2)}{2}\rfloor = \lfloor\frac{\lambda_G(s_1,s_2)}{2}\rfloor$, hence $\vec{G}$ is well-balanced. For any $s \in V(G) - a$, as $\vec{G}'$ is ℓ'-bounded, we have $d^+_{\vec{G}}(s) = d^+_{\vec{G}'}(s) \leq \ell'(s) = \ell(s)$. Finally, as $\vec{G}'$ is best-balanced, we have $d^+_{\vec{G}}(a) \leq d^+_{\vec{G}'}(a) \leq \lceil\frac{d_{G'}(a)}{2}\rceil = 8n + k = \ell(a)$. Hence $\vec{G}$ is ℓ-bounded.

Now suppose that there is an ℓ-bounded, well-balanced orientation of G. We obtain by Lemma 3 that there is also a convenient, ℓ-bounded, well-balanced orientation $\vec{G}$ of G. This yields $8n \leq d^+_{\vec{G}}(a) \leq 8n + k$.

We now create an orientation $\vec{G}'$ by giving every edge in $E(G)$ the orientation it has in $\vec{G}$, orienting $8n + k - d^+_{\vec{G}}(a)$ of the edges linking W and a from a to W and orienting all the remaining edges linking W and a from W to a. For any $(s_1, s_2) \in V(G) \times V(G)$, we have $\lambda_{\vec{G}'}(s_1, s_2) = \lambda_{\vec{G}}(s_1, s_2) \geq \lfloor\frac{\lambda_G(s_1,s_2)}{2}\rfloor = \lfloor\frac{\lambda_{G'}(s_1,s_2)}{2}\rfloor$. For any $(s_1, s_2) \in V(G') \times V(G')$ with $\{s_1, s_2\} \cap W \neq \emptyset$, we have $\lambda_{\vec{G}'}(s_1, s_2) \geq 0 = \lfloor\frac{1}{2}\rfloor = \lfloor\frac{\lambda_{G'}(s_1,s_2)}{2}\rfloor$. Hence, $\vec{G}'$ is well-balanced. For every $s \in V(G) - a$, we have $d^+_{\vec{G}'}(s) = d^+_{\vec{G}}(s) \leq \ell(s) = \ell'(s)$. Further, as $\vec{G}$ is convenient, we have $d^+_{\vec{G}'}(s) \in \{\lfloor\frac{d_G(s)}{2}\rfloor, \lceil\frac{d_G(s)}{2}\rceil\} = \{\lfloor\frac{d_{G'}(s)}{2}\rfloor, \lceil\frac{d_{G'}(s)}{2}\rceil\}$. For all $w \in W$, we have $d^+_{\vec{G}'}(w) \leq 1 = \ell(w)$ and $d^+_{\vec{G}'}(w) \in \{0, 1\} = \{\lfloor\frac{d_G(w)}{2}\rfloor, \lceil\frac{d_G(w)}{2}\rceil\}$. Finally, we have $d^+_{\vec{G}'}(a) = d^+_{\vec{G}}(a) + (8n + k - d^+_{\vec{G}}(a)) = 8n + k = \frac{d_{G'}(a)}{2} \leq \ell'(a)$. Hence $\vec{G}'$ is best-balanced and ℓ'-bounded. □

3.7 Conclusion

We here conclude the proof of Theorems 5 and 6. First observe that both UBWBO and UBBBO are clearly in NP. Next observe that the size of both (G, ℓ) and (G', ℓ') is polynomial in the size of (H, k). By Lemmas 2 to 4, we obtain that (G, ℓ) is a positive instance of UBWBO if and only if (H, k) is a positive instance of CVC. By Lemmas 2 to 5, we obtain that (G', ℓ') is a positive instance of UBBBO if and only if (H, k) is a positive instance of CVC. As CVC is NP-complete by Theorem 9, Theorems 5 and 6 follow.

Author Contributions The authors have equally contributed.

Funding Open Access funding enabled and organized by Projekt DEAL. No other funding is available.

Data availability No data is associated to this manuscript.

Declarations

Conflict of interest There are no competing interests.

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