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Adjacent vertex distinguishing edge coloring of IC-planar graphs

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Abstract

The adjacent vertex distinguishing edge coloring of a graph G is a proper edge coloring in which each pair of adjacent vertices is assigned different color sets. The smallest number of colors for which G has such a coloring is denoted by $\chi'_a(G)$. An important conjecture due to Zhang et al. (Appl Math Lett 15:623–626, 2002) asserts that $\chi'_a(G) \leq \Delta(G)+2$ for any connected graph G with order at least 6. By applying the discharging method, we show that this conjecture is true for any IC-planar graph G with $\Delta(G) \geq 16$.

Keywords IC-planar graph · Adjacent vertex distinguishing edge coloring · Discharging method

1 Introduction

Throughout this paper, we are only concerned with finite and simple graphs. For a plane graph G, let $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ be the vertex set, edge set, face set, maximum degree and minimum degree of G, respectively. For an arbitrary $x \in V(G) \cup F(G)$, let $d_G(x)$ denote the degree of x in G. Let $N_G(v)$ denote the set of neighbors of a vertex v in G. A vertex v satisfying $d_G(v) = k$ ($d_G(v) \geq k$, $d_G(v) \leq k$) is a *k-vertex* (*k^+-vertex*, *k^--vertex*). The *k-face* and *k^+-face* are defined similarly. For each $v \in V(G)$, let $d^k_G(v)$ denote the number of k-vertices adjacent to v in G. We call a 3-vertex $v \in V(G)$ *bad* if $d^3_G(v) = 1$ and *good* if $d^3_G(v) = 0$. Let $d^{3b}_G(v)$ and $d^{3g}_G(v)$ denote the number of bad and good 3-vertices adjacent to v in G, respectively. A 3-face (or cycle) $v_1v_2v_3$ is called a (k_1, k_2, k_3)-*face* (or *cycle*) if v_i is a k_i-vertex for all $1 \leq i \leq 3$. A 3-cycle is *bad* if it is incident with two 3-vertices. Any undefined notation can refer to (Bondy and Murty 1976).

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A *proper k-edge coloring* of a graph G is a mapping $\varphi : E(G) \to \{1, 2, \ldots, k\}$ such that $\varphi(e) \neq \varphi(e')$ for any two adjacent edges e and e' of G. For any $v \in V(G)$, let $C_\varphi(v) = \{\varphi(uv)|uv \in E(G)\}$ be the color set of v with respect to φ. For two adjacent vertices u and v, we call *u conflict with v* respect to φ if $C_\varphi(u) = C_\varphi(v)$. A proper k-edge coloring φ is *a k-adjacent vertex distinguishing edge coloring* (*k-avd-coloring* for short) provided that $C_\varphi(u) \neq C_\varphi(v)$ for all $uv \in E(G)$. The *adjacent vertex distinguishing edge chromatic index* of G, denoted by $\chi'_a(G)$, is the smallest k such that G has a k-avd-coloring. A graph without isolated edges is *normal*. Clearly, only normal graph can have avd-colorings. Thus, for avd-coloring, we only consider normal graphs.

Zhang et al. (2002) first introduced the concept of avd-coloring and put forward the following conjecture.

Conjecture 1 Zhang et al. (2002) *If G is a connected graph with order at least 6, then* $\chi'_a(G) \leq \Delta(G) + 2$.

Conjecture 1 was determined by Balister et al. (2007) for bipartite graphs and graphs with maximum degree 3. Horňák et al. (2014) showed that Conjecture 1 holds for planar graphs with maximum degree at least 12. Bonamy et al. (2013) verified that $\chi'_a(G) \leq \Delta(G) + 1$ for any planar graph G with $\Delta(G) \geq 12$. Wang and Huang (2015) proved that $\chi'_a(G) \leq \Delta(G) + 1$ for any planar graph G with $\Delta(G) \geq 16$ and $\chi'_a(G) = \Delta(G) + 1$ if and only if G contains two adjacent vertices of maximum degree.

A graph is *1-planar* if it can be drawn in the plane such that each edge is crossed by at most one other edge. Albertson (2008) first introduced the definition of IC-planar graph. A graph is *IC-planar* if it admits a drawing in the plane where each edge is crossed at most once and no two crossings are incident with the same vertex. Clearly, each IC-planar graph is 1-planar. The *associated plane graph* $G^\times$ of a 1-planar graph G is a plane graph obtained by turning all crossings of G into new 4-vertices. A vertex $v \in V(G^\times)$ is *false* if v is not a vertex of G and *real* otherwise. A face is *false* if it is incident with at least one false vertex. Clearly, for an associated plane graph $G^\times$ of an IC-planar graph G, each real vertex in $G^\times$ is adjacent to at most one false vertex and incident with at most two false 3-faces in $G^\times$. In the following, we always assume that every IC-planar graph is drawn in a plane such that the number of crossings is as few as possible.

Lemma 1 Zhang and Wu (2011) *Let G be a 1-plane graph and* $G^\times$ *be the associated plane graph of G. If* $d_G(u) = 3$ *and v is a false vertex of* $G^\times$*, then either* $uv \notin E(G^\times)$ *or uv is not incident with two 3-faces.*

In this paper, we will prove that Conjecture 1 is true for any IC-planar graph with maximum degree at least 16, which can be expressed more concisely as follows:

Theorem 1 *Let G be an IC-planar graph, then* $\chi'_a(G) \leq \max\{\Delta(G) + 2, 18\}$.

2 The proof of Theorem 1

We will prove Theorem 1 by contradiction. Let G be a counterexample to Theorem 1 minimizing $|V(G)|+|E(G)|$. Clearly, G is a connected graph. Let $t_G = \max\{\Delta(G)+2, 18\}$ and $C = \{1, 2, \ldots, t_G\}$. Then $\{1, 2, \ldots, 18\} \subseteq C$. First we will prove the following claims.

Claim 1 *There is no edge $uv \in E(G)$ with $d_G(u) = 1$ and $d_G(v) \leq 9$.*

Proof Assume, to the contrary, that G contains an edge uv with $d_G(u) = 1$ and $d_G(v) \leq 9$. We have $d_G(v) \geq 2$ because G is normal. Let $H = G - u$. If H contains only one edge, then we color this edge with 1 and color uv with 2 to obtain a t_G-avd-coloring of G, a contradiction. If H contains at least two edges, H has a t_G-avd-coloring φ with the color set C by the minimality of G. Note that v has at most eight conflict vertices. Hence we can color uv with a color in $C \setminus C_\varphi(v)$ such that v does not conflict with its neighbors, which yields a t_G-avd-coloring of G, a contradiction. □

Remark 1 Claim 1 implies that for an arbitrary $e \in E(G)$, $H = G - e$ is normal. Therefore $\chi'_a(H) \leq t_G$ by the minimality of G.

Remark 2 In the following, if $d_G(v) = k$, set $N_G(v) := \{v_1, v_2, \ldots, v_k\}$.

Claim 2 *Let v be a k-vertex of G with $2 \leq k \leq 6$, then $d^k_G(v) \leq 1$.*

Proof Assume, to the contrary, that G contains a k-vertex v ($2 \leq k \leq 6$) satisfying $d^k_G(v) \geq 2$. We prove the case that $k = 6$ (the proof can be given similarly and simply for $2 \leq k \leq 5$). Assume that $d_G(v_1) = d_G(v_2) = 6$. Let $N_G(v_1) = \{v, w_1, w_2, w_3, w_4, w_5\}$. Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. Without loss of generality (W.l.o.g.), $\varphi(vv_i) = i - 1$ for $2 \leq i \leq 6$ and $\varphi(v_1w_i) = a_i$ for $1 \leq i \leq 5$. We consider the next three cases.

Case 1: $3 \leq |\{a_1, a_2, \ldots, a_5\} \cap \{1, 2, \ldots, 5\}| \leq 5$. If $|\{a_1, a_2, \ldots, a_5\} \cap \{1, 2, \ldots, 5\}| = 5$, then we recolor vv_2 with a color in $C \setminus (C_\varphi(v) \cup C_\varphi(v_2))$ such that v_2 does not conflict with its neighbors. So we may assume that $3 \leq |\{a_1, a_2, \ldots, a_5\} \cap \{1, 2, \ldots, 5\}| \leq 4$. Hence we can color vv_1 with a color in $C \setminus (C_\varphi(v) \cup C_\varphi(v_1))$ such that v and v_1 do not conflict with their neighbors, which yields a t_G-avd-coloring of G, a contradiction.

Case 2: $1 \leq |\{a_1, a_2, \ldots, a_5\} \cap \{1, 2, \ldots, 5\}| \leq 2$. Set $|\{a_1, a_2, \ldots, a_5\} \cap \{1, 2, \ldots, 5\}| = l$, then $1 \leq l \leq 2$. W.l.o.g., $a_i = i$ for $1 \leq i \leq l$ and $a_i = i - l + 5$ for $l + 1 \leq i \leq 5$. Suppose that vv_1 cannot be colored without causing conflicts, say, $C_\varphi(v_i) = \{1, 2, 3, 4, 5, i - l + 9\}$ for $2 \leq i \leq 6$ and $C_\varphi(w_i) = \{1, 6, 7, 8, 16 - 7l, i - l + 15\}$ for $1 \leq i \leq l + 3$. We recolor vv_2 with a color in $\{13, 14, \ldots, 18\}$ such that v_2 does not conflict with its neighbors, then we color vv_1 with a color in $\{11, 12\}$ such that v_1 does not conflict with its neighbors, which yields a t_G-avd-coloring of G, a contradiction.

Case 3: $|\{a_1, a_2, \ldots, a_5\} \cap \{1, 2, \ldots, 5\}| = 0$. W.l.o.g., $a_i = i + 5$ for $1 \leq i \leq 5$. Suppose that vv_1 cannot be colored without causing conflicts, say, $C_\varphi(v_i) = \{1, 2, 3, 4, 5, i + 9\}$ for $2 \leq i \leq 6$ and $C_\varphi(w_i) = \{6, 7, 8, 9, 10, i + 15\}$ for

$1 \leq i \leq 3$, or $C_\varphi(v_i) = \{1, 2, 3, 4, 5, i+9\}$ for $2 \leq i \leq 5$ and $C_\varphi(w_i) = \{6, 7, 8, 9, 10, i+14\}$ for $1 \leq i \leq 4$. If $C_\varphi(v_i) = \{1, 2, 3, 4, 5, i+9\}$ for $2 \leq i \leq 6$ and $C_\varphi(w_i) = \{6, 7, 8, 9, 10, i+15\}$ for $1 \leq i \leq 3$, then we recolor vv_2 with a color in $\{6, 7, 8, 16, 17, 18\}$ such that v_2 does not conflict with its neighbors, and color vv_1 with a color in $\{12, 13, 14\}$ such that v_1 does not conflict with w_4 and w_5, which yields a t_G-avd-coloring of G, a contradiction. If $C_\varphi(v_i) = \{1, 2, 3, 4, 5, i+9\}$ for $2 \leq i \leq 5$ and $C_\varphi(w_i) = \{6, 7, 8, 9, 10, i+14\}$ for $1 \leq i \leq 4$, then we recolor vv_2 with a color in $\{6, 7, 8, 9, 10, 18\}$ such that v_2 does not conflict with its neighbors, and color vv_1 with a color in $\{12, 13, 14\}$ such that v and v_1 do not conflict with their neighbors, which yields a t_G-avd-coloring of G, a contradiction. □

Claim 3 *There is no edge $vv_1 \in E(G)$ with $2 \leq d_G(v_1) \leq 6$ and $d_G(v_1) + 1 \leq d_G(v) \leq 9$.*

Proof Assume, to the contrary, that G contains an edge vv_1 with $2 \leq d_G(v_1) \leq 6$ and $d_G(v_1) + 1 \leq d_G(v) \leq 9$. We prove the case that $d_G(v_1) = 6$ and $d_G(v) = 9$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. W.l.o.g., $\varphi(vv_i) = i - 1$ for $2 \leq i \leq 9$ and $C_\varphi(v_1) \subseteq \{1, 2, \ldots, 13\}$. By Claim 2, every 6-vertex has at most one conflict vertex. Suppose that vv_1 cannot be colored without causing conflicts, say, $C_\varphi(v_i) = \{1, 2, \ldots, 8, i+12\}$ for $2 \leq i \leq 5$ and $C_\varphi(v_1) = \{9, 10, ..., 13\}$. Without considering the conflict of v, for any given integer i $(2 \leq i \leq 5)$, we select $\{b_i, d_i\}$ from $\{9, 10, \ldots, 18\} \setminus \{i+12\}$ to recolor vv_i and color vv_1 such that v_i and v_1 do not conflict with their neighbors. $\{b_i, d_i\}$ has at least two selected ways. Since i has four possibilities, we have at least $2 \times 4 = 8$ ways such that v_1 does not conflict with its neighbors and v does not conflict with v_2, v_3, v_4 and v_5, while v has at most four conflict vertices other than v_2, v_3, v_4 and v_5. So we can obtain a t_G-avd-coloring of G, a contradiction. □

Claim 4 *Let v be a k-vertex of G with $10 \leq k \leq 11$, then $d_G^{(16-k)^-}(v) \leq 1$.*

Proof Assume, to the contrary, that G contains a k-vertex v $(10 \leq k \leq 11)$ satisfying $d_G^{(16-k)^-}(v) \geq 2$. Suppose that $d_G(v_1) = d_G(v_2) = 16 - k$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. W.l.o.g., $\varphi(vv_i) = i - 1$ for $2 \leq i \leq k$. Clearly, $|C_\varphi(v_i) \cap \{k, k+1, \ldots, 18\}| \leq 15 - k$ for $1 \leq i \leq 2$. By Claim 2, every 6^--vertex has at most one conflict vertex. If v_i has a conflict vertex w_i, and $|C_\varphi(v_i) \cap \{k, k+1, \ldots, 18\}| = 15 - k$ for $1 \leq i \leq 2$, then we recolor $v_i w_i$ with a color in $\{2, 3, \ldots, 9\} \setminus C_\varphi(w_i)$. Without considering the conflict of v, we have the following two types of proper colorings. (a): We color vv_1 with a color in $\{k, k+1, \ldots, 18\}$ such that v_1 does not conflict with its neighbors. There are at least four available colors. (b): We select $\{b_1, b_2\}$ from $\{k, k+1, \ldots, 18\}$ to recolor vv_2 and color vv_1 such that v_2 and v_1 do not conflict with their neighbors. $\{b_1, b_2\}$ has at least $\frac{4\times 3}{2} = 6$ selected ways. Hence we have at least $4 + 6 = 10$ ways, while v has at most $k - 2 \leq 9$ conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction. □

Claim 5 *Let v be a 12-vertex of G, then $d_G^{3^-}(v) \leq 1$.*

Proof Assume, to the contrary, that G contains a 12-vertex v satisfying $d_G^{3^-}(v) \geq 2$. Suppose that $d_G(v_1) = d_G(v_2) = 3$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. W.l.o.g., $\varphi(vv_i) = i-1$ for $2 \leq i \leq 12$. Clearly, $|C_\varphi(v_i) \cap \{12, 13, \ldots, 18\}| \leq 2$ for $1 \leq i \leq 2$. By Claim 2, each 3-vertex has at most one conflict vertex. If v_i has a conflict vertex w_i for $1 \leq i \leq 2$, we assume that $\varphi(v_i w_i) \notin \{12, 13, \ldots, 18\}$ (if $\varphi(v_i w_i) \in \{12, 13, \ldots, 18\}$, then we recolor $v_i w_i$ with a color in $\{2, 3, \ldots, 11\} \setminus (C_\varphi(v_i) \cup C_\varphi(w_i))$ to satisfy this condition). Without considering the conflict of v, we have the following two types of proper colorings. (a): We color vv_1 with a color in $\{12, 13, \ldots, 18\}$ such that v_1 does not conflict with its neighbors. There are at least five available colors. (b): We select $\{b_1, b_2\}$ from $\{12, 13, \ldots, 18\}$ to recolor vv_2 and color vv_1 such that v_2 and v_1 do not conflict with their neighbors. $\{b_1, b_2\}$ has at least $\frac{5\times 4}{2} = 10$ selected ways. Hence we have at least $5 + 10 = 15$ ways, while v has at most ten conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction. □

Claim 6 *Let v be a k-vertex of G with $11 \leq k \leq 12$, then $d_G^{6^-}(v) \leq 3k - 31$.*

Proof Assume, to the contrary, that G contains a k-vertex v $(11 \leq k \leq 12)$ satisfying $d_G^{6^-}(v) \geq 3k - 30$. Suppose that $d_G(v_i) = 6$ for $1 \leq i \leq 3k - 30$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. W.l.o.g., $\varphi(vv_i) = i - 1$ for $2 \leq i \leq k$. Clearly, $|C_\varphi(v_i) \cap \{k, k+1, \ldots, 18\}| \leq 5$ for $1 \leq i \leq 3k - 30$. By Claim 2, each 6-vertex has at most one conflict vertex. If v_i has a conflict vertex w_i, and $|C_\varphi(v_i) \cap \{k, k+1, \ldots, 18\}| = 5$ for $1 \leq i \leq 3k - 30$, then we recolor $v_i w_i$ with a color in $\{3k-30, 3k-29, \ldots, k-1\} \setminus C_\varphi(w_i)$. Without considering the conflict of v, we have the following two types of proper colorings. (a): We color vv_1 with a color in $\{k, k+1, \ldots, 18\}$ such that v_1 does not conflict with its neighbors. There are at least $14 - k$ available colors. (b): For any given integer i $(2 \leq i \leq 3k - 30)$, we select $\{b_i, d_i\}$ from $\{k, k+1, \ldots, 18\}$ to recolor vv_i and color vv_1 such that v_i and v_1 do not conflict with their neighbors. $\{b_i, d_i\}$ has at least $\frac{(14-k)\times(13-k)}{2}$ selected ways. Since i has $3k - 31$ possibilities, we have at least $\frac{(14-k)\times(13-k)}{2} \times (3k - 31) = 17 - k$ different coloring ways. Hence we have at least $14 - k + 17 - k = 31 - 2k$ ways, while v has at most $k - (3k - 30) = 30 - 2k$ conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction. □

Claim 7 *Let v be a k-vertex of G with $13 \leq k \leq 14$, then the following statements hold.*

(1) $d_G^{2^-}(v) \leq k - 12$;

(2) *If* $d_G^{m^-}(v) \geq 1$ *for* $m \leq 18 - k$, *then* $d_G^k(v) \geq (19 - k - m)d_G^{(19-k)^-}(v) + 1$.

Proof (1) Assume, to the contrary, that G contains a k-vertex v $(13 \leq k \leq 14)$ satisfying $d_G^{2^-}(v) \geq k - 11$. Suppose that $d_G(v_i) = 2$ for $1 \leq i \leq k - 11$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. W.l.o.g., $\varphi(vv_i) = i - 1$ for $2 \leq i \leq k$. Clearly, $|C_\varphi(v_i) \cap \{k, k+1, \ldots, 18\}| \leq 1$ for $1 \leq i \leq k - 11$. By Claim 2, each 2-vertex has at most one conflict vertex. If v_i has a conflict vertex w_i for $1 \leq i \leq k-11$, we assume that $\varphi(v_iw_i) \notin \{k, k+1, \ldots, 18\}$ (if $\varphi(v_iw_i) \in \{k, k+1, \ldots, 18\}$, then we recolor v_iw_i with a color in $\{3, 4, \ldots, 12\} \setminus (C_\varphi(v_i) \cup C_\varphi(w_i))$ to satisfy this condition). Without considering the conflict of v, we have the following two types of proper colorings. (a): We color vv_1 with a color in $\{k, k+1, \ldots, 18\}$ such that v_1 does not conflict with its neighbors. There are at least $18 - k \geq 4$ available colors. (b): For any given integer i $(2 \leq i \leq k - 11)$, we select $\{b_i, d_i\}$ from $\{k, k+1, \ldots, 18\}$ to recolor vv_i and color vv_1 such that v_i and v_1 do not conflict with their neighbors. $\{b_i, d_i\}$ has at least $\frac{(18-k)(17-k)}{2}$ selected ways. Since i has $k - 12$ possibilities, we have at least $\frac{(18-k)(17-k)}{2} \times (k - 12) \geq 10$ different coloring ways. Hence we have at least $4 + 10 = 14$ ways, while v has at most eleven conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction.

(2) Assume, to the contrary, that there is a k-vertex $v \in V(G)$ $(13 \leq k \leq 14)$ and an integer m $(m \leq 18 - k)$ satisfying $d_G^{m^-}(v) \geq 1$, where $d_G^k(v) \leq (19 - k - m)d_G^{(19-k)^-}(v)$. Set $d_G^{(19-k)^-}(v) = l$. W.l.o.g., $d_G(v_1) = m$ and $d_G(v_i) \leq 19 - k$ for $1 \leq i \leq l$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. Suppose that $\varphi(vv_i) = i - 1$ for $2 \leq i \leq k$. Clearly, $|C_\varphi(v_i) \cap \{k, k+1, \ldots, 18\}| \leq 18 - k$ for $1 \leq i \leq l$. By Claim 2, each 6^--vertex has at most one conflict vertex. If v_i has a conflict vertex w_i, and $|C_\varphi(v_i) \cap \{k, k+1, \ldots, 18\}| = d_G(v_i) - 1$ for $1 \leq i \leq l$, then we recolor v_iw_i with a color in $\{7, 8, \ldots, 12\} \setminus C_\varphi(w_i)$. Without considering the conflict of v, we have the following two types of proper colorings. (a): We color vv_1 with a color in $\{k, k+1, \ldots, 18\}$ such that v_1 does not conflict with its neighbors. There are at least $20 - k - m$ available colors. (b): For any given integer i $(2 \leq i \leq l)$, we select $\{b_i, d_i\}$ from $\{k, k+1, \ldots, 18\}$ to recolor vv_i and color vv_1 such that v_i and v_1 do not conflict with their neighbors. $\{b_i, d_i\}$ has at least $19-k-m$ selected ways. Since i has $l - 1$ possibilities, we have at least $(19 - k - m)(l - 1)$ different coloring ways. Hence we have at least $(20-k-m)+(19-k-m)(l-1) = (19-k-m)l+1$ ways, while v has at most $(19-k-m)l$ conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction. □

Claim 8 *Let v be a 15-vertex of G, then the following statements hold.*
(1) $d_G^{2^-}(v) \leq 3$;
(2) *If* $d_G^{2^-}(v) \geq 1$, *then* $d_G^{3^-}(v) \leq 4$;
(3) *If* $d_G^{m^-}(v) \geq 1$ *for* $m \leq 3$, *then* $d_G^{15}(v) \geq (4 - m)d_G^{4^-}(v) + 1$;
(4) *If v is incident with a bad 3-cycle, then* $d_G^{15}(v) \geq 9$.

Proof (1) Assume, to the contrary, that G contains a 15-vertex v satisfying $d_G^{2^-}(v) \geq 4$. Suppose that $d_G(v_i) = 2$ for $1 \leq i \leq 4$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. Suppose that $\varphi(vv_i) = i - 1$ for $2 \leq i \leq 15$. Clearly, $|C_\varphi(v_i) \cap \{15, 16, 17, 18\}| \leq 1$ for $1 \leq i \leq 4$. By Claim 2, each 2-vertex has at most one conflict vertex. If v_i has a conflict vertex w_i for $1 \leq i \leq 4$, we assume that $\varphi(v_iw_i) \notin \{15, 16, 17, 18\}$ (if $\varphi(v_iw_i) \in \{15, 16, 17, 18\}$, then we recolor v_iw_i with a color in $\{4, 5, \ldots, 14\} \setminus (C_\varphi(v_i) \cup C_\varphi(w_i))$ to satisfy this condition). Without considering the conflict of v, we have the following two types of proper colorings.

(a): We color vv_1 with a color in {15, 16, 17, 18} such that v_1 does not conflict with its neighbors. There are at least three available colors. (b): For any given integer i $(2 \leq i \leq 4)$, we select $\{b_i, d_i\}$ from {15, 16, 17, 18} to recolor vv_i and color vv_1 such that v_i and v_1 do not conflict with their neighbors. $\{b_i, d_i\}$ has at least three selected ways. Since i has three possibilities, we have at least $3 \times 3 = 9$ different coloring ways. Hence we have at least $3 + 9 = 12$ ways, while v has at most eleven conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction.

(2) Assume, to the contrary, that G contains a 15-vertex v satisfying $d_G^{2^-}(v) \geq 1$, where $d_G^{3^-}(v) \geq 5$. Suppose that $d_G(v_1) = 2$ and $d_G(v_i) = 3$ for $2 \leq i \leq 5$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. W.l.o.g., $\varphi(vv_i) = i - 1$ for $2 \leq i \leq 15$. Clearly, $|C_\varphi(v_i) \cap \{15, 16, 17, 18\}| \leq 2$ for $1 \leq i \leq 5$. By Claim 2, each 3^--vertex has at most one conflict vertex. If v_i has a conflict vertex w_i for $1 \leq i \leq 5$, we assume that $\varphi(v_i w_i) \notin \{15, 16, 17, 18\}$ (if $\varphi(v_i w_i) \in \{15, 16, 17, 18\}$, then we recolor $v_i w_i$ with a color in $\{8, 9, \ldots, 14\} \setminus (C_\varphi(v_i) \cup C_\varphi(w_i))$ to satisfy this condition). Without considering the conflict of v, we have the following two types of proper colorings. (a): We color vv_1 with a color in {15, 16, 17, 18} such that v_1 does not conflict with its neighbors. There are at least three available colors. (b): For any given integer i $(2 \leq i \leq 5)$, we select $\{b_i, d_i\}$ from {15, 16, 17, 18} to recolor vv_i and color vv_1 such that v_i and v_1 do not conflict with their neighbors. $\{b_i, d_i\}$ has at least two selected ways. Since i has four possibilities, we have at least $2 \times 4 = 8$ different coloring ways. Hence we have at least $3 + 8 = 11$ ways, while v has at most ten conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction.

(3) Assume, to the contrary, that there is a 15-vertex $v \in V(G)$ and an integer m $(m \leq 3)$ satisfying $d_G^{m^-}(v) \geq 1$, where $d_G^{15}(v) \leq (4 - m) d_G^{4^-}(v)$. Set $d_G^{4^-}(v) = l$. Suppose that $d_G(v_1) = m$ and $d_G(v_i) \leq 4$ for $1 \leq i \leq l$ (the proof can be given similarly and simply for other cases). Let $H = G - vv_1$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. Suppose that $\varphi(vv_i) = i - 1$ for $2 \leq i \leq 15$. Clearly, $|C_\varphi(v_i) \cap \{15, 16, 17, 18\}| \leq 3$ for $1 \leq i \leq l$. By Claim 2, each 4^--vertex has at most one conflict vertex. If v_i has a conflict vertex w_i for $1 \leq i \leq l$, we assume that $\varphi(v_i w_i) \notin \{15, 16, 17, 18\}$ (if $\varphi(v_i w_i) \in \{15, 16, 17, 18\}$, then we recolor $v_i w_i$ with a color in $\{8, 9, \ldots, 14\} \setminus (C_\varphi(v_i) \cup C_\varphi(w_i))$ to satisfy this condition). Without considering the conflict of v, we have the following two types of proper colorings. (a): We color vv_1 with a color in {15, 16, 17, 18} such that v_1 does not conflict with its neighbors. There are at least $5 - m$ available colors. (b): For any given integer i $(2 \leq i \leq l)$, we select $\{b_i, d_i\}$ from {15, 16, 17, 18} to recolor vv_i and color vv_1 such that v_i and v_1 do not conflict with their neighbors. $\{b_i, d_i\}$ has at least $4 - m$ selected ways. Since i has $l - 1$ possibilities, we have at least $(4 - m)(l - 1)$ different coloring ways. Hence we have at least $(5 - m) + (4 - m)(l - 1) = (4 - m)l + 1$ ways, while v has at most $(4 - m)l$ conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction.

(4) Assume, to the contrary, that there exists a 15-vertex $v \in V(G)$ incident with a bad 3-cycle vv_1v_2 $(d_G(v_1) = d_G(v_2) = 3)$, where $d_G^{15}(v) \leq 8$. Let w_i $(1 \leq i \leq 2)$ be the neighbor of v_i other than v, v_{3-i}. Let $H = G - v_1v_2$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. By Claim 2, v_i $(1 \leq i \leq 2)$ has exactly one conflict

vertex. If $C_{\varphi}(v_1) \neq C_{\varphi}(v_2)$, then we color v_1v_2 with a color in $C \setminus (C_{\varphi}(v_1) \cup C_{\varphi}(v_2))$ to get a t_G-avd-coloring of G, a contradiction. If $C_{\varphi}(v_1) = C_{\varphi}(v_2)$, w.l.o.g., $\varphi(vv_1) = \varphi(v_2w_2) = 1, \varphi(vv_2) = \varphi(v_1w_1) = 2$ and $\varphi(vv_i) = i$ for $3 \le i \le 15$. Without considering the conflict of v, we have the following two types of proper colorings. (a): For any given integer i $(1 \le i \le 2)$, we recolor vv_i with an arbitrary color in $\{16, 17, 18\}$ and color v_1v_2 with 3. Since i has two possibilities, we have $3 \times 2 = 6$ different coloring ways. (b): We select $\{b_1, b_2\}$ from $\{16, 17, 18\}$ to recolor vv_1 and vv_2, and color v_1v_2 with 3. $\{b_1, b_2\}$ has three selected ways. Hence we have $6+3 = 9$ ways, while v has at most eight conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction. □

Claim 9 *Let v be a k-vertex of G with $k \ge 14$, then v is incident with at most one bad 3-cycle.*

Proof Assume, to the contrary, that there exists a k-vertex $v \in V(G)$ $(k \ge 14)$ incident with two bad 3-cycles vv_1v_2, vv_3v_4, where $d_G(v_i) = 3$ for $1 \le i \le 4$. Let w_i be the neighbor of v_i for $1 \le i \le 4$. Let $H = G - v_1v_2$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. By Claim 2, each 3-vertex has at most one conflict vertex. If $C_{\varphi}(v_1) \neq C_{\varphi}(v_2)$, then we color v_1v_2 with an arbitrary color in $C \setminus (C_{\varphi}(v_1) \cup C_{\varphi}(v_2))$ to yield a t_G-avd-coloring of G, a contradiction. If $C_{\varphi}(v_1) = C_{\varphi}(v_2)$, w.l.o.g., $\varphi(vv_1) = \varphi(v_2w_2) = 1, \varphi(vv_2) = \varphi(v_1w_1) = 2$ and $\varphi(vv_i) = i$ for $3 \le i \le k$. Note that $|\{\varphi(v_3w_3), \varphi(v_4w_4)\} \cap \{3, 4\}| \le 1$, w.l.o.g., $\varphi(v_4w_4) \neq 3$. Clearly, $|\{\varphi(v_4w_4)\} \cap \{1, 2\}| \le 1$, w.l.o.g., $\varphi(v_4w_4) \neq 1$. We first delete the color of v_3v_4, switch the colors of vv_1 and vv_4, then color v_1v_2, v_3v_4 properly to yield a t_G-avd-coloring of G, a contradiction. □

Claim 10 *Let v be a k-vertex of G with $k \ge 16$. If v is incident with a bad 3-cycle, then $d_G^k(v) \ge 2d_G^{4^-}(v) + 1$.*

Proof Assume, to the contrary, that there exists a k-vertex $v \in V(G)$ $(k \ge 16)$ incident with a bad 3-cycle vv_1v_2 $(d_G(v_1) = d_G(v_2) = 3)$, where $d_G^k(v) \le 2d_G^{4^-}(v)$. Let w_i $(1 \le i \le 2)$ be the neighbor of v_i other than v, v_{3-i}. Set $d_G^{4^-}(v) = m$. Suppose that $d_G(v_i) \le 4$ for $1 \le i \le m$. Let $H = G - v_1v_2$, by Remark 1, H has a t_G-avd-coloring φ with the color set C. By Claim 2, each 4^--vertex has at most one conflict vertex. If $C_{\varphi}(v_1) \neq C_{\varphi}(v_2)$, then we color v_1v_2 with an arbitrary color in $C \setminus (C_{\varphi}(v_1) \cup C_{\varphi}(v_2))$ to yield a t_G-avd-coloring of G, a contradiction. If $C_{\varphi}(v_1) = C_{\varphi}(v_2)$, w.l.o.g., $\varphi(vv_1) = \varphi(v_2w_2) = 1, \varphi(vv_2) = \varphi(v_1w_1) = 2$ and $\varphi(vv_i) = i$ for $3 \le i \le k$. Clearly, $|C_{\varphi}(v_i) \cap \{1, 2, k+1, k+2\}| \le 3$ for $1 \le i \le m$. If v_i has a conflict vertex w_i for $3 \le i \le m$, we assume that $\varphi(v_iw_i) \notin \{1, 2, k+1, k+2\}$ (if $\varphi(v_iw_i) \in \{1, 2, k+1, k+2\}$, then we recolor v_iw_i with a color in $\{k-6, k-5, \ldots, k\} \setminus (C_{\varphi}(v_i) \cup C_{\varphi}(w_i))$ to satisfy this condition). Without considering the conflict of v, we have the following three types of proper colorings. (a): For any given integer i $(1 \le i \le 2)$, we recolor vv_i with an arbitrary color in $\{k+1, k+2\}$ and color v_1v_2 with 3. Since i has two possibilities, we have $2 \times 2 = 4$ different coloring ways. (b): We recolor vv_i with $k+i$ for $1 \le i \le 2$ and color v_1v_2 with 3. (c): For any given integer i $(3 \le i \le m)$, we recolor vv_i with b_i in $\{1, 2, k+1, k+2\}$ such that v_i does not conflict with its neighbors. If $b_i \in \{1, 2\}$,

Table 1 The relation between $d_G(v)$ and $d_H(v)$

$d_G(v)$	$3 \le d_G(v) \le 9$	10	11	12	13	14	15	16	17	≥ 18
$d_H(v)$	$= d_G(v)$	≥ 9	≥ 10	≥ 11	≥ 12	≥ 12	≥ 12	≥ 9	≥ 9	≥ 10

then we recolor vv_{b_i} with $k+1$ or $k+2$ and color v_1v_2 with 3, so there are two coloring ways. If $b_i \in \{k+1, k+2\}$, then we recolor vv_1 or vv_2 with a color in $\{k+1, k+2\} \setminus \{b_i\}$ and color v_1v_2 with 3, so there are two ways. Since i has $m-2$ possibilities, we have $2(m-2)$ ways. Hence we have $4+1+2(m-2) = 2m+1$ ways, while v has at most $2m$ conflict vertices. So we can obtain a t_G-avd-coloring of G, a contradiction. □

Claim 11 Yan et al. (2012) *Let v be a k-vertex of G with $k \ge 16$. If $d_G^{2^-}(v) \ge 1$, then $d_G^{3^-}(v) \le \lceil \frac{k}{2} \rceil - 1$ and $d_G^k(v) \ge d_G^{3^-}(v) + 1$.*

Let H be one of the connected component of the graph which is obtained from G by deleting all 2^--vertices. By Claims 1, 3–5, 7–8, 11, the relation between $d_G(v)$ and $d_H(v)$ is as in Table 1.

By Table 1, we deduce that $\delta(H) \ge 3$, and for any $v \in V(H)$, we have $d_H^k(v) = d_G^k(v)$, where $3 \le k \le 6$. Let $H^\times$ be the associated plane graph of H. By Claims 2–4, 11 and Table 1, every 3-face of $H^\times$ is one of the following types:

Type I: (3, 3, 4)-faces, (4, 4, 4)-faces;

Type II: (3, 3, 10^+)-faces, (3, 4, 10^+)-faces, (4, 4, 9^+)-faces, (4, 5, 9^+)-faces;

Type III: (3, 10^+, 10^+)-faces, (4, 5, 5)-faces, (4, 6, 6)-faces, (4, 6, 9^+)-faces, (4, 7^+, 7^+)-faces, (5, 5, 9^+)-faces, (5, 9^+, 9^+)-faces, (6, 6, 9^+)-faces, (6, 9^+, 9^+)-faces;

Type IV: (7^+, 7^+, 7^+)-faces.

Let c_f be the false vertex incident with a false 3-face f, and $N_{\bar{f}}(c_f)$ be the set of neighbors of c_f which are not incident with f. f is the *corresponding face* of the vertices in $N_{\bar{f}}(c_f)$. By Claims 2–3, v has at most one corresponding 3-face of **Type I**. A vertex v is of *Type I* if it has a corresponding 3-face of **Type I**. Let $n_i(v)$ be the number of 3-faces of **Type *i*** incident with v, $i \in \{\text{II}, \text{III}, \text{IV}\}$. Let $n_{4^+}(v)$ be the number of 4^+-faces incident with v in $H^\times$.

By Euler's formula $|V(H^\times)| - |E(H^\times)| + |F(H^\times)| = 2$, we have:

$$\sum_{v \in V(H^\times)} (d_{H^\times}(v) - 4) + \sum_{f \in F(H^\times)} (d_{H^\times}(f) - 4) = -8$$

Next, we will apply the discharging method to derive a contradiction. We define the initial charge function $w(x) = d_{H^\times}(x) - 4$ for $x \in V(H^\times) \cup F(H^\times)$, and design discharging rules to redistribute charges. Let w' be the new charge after the discharging process, then we will show that $w'(x) \ge 0$ for $x \in V(H^\times) \cup F(H^\times)$, which leads to a contradiction.

The discharging rules are defined as follows. In the following rules, the degree of a vertex refers to its degree in H.

R1: Each 3-face f of **Type I** gets $\frac{1}{2}$ from every 9^+-vertex in $N_{\bar{f}}(c_f)$ (by Claims 2–3, f is false and $N_{\bar{f}}(c_f)$ consists of two 9^+-vertices);

R2: Each 3-face of **Type II** gets 1 from its incident 9^+-vertex;

R3: Each of $(5, 9^+, 9^+)$-faces and $(6, 9^+, 9^+)$-faces gets $\frac{1}{2}$ from every incident 9^+-vertex, and each other 3-face of **Type III** gets $\frac{1}{2}$ from every incident 5^+-vertex;

R4: Each 3-face of **Type IV** gets $\frac{1}{3}$ from every incident 7^+-vertex;

R5: Each good 3-vertex gets $\frac{1}{3}$ from every adjacent 10^+-vertex in H, and each bad 3-vertex gets $\frac{1}{2}$ from every adjacent 10^+-vertex in H.

We first verify the new charge of $f \in F(H^\times)$.

• $d_{H^\times}(f) = 3$. By **R1–R4**, $w'(f) \geq 0$.

• $d_{H^\times}(f) \geq 4$. The charge remains unchanged, $w'(f) = d_{H^\times}(f) - 4 \geq 0$.

Next, we verify the new charge of $v \in V(H^\times)$. For each real vertex $v \in V(H^\times)$, we have $d_{H^\times}(v) = d_G(v) - d_G^{2^-}(v)$.

• $d_{H^\times}(v) = 3$. By Claims 2–4 and Table 1, $d_H^{9^-}(v) = d_H^3(v) \leq 1$. If v is good, then $d_H^{10^+}(v) = 3$, otherwise $d_H^{10^+}(v) = 2$. By **R5**, $w'(v) \geq 3-4+\min\{\frac{1}{3}\times 3, \frac{1}{2}\times 2\} = 0$.

• $d_{H^\times}(v) = 4$. No rule applies to v, then $w'(v) = 4 - 4 = 0$.

• $d_{H^\times}(v) = 5$. By Claims 2–3 and Table 1, $d_H^{8^-}(v) = d_H^5(v) \leq 1$. By **R3**, only $(4, 5, 5)$-faces and $(5, 5, 9^+)$-faces incident with v get charges from v. There are at most two such faces incident with v. By **R3**, $w'(v) \geq 5 - 4 - \frac{1}{2} \times 2 = 0$.

• $d_{H^\times}(v) = 6$. By Claims 2–3 and Table 1, $d_H^{8^-}(v) = d_H^6(v) \leq 1$. By **R3**, only $(4, 6, 6)$-faces, $(4, 6, 9^+)$-faces and $(6, 6, 9^+)$-faces incident with v get charges from v. There are at most four such faces incident with v. By **R3**, $w'(v) \geq 6-4-\frac{1}{2}\times 4 = 0$.

• $7 \leq d_{H^\times}(v) \leq 8$. By Claim 3 and Table 1, $d_H^{6^-}(v) = 0$ and v is not of Type I. Thus we have $n_{\mathrm{III}}(v) \leq 2$. By **R3–R4**, $w'(v) \geq d_{H^\times}(v)-4-\frac{1}{2}\times 2-\frac{1}{3}\times (d_{H^\times}(v) - 2) = \frac{2d_{H^\times}(v)-13}{3} > 0$.

• $d_{H^\times}(v) = 9$. We first give the following fact.

Fact 1 If $d_{H^\times}(v) = 9$, then $d_H^3(v) = 0$ and $d_H^{6^-}(v) \leq 1$.

Proof By Table 1, we have $d_G(v) \in \{9, 10, 16, 17\}$. If $d_G(v) = 9$, by Claim 3, $d_H^{6^-}(v) = 0$. If $d_G(v) = 10$, then $d_G^{2^-}(v) = 1$. By Claim 4, $d_H^{6^-}(v) = 0$. If $d_G(v) = k$ $(16 \leq k \leq 17)$, then $d_G^{2^-}(v) = k - 9$. By Claim 11, $d_G^{3^-}(v) \leq \lceil\frac{k}{2}\rceil - 1 = k - 9$ and $d_G^k(v) \geq d_G^{3^-}(v) + 1$. Thus $d_H^3(v) = 0$ and $d_H^{6^-}(v) \leq k - d_G^{2^-}(v) - d_G^k(v) \leq k - (k-9) - (k-8) \leq 1$. □

By Fact 1, if v is of Type I, then $n_{\mathrm{II}}(v) = 0$, otherwise $n_{\mathrm{II}}(v) \leq 1$. By **R1–R4**, $w'(v) \geq 9 - 4 - \max\{\frac{1}{2} + \frac{1}{2} \times 9, 1 + \frac{1}{2} \times 8\} = 0$.

• $d_{H^\times}(v) = 10$. We first give the following fact.

Fact 2 If $d_{H^\times}(v) = 10$, then $d_H^3(v) \leq 1$ and $d_H^{6^-}(v) \leq 3$.

Proof By Table 1, we have $d_G(v) \in \{10, 11\}$ or $d_G(v) \geq 16$. If $d_G(v) = 10$, by Claim 4, $d_H^{6^-}(v) \leq 1$. If $d_G(v) = 11$, then $d_G^{2^-}(v) = 1$. By Claims 4 and 6, $d_G^{5^-}(v) \leq 1$ and $d_G^{6^-}(v) \leq 2$. Thus $d_H^{3^-}(v) = 0$ and $d_H^{6^-}(v) \leq 1$. If $d_G(v) = k$ $(k \geq 16)$, then

$d_G^{2^-}(v) = k - 10$. By Claim 11, $d_G^{3^-}(v) \le \lceil \frac{k}{2} \rceil - 1$ and $d_G^k(v) \ge d_G^{3^-}(v) + 1 \ge d_G^{2^-}(v)+1$. Thus $d_H^3(v) \le \lceil \frac{k}{2} \rceil - 1 - (k-10) \le 1$ and $d_H^{6^-}(v) \le k - d_G^{2^-}(v) - d_G^k(v) \le k - (k-10) - (k-9) \le 3$. □

By Fact 2, if v is of Type I, then $n_{\text{II}}(v) \le 1$ and $n_{\text{III}}(v) \le 5$; otherwise we have either $n_{\text{II}}(v) \le 1$, or $n_{\text{II}}(v) = 2$ and $n_{\text{III}}(v) \le 4$. Noting that $d_H^3(v) \le 1$, by **R1–R5**, we have $w'(v) \ge 10 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 5 + \frac{1}{3} \times 4, 1 + \frac{1}{2} \times 9, 1 \times 2 + \frac{1}{2} \times 4 + \frac{1}{3} \times 4\} - \frac{1}{2} = 0$.

• $d_{H^\times}(v) = 11$. We first give the following fact.

Fact 3 If $d_{H^\times}(v) = 11$, then $d_H^3(v) \le 2$ and $d_H^{6^-}(v) \le 5 - d_H^3(v)$.

Proof By Table 1, we have $d_G(v) \in \{11, 12\}$ or $d_G(v) \ge 16$. If $d_G(v) = 11$, by Claims 4 and 6, $d_H^{3^-}(v) \le 1$ and $d_H^{6^-}(v) \le 2$. If $d_G(v) = 12$, then $d_G^{2^-}(v) = 1$. By Claims 5–6, $d_G^{3^-}(v) \le 1$ and $d_G^{6^-}(v) \le 5$. Thus $d_H^3(v) = 0$ and $d_H^{6^-}(v) \le 4$. If $d_G(v) = k$ ($k \ge 16$), then $d_G^{2^-}(v) = k - 11$. By Claim 11, $d_G^{3^-}(v) \le \lceil \frac{k}{2} \rceil - 1$ and $d_G^k(v) \ge d_G^{3^-}(v) + 1$. Thus $d_H^3(v) = d_G^3(v) \le \lceil \frac{k}{2} \rceil - 1 - (k-11) \le 2$ and $d_H^{6^-}(v) \le k - d_G^{2^-}(v) - d_G^k(v) \le k - (k-11) - (k - 10 + d_G^3(v)) \le 5 - d_H^3(v)$. □

- $d_H^3(v) \ne 0$. By Fact 3, if v is of Type I, then $n_{\text{II}}(v) \le 1$ and $n_{\text{III}}(v) \le 7$; otherwise we have either $n_{\text{II}}(v) \le 1$ or $n_{\text{II}}(v) = 2$ and $n_{\text{III}}(v) \le 6$. Noting that $d_H^3(v) \le 2$, by **R1–R5**, we have $w'(v) \ge 11 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 7 + \frac{1}{3} \times 3, 1 + \frac{1}{2} \times 10, 1 \times 2 + \frac{1}{2} \times 6 + \frac{1}{3} \times 3\} - \frac{1}{2} \times 2 = 0$.

- $d_H^3(v) = 0$. By Fact 3, if v is of Type I, then $n_{\text{II}}(v) \le 2$, otherwise $n_{\text{II}}(v) \le 3$. By **R1–R4**, $w'(v) \ge 11 - 4 - \max\{\frac{1}{2} + 1 \times 2 + \frac{1}{2} \times 9, 1 \times 3 + \frac{1}{2} \times 8\} = 0$.

• $d_{H^\times}(v) = 12$. We first give the following fact.

Fact 4 If $d_{H^\times}(v) = 12$, then either $d_H^3(v) \le 1$ and $d_H^{5^-}(v) \le 7 - d_H^3(v)$, or $2 \le d_H^3(v) \le 3$ and $d_H^{6^-}(v) \le 7 - d_H^3(v)$.

Proof By Table 1, we have $d_G(v) \ge 12$. (a): $d_G(v) = 12$. By Claims 5–6, $d_H^3(v) \le 1$ and $d_H^{5^-}(v) \le 5$. So, in this case, Fact 4 holds. (b): $d_G(v) = k$ ($13 \le k \le 14$). Then $d_G^{2^-}(v) = k - 12 > 0$, by Claim 7(2), let $m = 2$, we have $d_G^k(v) \ge (17 - k)d_G^{(19-k)^-}(v) + 1$. Noting that $d_G^{(19-k)^-}(v) + d_G^k(v) \le k$, we get that $d_H^{(19-k)^-}(v) = d_G^{(19-k)^-}(v) - d_G^{2^-}(v) \le \lfloor \frac{k-1}{18-k} \rfloor - (k-12) = 1$. So, in this case, Fact 4 holds. (c): $d_G(v) = 15$, then $d_G^{2^-}(v) = 3$. By Claim 8(2), $d_H^3(v) = d_G^{3^-}(v) - d_G^{2^-}(v) \le 1$. By Claim 8(3), let $m = 2$, we have $d_G^{15}(v) \ge 2d_G^{4^-}(v) + 1$. Thus $d_H^{5^-}(v) \le d_G(v) - d_G^{2^-}(v) - d_G^{15}(v) \le 14 - 3d_G^{2^-}(v) = 5$. So, in this case, Fact 4 holds. (d): $d_G(v) = k$ ($k \ge 16$), then $d_G^{2^-}(v) = k - 12$. By Claim 11, $d_G^{3^-}(v) \le \lceil \frac{k}{2} \rceil - 1$ and $d_G^k(v) \ge d_G^{3^-}(v) + 1$. Thus $d_H^3(v) = d_G^3(v) \le \lceil \frac{k}{2} \rceil - 1 - (k-12) \le 3$ and $d_H^{6^-}(v) \le k - d_G^{2^-}(v) - d_G^k(v) \le k - (k-12) - (k - 11 + d_G^3(v)) \le 7 - d_H^3(v)$. So, in this case, Fact 4 holds. □

- $d_H^3(v) = 3$ and $d_H^{6^-}(v) \le 4$. If v is of Type I, by Lemma 1, we have $n_{\text{II}}(v) \le 1$ and $n_{4^+}(v) \ge 1$; otherwise we have either $n_{\text{II}}(v) \le 1$, or $n_{\text{II}}(v) = 2$ and $n_{\text{III}}(v) \le 6$. By

R1–R5, $w'(v) \geq 12-4-\max\{\frac{1}{2}+1+\frac{1}{2}\times 10, 1+\frac{1}{2}\times 11, 1\times 2+\frac{1}{2}\times 6+\frac{1}{3}\times 4\}-\frac{1}{2}\times 3 = 0$.

- $d_H^3(v) = 2$ and $d_H^{6^-}(v) \leq 5$. If v is of Type I, then either $n_{\text{II}}(v) \leq 1$, or $n_{\text{II}}(v) = 2$ and $n_{\text{III}}(v) \leq 6$; otherwise we have either $n_{\text{II}}(v) \leq 2$, or $n_{\text{II}}(v) = 3$ and $n_{\text{III}}(v) \leq 6$. By **R1–R5**, $w'(v) \geq 12 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 11, \frac{1}{2} + 1 \times 2 + \frac{1}{2} \times 6 + \frac{1}{3} \times 4, 1 \times 2 + \frac{1}{2} \times 10, 1 \times 3 + \frac{1}{2} \times 6 + \frac{1}{3} \times 3\} - \frac{1}{2} \times 2 = 0$.

- $d_H^3(v) \leq 1$. By Fact 4, if v is of Type I, then $n_{\text{II}}(v) \leq 3 - d_H^3(v)$, otherwise $n_{\text{II}}(v) \leq 4 - d_H^3(v)$. By **R1–R5**, $w'(v) \geq 12 - 4 - \max\{\frac{1}{2} + 1 \times (3 - d_H^3(v)) + \frac{1}{2} \times (12 - (3 - d_H^3(v))), 1 \times (4 - d_H^3(v)) + \frac{1}{2} \times (12 - (4 - d_H^3(v)))\} - \frac{1}{2}d_H^3(v) = 0$.

• $d_{H^\times}(v) = 13$. We first give the following fact.

Fact 5 If $d_{H^\times}(v) = 13$, then $d_H^3(v) \leq 4$ and $d_H^{5^-}(v) \leq 9 - d_H^3(v)$. Furthermore, if $2 \leq d_H^3(v) \leq 4$ and $d_H^{5^-}(v) \geq 7 - d_H^3(v)$, then v is not incident with any bad 3-cycle.

Proof By Table 1, we have $d_G(v) \geq 13$. (a): $d_G(v) = 13$. If $d_G^3(v) \geq 1$, by Claim 7(2), $d_G^{13}(v) \geq 3d_G^{6^-}(v) + 1$. Noting that $d_G^{6^-}(v) + d_G^{13}(v) \leq 13$, we have $d_H^{5^-}(v) \leq d_G^{6^-}(v) \leq 3$. If $d_G^3(v) = 0$ and $d_G^{5^-}(v) \geq 1$, by Claim 7(2), $d_G^{13}(v) \geq d_G^{5^-}(v) + 1$. Noting that $d_G^{5^-}(v) + d_G^{13}(v) \leq 13$, we have $d_H^{5^-}(v) \leq d_G^{5^-}(v) \leq 6$. So, in this case, Fact 5 holds. (b): $d_G(v) = 14$, then $d_G^{2^-}(v) = 1$. By Claim 7(2), let $m = 2$, we have $d_G^{14}(v) \geq 3d_G^{5^-}(v) + 1$. Noting that $d_G^{5^-}(v) + d_G^{14}(v) \leq 14$, we get that $d_H^{5^-}(v) = d_G^{5^-}(v) - d_G^{2^-}(v) \leq 3 - 1 = 2$. So, in this case, Fact 5 holds. (c): $d_G(v) = 15$, then $d_G^{2^-}(v) = 2$. By Claim 8(2), we have $d_H^3(v) = d_G^{3^-}(v) - d_G^{2^-}(v) \leq 2$. By Claim 8(3), let $m = 2$, we have $d_G^{15}(v) \geq 2d_G^{4^-}(v) + 1$. Thus $d_H^{5^-}(v) \leq d_G(v) - d_G^{2^-}(v) - d_G^{15}(v) \leq 14 - 3d_G^{2^-}(v) - 2d_G^3(v) = 8 - 2d_H^3(v)$. So, in this case, Fact 5 holds. (d): $d_G(v) = k\ (k \geq 16)$, then $d_G^{2^-}(v) = k - 13$. By Claim 11, $d_G^{3^-}(v) \leq \lceil \frac{k}{2} \rceil - 1$ and $d_G^k(v) \geq d_G^{3^-}(v) + 1$. Thus $d_H^3(v) = d_G^3(v) \leq \lceil \frac{k}{2} \rceil - 1 - (k - 13) \leq 4$ and $d_H^{5^-}(v) \leq k - d_G^{2^-}(v) - d_G^k(v) \leq k - (k - 13) - (k - 12 + d_G^3(v)) \leq 9 - d_H^3(v)$. Furthermore, suppose that $2 \leq d_H^3(v) \leq 4$ and $d_H^{5^-}(v) \geq 7 - d_H^3(v)$. Assume that v is incident with a bad 3-cycle, by Claim 10, $d_G^k(v) \geq 2d_G^{4^-}(v) + 1$. Noting that $d_G^{2^-}(v) + d_H^{5^-}(v) + d_G^k(v) - k \leq 0$, while $d_G^{2^-}(v) + d_H^{5^-}(v) + d_G^k(v) - k \geq k - 13 + 7 - d_H^3(v) + 2(k - 13) + 2d_H^3(v) + 1 - k > 2k - 31 > 0$, a contradiction. So, in this case, Fact 5 holds. □

- $d_H^3(v) = 4$ and $d_H^{5^-}(v) \leq 5$. By Fact 5, v is not incident with any bad 3-cycle. If v is of Type I, by Lemma 1, $n_{\text{II}}(v) = 0$, otherwise $n_{\text{II}}(v) \leq 1$. By **R1–R5**, $w'(v) \geq 13 - 4 - \max\{\frac{1}{2} + \frac{1}{2} \times 13, 1 + \frac{1}{2} \times 12\} - \frac{1}{2} \times 4 = 0$.

- $d_H^3(v) = 3$. By Fact 5, $d_H^{5^-}(v) = 3$, or $4 \leq d_H^{5^-}(v) \leq 6$ and v is not incident with any bad 3-cycle. If v is of Type I, then $n_{\text{II}}(v) \leq 1$, otherwise $n_{\text{II}}(v) \leq 2$. By **R1–R5**, $w'(v) \geq 13 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 12, 1 \times 2 + \frac{1}{2} \times 11\} - \frac{1}{2} \times 3 = 0$.

- $d_H^3(v) = 2$. By Fact 5, $d_H^{5^-}(v) \leq 4$, or $5 \leq d_H^{5^-}(v) \leq 7$ and v is not incident with any bad 3-cycle. If v is of Type I, then $n_{\text{II}}(v) \leq 2$, otherwise $n_{\text{II}}(v) \leq 3$. By **R1–R5**, $w'(v) \geq 13 - 4 - \max\{\frac{1}{2} + 1 \times 2 + \frac{1}{2} \times 11, 1 \times 3 + \frac{1}{2} \times 10\} - \frac{1}{2} \times 2 = 0$.

- $d_H^3(v) \leq 1$. By Fact 5, if v is of Type I, then $n_{\text{II}}(v) \leq 4 - d_H^3(v)$, otherwise $n_{\text{II}}(v) \leq 5 - d_H^3(v)$. By **R1–R5**, $w'(v) \geq 13 - 4 - \max\{\frac{1}{2} + 1 \times (4 - d_H^3(v)) + \frac{1}{2} \times (13 - (4 - d_H^3(v))), 1 \times (5 - d_H^3(v)) + \frac{1}{2} \times (13 - (5 - d_H^3(v)))\} - \frac{1}{2} \times d_H^3(v) = 0$.

• $d_{H^\times}(v) = 14$. We first give the following fact.

Fact 6 If $d_{H^\times}(v) = 14$, then either $d_H^3(v) = 0$, or $1 \leq d_H^3(v) \leq 5$ and $d_H^{5^-}(v) \leq 11 - d_H^3(v)$. Furthermore, if $d_H^3(v) \geq 4$ and $d_H^{5^-}(v) \geq 5$, or $2 \leq d_H^3(v) \leq 3$ and $d_H^{5^-}(v) \geq 6$, then v is not incident with any bad 3-cycle.

Proof By Table 1, we have $d_G(v) \geq 14$. (a): $d_G(v) = 14$. If $d_G^3(v) \geq 1$, by Claim 7(2), $d_G^{14}(v) \geq 2d_G^{5^-}(v) + 1$. Noting that $d_G^{5^-}(v) + d_G^{14}(v) \leq 14$, we have $d_H^{5^-}(v) \leq d_G^{5^-}(v) \leq 4$. So, in this case, Fact 6 holds. (b): $d_G(v) = 15$, then $d_G^{2^-}(v) = 1$. By Claim 8(2), we have $d_H^3(v) = d_G^{3^-}(v) - d_G^{2^-}(v) \leq 3$. By Claim 8(3), let $m = 2$, we have $d_G^{15}(v) \geq 2d_G^{4^-}(v) + 1$. Thus $d_H^{5^-}(v) \leq d_G(v) - d_G^{2^-}(v) - d_G^{15}(v) \leq 14 - 3d_G^{2^-}(v) - 2d_G^3(v) - 2d_G^4(v) \leq 11 - 2d_H^3(v)$, which implies that $d_H^3(v) \leq 3$. Furthermore, if $d_H^{5^-}(v) \geq 6$, by $d_G^{2^-}(v) + d_H^{5^-}(v) + d_G^{15}(v) \leq 15$ and Claim 8(4), v is not incident with any bad 3-cycle. So, in this case, Fact 6 holds. (c): $d_G(v) = k$ $(k \geq 16)$, then $d_G^{2^-}(v) = k - 14$. By Claim 11, $d_G^{3^-}(v) \leq \lceil \frac{k}{2} \rceil - 1$ and $d_G^k(v) \geq d_G^{3^-}(v) + 1$. Thus $d_H^3(v) = d_G^3(v) \leq \lceil \frac{k}{2} \rceil - 1 - (k - 14) \leq 5$ and $d_H^{5^-}(v) \leq k - d_G^{2^-}(v) - d_G^k(v) \leq k - (k - 14) - (k - 13 + d_G^3(v)) \leq 11 - d_H^3(v)$. Furthermore, suppose that $d_H^3(v) \geq 4$ and $d_H^{5^-}(v) \geq 5$, or $2 \leq d_H^3(v) \leq 3$ and $d_H^{5^-}(v) \geq 6$. Assume that v is incident with a bad 3-cycle, by Claim 10, $d_G^k(v) \geq 2d_G^{4^-}(v) + 1$. Noting that $d_G^{2^-}(v) + d_H^{5^-}(v) + d_G^k(v) - k \leq 0$, while $d_G^{2^-}(v) + d_H^{5^-}(v) + d_G^k(v) - k \geq d_G^{2^-}(v) + d_H^{5^-}(v) + 2d_G^{4^-}(v) + 1 - k \geq 3d_G^{2^-}(v) + 2d_H^3(v) + d_H^{5^-}(v) + 1 - k \geq 3(k - 14) + \min\{2 \times 4 + 5, 2 \times 2 + 6\} + 1 - k = 2k - 31 > 0$, a contradiction. So, in this case, Fact 6 holds. □

By Fact 6, we consider the following cases.

- $d_H^3(v) = 5$ and $d_H^{5^-}(v) \leq 6$, or $d_H^3(v) = 4$ and $5 \leq d_H^{5^-}(v) \leq 7$. By Fact 6, v is not incident with any bad 3-cycle. If v is of Type I, by Lemma 1, we have $n_{\text{II}}(v) \leq 5 - d_H^3(v)$, otherwise $n_{\text{II}}(v) \leq 6 - d_H^3(v)$. By **R1–R5**, $w'(v) \geq 14 - 4 - \max\{\frac{1}{2} + 1 \times (5 - d_H^3(v)) + \frac{1}{2} \times (14 - (5 - d_H^3(v))), 1 \times (6 - d_H^3(v)) + \frac{1}{2} \times (14 - (6 - d_H^3(v)))\} - \frac{1}{2} d_H^3(v) = 0$.

- $d_H^3(v) = d_H^{5^-}(v) = 4$. If v is of Type I, by Lemma 1, we have $n_{\text{II}}(v) \leq 1$, otherwise, by Claim 9, we have $n_{\text{II}}(v) \leq 2$. By **R1–R5**, $w'(v) \geq 14 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 13, 1 \times 2 + \frac{1}{2} \times 12\} - \frac{1}{2} \times 4 = 0$.

- $2 \leq d_H^3(v) \leq 3$ and $d_H^{5^-}(v) \leq 5$, or $d_H^3(v) = 3$ and $6 \leq d_H^{5^-}(v) \leq 8$ and v is not incident with any bad 3-cycle. If v is of Type I, then $n_{\text{II}}(v) \leq 2$, otherwise $n_{\text{II}}(v) \leq 3$. Noting that $d_H^3(v) \leq 3$, by **R1–R5**, we have $w'(v) \geq 14 - 4 - \max\{\frac{1}{2} + 1 \times 2 + \frac{1}{2} \times 12, 1 \times 3 + \frac{1}{2} \times 11\} - \frac{1}{2} \times 3 = 0$.

- $d_H^3(v) = 2$, $6 \leq d_H^{5^-}(v) \leq 9$ and v is not incident with any bad 3-cycle. If v is of Type I, then $n_{\text{II}}(v) \leq 3$, otherwise $n_{\text{II}}(v) \leq 4$. By **R1–R5**, $w'(v) \geq 14 - 4 - \max\{\frac{1}{2} + 1 \times 3 + \frac{1}{2} \times 11, 1 \times 4 + \frac{1}{2} \times 10\} - \frac{1}{2} \times 2 = 0$.

- $d_H^3(v) = 1$ and $d_H^{5^-}(v) \le 10$. If v is of Type I, then $n_{\rm II}(v) \le 4$, otherwise $n_{\rm II}(v) \le 5$. By **R1–R5**, $w'(v) \ge 14-4-\max\{\frac{1}{2}+1\times4+\frac{1}{2}\times10, 1\times5+\frac{1}{2}\times9\}-\frac{1}{2} = 0$.
- $d_H^3(v) = 0$. Then $n_{\rm II}(v) \le 5$, or $n_{\rm II}(v) = 6$ and $n_{4^+}(v) \ge 1$, or $n_{\rm II}(v) = 7$ and $n_{4^+}(v) \ge 5$ by Claims 2–3. By **R1–R4**, $w'(v) \ge 14 - 4 - \frac{1}{2} - \max\{1 \times 5 + \frac{1}{2} \times 9, 1 \times 6 + \frac{1}{2} \times 7, 1 \times 7 + \frac{1}{2} \times 2\} = 0$.

Remark 3 For any 15^+-vertex $v \in V(H^\times)$, if v is not incident with any bad 3-cycle and $d_H^{3b}(v) \ge 2$, then $n_{4^+}(v) \ge 1$.

- $d_{H^\times}(v) = 15$. We first give the following fact.

Fact 7 If $d_{H^\times}(v) = 15$, then either $d_H^3(v) = 0$, or $1 \le d_H^3(v) \le 7$ and $d_H^{6^-}(v) \le 14-d_H^3(v)$. Furthermore, if $d_H^3(v) \ge 3$ and $d_H^{6^-}(v) \ge 7$, or $d_H^3(v) = 2$ and $d_H^{6^-}(v) \ge 9$, then v is not incident with any bad 3-cycle.

Proof By Table 1, we have $d_G(v) \ge 15$. (a): $d_G(v) = 15$. If $d_G^3(v) \ge 1$, by Claim 8(3), let $m = 3$, we have $d_G^{15}(v) \ge d_G^{4^-}(v) + 1$. Noting that $d_H^3(v) \le d_G^{4^-}(v) \le d_G^{6^-}(v) \le d_G(v) - d_G^{15}(v)$, we have $d_H^3(v) \le 7$ and $d_H^{6^-}(v) \le 14 - d_H^3(v)$. Furthermore, if $d_H^{6^-}(v) \ge 7$, by $d_G^{6^-}(v) + d_G^{15}(v) \le 15$ and Claim 8(4), v is not incident with any bad 3-cycle. So, in this case, Fact 7 holds. (b): $d_G(v) = k$ ($k \ge 16$), then $d_G^{2^-}(v) = k - 15$. By Claim 11, $d_G^{3^-}(v) \le \lceil\frac{k}{2}\rceil - 1$ and $d_G^k(v) \ge d_G^{3^-}(v) + 1$. Thus $d_H^3(v) = d_G^3(v) \le \lceil\frac{k}{2}\rceil - 1 - (k - 15) \le 6$ and $d_H^{6^-}(v) \le k - d_G^{2^-}(v) - d_G^k(v) \le k - (k - 15) - (k - 14 + d_G^3(v)) < 14 - d_H^3(v)$. Furthermore, suppose that $d_H^3(v) \ge 3$ and $d_H^{6^-}(v) \ge 7$, or $d_H^3(v) = 2$ and $d_H^{6^-}(v) \ge 9$. Assume that v is incident with a bad 3-cycle, by Claim 10, $d_G^k(v) \ge 2d_G^{4^-}(v)+1$. Noting that $d_G^{2^-}(v)+d_H^{6^-}(v)+d_G^k(v)-k \le 0$, while $d_G^{2^-}(v) + d_H^{6^-}(v) + d_G^k(v) - k \ge d_G^{2^-}(v) + d_H^{6^-}(v) + 2d_G^{4^-}(v) + 1 - k \ge 3d_G^{2^-}(v) + 2d_H^3(v)+d_H^{6^-}(v)+1-k \ge 3(k-15)+\min\{2\times3+7, 2\times2+9\}+1-k = 2k-31 > 0$, a contradiction. So, in this case, Fact 7 holds. □

By Fact 7, we consider the following cases.
- $d_H^3(v) = d_H^{6^-}(v) = 7$, and v is not incident with any bad 3-cycle. If v is of Type I, by Lemma 1, we have $n_{\rm II}(v) = 0$ and $n_{4^+}(v) \ge 1$; otherwise we have $n_{\rm II}(v) \le 1$ and either $d_H^{3b}(v) \le 1$ or $n_{4^+}(v) \ge 1$ by Remark 3. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + \frac{1}{2} \times 14 + \frac{1}{2} \times 7, 1 + \frac{1}{2} \times 14 + \frac{1}{2} + \frac{1}{3} \times 6, 1 + \frac{1}{2} \times 13 + \frac{1}{2} \times 7\} = 0$.
- $d_H^3(v) = d_H^{6^-}(v) = 6$. By Claim 9, v is incident with at most one bad 3-cycle. If v is of Type I, by Lemma 1, then $n_{\rm II}(v) \le 1$ and $n_{4^+}(v) \ge 1$; otherwise we have either $n_{\rm II}(v) \le 1$, or $n_{\rm II}(v) = 2$ and $n_{\rm III}(v) \le 10$. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 13, 1 + \frac{1}{2} \times 14, 1 \times 2 + \frac{1}{2} \times 10 + \frac{1}{3} \times 3\} - \frac{1}{2} \times 6 = 0$.
- $d_H^3(v) = 6$, $7 \le d_H^{6^-}(v) \le 8$, and v is not incident with any bad 3-cycle. If v is of Type I, we have $n_{\rm II}(v) \le 1$ and either $d_H^{3b}(v) \le 1$ or $n_{4^+}(v) \ge 1$ by Remark 3; otherwise we have $n_{\rm II}(v) \le 2$ and either $d_H^{3b}(v) \le 1$ or $n_{4^+}(v) \ge 1$ by Remark 3. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 14 + \frac{1}{2} + \frac{1}{3} \times 5, \frac{1}{2} + 1 + \frac{1}{2} \times 13 + \frac{1}{2} \times 6, 1 \times 2 + \frac{1}{2} \times 13 + \frac{1}{2} + \frac{1}{3} \times 5, 1 \times 2 + \frac{1}{2} \times 12 + \frac{1}{2} \times 6\} = 0$.

- $d^3_H(v) = 5$ and $d^{6^-}_H(v) \le 6$. By Claim 9, v is incident with at most one bad 3-cycle. If v is of Type I, then $n_{\mathrm{II}}(v) \le 1$, otherwise $n_{\mathrm{II}}(v) \le 2$. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 14, 1 \times 2 + \frac{1}{2} \times 13\} - \frac{1}{2} \times 5 = 0$.

- $d^3_H(v) = 5$, $7 \le d^{6^-}_H(v) \le 9$, and v is not incident with any bad 3-cycle. If v is of Type I, then either $n_{\mathrm{II}}(v) \le 1$, or $n_{\mathrm{II}}(v) = 2$ and $n_{4^+}(v) \ge 1$ by Claims 2–3; otherwise we have $n_{\mathrm{II}}(v) \le 3$ and either $d^{3b}_H(v) \le 1$ or $n_{4^+}(v) \ge 1$ by Remark 3. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 + \frac{1}{2} \times 14 + \frac{1}{2} \times 5, \frac{1}{2} + 1 \times 2 + \frac{1}{2} \times 12 + \frac{1}{2} \times 5, 1 \times 3 + \frac{1}{2} \times 12 + \frac{1}{2} + \frac{1}{3} \times 4, 1 \times 3 + \frac{1}{2} \times 11 + \frac{1}{2} \times 5\} = 0$.

- $3 \le d^3_H(v) \le 4$ and $d^{6^-}_H(v) \le 6$. By Claim 9, v is incident with at most one bad 3-cycle. If v is of Type I, then $n_{\mathrm{II}}(v) \le 2$, otherwise $n_{\mathrm{II}}(v) \le 3$. Noting that $d^3_H(v) \le 4$, by **R1–R5**, we have $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 \times 2 + \frac{1}{2} \times 13, 1 \times 3 + \frac{1}{2} \times 12\} - \frac{1}{2} \times 4 = 0$.

- $3 \le d^3_H(v) \le 4$, $7 \le d^{6^-}_H(v) \le 14 - d^3_H(v)$, and v is not incident with any bad 3-cycle. If v is of Type I, then either $n_{\mathrm{II}}(v) \le 6 - d^3_H(v)$, or $n_{\mathrm{II}}(v) = 7 - d^3_H(v)$ and $n_{4^+}(v) \ge 1$ by Claims 2–3; otherwise we have either $n_{\mathrm{II}}(v) \le 7 - d^3_H(v)$, or $n_{\mathrm{II}}(v) = 8 - d^3_H(v)$ and $n_{4^+}(v) \ge 1$ by Claims 2–3. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 \times (6 - d^3_H(v)) + \frac{1}{2} \times (15 - (6 - d^3_H(v))), \frac{1}{2} + 1 \times (7 - d^3_H(v)) + \frac{1}{2} \times (14 - (7 - d^3_H(v))), 1 \times (7 - d^3_H(v)) + \frac{1}{2} \times (15 - (7 - d^3_H(v))), 1 \times (8 - d^3_H(v)) + \frac{1}{2} \times (14 - (8 - d^3_H(v)))\} - \frac{1}{2} d^3_H(v) = 0$.

- $d^3_H(v) = 2$ and $d^{6^-}_H(v) \le 8$. If v is of Type I, then $n_{\mathrm{II}}(v) \le 4$, otherwise $n_{\mathrm{II}}(v) \le 5$. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 \times 4 + \frac{1}{2} \times 11, 1 \times 5 + \frac{1}{2} \times 10\} - \frac{1}{2} \times 2 = 0$.

- $d^3_H(v) = 2$, $9 \le d^{6^-}_H(v) \le 12$, and v is not incident with any bad 3-cycle. If v is of Type I, then either $n_{\mathrm{II}}(v) \le 4$, or $n_{\mathrm{II}}(v) = 5$ and $n_{4^+}(v) \ge 1$ by Claims 2–3; otherwise we have either $n_{\mathrm{II}}(v) \le 5$, or $n_{\mathrm{II}}(v) = 6$ and $n_{4^+}(v) \ge 1$ by Claims 2–3. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 \times 4 + \frac{1}{2} \times 11, \frac{1}{2} + 1 \times 5 + \frac{1}{2} \times 9, 1 \times 5 + \frac{1}{2} \times 10, 1 \times 6 + \frac{1}{2} \times 8\} - \frac{1}{2} \times 2 = 0$.

- $d^3_H(v) \le 1$ and $d^{6^-}_H(v) \le 15 - 2d^3_H(v)$. If v is of Type I, then either $n_{\mathrm{II}}(v) \le 6 - d^3_H(v)$, or $n_{\mathrm{II}}(v) = 7 - d^3_H(v)$ and $n_{4^+}(v) \ge 1$ by Claims 2–3; otherwise we have either $n_{\mathrm{II}}(v) \le 7 - d^3_H(v)$, or $n_{\mathrm{II}}(v) = 8 - d^3_H(v)$ and $n_{4^+}(v) \ge 1$ by Claims 2–3. By **R1–R5**, $w'(v) \ge 15 - 4 - \max\{\frac{1}{2} + 1 \times (6 - d^3_H(v)) + \frac{1}{2} \times (15 - (6 - d^3_H(v))), \frac{1}{2} + 1 \times (7 - d^3_H(v)) + \frac{1}{2} \times (14 - (7 - d^3_H(v))), 1 \times (7 - d^3_H(v)) + \frac{1}{2} \times (15 - (7 - d^3_H(v))), 1 \times (8 - d^3_H(v)) + \frac{1}{2} \times (14 - (8 - d^3_H(v)))\} - \frac{1}{2} d^3_H(v) = 0$.

• $d_{H^\times}(v) = k$ ($k \ge 16$). By Claim 2 and Table 1, every 5^--vertex has at most one conflict vertex.

- $d^3_H(v) = 0$. (a): $3n_{\mathrm{II}}(v) \le k + 5$. By **R1–R4**, $w'(v) \ge k - 4 - \frac{1}{2} - 1 \times n_{\mathrm{II}}(v) - \frac{1}{2} \times (k - n_{\mathrm{II}}(v)) \ge \frac{k-16}{3} \ge 0$. (b): $3n_{\mathrm{II}}(v) > k + 5$. Note that a 3-face of **Type II** is incident with two 5^--vertices. If v is not adjacent to any false vertex, then $d^{9^+}_H(v) \le k - 2n_{\mathrm{II}}(v)$ and $n_{4^+}(v) \ge n_{\mathrm{II}}(v) - d^{9^+}_H(v)$; otherwise we have $d^{9^+}_H(v) \le k - 2n_{\mathrm{II}}(v) + 2$ and $n_{4^+}(v) \ge n_{\mathrm{II}}(v) - d^{9^+}_H(v) - 1$. Thus $n_{4^+}(v) \ge \min\{n_{\mathrm{II}}(v) - (k - 2n_{\mathrm{II}}(v)), n_{\mathrm{II}}(v) - (k - 2n_{\mathrm{II}}(v) + 2) - 1\} = 3n_{\mathrm{II}}(v) - k - 3$. By **R1–R4**, $w'(v) \ge k - 4 - \frac{1}{2} - 1 \times n_{\mathrm{II}}(v) - \frac{1}{2} \times (k - n_{\mathrm{II}}(v) - n_{4^+}(v)) = \frac{1}{2}(k - n_{\mathrm{II}}(v) + n_{4^+}(v) - 9) \ge \frac{1}{2}(k - n_{\mathrm{II}}(v) + (3n_{\mathrm{II}}(v) - k - 3) - 9) = n_{\mathrm{II}}(v) - 6 > \frac{k+5}{3} - 6 > 0$.

- $d_H^3(v) \geq 1$ and v is incident with a bad 3-cycle. (a): If v is of Type I, then v is not incident with any $(4, 5, 16^+)$-face. By Lemma 1 and Claim 9, we have $n_{\text{II}}(v) \leq \frac{d_H^4(v)}{2}+1$. (b): If v is not of Type I. Noting that v is incident with at most two $(4, 5, 16^+)$-faces, we have $n_{\text{II}}(v) \leq \frac{d_H^4(v)}{2}+3$. By Claim 10, $d_H^3(v)+d_H^4(v) \leq d_G^{4^-}(v) \leq \frac{k-1}{3}$. By **R1–R5**, $w'(v) \geq k-4-\max\{\frac{1}{2}+1\times(\frac{d_H^4(v)}{2}+1)+\frac{1}{2}\times(k-\frac{d_H^4(v)}{2}-1), 1\times(\frac{d_H^4(v)}{2}+3)+\frac{1}{2}\times(k-\frac{d_H^4(v)}{2}-3)\}-\frac{1}{2}d_H^3(v) = \frac{k}{2}-\frac{d_H^3(v)}{2}-\frac{d_H^4(v)}{4}-\frac{11}{2} \geq \frac{k-11}{2}-\frac{d_H^3(v)+d_H^4(v)}{2} \geq \frac{k-11}{2}-\frac{k-1}{6} = \frac{k-16}{3} \geq 0$.

- $d_H^3(v) \geq 1$ and v is not incident with any bad 3-cycle.

- - $n_{4^+}(v) = 0$. Then $3n_{\text{II}}(v) + 2d_H^3(v) - 4 \leq k$. By Remark 3, $d_H^{3b}(v) \leq 1$. By **R1–R5**, $w'(v) \geq k-4-\frac{1}{2}-1\times n_{\text{II}}(v)-\frac{1}{2}\times(k-n_{\text{II}}(v))-(\frac{1}{2}+\frac{1}{3}(d_H^3(v)-1)) = \frac{k}{2}-\frac{1}{6}(3n_{\text{II}}(v)+2d_H^3(v))-\frac{14}{3} \geq \frac{k}{2}-\frac{k+4}{6}-\frac{14}{3} = \frac{k-16}{3} \geq 0$.

- - $n_{4^+}(v) \geq 1$ and $3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor \leq k+4$. Note that v is not incident with any bad 3-cycle. If v is of Type I, $n_{4^+}(v) \geq \lceil\frac{d_H^{3b}(v)}{2}\rceil$; otherwise we have $n_{4^+}(v) \geq \lceil\frac{d_H^{3b}(v)}{2}\rceil - 1$. By **R1–R5**, $w'(v) \geq k-4-\max\{\frac{1}{2}+1\times n_{\text{II}}(v)+\frac{1}{2}\times(k-n_{\text{II}}(v)-\lceil\frac{d_H^{3b}(v)}{2}\rceil), 1\times n_{\text{II}}(v)+\frac{1}{2}\times(k-n_{\text{II}}(v)-(\lceil\frac{d_H^{3b}(v)}{2}\rceil-1))\}-(\frac{1}{3}\times d_H^{3g}(v)+\frac{1}{2}\times d_H^{3b}(v)) = \frac{k-9}{2}-\frac{n_{\text{II}}(v)}{2}-\frac{d_H^{3g}(v)}{3}-\frac{d_H^{3b}(v)}{2}+\frac{1}{2}\lceil\frac{d_H^{3b}(v)}{2}\rceil = \frac{k-9}{2}-\frac{1}{6}(3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor) \geq \frac{k-9}{2}-\frac{k+4}{6} = \frac{2k-31}{6} > 0$.

- - $n_{4^+}(v) \geq 1$ and $3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor \geq k+5$. Note that v is not incident with any bad 3-cycle. If v is of Type I, $n_{4^+}(v) \geq 3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor-(k+5)+\lceil\frac{d_H^{3b}(v)}{2}\rceil$; otherwise we have $n_{4^+}(v) \geq 3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor-(k+5)+(\lceil\frac{d_H^{3b}(v)}{2}\rceil-1)$. By **R1–R5**, $w'(v) \geq k-4-\max\{\frac{1}{2}+1\times n_{\text{II}}(v)+\frac{1}{2}\times(k-n_{\text{II}}(v)-(3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor-(k+5)+\lceil\frac{d_H^{3b}(v)}{2}\rceil)), 1\times n_{\text{II}}(v)+\frac{1}{2}\times(k-n_{\text{II}}(v)-(3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor-(k+5)+\lceil\frac{d_H^{3b}(v)}{2}\rceil-1))\}-(\frac{1}{3}\times d_H^{3g}(v)+\frac{1}{2}\times d_H^{3b}(v)) = n_{\text{II}}(v)+\frac{2}{3}d_H^{3g}(v)+\frac{1}{2}\lceil\frac{d_H^{3b}(v)}{2}\rceil+\frac{3}{2}\lfloor\frac{d_H^{3b}(v)}{2}\rfloor-\frac{d_H^{3b}(v)}{2}-7 = \frac{1}{3}(3n_{\text{II}}(v)+2d_H^{3g}(v)+3\lfloor\frac{d_H^{3b}(v)}{2}\rfloor)-7 \geq \frac{k+5}{3}-7 \geq 0$.

In conclusion, the new charge of $x \in V(H^\times) \cup F(H^\times)$ is nonnegative, a contradiction. The proof of Theorem 1 is done.

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