The effect of vertex and edge deletion on the edge metric dimension of graphs

Meiqin Wei[1] · Jun Yue[2] · Lily Chen[3]

Abstract

Let $G = (V(G), E(G))$ be a connected graph. A set of vertices $S \subseteq V(G)$ is an edge metric generator of G if any pair of edges in G can be distinguished by their distance to a vertex in S. The edge metric dimension $edim(G)$ is the minimum cardinality of an edge metric generator of G. In this paper, we first give a sharp bound on $edim(G - e) - edim(G)$ for a connected graph G and any edge $e \in E(G)$. On the other hand, we show that the value of $edim(G) - edim(G - e)$ is unbounded for some graph G and some edge $e \in E(G)$. However, for a unicyclic graph H, we obtain that $edim(H) - edim(H - e) \leq 1$, where e is an edge of the unique cycle in H. And this conclusion generalizes the result on the edge metric dimension of unicyclic graphs given by Knor et al. Finally, we construct graphs G and H such that both $edim(G) - edim(G - u)$ and $edim(H - v) - edim(H)$ can be arbitrarily large, where $u \in V(G)$ and $v \in V(H)$.

Keywords Edge metric generator · Vertex deletion · Edge deletion · Edge metric dimension

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1 Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. We follow the notation and terminology of Bondy and Murty (2008) and Diestel (2005). The concept of metric dimension was introduced independently by Harary and Melter (1976) and Slater (1975) . Let $G = (V(G), E(G))$ be a connected graph and $\overline{G}$ be its complement graph. For any pair of vertices $x, y \in V(G)$, the *distance* between x and y, which is denoted by $d_G(x, y)$, is the length of a shortest $x - y$ path in G. A $x - y$ path with length $d_G(x, y)$ in G is a $x - y$ geodesic. A vertex $v \in V(G)$ is said to distinguish vertices u and w if $d_G(u, v) \neq d_G(w, v)$. A vertex set $S \subseteq V(G)$ is a *metric generator* of G if any pair of vertices in G can be distinguished by some element of S. The minimum cardinality of a metric generator is the *metric dimension* of G, denoted by $dim(G)$. In Harary and Melter (1976), Harary and Melter determined the exact values of the metric dimension for trees, grid graphs and characterized graphs with small metric dimension. In addition, they obtained an approximation algorithm for the metric dimension problem. Since then, it has been studied in various directions by numerous researchers. For more details we refer to Beaudou et al. (2018); Cáceres et al. (2007); Jiang and Polyanskii (2019); Tillquist et al. (Tillquist et al.).

Nowadays there exist some different kinds of metric generators in graphs, but there are still quite a lot of other points of view which are still not completely taken into account while describing a graph throughout these metric generators. Inspired by this, (Kelenc et al. 2018) recently proposed the concept of an edge metric generator of a graph. For any $v \in V(G)$, $e = xy \in E(G)$, the distance between them is defined as $d_G(e, v) = \min\{d_G(x, v), d_G(y, v)\}$. For two edges $e_1, e_2 \in E(G)$, if $d_G(e_1, v) \neq d_G(e_2, v)$, then we say the vertex v distinguishes e_1 and e_2. A set of vertices $S \subseteq V(G)$ is an *edge metric generator* of G if any pair of edges can be distinguished by some element of S. The minimum cardinality of an edge metric generator of G is the *edge metric dimension* of G and is denoted by $edim(G)$. Additionally, an *edge metric basis* of G is an edge metric generator with cardinality $edim(G)$. In Kelenc et al. (2018), they proved that the problem of determining the edge metric dimension of a connected graph is NP-hard. In Kelenc et al. (2018); Wei et al. (2020); Zubrilina (2018); Geneson (2020); Zhu et al. (2019), the authors characterized the graphs G of order n with $edim(G) = n - 1, n - 2$ or 2. (Peterin and Yero 2020) investigated the edge metric dimension of the join, lexicographic and corona product of graphs. For more results and details, we refer to Beaudou et al. (2018); Zhu et al. (2019); Zubrilina (2018); Peterin and Yero (2020); Knor et al. (2021); Geneson (2020); Wei et al. (2020); Jiang and Polyanskii (2019); Cáceres et al. (2007); Kuziak and Yero (Kuziak and Yero).

For any graph G, both the exact values of $dim(G)$ and $edim(G)$ and their computational complexity attract the attention of researchers. In addition, researchers also focus on the relationships between $dim(G)$, $edim(G)$ and some other graph parameters such as diameter, zero forcing number and so on. Furthermore, researchers pay attention to the effects of different graph operations on $dim(G)$ and $edim(G)$. Particularly, the question as to the effect of the deletion of a vertex or of an edge on $dim(G)$ was raised as a fundamental question in graph theory by Chartrand and (Chartrand and Zhang 2003) and further studied by Eroh et al. (2015). Inspired by their idea, we focus

on the question how $edim(G)$ will change after vertex or edge deletion. Recently, (Knor et al. 2021) and (Zubrilina 2018) showed that both $\frac{edim(G)}{dim(G)}$ and $\frac{dim(G)}{edim(G)}$ are not bounded for some certain graph G, which in a sense implies that the properties of $dim(G)$ and $edim(G)$ may be much different. In this paper, we characterize the difference between the edge metric dimension of the original graph and the resulting graph after vertex or edge deletion. Additionally, (Knor et al. 2021) determined the edge metric dimension of some unicyclic graphs in a certain class and our result on the edge metric dimension of any unicyclic graphs generalizes it.

This paper is organized as follows. In Sect. 2, we first prove that $edim(G - e) - edim(G) \le 2$ for any connected graph G and any edge $e \in E(G)$, and this bound is sharp for some certain graphs. Then we show that the value of $edim(G) - edim(G - e)$ is unbounded for some graph G and some edge $e \in E(G)$. However, for a unicyclic graph H and $e \in E(C)$, we conclude that $edim(H) - edim(H - e) \le 1$, where C is the unique induced cycle of H. In Sect. 3, we prove that there exist two graphs G and H such that both $edim(G) - edim(G - u)$ and $edim(H - v) - edim(H)$ can be arbitrarily large, where $u \in V(G)$ and $v \in V(H)$.

2 The effect of edge deletion on edge metric dimention of graphs

In this section, we focus on how the edge metric dimension of a graph changes upon deletion of an edge. Before giving our main results, we need the following definitions. Let $T = (V(T), E(T))$ be a tree and $v \in V(T)$. Define the equivalence relation R_v in the following way: for every two edges $e, f \in E(T)$, $eR_v f$ if and only if there is a path in T including both e and f but does not have v as an internal vertex. We call the subgraphs induced by the edges of the equivalence classes of $E(T)$ the *bridges* of T relative to v. Furthermore, if a bridge at v in T is a path, then the bridge is a *leg* at v and we denote the number of legs at v in T by $\ell_T(v)$.

Proposition 2.1 *Kelenc et al. (2018) Let $T = (V(T), E(T))$ be a tree. If T is not a path, then $edim(T) = \sum\limits_{\substack{v \in V(T) \\ \ell_T(v) > 1}} (\ell_T(v) - 1)$.*

Now our first result can be stated as follows.

Theorem 2.2 *For any graph G and any edge $e \in E(G)$, $edim(G - e) - edim(G) \le 2$. Moreover, this bound is sharp for certain graphs.*

Proof Note that $edim(G) = edim(G_1) + \cdots + edim(G_m)$, where $G_1, \ldots, G_m$ are all the connected components of G. Next we assume that G is connected. Let S be an edge metric basis of G and $e = uv$. If S is also an edge metric generator of $G - e$, then $edim(G - e) \le edim(G)$. Otherwise, set $S' = S \cup \{u, v\}$ and we will show that S' is an edge metric generator of $G - e$.

If $G - e$ has two components, denoted by G_1 and G_2, then $edim(G - e) = edim(G_1) + edim(G_2)$. Without loss of generality, we may assume that $u \in V(G_1)$ and $v \in V(G_2)$. For any pair of edges $e_1, e_2 \in E(G_1)$, if there exists $w \in V(G_2) \cap S$ such that $d_G(e_1, w) \ne d_G(e_2, w)$, then $d_G(e_1, u) \ne d_G(e_2, u)$ since

$d_G(e_1, w) = d_G(e_1, u) + d_G(u, w)$ and $d_G(e_2, w) = d_G(e_2, u) + d_G(u, w)$. That is to say, $d_{G_1}(e_1, u) \neq d_{G_1}(e_2, u)$. This implies that $edim(G_1) \leq |(S \cap V(G_1)) \cup \{u\}|$. With similar analysis, we can obtain that $edim(G_2) \leq |(S \cap V(G_2)) \cup \{v\}|$. Therefore $edim(G - e) \leq |S'| \leq edim(G) + 2$.

Now we suppose that $G - e$ is connected. Take any pair of edges $e_1 = ab, e_2 = xy \in E(G - e)$ that are unable to be distinguished by any element of S. Since S is an edge metric generator of G but not of $G - e$, there exists $z \in S$ such that $d_G(e_1, z) \neq d_G(e_2, z)$ and $d_{G-e}(e_1, z) = d_{G-e}(e_2, z)$. We will complete the proof in the following two cases.

Case I For one of the edges e_1 and e_2, say e_2, the distance to z is not changed after removing the edge e. That means $d_{G-e}(e_2, z) = d_G(e_2, z)$ and $d_{G-e}(e_1, z) > d_G(e_1, z)$. Thus $d_G(e_2, z) > d_G(e_1, z)$. We may assume that $d_G(e_1, z) = d_G(a, z)$ and $d_G(e_2, z) = d_G(x, z)$. It follows that $d_G(x, z) > d_G(a, z)$.

Since $d_{G-e}(e_1, z) > d_G(e_1, z) = d_G(a, z)$, then $d_{G-e}(a, z) > d_G(a, z)$. This implies that the edge e lies on every $z - a$ geodesic in G. Notice that if a geodesic in G from z to a traverses the edge e in the order u, v, then each geodesic in G from z to a traverses e in the order u, v. Without loss of generality, we may assume that every geodesic in G from z to a traverses the edge e in the order u, v. Thus we have $d_G(z, a) = d_G(z, v) + d_G(v, a)$. Now we will show that e_1, e_2 can be distinguished by v in $G - e$.

First we show that $d_{G-e}(e_1, v) < d_{G-e}(v, x)$. Since $d_G(z, v) + d_G(v, x) \geq d_G(z, x) > d_G(z, a) = d_G(z, v) + d_G(v, a)$, that is to say, $d_G(v, x) > d_G(v, a)$. It follows that $d_{G-e}(v, x) \geq d_G(v, x) > d_G(v, a) = d_{G-e}(v, a)$. If $d_{G-e}(e_1, v) = d_{G-e}(v, a)$, then we have $d_{G-e}(e_1, v) < d_{G-e}(v, x)$. If $d_{G-e}(e_1, v) < d_{G-e}(v, a)$, then $d_{G-e}(v, a) > d_{G-e}(v, b)$ and $d_G(z, v) + d_{G-e}(v, a) > d_G(z, v) + d_{G-e}(v, b)$. Since $d_{G-e}(v, a) = d_G(v, a)$ and $d_{G-e}(v, b) \geq d_G(v, b)$, $d_G(z, a) > d_G(z, v) + d_G(v, b) \geq d_G(z, b)$, which contradicts to the assumption that $d_G(z, a) \leq d_G(z, b)$.

Now we prove $d_{G-e}(e_1, v) < d_{G-e}(v, y)$, i.e., $d_{G-e}(v, a) < d_{G-e}(v, y)$. Suppose that $d_{G-e}(v, a) \geq d_{G-e}(v, y)$. We can get $d_G(z, v) + d_{G-e}(v, a) \geq d_G(z, v) + d_{G-e}(v, y)$. Since $d_{G-e}(v, a) = d_G(v, a)$ and $d_{G-e}(v, y) \geq d_G(v, y)$, we have $d_G(z, v) + d_G(v, a) \geq d_G(z, v) + d_G(v, y) \geq d_G(z, y) \geq d_G(z, x)$. That is to say, $d_G(z, a) \geq d_G(z, x)$, which contradicts to $d_G(z, a) < d_G(z, x)$. As a result, $d_{G-e}(v, a) < d_{G-e}(v, y)$ and it follows that $d_{G-e}(e_1, v) < d_{G-e}(v, y)$. Therefore, $d_{G-e}(e_1, v) < d_{G-e}(e_2, v)$ since $d_{G-e}(e_2, v) = \min\{d_{G-e}(v, x), d_{G-e}(v, y)\}$. This implies that e_1, e_2 can be distinguished by v. Thus $S' = S \cup \{u, v\}$ is an edge metric generator of $G - e$ and then $edim(G - e) \leq edim(G) + 2$.

Case II For both e_1 and e_2, the distances to z are increased after removing the edge e, that is to say, $d_{G-e}(e_1, z) > d_G(e_1, z)$ and $d_{G-e}(e_2, z) > d_G(e_2, z)$. Without loss of generality, we may assume that $d_G(e_1, z) = d_G(z, a)$ and $d_G(e_2, z) = d_G(z, x)$. It follows that $d_{G-e}(z, a) > d_G(z, a)$ and $d_{G-e}(z, x) > d_G(z, x)$. This implies that the edge e lies on every $z - a$ and every $z - x$ geodesic in G. Here we may assume that every geodesic in G from z to a traverses the edge e in the order u, v. Thus we know that every geodesic in G from z to x traverses the edge e also in the order u, v, since otherwise we can find a shorter path than its geodesic from z to x. As a result,
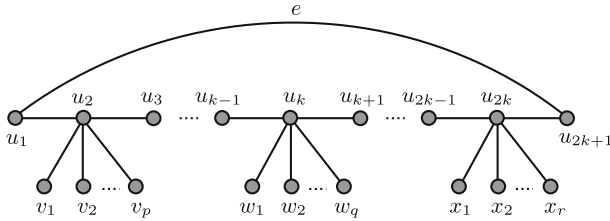
Fig. 1 The graph $G_{2k+1}^{p,q,r}$

$d_G(z, v) + d_G(v, a) = d_G(z, a) = d_G(e_1, z)$ and $d_G(z, v) + d_G(v, x) = d_G(z, x) = d_G(e_2, z)$.

Without loss of generality, we may assume that $d_G(e_1, z) > d_G(e_2, z)$ and hence $d_G(v, a) > d_G(v, x)$. We claim that the vertex v distinguishes the edges e_1 and e_2. Indeed, since there exist $v - a$ and $v - x$ geodesic in G that does not contain the edge e, $d_{G-e}(v, a) = d_G(v, a)$ and $d_{G-e}(v, x) = d_G(v, x)$. Therefore we have $d_{G-e}(v, a) > d_{G-e}(v, x)$. On the other hand, $d_G(z, v) + d_G(v, b) \geq d_G(z, b) \geq d_G(z, a) = d_G(z, v) + d_G(v, a)$, that is to say, $d_G(v, b) \geq d_G(v, a)$. It follows that $d_{G-e}(v, b) \geq d_G(v, b) \geq d_G(v, a) = d_{G-e}(v, a) > d_{G-e}(v, x)$. So we have $d_{G-e}(e_1, v) > d_{G-e}(v, x) \geq d_{G-e}(e_2, v)$, which implies that the edges e_1, e_2 can be distinguished by v in $G - e$. Therefore $S' = S \cup \{u, v\}$ is an edge metric generator of $G - e$ and thus $edim(G - e) \leq edim(G) + 2$, as desired. □

Finally, we show the sharpness of the bound by proving that $edim(G_{2k+1}^{p,q,r} - e) = edim(G_{2k+1}^{p,q,r}) + 2$, where the graph $G_{2k+1}^{p,q,r}$ is depicted in Figure 1. Let $C_{2k+1} = u_1 u_2 \ldots u_{2k+1}$ be an odd cycle of length $2k + 1$, where $k \geq 3$. Attach p, q, r pendant vertices to u_2, u_k, u_{2k} respectively, where $p, q, r \geq 2$. Denote the new graph as $G_{2k+1}^{p,q,r}$. It is easy to verify that $S_1 = \{v_2, \ldots, v_p, w_2, \ldots, w_q, x_2, \ldots, x_r\}$ is an edge metric basis of $G_{2k+1}^{p,q,r}$ and thus $edim(G_{2k+1}^{p,q,r}) = p+q+r-3$. And by Proposition 2.1, we have

$$edim(G_{2k+1}^{p,q,r} - e) = (\ell_{G_{2k+1}^{p,q,r}-e}(u_2)-1) + (\ell_{G_{2k+1}^{p,q,r}-e}(u_k)-1) + (\ell_{G_{2k+1}^{p,q,r}-e}(u_{2k})-1) =$$

$p + q + r - 1$. So we arrive at $edim(G_{2k+1}^{p,q,r} - e) - edim(G_{2k+1}^{p,q,r}) = 2$ and thus we complete the proof.

Let k be a positive integer. Let $E_1, E_2, \ldots, E_k$ be cycles of length 9 and $S_1, S_2, \ldots, S_k$ be stars with center vertices $g_1, g_2, \ldots, g_k$ and pendent vertex sets $\{h_1, c_1, d_1\}, \{h_2, c_2, d_2\}, \ldots, \{h_k, c_k, d_k\}$. Let $G_k = (V(G_k), E(G_k))$, depicted in Figure 2, where $V(G_k) = V(E_1) \cup \cdots \cup V(E_k) \cup V(S_1) \cup \cdots \cup V(S_k) \cup \{w_1, \ldots, w_k\} \cup \{A, B\}$ and $E(G_k) = E(E_1) \cup \cdots \cup E(E_k) \cup E(S_1) \cup \cdots \cup E(S_k) \cup \{e, u_i A, c_i w_i, d_i w_i, w_i B, v_i h_i : i \in [k]\}$.

Theorem 2.3 *Let G_k be the graph depicted in Figure 2. For any positive integer k, $edim(G_k) - edim(G_k - e) = k - 1$.*

Proof Let S be an edge metric generator of G_k and S' be an edge metric generator of $G_k - e$. Note that, for $1 \leq i \leq k$, $|S \cap \{c_i, d_i\}| \geq 1$ since $g_i c_i, g_i d_i$ are unable to be distinguished by any element of $V(G) \setminus \{c_i, d_i\}$. Similarly, $|S' \cap \{c_i, d_i\}| \geq 1$. Without loss of generality, we may assume that $\{c_1, c_2, \ldots, c_k\} \subseteq S \cap S'$. Set $S_0 = \{c_1, c_2, \ldots, c_k\}$.
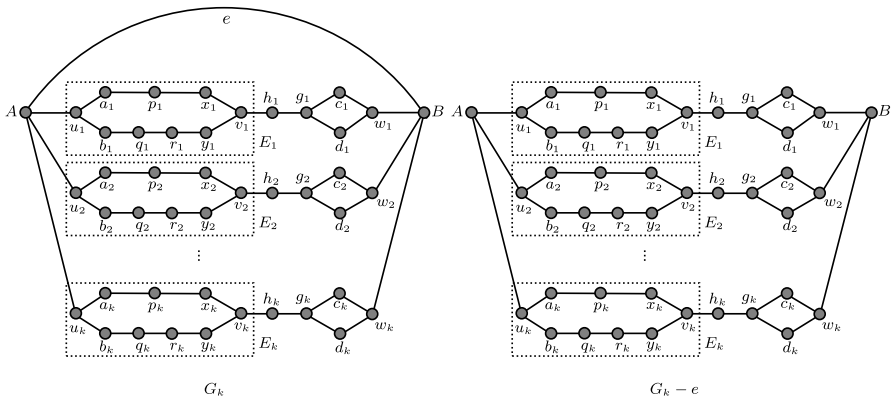
Fig. 2 The graphs G_k and $G_k - e$

First we will show that $edim(G_k) = 2k$. For $1 \leq i, j \leq k$, $d_{G_k}(c_j, u_i a_i) = d_{G_k}(c_j, u_i b_i) = 4$ and then $u_i a_i, u_i b_i$ are unable to be distinguished by any element of S_0. Simultaneously, $d_{G_k}(d_j, u_i a_i) = d_{G_k}(d_j, u_i b_i) = 4$, $d_{G_k}(w_j, u_i a_i) = d_{G_k}(w_j, u_i b_i) = 3$, $d_{G_k}(A, u_i a_i) = d_{G_k}(A, u_i b_i) = 1$, $d_{G_k}(B, u_i a_i) = d_{G_k}(B, u_i b_i) = 2$ and $d_{G_k}(g_j, u_i a_i) = d_{G_k}(g_j, u_i b_i) = 5$. If there exists some i such that $(V(E_i) \cup \{h_i\}) \cap S = \emptyset$, then $u_i a_i, u_i b_i$ are unable to be distinguished by any element of S, which contradicts that S is an edge metric generator of G_k. Therefore, we have $|(V(E_i) \cup \{h_i\}) \cap S| \geq 1$ for $1 \leq i \leq k$, which implies that $edim(G_k) \geq 2k$. On the other hand, $S_1 = S_0 \cup \{a_1, a_2, \ldots, a_k\}$ is an edge metric generator of G_k. Indeed, for any edges $e_1 = mn, e_2 = st \in E(G_k)$ such that $\{m, n, s, t\} \cap S_1 \neq \emptyset$ and $\{m, n\} \cap \{s, t\} = \emptyset$, then e_1, e_2 can be distinguished by any element of $\{m, n, s, t\} \cap S_1$; for any edges $e_1 = mn, e_2 = mt \in E(G_k)$ such that $n \in \{m, n, t\} \cap S_1$ or $t \in \{m, n, t\} \cap S_1$, then e_1, e_2 can be distinguished by n or t; if $e_1 = a_i p_i, e_2 = a_i u_i$ $(1 \leq i \leq k)$, then $d_{G_k}(c_i, e_1) = 5$ and $d_{G_k}(c_i, e_2) = 4$; if $e_1 = c_i g_i, e_2 = c_i w_i$ $(1 \leq i \leq k)$, then $d_{G_k}(a_i, e_1) = 5$ and $d_{G_k}(a_i, e_2) = 4$. Now we focus on the edges whose end vertices are both in $V(G) \setminus S_1$. For edges $e_1 = B w_i$, $e_2 = B w_j$ $(1 \leq i \neq j \leq k)$, we have $d_{G_k}(c_i, e_1) = 1$ and $d_{G_k}(c_i, e_2) = 2$. For edges $e_1 = A u_i, e_2 = A u_j$ $(1 \leq i \neq j \leq k)$, we have $d_{G_k}(a_i, e_1) = 1$ and $d_{G_k}(a_i, e_2) = 2$. And if $e_1 = e = AB, e_2 = v_1 h_1$, then $d_{G_k}(a_1, e_1) = 2$ and $d_{G_k}(a_1, e_2) = 3$. Next we will check that the edges which have the same distance to a_i can be distinguished by c_i or a_j, and also the edges which have the same distance to c_i can be distinguished by a_i or c_j, where $1 \leq i \neq j \leq k$. Indeed, if $e_1 = p_1 x_1$, $e_2 = u_1 A, e_3 = u_1 b_1$, then $d_{G_k}(c_1, e_1) = 4$, $d_{G_k}(c_1, e_2) = 3$, $d_{G_k}(c_1, e_3) = 4$, $d_{G_k}(a_2, e_1) = 5$ and $d_{G_k}(a_2, e_3) = 3$. For edges $e_1 = x_1 v_1, e_2 = b_1 q_1, e_3 = e = AB$, we have $d_{G_k}(c_1, e_1) = 3$, $d_{G_k}(c_1, e_2) = 5$ and $d_{G_k}(c_1, e_3) = 2$. For edges $e_1 = v_1 h_1$, $e_2 = v_1 y_1, e_3 = q_1 r_1$, we have $d_{G_k}(c_1, e_1) = 2, d_{G_k}(c_1, e_2) = 3$ and $d_{G_k}(c_1, e_3) = 5$. For edges $e_1 = g_1 h_1, e_2 = g_1 d_1, e_3 = d_1 w_1$ and $e_4 = B w_1$, we have $d_{G_k}(c_2, e_1) = 5$, $d_{G_k}(c_2, e_2) = 4$, $d_{G_k}(c_2, e_3) = 3$ and $d_{G_k}(c_2, e_4) = 2$. And for other pair of edges, one can easily check that they can also be distinguished by some element of S_1, which implies that S_1 is indeed an edge metric generator of G_k. This means $edim(G_k) \leq 2k$ and hence $edim(G_k) = 2k$.
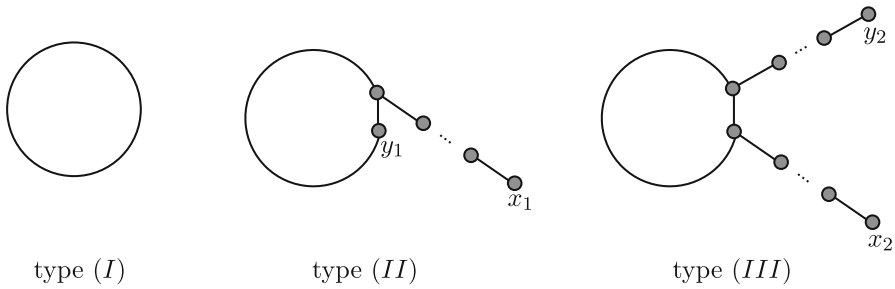
Fig. 3 The graphs $P_n + e$.

Next we will show that $edim(G_k - e) = k + 1$. For $1 \leq i \neq j \leq k$, $d_{G_k-e}(c_j, x_i v_i) = d_{G_k-e}(c_j, y_i v_i) = 7$, $d_{G_k-e}(c_i, x_i v_i) = d_{G_k-e}(c_i, y_i v_i) = 3$ and then $x_i v_i$, $y_i v_i$ are unable to be distinguished by any element of S_0. It follows that $edim(G_k - e) > |S_0| = k$, i.e., $edim(G_k - e) \geq k + 1$. Set $S_2 = S_0 \cup \{A\}$. One can check that S_2 is an edge metric generator of $G_k - e$, which implies that $edim(G_k - e) \leq k + 1$ and thus $edim(G_k - e) = k + 1$. Therefore, we have $edim(G_k) - edim(G_k - e) = k - 1$ and the theorem holds. □

In the rest of this section, we pay attention to the upper bound on $edim(G) - edim(G - e)$ for a unicyclic graph G.

Lemma 2.4 *Let P_n be a path of order n. If e is an edge of $\overline{P_n}$, then $edim(P_n + e) = edim(P_n) + 1 = 2$.*

Proof It is easy to verify that the graph $P_n + e$ has one of the following three types (see Fig. 3).

If $P_n + e$ has type (I), then $P_n + e \cong C_n$ and so $edim(P_n + e) = 2$.

When $P_n + e$ has type (II), we set $S = \{x_1, y_1\}$. We claim that S is an edge metric generator of $P_n + e$. Indeed, for any pair of edges $f_1, f_2 \in P_n + e$, it is impossible that $d(f_1, x_1) = d(f_2, x_1)$ and $d(f_1, y_1) = d(f_2, y_1)$ hold simultaneously. Thus $edim(P_n + e) \leq 2$. On the other side, any single vertex of $P_n + e$ is unable to distinguish all pairs of edges, which implies that $edim(P_n + e) \geq 2$. Consequently, $edim(P_n + e) = 2$.

Finally we consider the case that $P_n + e$ has type (III). Set $Q = \{x_2, y_2\}$. We will show that Q is an edge metric generator of $P_n + e$. One can check that for any pair of edges $g_1, g_2 \in P_n + e$, the equalities $d(g_1, x_2) = d(g_2, x_2)$ and $d(g_1, y_2) = d(g_2, y_2)$ are unable to hold at the same time. So we have $edim(P_n + e) \leq 2$. Simultaneously, $edim(P_n + e) \geq 2$ since any single vertex is not an edge metric generator of $P_n + e$. As a result, $edim(P_n + e) = 2$. □

Theorem 2.5 *Let T be a tree. If e is an edge of $\overline{T}$, then $edim(T + e) - edim(T) \leq 1$.*

Proof By Lemma 2.4, we know that $edim(T + e) = edim(T) + 1$ when T is a path. So we consider that T is a tree which is not a path. Obviously, $edim(T) \geq 2$. Denote the vertices of the unique cycle C_k in $T + e$ as $u_1, u_2, \ldots, u_k$ and the component of

$T + e - E(C_k)$ which contains u_i as T_i. Let S be an edge metric basis of T and set $S_i = S \cap V(T_i)$ for $1 \le i \le k$. Without loss of generality, we may assume $S_1 \ne \emptyset$.

If $S_i = \emptyset$ for $2 \le i \le k$, then $T - T_1$ is a path and thus $S \cup \{u_2\}$ or $S \cup \{u_k\}$ is an edge metric generator of $T + e$. Therefore, we have $edim(T + e) \le edim(T) + 1$. On the other hand, if there exists some i ($2 \le i \le k$) such that $S_i \ne \emptyset$, then we will complete the proof in two cases. If there exist i, j ($1 \le i < j \le k$) such that $S_i \ne \emptyset$, $S_j \ne \emptyset$ and $d_{T+e}(u_i, u_j) = \lfloor \frac{k}{2} \rfloor$, then take $x_0 \in V(C_k) \setminus \{u_i, u_j\}$ and set $S' = S \cup \{x_0\}$. Otherwise, we will take $x_0 = u_{\lfloor \frac{k}{2} \rfloor + 1}$ and set $S' = S \cup \{x_0\}$.

Next we will show that S' is an edge metric generator of $T + e$. Since the analyses of the two cases are similar, then we pay attention to the case that there exist i, j ($1 \le i < j \le k$) such that $S_i \ne \emptyset$, $S_j \ne \emptyset$ and $d_{T+e}(u_i, u_j) = \lfloor \frac{k}{2} \rfloor$. Set $S_0 = \{x_0, x_i, x_j\}$, where $x_i \in S_i$, $x_j \in S_j$ and thus $S_0 \subseteq S'$.

For any pair of edges $e_1, e_2 \in T_\ell (1 \le \ell \le k)$, there exists some $z \in S$ such that $d_T(e_1, z) \ne d_T(e_2, z)$. If $z \in V(T_\ell)$, then $d_{T+e}(e_1, z) = d_T(e_1, z)$, $d_{T+e}(e_2, z) = d_T(e_2, z)$ and thus $d_{T+e}(e_1, z) \ne d_{T+e}(e_2, z)$. If $z \in V(C_k)$, then $d_{T+e}(e_1, z) = d_{T+e}(e_1, u_\ell) + d_{T+e}(u_\ell, z)$ and $d_{T+e}(e_2, z) = d_{T+e}(e_2, u_\ell) + d_{T+e}(u_\ell, z)$. Since $d_T(e_1, z) = d_T(e_1, u_\ell) + d_T(u_\ell, z)$, $d_T(e_2, z) = d_T(e_2, u_\ell) + d_T(u_\ell, z)$ and $d_T(e_1, u_\ell) = d_{T+e}(e_1, u_\ell)$, $d_T(e_2, u_\ell) = d_{T+e}(e_2, u_\ell)$, we come to the conclusion that $d_{T+e}(e_1, z) \ne d_{T+e}(e_2, z)$. For the case $z \in V(T_{\ell'})(\ell' \ne \ell)$, we have $d_{T+e}(e_1, z) = d_{T+e}(e_1, u_\ell) + d_{T+e}(u_\ell, u_{\ell'}) + d_{T+e}(u_{\ell'}, z)$ and $d_{T+e}(e_2, z) = d_{T+e}(e_2, u_\ell) + d_{T+e}(u_\ell, u_{\ell'}) + d_{T+e}(u_{\ell'}, z)$. Note that $d_T(e_1, z) = d_T(e_1, u_\ell) + d_T(u_\ell, u_{\ell'}) + d_T(u_{\ell'}, z)$, $d_T(e_2, z) = d_T(e_2, u_\ell) + d_T(u_\ell, u_{\ell'}) + d_T(u_{\ell'}, z)$ and $d_T(e_1, u_\ell) = d_{T+e}(e_1, u_\ell)$, $d_T(e_2, u_\ell) = d_{T+e}(e_2, u_\ell)$. It follows that $d_{T+e}(e_1, z) \ne d_{T+e}(e_2, z)$.

Consider any two edges $f_1 \in T_\ell$, $f_2 \in T_{\ell'}$ ($1 \le \ell \ne \ell' \le k$). We denote the three segments of the cycle C_k as $x_0 C_k u_i$, $u_i C_k u_j$ and $u_j C_k x_0$. If $\{u_\ell, u_{\ell'}\} \subseteq x_0 C_k u_i$, without loss of generality, we may assume that $d_{T+e}(x_0, u_\ell) < d_{T+e}(x_0, u_{\ell'})$ and thus $d_{T+e}(u_i, u_\ell) > d_{T+e}(u_i, u_{\ell'})$. If further $d_{T+e}(f_1, x_0) = d_{T+e}(f_2, x_0)$, then $d_{T+e}(f_1, u_\ell) > d_{T+e}(f_2, u_{\ell'})$. Since $d_{T+e}(f_1, x_i) = d_{T+e}(f_1, u_\ell) + d_{T+e}(u_\ell, u_i) + d_{T+e}(u_i, x_i)$ and $d_{T+e}(f_2, x_i) = d_{T+e}(f_2, u_{\ell'}) + d_{T+e}(u_{\ell'}, u_i) + d_{T+e}(u_i, x_i)$, we have $d_{T+e}(f_1, x_i) > d_{T+e}(f_2, x_i)$. Similarly, we can obtain that f_1, f_2 can be distinguished by some element of S_0 for $\{u_\ell, u_{\ell'}\} \subseteq u_i C_k u_j$ or $\{u_\ell, u_{\ell'}\} \subseteq u_j C_k x_0$. Now we consider the case $u_\ell \in x_0 C_k u_i$ and $u_{\ell'} \in u_i C_k u_j$. If $d_{T+e}(f_1, u_\ell) = d_{T+e}(f_2, u_{\ell'})$ and $d_{T+e}(u_\ell, x_0) \ne d_{T+e}(u_{\ell'}, x_0)$, then we have $d_{T+e}(f_1, x_0) \ne d_{T+e}(f_2, x_0)$ since $d_{T+e}(f_1, x_0) = d_{T+e}(f_1, u_\ell) + d_{T+e}(u_\ell, x_0)$, $d_{T+e}(f_2, x_0) = d_{T+e}(f_2, u_{\ell'}) + d_{T+e}(u_{\ell'}, x_0)$. For the case that $d_{T+e}(f_1, u_\ell) = d_{T+e}(f_2, u_{\ell'})$ and $d_{T+e}(x_0, u_\ell) = d_{T+e}(x_0, u_{\ell'})$, we have $d_{T+e}(f_1, x_j) > d_{T+e}(f_2, x_j)$ since $d_{T+e}(f_1, x_j) = d_{T+e}(f_1, u_\ell) + d_{T+e}(u_\ell, u_j) + d_{T+e}(u_j, x_j)$, $d_{T+e}(f_2, x_j) = d_{T+e}(f_2, u_{\ell'}) + d_{T+e}(u_{\ell'}, u_j) + d_{T+e}(u_j, x_j)$ and $d_{T+e}(u_\ell, u_j) = d_{T+e}(u_\ell, x_0) + d_{T+e}(x_0, x_j) > d_{T+e}(u_{\ell'}, u_j)$. With similar analysis, for the cases $d_{T+e}(f_1, u_\ell) > d_{T+e}(f_2, u_{\ell'})$, $d_{T+e}(u_i, u_\ell) = d_{T+e}(u_i, u_{\ell'})$ and $d_{T+e}(f_1, u_\ell) > d_{T+e}(f_2, u_{\ell'})$, $d_{T+e}(u_i, u_\ell) > d_{T+e}(u_i, u_{\ell'})$, we have $d_{T+e}(f_1, x_i) > d_{T+e}(f_2, x_i)$. Meanwhile, if $d_{T+e}(f_1, u_\ell) > d_{T+e}(f_2, u_{\ell'})$ and also $d_{T+e}(u_i, u_\ell) < d_{T+e}(u_i, u_{\ell'})$, then $d_{T+e}(u_j, u_\ell) > d_{T+e}(u_j, u_{\ell'})$ and thus $d_{T+e}(f_1, x_j) > d_{T+e}(f_2, x_j)$. Furthermore, we can similarly obtain that f_1, f_2 can be distinguished by some element of S_0 when
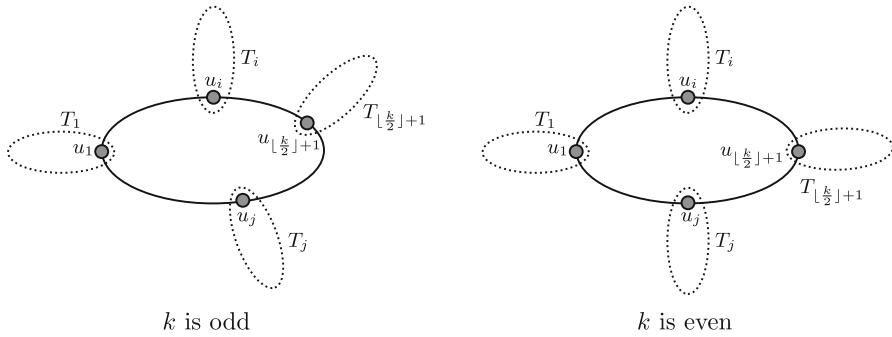
Fig. 4 The graph $T + e$.

$d_{T+e}(f_1, u_\ell) < d_{T+e}(f_2, u_{\ell'})$. For the cases $u_\ell \in u_i C_k u_j$ and $u_{\ell'} \in u_j C_k x_0$ or $u_\ell \in u_j C_k x_0$ and $u_{\ell'} \in x_0 C_k u_i$, the analysis is similar to that of $u_\ell \in x_0 C_k u_i$ and $u_{\ell'} \in u_i C_k u_j$, so we omit the details here.

For any pair of edges g_1, g_2, if $g_1, g_2 \in C_k$, then it is easy to check that g_1, g_2 can be distinguished by some element of S_0 and so we consider the case $g_1 \in C_k$, $g_2 \in T_\ell$ ($1 \leq \ell \leq k$). If g_1 lies on one shortest path from w to u_ℓ in $T+e$, where $w \in S_0$, then $d_{T+e}(g_1, w) < d_{T+e}(g_2, w)$. Otherwise, we claim that g_1, g_2 can be distinguished by some element of S_0. Here we take the case $g_1 \in u_j C_k x_0$ and $u_\ell \in u_i C_k u_j$ for example, and the other cases are similar. If $d_{T+e}(g_1, x_j) \neq d_{T+e}(g_2, x_j)$, then we finish the proof. Otherwise, $d_{T+e}(g_1, u_j) = d_{T+e}(g_2, u_j) = d_{T+e}(g_2, u_\ell) + d_{T+e}(u_\ell, u_j)$ since $d_{T+e}(g_1, x_j) = d_{T+e}(g_1, u_j) + d_{T+e}(u_j, x_j)$ and $d_{T+e}(g_2, x_j) = d_{T+e}(g_2, u_j) + d_{T+e}(u_j, x_j)$. If $d_{T+e}(g_2, u_\ell) \neq 0$, then we have $d_{T+e}(g_1, u_j) > d_{T+e}(u_\ell, u_j)$ and thus $d_{T+e}(g_1, u_i) < d_{T+e}(u_\ell, u_i)$. This implies that $d_{T+e}(g_1, x_i) < d_{T+e}(g_2, x_i)$. If $d_{T+e}(g_2, u_\ell) = 0$, then $d_{T+e}(g_1, u_j) = d_{T+e}(u_\ell, u_j)$ and thus $d_{T+e}(g_1, x_0) < d_{T+e}(u_\ell, x_0) = d_{T+e}(g_2, x_0)$. Therefore, S' is an edge metric generator of $T + e$ and thus we complete the proof. □

Consequently, if G is a unicyclic graph and e is an edge on its unique cycle, then by Lemma 2.4 and Theorem 2.5, we have $edim(G) - edim(G - e) \leq 1$.

In summary, we come to the following conclusion.

Theorem 2.6 *There exist some graph G and some edge $e \in E(G)$ such that $edim(G) - edim(G - e)$ can be arbitrarily large. While for any unicyclic graph H, $edim(H) - edim(H - e) \leq 1$, where e is an edge of the unique cycle in H.*

3 The effect of vertex deletion on edge metric dimention of graphs

The join graph $G + H$ is the graph obtained from G and H by adding all possible edges between any vertex of G and any vertex of H. Obviously, the wheel graph $W_{1,n}$ is isomorphic to $C_n + K_1$ and we refer to the vertex of degree n in $W_{1,n}$ as its center vertex.
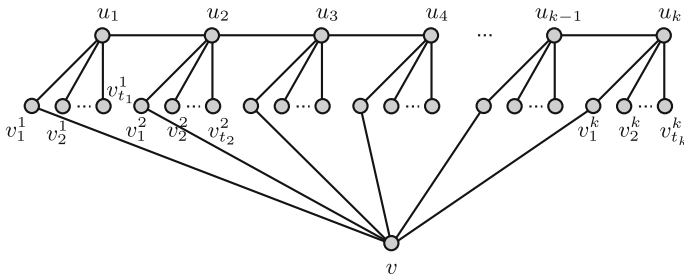
Fig. 5 The graph $G_{k,t}$.

Proposition 3.1 *Kelenc et al. (2018) If $W_{1,n}$ is a wheel graph, then*

$$edim(W_{1,n}) = \begin{cases} n & if\ n = 3, 4; \\ n-1 & if\ n \geq 5. \end{cases}$$

Proposition 3.2 *Kelenc et al. (2018) For any integer $n \geq 3$, $edim(C_n) = 2$.*

From Propositions 3.1 and 3.2, we can obtain the following observation immediately.

Observation 3.3 *Let $k \geq 2$ be an positive integer. If v_0 is the center vertex of the wheel graph $W_{1,k+3}$, then $edim(W_{1,k+3}) - edim(W_{1,k+3} - v_0) = k$.*

Let $k \geq 8, t \geq 24$ be positive integers and $P = u_1 u_2 \ldots u_k$ be a path. For $1 \leq i \leq k$, attach t_i (≥ 3) pendant vertices $v_1^i, v_2^i, \ldots, v_{t_i}^i$ to u_i and denote the resulting graph by $H_{k,t}$, where $t = \sum_{i=1}^{k} t_i$. Let $G_{k,t}$ be the graph with vertex set $V(H_{k,t}) \cup \{v\}$ and edge set $E(H_{k,t}) \cup \{vv_1^1, vv_1^2, \ldots, vv_1^k\}$ (see Fig. 5).

Lemma 3.4 *Let $k \geq 8$, $t \geq 24$ be positive integers. If $v \in V(G_{k,t})$ is the vertex as depicted in Figure 5, then $edim(G_{k,t} - v) - edim(G_{k,t}) = k$.*

Proof Since $G_{k,t} - v$ is a tree, we have $edim(G_{k,t} - v) = t - k$ by Proposition 2.1. Next we will show that $edim(G_{k,t}) = t - 2k$, which implies that $edim(G_{k,t} - v) - edim(G_{k,t}) = k$. Set $S = \{v_3^1, \ldots, v_{t_1}^1\} \cup \{v_3^2, \ldots, v_{t_2}^2\} \cup \cdots \cup \{v_3^k, \ldots, v_{t_k}^k\}$. For any pair of edges $e = x_1 y_1$, $f = x_2 y_2 \in E(G_{k,t})$, if $\{x_1, x_2, y_1, y_2\} \cap S \neq \emptyset$, then e, f can be distinguished by any element of $S \cap \{x_1, x_2, y_1, y_2\}$. Hence we can assume that $e, f \in \{u_i u_{i+1}, vv_1^i, u_i v_1^i, u_i v_2^i : 1 \leq i \leq k\}$.

If $e = u_i u_{i+1}$, $f = u_j u_{j+1}$ $(i < j)$, then $d(e, v_3^i) = 1, d(f, v_3^i) \geq 2$ and e, f can be distinguished by $v_3^i \in S$. If $e = vv_1^i$, $f = vv_1^j$, then $d(e, v_3^i) = 2, d(f, v_3^i) = 3$ and e, f can be distinguished by $v_3^i \in S$. For $e = u_i v_1^i$, $f = u_j v_1^j$, we have that e, f can be distinguished by $v_3^i \in S$ since $d(e, v_3^i) = 1$ and $d(f, v_3^i) \geq 2$. And for $e = u_i v_2^i$, $f = u_j v_2^j$, one can easily verify that $d(e, v_3^i) = 1, d(f, v_3^i) \geq 2$ and then e, f can be distinguished by $v_3^i \in S$.

For the case that $e = u_i u_{i+1}$, $f = v v_1^i$, we have $d(e, v_3^i) = 1, d(f, v_3^i) = 2$ and e, f can be distinguished by $v_3^i \in S$. If $e = u_i u_{i+1}$, $f = v v_1^j$ $(i \neq j)$, then $d(e, v_3^i) = 1$, $d(f, v_3^i) = 3$ and e, f can be distinguished by $v_3^i \in S$. If $e = u_i u_{i+1}$, $f = u_i v_1^i$ or $e = u_i u_{i+1}$, $f = u_i v_2^i$, then $d(e, v_3^{i+1}) = 1, d(f, v_3^{i+1}) = 2$ and e, f can be distinguished by $v_3^{i+1} \in S$. For $e = u_i u_{i+1}$, $f = u_j v_1^j (i \neq j)$ or $e = u_i u_{i+1}$, $f = u_j v_2^j (i \neq j)$, we have that e, f can be distinguished by $v_3^i \in S$ since $d(e, v_3^i) = 1$ and $d(f, v_3^i) \geq 2$.

Next we come to the case that $e = v v_1^i$, $f = u_i v_1^i$ or $e = v v_1^i$, $f = u_i v_2^i$. Obviously, $d(e, v_3^i) = 2$, $d(f, v_3^i) = 1$ and thus e, f can be distinguished by $v_3^i \in S$. If $e = v v_1^i$, $f = u_j v_1^j (i \neq j)$ or $e = v v_1^i$, $f = u_j v_2^j (i \neq j)$, then $d(e, v_3^i) = 3, d(f, v_3^i) = 1$ and e, f can be distinguished by $v_3^i \in S$. For $e = u_i v_1^i$, $f = u_i v_2^i$, we have $d(e, v_3^{i+4}) = 4, d(f, v_3^{i+4}) = 5$ for $i \leq k - 4$ and $d(e, v_3^{i-4}) = 4, d(f, v_3^{i-4}) = 5$ for $k - 3 \leq i \leq k$ and thus e, f can be distinguished by $v_3^{i+4} \in S$ (or $v_3^{i-4} \in S$). Finally, for $e = u_i v_1^i$, $f = u_j v_2^j (i \neq j)$, we obtain that e, f can be distinguished by $v_3^i \in S$ since $d(e, v_3^i) = 1, d(f, v_3^i) \geq 2$.

Therefore, S is an edge metric generator of $G_{k,t}$ and thus $edim(G_{k,t}) \leq t - 2k$. On the other hand, let W be an arbitrary edge metric generator of $G_{k,t}$. If there exist some i ($1 \leq i \leq k$) such that $|W \cap \{v_2^i, \ldots, v_{t_i}^i\}| < t_i - 2$, then the edges $u_i v_a^i$ and $u_i v_b^i$ are unable to be distinguished by any element of W, where $v_a^i, v_b^i \in \{v_2^i, \ldots, v_{t_i}^i\} \backslash W$. We arrive at a contradiction and thus $edim(G_{k,t}) \geq t - 2k$. Therefore, $edim(G_{k,t}) = t - 2k$ and then $edim(G_{k,t} - v) - edim(G_{k,t}) = k$, as desired. $\qquad \square$

From Observation 3.3 and Lemma 3.4, we arrive at the following conclusion.

Theorem 3.5 *For any positive integer k, there exist graphs G and H such that both $edim(G) - edim(G - u)$ and $edim(H - v) - edim(H)$ can be at least k, where $u \in V(G)$ and $v \in V(H)$.*

4 Concluding remarks

In this paper, we analyze what happens with the edge metric dimension if one edge or one vertex is deleted. For any graph G and any edge $e \in E(G)$, $edim(G - e) - edim(G) \leq 2$. On the other hand, the value of $edim(G) - edim(G - e)$ is unbounded for some graph G and some edge $e \in E(G)$. However, for a unicyclic graph H, $edim(H) - edim(H - e) \leq 1$, where e is an edge of the unique cycle in H. Additionally, the difference between the edge metric dimension of the original graph and the resulting graph after vertex deletion can be arbitrarily large. And as a future work, it's interesting to think about whether it is possible to bound the corresponding difference between the edge metric dimension of the original graph and the resulting graph using the vertex degree.

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