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Improved stretch factor of Delaunay triangulations of points in convex position

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Abstract
Let S be a set of n points in the plane, and let $DT(S)$ be the planar graph of the Delaunay triangulation of S. For a pair of points $a, b \in S$, denote by $|ab|$ the Euclidean distance between a and b. Denote by $DT(a, b)$ the shortest path in $DT(S)$ between a and b, and let $|DT(a, b)|$ be the total length of $DT(a, b)$. Dobkin et al. were the first to show that $DT(S)$ can be used to approximate the complete graph of S in the sense that the stretch factor $\frac{|DT(a,b)|}{|ab|}$ is upper bounded by $((1+\sqrt{5})/2)\pi \approx 5.08$. Recently, Xia improved this factor to 1.998. Amani et al. have also shown that if the points of S are in *convex position* (i.e., they form the vertices of a convex polygon), then a planar graph with these vertices can be constructed such that its stretch factor is 1.88. In this paper, we prove that if the points of S are in convex position, then the stretch factor of $DT(S)$ is less than 1.84, improving upon the previously known factors of Delaunay triangulations or planar graphs in the convex case.

Keywords Computational geometry · Delaunay triangulations · Stretch factor · Convex polygons

1 Introduction

Let S be a set of n points in the plane, and let $G(S)$ be such a graph that each vertex corresponds to a point in S and the weight of an edge is the Euclidean distance

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between its two endpoints. For a pair of points p, q in the plane, denote by pq the line segment connecting p and q, and $|pq|$ the Euclidean distance between p and q. Denote by $G(a, b)$ the shortest path in $G(S)$ between two points $a, b \in S$, and $|G(a, b)|$ the total length of path $G(a, b)$. The graph $G(S)$ is said to approximate the complete graph of S if $\frac{|G(a,b)|}{|ab|}$, called the *stretch factor* of $G(S)$, is upper bounded by a constant, independent of S and n. It is then desirable to identify classes of graphs that approximate complete graphs well and have only $O(n)$ edges, as these graphs have potential applications in geometric network design problems (Eppstein 2000; Narasimhan and Smid 2007).

Denote by $DT(S)$ the planar graph of the Delaunay triangulation of S (de Berg et al. 2008). Dobkin et al. (1990) were the first to give a stretch factor $((1 + \sqrt{5})/2)\pi \approx 5.08$) of Delaunay triangulations to complete graphs. Later, Keil and Gutwin (1992) improved it to $2\pi/(3\cos(\pi/6)) \approx 2.42$, and Cui et al. (2011) showed that the stretch factor of $DT(S)$ for a set of points in convex position is 2.33. A set of points is said to be in *convex position*, if all points form the vertices of a convex polygon. Currently, the best result is due to Xia (2013), who proved that the stretch factor of $DT(S)$ is 1.998. Determining the best possible stretch factor of Delaunay triangulations has been a long-standing open problem in computational geometry (Bose and Smid 2013). On the other hand, Xia and Zhang (2011) gave a lower bound 1.5932 on the stretch factor of $DT(S)$.

Amani et al. (2016) have also constructed a planar graph, whose vertices are in convex position, such that its stretch factor is 1.88. Notice that the planar graph studied by Amani et al. is *not* the Delaunay triangulation of the given point set. The lower bound on the stretch factor of planar graphs in the convex case is 1.41611 (Bose and Smid 2013).

In this paper, we prove that the stretch factor of $DT(S)$ for a set of points in convex position is 1.84. This improves upon the previously known factor 1.998 for Delaunay triangulations of points in convex position (clearly, the result of Xia 2013 works for a set of points in convex position). It also gives an improvement upon the stretch factor 1.88 for planar graphs with vertices in convex position, as Delaunay triangluations are planar. Our result is obtained by investigating some geometric properties of $DT(S)$ and showing that there exists a convex chain between a and b in $DT(S)$ such that it is either contained in a semicircle of diameter ab, or enclosed by segment ab and a simple (convex) chain that consists of a circular arc and a few line segments. The total length of the simple chain is less than $1.84|ab|$.

2 Preliminaries

Assume that no four points of S are on the boundary of a circle in the plane, and no three points of S are on a line. The *Voronoi diagram* for S, denoted by $Vor(S)$, is a partition of the plane into regions, each containing exactly one point in S, such that for each point $p \in S$, every point within its corresponding region, denoted by $Vor(p)$, is closer to p than to any other point of S (de Berg et al. 2008). The boundaries of these Voronoi regions form a planar graph. The *Delaunay triangulation* of S, denoted by

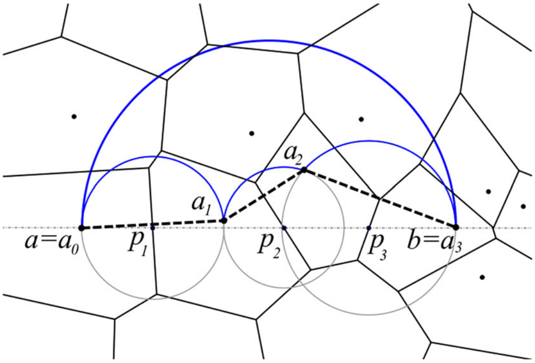

Fig. 1 A one-sided, direct path from a to b

$DT(S)$, is the straight-line dual of the Voronoi diagram for S; that is, we connect a pair of points in S if and only if they share a Voronoi boundary.

For a pair of points $a, b \in S$, denote by $DT(a, b)$ the shortest path in $DT(S)$ between a and b, in the Euclidean metric, and $|DT(a, b)|$ the total length of path $DT(a, b)$. The *stretch factor* of $DT(S)$ is then the maximum value $\frac{|DT(a,b)|}{|ab|}$ among all point pairs (a, b).

Let us review an important idea of Dobkin et al.'s work (Dobkin et al. 1990). Denote by $a = a_0, a_1, \ldots, a_m = b$ the sequence of points of S, whose Voronoi regions intersect segment ab. See Fig. 1. The path obtained in this way is called the *direct path from a to b* (Dobkin et al. 1990).

For ease of presentation, denote by $dp(a, b)$ the direct path from a to b in $DT(S)$. Path $dp(a, b)$ is said to be *one-sided* if all points of the path are to the same side of the line through a and b, including the special case that $dp(a, b)$ consists of a single edge ab. See Fig. 1. If $dp(a, b)$ is one-sided, then it has length at most $\pi|ab|/2$.

Lemma 1 *(Dobkin et al. 1990) If path $dp(a, b)$ is one-sided, then it has length at most $\pi|ab|/2$.*

Let p_i be the intersection point of ab with the Voronoi edge between $Vor(a_{i-1})$ and $Vor(a_i)$, for $1 \leq i \leq m$. It follows from the definition of the Voronoi diagram that p_i is the center of a circle that passes through a_{i-1} and a_i but contains no points of S in its *interior*. See Fig. 1. All points of path $dp(a, b)$ are thus contained in the circle of diameter ab.

A simple property of the one-sided path $dp(a, b)$ is that all points p_i ($1 \leq i \leq m$) are monotone on segment ab (Dobkin et al. 1990). See also Fig. 1. A more general result than Lemma 1 is the following.

Lemma 2 *(Dobkin et al. 1990) Let $C_1, C_2, \ldots, C_k$ be the circles all centered on a same line such that $U = \bigcup_{1 \leq i \leq k} C_i$ is connected. The boundary of U has length at most $\pi|ab|$ and is contained in the circle of diameter ab, where a and b are two extreme endpoints of U on the line.*[1]

[1] From the proof (Dobkin et al. 1990, Lemma 2), the boundary of U is also contained in the circle of diameter ab.

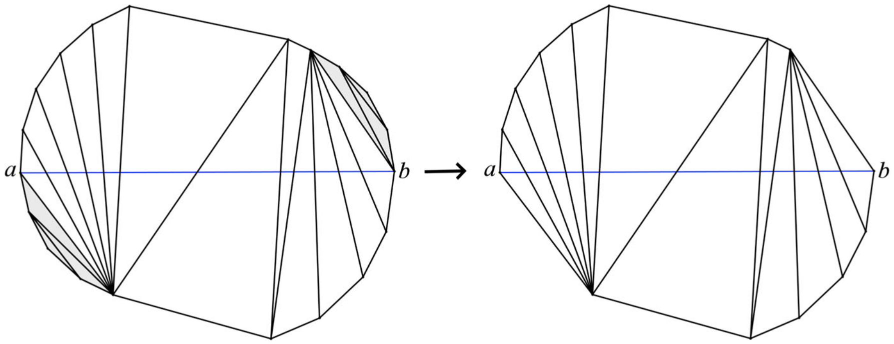

Fig. 2 All triangles of $DT(S)$ are assumed to properly intersect ab

3 The main result

Assume that the set S of given points is in convex position. For a point p in the plane, denote the coordinates of p by $p.x$ and $p.y$, respectively. Assume that both a and b are on the x-axis, with $a.x < b.x$. Let C be the circle of diameter ab, and let o be the center of C. The *bisector* of two points p and q, denoted by $B_{p,q}$, is the perpendicular line through the middle point of segment pq.

Let $CH(S)$ be the convex hull of points of S, i.e., the boundary of the smallest convex polygon containing all points of S (de Berg et al. 2008). Denote by $SA[a, b]$ (resp. $SB[a, b]$) the polygonal chain of $CH(S)$, which is above (resp. below) the line through a and b. For a point $p \in SA[a, b]$, denote by $SA[a, p]$ and $SA[p, b]$ the chains of $SA[a, b]$ from a to p and from p to b, respectively. Analogously, $SB[a, q]$ (resp. $SB[q, b]$), $q \in SB[a, b]$, represents the polygonal chain of $SB[a, b]$ from a to q (resp. from q to b). Also, denote by $SA(a, b)$ and $SB(a, b)$ the open chains of $SA[a, b]$ and $SB[a, b]$, respectively.

We say segment ab *properly* intersects a Delaunay triangle if it goes across the interior of the triangle (i.e., ab does not intersect only at a vertex of the triangle). If a Delaunay triangle does not properly intersect ab, then at least one of its vertices (and two edges incident to that vertex) can be deleted from $DT(S)$, without affecting the value of $\frac{|DT(a,b)|}{|ab|}$ (see Fig. 2). Then, the following observation can be made.

Observation 1 *For any two points $a, b \in S$, one can assume that ab properly intersects all triangles of $DT(S)$, in evaluating the value of $\frac{|DT(a,b)|}{|ab|}$.*

Assume below that ab properly intersects all triangles of $DT(S)$, see the right of Fig. 2. We will show that $\min\{\frac{|SA[a,b]|}{|ab|}, \frac{|SB[a,b]|}{|ab|}\} \le 1.84$. The choice of $SA[a, b]$ or $SB[a, b]$ depends on whether path $dp(a, b)$ intersects segment ab an odd number of times or not. The following results are obtained in this paper.

Lemma 3 *Suppose that the first and last segments of path $dp(a, b)$ are below and above the line through a and b, respectively. Then, there exists an angle α such that (i) $|DT(a, b)|/|ab| \le \sin(\alpha) + \pi\cos(\alpha)/2$, $\pi/4 \le \alpha < \pi/2$, (ii) $|DT(a, b)|/|ab| \le \sin(\alpha) + \cos(\alpha)(\cos(\alpha) + \alpha)$, $0 < \alpha < \pi/4$, (iii) $|DT(a, b)|/|ab| \le \sin(\alpha) +$*

$\cos(\alpha)(\sin(\alpha) + \pi/2 - \alpha)$, $\pi/6 \le \alpha < \pi/4$, *or (iv)* $|DT(a,b)|/|ab| \le \sin(\alpha) + \cos(\alpha)(2\sin(\alpha) + \pi/2 - 2\alpha)$, $0 < \alpha < \pi/6$.

Lemma 4 *Suppose that the first and last segments of path $dp(a,b)$ are to the same side of the line through a and b. Then, there exists an angle β such that $|DT(a,b)|/|ab| \le \beta + \cos(\beta)(3\sin(\beta) + \pi/2 - 3\beta)$, $0 < \beta < \pi/6$.*

The main result of this paper can then be summarized in the following theorem.

Theorem 1 *Suppose that the set S of given points is in convex position, and a and b are two points of S. In the Delaunay triangulation of S, there is a path from a to b such that its length is less than* $1.84|ab|$.

Proof Suppose that path $dp(a,b)$ is not one-sided; otherwise, $|DT(a,b)| \le \pi|ab|/2$. Let $f_1(\alpha) = \sin(\alpha) + \pi\cos(\alpha)/2, \alpha \in [\pi/4, \pi/2)$, $f_2(\alpha) = \sin(\alpha) + \cos(\alpha)(\cos(\alpha) + \alpha)$, $\alpha \in (0, \pi/4)$, $f_3(\alpha) = \sin(\alpha) + \cos(\alpha)(\sin(\alpha) + \pi/2 - \alpha)$, $\alpha \in [\pi/6, \pi/4)$, $f_4(\alpha) = \sin(\alpha) + \cos(\alpha)(2\sin(\alpha) + \pi/2 - 2\alpha)$, $\alpha \in (0, \pi/6)$, and $f_5(\beta) = \beta + \cos(\beta)(3\sin(\beta) + \pi/2 - 3\beta)$, $\beta \in (0, \pi/6)$. It follows from Lemmas 3 and 4 that $|DT(a,b)|/|ab| \le \max\{\pi/2, f_1(\alpha), f_2(\alpha), f_3(\alpha), f_4(\alpha), f_5(\beta)\}$, for all possible values α and β. Figure 3 shows five functions, which are produced using *gnuplot*. (By definition, f_4 is very close to f_5.) For each $1 \le i \le 5$, we can obtain $f_i < 1.84$ by considering the variable's value(s) satisfying $f_i' = 0$ and two extreme values of variable α or β. Actually, $f_3(\pi/6)$ gives the maximum value among all considered functions (see also Fig. 3). □

4 Proof of Lemma 3

Assume that neither $SA[a,b]$ nor $SB[a,b]$ is completely contained in the circle C of diameter ab; otherwise, $|DT(a,b)| \le \pi|ab|/2$. Denote by ac and bd the first and last segments of path $dp(a,b)$, as viewed from a, respectively. Then, both ac and bd are contained in C (Dobkin et al. 1990). See Fig. 4. Extend segments ac and bd until they touch the boundary of C, say, at points c' and d' respectively. Since $\angle bc'a = \angle ad'b = \pi/2$, either $\angle c'ad'$ or $\angle d'bc'$ is at least $\pi/2$. In the following, assume that $\angle d'bc' \ge \pi/2$, or equivalently, $\angle dbc' \ge \pi/2$.

Let i be the intersection point of C with $B_{b,d}$, which is vertically below ac. Since $B_{b,d}$ is perpendicular to bd, and since $\angle bc'a = \pi/2$ and $\angle dbc' \ge \pi/2$, $B_{b,d}$ properly intersects ac'. Hence, $i \ne c'$, and point i is outside of $CH(S)$.

Denote by H the semicircle of diameter bi, which is vertically below bi (Fig. 4). We show below that $SB[a,b]$ is contained in the region bounded by ab, ai and H, and then give a method to bound the total length of $SB[a,b]$.

4.1 *SB*[*a*, *b*] is contained in the region bounded by *ab*, *ai* and *H*

Let e be the first vertex of $SB[a,b]$, which is outside of C, as viewed from a. From the definition of i, point e is vertically below bi. Then, $Vor(e)$ is adjacent to

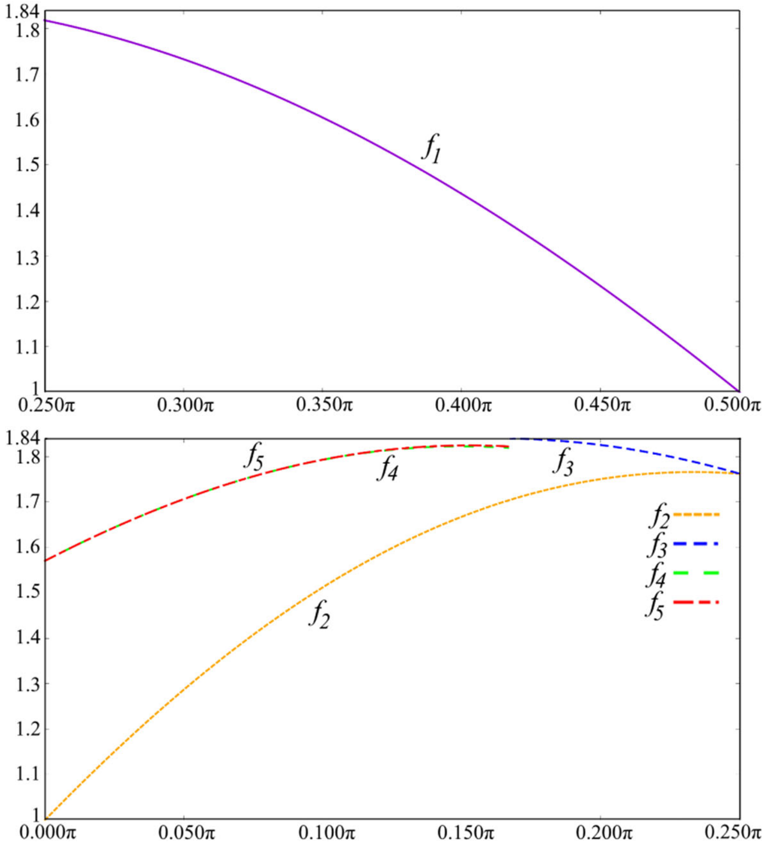

Fig. 3 Illustrating the proof of Theorem 1

some regions $Vor(q)$, $q \in SA[a, b] \cap S$ (Observation 1). Let f be the first vertex of $SA[a, b]$ such that $Vor(e)$ and $Vor(f)$ are adjacent. Denote by $\mathcal{R}$ the chain formed by the common edges of $Vor(p)$ and $Vor(q)$, $p \in SB[e, b) \cap S$ and $q \in SA[f, b) \cap S$ (see Fig. 4).

We claim that $\mathcal{R}$ is vertically above (or on) $B_{b,d}$. If f happens to be d, then $\mathcal{R}$ is vertically above $B_{b,d}$ (Fig. 5). Consider below the situation in which f differs from d. If f is adjacent to d, then from Observation 1, there is a point $g \in SB[e, b) \cap S$ such that $Vor(f)$, $Vor(d)$ and $Vor(g)$ share a Voronoi vertex, say, w. Clearly, w is the rightmost (resp. leftmost) vertex of $Vor(f)$ (resp. $Vor(d)$). Since the common edge between $Vor(f)$ and $Vor(g)$ is on $B_{f,g}$, from the definition and convexity of Voronoi regions, $B_{f,g}$ properly intersects $Vor(d)$. See the bottom of Fig. 4. Thus, $B_{f,g}$ intersects $B_{b,d}$ at a point that is to the right of w. Since $Vor(f)$ is vertically above $B_{f,g}$, it is above $B_{b,d}$, too. Then, $\mathcal{R}$ is vertically above $B_{b,d}$. In the case that f is not adjacent to d, a similar argument on each pair of consecutive vertices of $SA[f, b)$ can show that $\mathcal{R}$ is vertically above $B_{b,d}$. Our claim is thus proved.

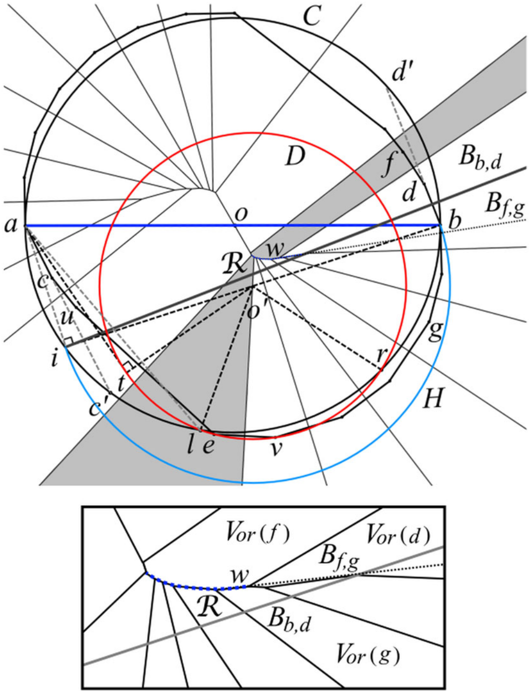

Fig. 4 $SB[a, b]$ is contained in the region bounded by ab, ai and H

From the convexity of S, all regions of $Vor(S)$ are unbounded. From our claim, all the regions $Vor(p)$, $p \in SB[e, b] \cap S$, then intersect $B_{b,d}$. Since bi is vertically below $B_{b,d}$, it intersects regions $Vor(p)$, $p \in SB[e, b] \cap S$, too.

Let $u, v \in S$ be the points immediately before and after e in $SB[a, b]$, respectively. (Note that u and v may be identical to c and b, respectively.) Since $v \in SB[e, b]$, segment bi then intersects the edge between $Vor(e)$ and $Vor(v)$ at a point, say, o'. See Fig. 4. Let D be the circle of radius $|o'e|$, centered at point o'.

We now show that i is on or outside of D. Since o' is an interior point of the edge between $Vor(e)$ and $Vor(v)$, except for e and v, all other vertices of S are outside of D (de Berg et al. 2008). Thus, the radius of D is smaller than that of C. Since o' and e are inside and outside of C respectively, C and D intersect. Denote by l and r the left and right intersection points between C and D, respectively. See Fig. 4. The circular sector of C bounded by ol, or and the arc $\widehat{lr}$ of C, which is to right of the line through a and l, then contains point o'. So, o' is to the right of the line through a and l.

Consider the tangent from a downward to D. Denote by t the found tangent point, see Fig. 4. Since l is the left common point of C and D, point t is to the left of the line through a and l, and lies in the interior of C. Then, point t is on or vertically below bi; otherwise, $\angle ato' > \angle aio' = \pi/2$, a contradiction. Hence, at intersects bi, and i is on or outside of D. The intersection point of $B_{i,e}$ with bi is thus to the left of or identical to o'.

Lemma 2 can then be applied to the collection of the circles, which are centered on bi and pass through the point pair (i, e) and all pairs of consecutive vertices of

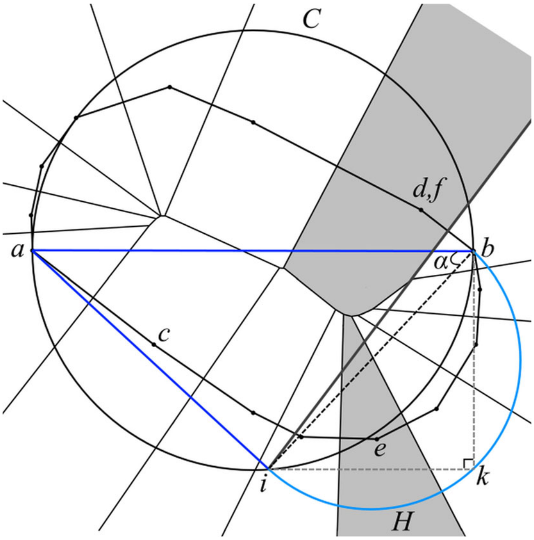

Fig. 5 Illustrating the case $\alpha \geq \pi/4$

$SB[e, b]$. Therefore, all points of $SB[e, b]$ are contained in H. Since any point of $SB[a, b]$ is to the right of (or on) the line through a and i, chain $SB[a, b]$ is then contained in the region bounded by ab, ai and H. See Fig. 4.

4.2 Bounding the total length of *SB*[*ab*]

Let $\alpha = \angle abi$, $0 < \alpha < \pi/2$. Denote by k the intersection point of H with the horizontal line through i. Then, $\angle bki = \pi/2$ and $\angle bik = \alpha$. We distinguish the following situations.

Case 1. $\pi/4 \leq \alpha < \pi/2$. In this case, $|ai| = \sin(\alpha)|ab|$ and $|bi| = \cos(\alpha)|ab|$. See Fig. 5. A simple argument (as in Hershberger and Suri 1998) then shows that the length of $SB[a, b]$ is less than $(\sin(\alpha) + \pi\cos(\alpha)/2)|ab|$. Thus, we have (i) $|DT(a, b)| \leq |SB[a, b]| \leq (\sin(\alpha) + \pi\cos(\alpha)/2)|ab|$.

Case 2. $0 < \alpha < \pi/4$. We further distinguish two different situations.

Case 2.1. The whole chain $SB[a, b]$ is vertically above the line through i and k. In this case, $SB[a, b]$ is contained in the convex region bounded by ba, ai, ik and the arc $\widehat{kb}$ of H. Since $|ik| = \cos^2(\alpha)|ab|$ and $|\widehat{kb}| = \alpha\cos(\alpha)|ab|$, we have (ii) $|DT(a, b)| \leq |SB[a, b]| \leq (\sin(\alpha) + \cos(\alpha)(\cos(\alpha) + \alpha))|ab|$.

Case 2.2. A portion of $SB[a, b]$ is vertically below the line through i and k. To bound the total length of $SB[a, b]$, we draw a tangent from point i to the portion of $SB[a, b]$ contained in H. The tangent intersects H at a point, say, n ($\neq i$). Since n is vertically below ik, segment bn intersects C at a point, say, m ($\neq b$). See the left of Fig. 6.

Let $\gamma = \angle ibm$. Our second claim is that $\gamma > \alpha$. Since n and m are on H and C respectively, $\angle bni = \angle bma = \pi/2$. Two segments am and in are thus parallel. From the definition of point c', segment in intersects ac', and it thus intersects C at a point

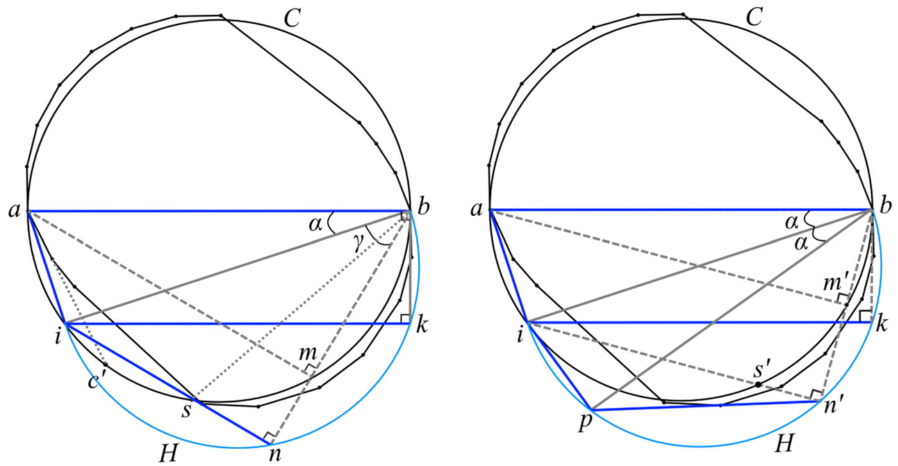

Fig. 6 In the case $\alpha < \pi/6$, two shortcuts can be introduced in H

s ($\neq i$). See the left of Fig. 6. Hence, two circular arcs $\widehat{ai}$ and $\widehat{ms}$ of C are of the same length. Therefore, $\angle sbm = \alpha$, and $\gamma > \alpha$.

Let p be the point on H such that $\angle ibp = \alpha$, see Fig. 6. Since $\gamma > \alpha$, segment ip does not intersect $SB[a, b]$. If $\pi/6 \leq \alpha < \pi/4$, then ip can be used to cut off its corresponding arc of H. Since $|ip| = \cos(\alpha)\sin(\alpha)|ab|$, we have (iii) $|DT(a, b)| \leq |SB[ab]| \leq (\sin(\alpha) + \cos(\alpha)(\sin(\alpha) + \pi/2 - \alpha))|ab|$.

Finally, consider the situation in which $\alpha < \pi/6$. Denote by n' the intersection point of H with the tangent from p rightward to $SB[a, b]$. If n' is vertically above ik, then $\angle pbn' > \pi/2 - 2\alpha > \pi/6 > \alpha$. Thus, we can draw another chord pn' of at least length $\cos(\alpha)\sin(\alpha)|ab|$ to cut off its corresponding arc of H. Suppose now that point n' is vertically below ik, see the right of Fig. 6. Since n' is vertically below ik, segment bn' intersects C at a point m' ($\neq b$), and segment in' intersects C at a point s' ($\neq i$). See the right of Fig. 6. As discussed above, we also have $\angle pbn' > \alpha$. Again, the chord pn' of at least length $\cos(\alpha)\sin(\alpha)|ab|$ can be introduced to cut off its corresponding arc of H. In summary, we have (iv) $|DT(a, b)| \leq |SB[ab]| \leq (\sin(\alpha) + \cos(\alpha)(2\sin(\alpha) + \pi/2 - 2\alpha))|ab|$.

The above analysis handles all possible situations. Our proof is thus complete.

5 Proof of Lemma 4

From the lemma assumption, path $dp(a, b)$ intersects segment ab an even number of times. Assume without loss of generality that the first and last segments of path $dp(a, b)$ are vertically above ab. Assume also that $SA[a, b]$ is not wholly contained in C; otherwise, $|DT(a, b)|/|ab| \leq \pi/2$.

Let c be the first point of path $dp(a, b)$ such that c and its next point on $dp(a, b)$ are above and below ab, respectively. Also, let d be the last point of $dp(a, b)$ such that d and its previous point on $dp(a, b)$ are above and below ab, respectively. See Fig. 7.

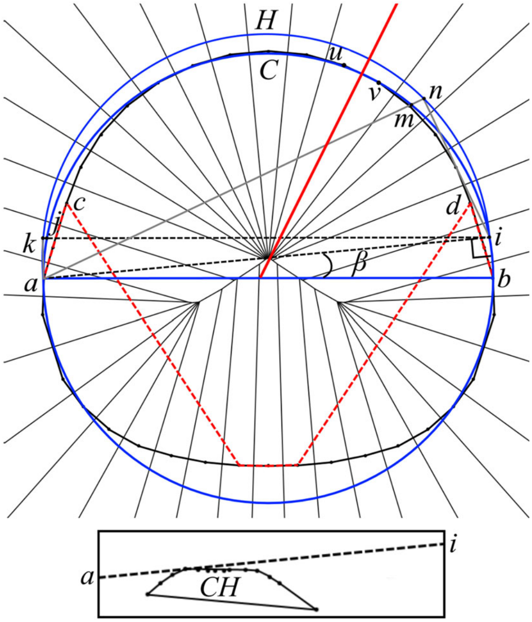

Fig. 7 Illustration of the proof of Lemma 4

Then, c and d are contained in C (Dobkin et al. 1990). Note that $SA(c, d)$ consists of at least two vertices; otherwise, since both $Vor(c)$ and $Vor(d)$ intersect ab, the only vertex of $SA(c, d)$, c and d form a Delaunay triangle, contradicting the assumption made in Observation 1.

Without loss of generality, assume that uv is an edge of $SA[c, d]$ such that u and v are outside of and inside C respectively, the slope of $B_{u,v}$ is positive and $SA[v, b]$ is contained in C. See Fig. 7. (The other possible situation in which $u'v'$ is an edge of $SA[c, d]$ such that u' and v' are inside and outside of C respectively, the slope of $B_{u',v'}$ is negative and $SA[a, u']$ is contained in C can be dealt with analogously.)

Let CH be the convex hull of the Voronoi vertices whose y-coordinates are positive. (Recall that a and b are on the x-axis.) Consider the tangent from a to CH, which is vertically above CH. See Fig. 7. Denote by i the intersection point of the tangent with C.

We first claim that segment uv intersects ai, or it is vertically above the line through a and i. Assume that our claim is not true. So, uv and ai do not intersect, and both u and v are vertically below the line through a and i. Since the slope of $B_{u,v}$ is positive and v is contained in C, the line through u and v intersects bi and ab as well, contradicting the convexity of S. The claim is proved. From the convexity of S and the definition of uv, our claim also implies that $u.y > i.y$.

From the above claim and the definition of point v, segment ai properly intersects $SA[a, b]$. Hence, i is outside of $CH(S)$, see Fig. 7. If ai intersects uv, then it intersects the edge between $Vor(u)$ and $Vor(v)$. Otherwise, uv is vertically above the line through a and i. As v lies in C, from the convexity of S, segment uv is to the left of the line through b and i. Since the Voronoi vertex on $B_{u,v}$ is contained in CH and

ai is vertically above CH, segment ai thus intersects the edge between $Vor(u)$ and $Vor(v)$, too. From the definition of uv, we can conclude that ai intersects regions $Vor(p)$, for all points $p \in SA[a, v] \cap S$.

Denote by H the semicircle of diameter ai, which is vertically above ai. As shown in Sect. 4, $SA[a, u]$ (or $SA[a, v]$ if v is vertically above ai) is completely contained in H. See Fig. 7. From the assumption that $SA[v, b]$ is contained in C, chain $SA[a, b]$ is contained in the region bounded by H, ab and the arc $\widehat{bi}$ of C.

Let $\beta = \angle bai$. Denote by j and k the intersection points of C and H with the horizontal line through point i, respectively. See Fig. 7. Draw a tangent from i to the portion of $SA[a, b]$, which is vertically above ai. Denote by n the intersection point of the tangent with H, see Fig. 7. Let m be the intersection point of C with an. As in the proof of Lemma 3, we can show that $\angle iam > \beta$. Since $u.y > i.y$, we have $m.y > i.y$ $(= j.y)$. Hence, $\angle bai + \angle iam + \angle abj < \pi/2$. Since $\angle abj = \beta$ and $\angle iam > \beta$, we obtain $\beta < \pi/6$.

As in the proof of Lemma 3, a chain of two segments of length $\cos(\beta)\sin(\beta)|ab|$, starting from point i, can then be introduced in H. Moreover, since $\angle bak = \pi/2$, segment ak is tangent to C and does not intersect $SA[a, b]$. Since $\beta < \pi/6$, the third cut ak of length $\cos(\beta)\sin(\beta)|ab|$ can be further introduced in H. By noticing that $|\widehat{bi}| = \beta|ab|$, we obtain $|DT(a, b)| \le (\beta + \cos(\beta)(3\sin(\beta) + \pi/2 - 3\beta))|ab|$.

6 Concluding remarks

We have shown that the stretch factor of the Delaunay triangulation of a set of points in convex position is less than 1.84. The same stretch factor might hold for the set of points in general position, too. A new difficulty is that the considered path between a and b is generally not convex, which needs be examined more. Also, it is a challenging open problem to reduce the stretch factor of $DT(S)$ further, even for a set of points in convex position, so as to close the gap to its lower bound (roughly about 1.60).

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Declarations

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