Journal of Combinatorial Optimization (2021) 41:762–779
https://doi.org/10.1007/s10878-021-00718-0

Digraphs that contain at most *t* distinct walks of a given length with the same endpoints

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Accepted: 22 February 2021 / Published online: 4 March 2021

Abstract
Let n, k, t be positive integers. What is the maximum number of arcs in a digraph on n vertices in which there are at most t distinct walks of length k with the same endpoints? Determine the extremal digraphs attaining the maximum number. When $t = 1$, the problem has been studied by Wu, by Huang and Zhan, by Huang, Lyu and Qiao, by Lyu in four papers, and they solved all the cases but $k = 3$. For $t \geq 2$, Huang and Lyu proved that the maximum number is equal to $n(n-1)/2$ and the extremal digraph is the transitive tournament when $n \geq 6t + 2$ and $k \geq n - 1$. They also discussed the maximum number for the case $n = k + 2, k + 3, k + 4$. In this paper, we solve the problem for the case $k \geq 6t + 1$ and $n \geq k + 5$, and we also characterize the structures of the extremal digraphs for $n = k + 2, k + 3, k + 4$.

Keywords Turán problem · Digraph · Walk · Tournament

Mathematics Subject Classification 05C35 · 05C20

1 Introduction

Given a family of graphs $\mathscr{F}$, what is the maximum number of edges in a graph on n vertices if it does not contain any member of $\mathscr{F}$ as a subgraph? Turán (1941), Turán (1954) determined the maximum number of edges of graphs on n vertices which do not contain a complete graph, and also determined the unique graph attaining that maximum. Most of the previous results in Turán type extremal graph theory concern undirected graphs and only a few extremal problems on digraphs have been investigated; see (Bollobás 1995; Brown et al. 1973, 1985; Brown and Harary 1970; Brown and Simonovits 2002; Huang and Lyu 2020a, b; Jacob and Meyniel 1983; Maurer et al. 1980; Scott 2000). In this paper we study an extremal problem on digraphs.

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The systematic investigation of digraph extremal problem was initiated by Brown and Harary (1970). This is where the area started and it covers all the cases where the excluded subgraph has at most 4 vertices. Maurer et al. (1980) studied the extremal subgraphs of the transitive tournament which contain at most one directed path from x to y with $x \neq y$. In this paper, we consider a problem posed in Huang and Lyu (2020a):

Problem 1.1 Let n, k, t be positive integers. What is the maximum number of arcs in a digraph on n vertices in which there are at most t distinct walks of length k with the same endpoints? Determine the extremal digraphs attaining the maximum number.

When $t = 1$, the above problem is the same as the unsolved problem 20 in (Zhan 2013, p.234). In Huang et al. (2019), Huang and Zhan (2011), Lyu (2020), Wu (2010) the authors solved all the cases but $k = 3$. For $t \geq 2$, Huang and Lyu (2020a) prove that the maximum number is equal to $n(n-1)/2$ and the extremal digraph is the transitive tournament when $k \geq n - 1 \geq 6t + 1$. They also discussed the maximum number for the case $k \geq 6t + 1$ and $n \in \{k + 2, k + 3, k + 4\}$. The most interesting case of Problem 1.1 is that n is sufficiently large and t is fixed. In this paper, under this condition we prove that the extremal digraphs are some particular k-partite transitive tournaments for $k \geq 6t + 1$, and the case that k is close to n is also solved. For the case $k \leq 6t$, the maximum number is difficult to determine and we leave it for future research. Generally speaking, t substituting 1 leads a deeper problem. We follow the similar approach with the approach in Huang et al. (2019) but different strategy to present the general results.

We consider digraphs without multiple arcs but allowing loops. We abbreviate directed walks and directed cycles as walks and cycles, respectively. The number of vertices in a digraph is called its *order* and the number of arcs its *size*. A p-cycle is a cycle of length p and a 1-cycle is a loop. Similarly, we define p-walk. Given a family of digraphs $\mathscr{F}$, we say a digraph D is $\mathscr{F}$*-free* if D contains no subgraph from $\mathscr{F}$. Let $ex(n, \mathscr{F})$ be the maximum size of $\mathscr{F}$-free digraphs of order n and $EX(n, \mathscr{F})$ be the set of $\mathscr{F}$-free digraphs of order n with size $ex(n, \mathscr{F})$. Given two positive integers k, t, denote by $\mathscr{F}_{k,t}$ the family of digraphs consisting of t different walks of length k with the same initial vertex and the same terminal vertex. The Problem 1.1 is equivalent to the following

Problem 1.2 Given positive integers n, k, t, determine $ex(n, \mathscr{F}_{k,t+1})$ and $EX(n, \mathscr{F}_{k,t+1})$.

Given any positive integer t, we always assume $k \geq 6t + 1$ and $t_0 = \lfloor \log_2 t \rfloor$. In Huang and Lyu (2020a), the authors determined $ex(n, \mathscr{F}_{k,t+1})$ for $n = k + 1, \dots, k+4$. They also characterized $EX(n, \mathscr{F}_{k,t+1})$ for $n = k+1$. We will determine $ex(n, \mathscr{F}_{k,t+1})$ for $n \geq k + 5$ and characterize $EX(n, \mathscr{F}_{k,t+1})$ for $n \geq k + 2$. The rest of this paper is organized as follows. Section 2 presents our main result Theorem 2.2, which determines $ex(n, \mathscr{F}_{k,t+1})$ for $n \geq k + 4 + t_0$ and characterizes $EX(n, \mathscr{F}_{k,t+1})$ for $n \geq k + 5 + t_0$. Section 3 and Section 4 present the exact values of $ex(n, \mathscr{F}_{k,t+1})$ for $n = k + 5, \dots, k + 4 + t_0$ as well as the characterization of $EX(n, \mathscr{F}_{k,t+1})$ for $n = k + 2, \dots, k + 4 + t_0$. Section 5 presents the proof of Theorem 2.2.

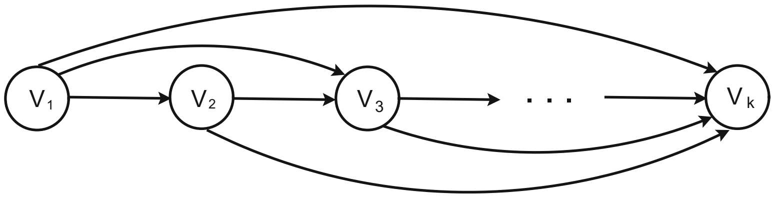

Fig. 1 (s, k, t)-transitive tournament

2 Main result

In order to present our main result, we need the following notations and definitions. For a digraph $D = (\mathcal{V}, \mathcal{A})$, we denote by $a(D)$ the size of D. Given an arbitrary subset X of $\mathcal{V}$, the subgraphs of D induced by X and $\mathcal{V}\backslash X$ are denoted by $D[X]$ and $D - X$, respectively. For convenience, if a set $X = \{x\}$ it will be abbreviated as x. For $i, j \in \mathcal{V}$, if there is an arc from i to j, then we say j is a *successor* of i, and i is a *predecessor* of j. The notation $i \to j$ means there is an arc from i to j; $i \nrightarrow j$ means there exists no arc from i to j. Given $S, T \subset \mathcal{V}$, the notation $S \to T$ means there is an arc from each vertex of S to each vertex of T; $S \nrightarrow T$ means there is no arc from S to T. If $S = \{s\}$, we write $s \to T$ and $s \nrightarrow T$. Analogously, if $T = \{t\}$, we write $S \to t$ and $S \nrightarrow t$.

A digraph $D = (\mathcal{V}, \mathcal{A})$ is said to be *transitive* if for any three vertices $x, y, z \in \mathcal{V}$, $x \to y$ and $y \to z$ indicates $x \to z$. Recall that a tournament is an orientation of the complete graph. We denote by T_n the transitive tournament with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, j) : 1 \le i < j \le n\}$.

For a digraph $D = (\mathcal{V}, \mathcal{A})$ with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, a *blow-up* of D is obtained by replacing every vertex v_i with a finite collection of copies of v_i, denoted by V_i, so that (x, y) is an arc for $x \in V_i$ and $y \in V_j$ if and only if $(v_i, v_j) \in \mathcal{A}$. Note that each V_i is an independent set.

Suppose s, k, t are nonnegative integers with $t < k$. We call a digraph D of order $sk + t$ an (s, k, t)-*transitive tournament* if it is a blow-up of T_k obtained by replacing each vertex of T_k with a collection of s or $s+1$ copies. An (s, k, t)-transitive tournament has the following diagram (Fig. 1)
where each vertex partition V_i contains s or $s + 1$ vertices and there is an arc from vertex x to vertex y if and only if $x \in V_i$, $y \in V_j$ with $i < j$. If $|V_1| = s + 1$ the (s, k, t)-transitive tournament is *initial*; if $|V_k| = s + 1$ it is *terminal*.

In Huang and Lyu (2020a), the authors presented the following result.

Theorem 2.1 *Let n, k, t be positive integers with $n \ge 6t + 2$ and $k \ge n - 1$. Then*

$$ex(n, \mathscr{F}_{k,t+1}) = \frac{n(n-1)}{2},$$

and $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is a transitive tournament.

They also determined $ex(n, \mathscr{F}_{k,t+1})$ for $k \ge 6t + 1$ and $n = k + 2, k + 3, k + 4$. Now we are ready to state our main result.

Theorem 2.2 *Let k, t, t_0 be nonegative integers with $k \geq 6t+1$ and $t_0 = \lfloor \log_2 t \rfloor$. Let n, s, r be nonnegative integers with $n = sk + r$ and $r < k$. If $n \geq k + t_0 + 4$, then*

$$ex(n, \mathscr{F}_{k,t+1}) = \binom{n}{2} - \binom{s}{2}k - sr. \tag{2.1}$$

Moreover, if $n \geq k + t_0 + 5$, $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is an (s, k, r)-transitive tournament.

We will also determine $EX(n, \mathscr{F}_{k,t+1})$ for $n \in \{k+2, \ldots, k+4+t_0\}$, and $ex(n, \mathscr{F}_{k,t+1})$ for $n \in \{k+5, \ldots, k+3+t_0\}$. For the case $k \leq 6t$, our result does not always hold. We can see the example in Huang et al. (2019), which illustrates Theorem 2.2 is not true even when $t = 1$ and $k = 3$. We leave the cases $k \leq 6t$ of Problem 1.2 for future work.

Remark (Huang et al. 2019, Theorem 1) implies Theorem 2.2 holds if $t = 1$. Hence, throughout the article we always assume $t \geq 2$. In addition, we also assume $k \geq 6t+1$ and $t_0 = \lfloor \log_2 t \rfloor$ if no otherwise statement.

3 $ex(n, \mathscr{F}_{k,t+1})$ and $EX(n, \mathscr{F}_{k,t+1})$ for $n \leq k + t_0 + 3$

For convenience, we always use $\langle n \rangle = \{1, \ldots, n\}$ to denote the vertex set of a digraph D of order n unless otherwise stated. We need the following lemmas.

Lemma 3.1 *(Huang et al. 2019) Let $n \geq 3$ and p be nonnegative integers, and let D be a digraph on n vertices. Given $q \geq 0$ such that $p(n-1)/2 + q$ is a positive integer, if*

$$a(D - i) \leq \frac{(n-1)(n-2)}{2} - \frac{p(n-1)}{2} - q \quad for\ all\ i \in \langle n \rangle,$$

then

$$a(D) \leq \frac{n(n-1)}{2} - \frac{p(n+1)}{2} - q - 1. \tag{3.1}$$

Two distinct cycles are said to be *joint* if they have a common vertex.

Lemma 3.2 *(Huang and Lyu 2020a) Let n, t be positive integers and D be a digraph of order n. If an m_1-cycle and an m_2-cycle in D are joint or connected by an arc, then D is not $\mathscr{F}_{k,t+1}$-free for all $k \geq tL + 1$, where L is the least common multiple of m_1 and m_2.*

Given a digraph $D = (\mathcal{V}, \mathcal{A})$, denote by

$$N^+(u) = \{x \in \mathcal{V} | (u, x) \in \mathcal{A}\} \text{ and } N^-(u) = \{x \in \mathcal{V} | (x, u) \in \mathcal{A}\}$$

the sets of successors and predecessors of a vertex u. The outdegree and indegree of u are $d^+(u) \equiv |N^+(u)|$ and $d^-(u) \equiv |N^-(u)|$, respectively. Let $d(u)$ be the number of arcs incident with a vertex u. Then $d(u) = a(D) - a(D-u)$ and

$$d(u) = \begin{cases} d^+(u) + d^-(u) - 1, & u \to u; \\ d^+(u) + d^-(u), & \text{otherwise.} \end{cases}$$

Lemma 3.3 *(Huang and Lyu 2020a) Let $D = (\mathcal{V}, \mathcal{A})$ be a digraph of order n and let t be a positive integer. Suppose $v \in \mathcal{V}$ is contained in a cycle. If $D[N^+(v)]$ or $D[N^-(v)]$ contains a transitive tournament of order $t+2$, then D is not $\mathscr{F}_{k,t+1}$-free for $k \geq 2$.*

Now we present the main result of this section.

Theorem 3.4 *Let n, k, t be positive integers with $k \geq 6t+1$ and let $t_0 = \lfloor \log_2 t \rfloor$. If $k+1 \leq n \leq k+t_0+3$, we have*

$$ex(n, \mathscr{F}_{k,t+1}) = \frac{n(n-1)}{2} - n + k + 1. \tag{3.2}$$

Moreover,

(1) *If $k+1 \leq n \leq k+t_0+1$, $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is a $(1, k+1, n-k-1)$-transitive tournament;*
(2) *If $n = k+t_0+2$, $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is an initial or a terminal $(1, k+1, t_0+1)$-transitive tournament;*
(3) *If $n = k+t_0+3$, $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is a $(1, k+1, t_0+2)$-transitive tournament both initial and terminal.*

Proof We follows the strategy as follows. First we prove (3.2). Then we show that every digraph $H \in EX(n, \mathscr{F}_{k,t+1})$ has an induced subgraph in $EX(n-1, \mathscr{F}_{k,t+1})$, say H'. By the induction hypothesis, the structure of H' is clear. At last, we add a vertex to H' to get the structure of H. Let D be an $\mathscr{F}_{k,t+1}$-free digraph of order n. We use induction on the order of D. By Theorem 2.1, Theorem 3.4 holds for $n = k+1$. Assume Theorem 3.4 holds for $n = k+1, k+2, \ldots, k+\tau$, where $2 \leq \tau \leq t_0+2$. Now we consider $n = k+\tau+1$. It is clear that $D-i$ is $\mathscr{F}_{k,t+1}$-free for each $i \in \mathcal{V}$. By the induction hypothesis $a(D-i) \leq ex(n-1, \mathscr{F}_{k,t+1})$. Applying Lemma 3.1 we obtain

$$a(D) \leq \frac{n(n-1)}{2} - \tau.$$

On the other hand, when $\tau \leq t_0$, it is easily seen that all $(1, k+1, \tau)$-transitive tournaments are $\mathscr{F}_{k,t+1}$-free with size $n(n-1)/2 - \tau$. When $\tau = t_0+1$, an initial or a terminal $(1, k+1, \tau)$-transitive tournament is $\mathscr{F}_{k,t+1}$-free with size $n(n-1)/2 - \tau$. When $\tau = t_0+2$, a $(1, k+1, \tau)$-transitive tournament both initial and terminal is $\mathscr{F}_{k,t+1}$-free with size $n(n-1)/2 - \tau$. Thus we get (3.2) and the sufficiency parts of (1), (2), (3).

Let $D \in EX(n, \mathscr{F}_{k,t+1})$. By Lemma 3.1 and (3.2), there exists $i_0 \in \mathcal{V}(D)$ such that

$$a(D - i_0) = \frac{(n-1)(n-2)}{2} - \tau + 1.$$

By the induction hypothesis, $D - i_0$ is a $(1, k+1, \tau - 1)$-transitive tournament.

Let $\{i_1, \ldots, i_{\tau-1}\}$ be an arbitrary $(\tau - 1)$-subset of $\langle k+1 \rangle$. Denote by

$$V_i = \{i, i+k+1\} \text{ for } i \in \{i_1, \ldots, i_{\tau-1}\}$$

and

$$V_i = \{i\} \text{ for } i \in \langle k+1 \rangle \setminus \{i_1, \ldots, i_{\tau-1}\}.$$

Without loss of generality, we assume that D is a digraph with vertex set

$$\mathcal{V} = \bigcup_{i=1}^{k} V_i \cup \{i_0\} = \{1, 2, \ldots, k+1, i_1 + k + 1, \ldots, i_{\tau-1} + k + 1, i_0\}$$

such that there is an arc $x \to y$ in $D - i_0$ if and only if $x \in V_i$, $y \in V_j$ with $i < j$.

Since $d(i_0) = n - 2 = k + \tau - 1$, we have

$$d^+(i_0) + d^-(i_0) \geq 6t + 1 \geq 2t + 3 + 2t_0.$$

Then either i_0 has $t+2$ predecessors in $\langle k+1 \rangle$ or it has $t+2$ successors in $\langle k+1 \rangle$. By Lemma 3.3, we have $i_0 \nrightarrow i_0$ and i_0 is not contained in any 2-cycle. Recalling $d(i_0) = n - 2$, i_0 is adjacent with all vertices in $\mathcal{V} \setminus \{i_0\}$ but one, say j_0.

Let $s' \in \langle k+1 \rangle$ be the largest integer such that $s' \to i_0$ and $t' \in \langle k+1 \rangle$ be the smallest integer such that $i_0 \to t'$. Here we let $s' = 0$ if $\langle k+1 \rangle \nrightarrow i_0$ and $t' = k+2$ if $i_0 \nrightarrow \langle k+1 \rangle$. By Lemma 3.3, $t' > s'$. If $t' = s' + 1$, we have the following walk of length $k+1$:

$$w : 1 \to 2 \to \cdots \to s' \to i_0 \to s' + 1 \to \cdots \to k + 1.$$

Note that if $s' = 0$, w begins at i_0; if $t' = k+2$, w ends at i_0. We obtain w_i from w by deleting i and joining its predecessor to its successor. There exists a set $\{u_1, u_2, \ldots, u_{t+1}\} \subset \langle k \rangle \setminus \{1\}$ such that $s', t' \notin \{u_1, u_2, \ldots, u_{t+1}\}$. It is easily seen that $w_{u_1}, \ldots, w_{u_{t+1}}$ are $t+1$ distinct walks of length k sharing the same endpoints, a contradiction. Hence $t' > s' + 1$. Combining this with $d(i_0) = n - 2$, we obtain

$$t' = s' + 2 \text{ with } s' \in \{0\} \cup \langle k \rangle. \tag{3.3}$$

It follows that $j_0 = s' + 1$ and

$$\{1, \ldots, s'\} \to n \to \{t', \ldots, k+1\}.$$

Here $s' = 0$ means $\{1, \ldots, s'\}$ is empty and $s' = k$ means $\{t', \ldots, k+1\}$ is empty.

Note that i_0 is adjacent with each vertex of $\{k+1+i_1, \ldots, k+1+i_{\tau-1}\}$. We assert that for each $i \in \{i_1, \ldots, i_{\tau-1}\}$, if $i \to i_0$ we have $k+1+i \to i_0$. Otherwise we assume $i_0 \to k+1+i$. Replacing the role of i with $i+k+1$ we get $t' \neq s'+2$, contradicting (3.3). Similarly, if $i_0 \to i$ we have $i_0 \to k+1+i$ and if $i = j_0$ there is no arc between i_0 and $i+k+1$. Recalling i_0 is adjacent with each vertex of $\{k+1+i_1, \ldots, k+1+i_{\tau-1}\}$, we get $j_0 \notin \{i_1, \ldots, i_{\tau-1}\}$. Therefore, D is a $(1, k+1, \tau)$-transitive tournament. Then we leave the following two cases to discuss.

Case 1. $\tau = t_0 + 1$. We assert that D is initial or terminal. Otherwise $\{1, k+1\} \nsubseteq \{j_0, i_1 \ldots, i_{\tau-1}\}$. There is a k-walk w' as follows.

$$1 \to 2 \to \cdots \to k+1.$$

For each $i \in \{j_0, i_1 \ldots, i_{\tau-1}\}$, we could obtain new k-walks from w' by replacing i with $k+1+i$. Note that we may replace j_0 with i_0. Hence there are 2^{t_0+1} walks of length k with the same endpoints, a contradiction.

Case 2. $\tau = t_0 + 2$. Applying the same arguments as above, we obtain that D is both initial and terminal. This completes the proof. □

4 $ex(n, \mathscr{F}_{k,t+1})$ and $EX(n, \mathscr{F}_{k,t+1})$ for $n = k + t_0 + 4$

To present the main result of this section, we need the following lemmas.

Lemma 4.1 *Let D be a digraph such that $D - i_0$ is a blow-up of T_{k+1} with vertex partition*

$$\mathcal{V} \setminus \{i_0\} = \bigcup_{i=1}^{k+1} V_i,$$

where $i_0 \in \mathcal{V}(D)$. In $D - i_0$ we have $x \to y$ for $x \in V_i$, $y \in V_j$ if and only if $i < j$. If $k \geq t+2$ and D is $\mathscr{F}_{k,t+1}$-free, then $V_{k+1} \nrightarrow i_0 \nrightarrow V_1$.

Proof Without loss of generality, we assume that $i \in V_i$ for $i \in \langle k+1 \rangle$. Suppose there is $x \in V_1$ such that $i_0 \to x$, we obtain a walk of length $k+1$ as follows.

$$w : i_0 \to x \to 2 \to \cdots \to k \to k+1.$$

For each $j \in \langle k \rangle \setminus \{1\}$, we could obtain a new walk w_j of length k from w by deleting j and joining its predecessor to its successor. Since $|\langle k \rangle \setminus \{1\}| \geq t+1$, there are $t+1$ distinct walks of length k with the same endpoints, a contradiction. Hence $i_0 \nrightarrow V_1$. Similarly, $V_{k+1} \nrightarrow i_0$. □

Let k be a positive integer and let α be an arbitrary subset of $\langle k \rangle \setminus \{1\}$. Denote by $H(k, \alpha)$ the digraph with vertex set

$$\mathcal{V} = \bigcup_{i \in \langle k+1 \rangle} V_i,$$

where $V_i = \{i, (k+1)+i\}$ for $i \in \alpha \cup \{1, k+1\}$ and $V_i = \{i\}$ for $i \in \langle k\rangle \backslash (\{1\} \cup \alpha)$, such that $x \rightarrow y$ if and only if $x \in V_i$, $y \in V_j$ with $i < j$. Note that $H(k, \alpha)$ is a $(1, k+1, |\alpha|+2)$-transitive tournament both initial and terminal.

Now we define several new classes of digraphs. Given $i \in \alpha \cup \{1, k+1\}$, let $H_i(k, \alpha)$ be the digraph obtained from $H(k, \alpha)$ by adding a vertex $2k+2+i$, the arcs $2k+2+i \rightarrow V_j$ for $i < j$ and the arcs $V_j \rightarrow 2k+2+i$ for $i > j$; given $i \in \langle k-2\rangle$, let $H_u(k, i, \alpha)$ be the digraph obtained from $H(k, \alpha)$ by adding a vertex u, and the arcs $V_j \rightarrow u$ for $j \leq i$, $u \rightarrow V_j$ for $j \geq i+3$.

Lemma 4.2 *Let k, t, t_0 be nonegative integers with $t_0 = \lfloor \log_2 t \rfloor$, $k \geq 6t+1$ and let α be an arbitrary t_0-subset of $\langle k\rangle \setminus \{1\}$. Let D be an $\mathscr{F}_{k,t+1}$-free digraph with $D - i_0 = H(k, \alpha)$. If there exist $i, j \in \langle k+1\rangle$ with $x \in V_i$, $y \in V_j$ such that $x \rightarrow i_0 \rightarrow y$, then $j \geq i+2$.*

Proof Suppose there exist $x \in V_i$, $y \in V_j$ with $j \leq i+2$ such that $x \rightarrow i_0 \rightarrow y$. Since D is $\mathscr{F}_{k,t+1}$-free, by Lemma 4.1 we have $i \leq k$ and $j \geq 2$. Without loss of generality, we let $i \rightarrow i_0 \rightarrow j$. If $j \leq i$, there is a walk of length $k+2$ as follows.

$$1 \rightarrow \cdots \rightarrow i \rightarrow i_0 \rightarrow j \rightarrow i+1 \rightarrow \cdots \rightarrow k+1.$$

Note that $k \geq 6t+1$. There are more than $6t-3$ vertices in $\langle k\rangle \setminus \{1, i, j\}$. We could obtain more than t walks from the walk above by deleting any pair of these vertices and adding two arcs from their predecessors to their successors respectively, a contradiction. From the above discussion, if $i, j \in \alpha$, we have $i+k+1 \rightarrow i_0$, $i_0 \rightarrow j+k+1$.

If $j = i+1$, there is a walk of length $k+1$ as follows.

$$1 \rightarrow \cdots \rightarrow i \rightarrow i_0 \rightarrow i+1 \rightarrow \cdots \rightarrow k+1.$$

There are more than t vertices in $\langle k\rangle \setminus \{1, i, i+1\}$. We could obtain more than t walks from the walk above by deleting any one of vertices in $\langle k\rangle \setminus \{1, i, i+1\}$ and adding an arc from its predecessor to its successor, a contradiction. Hence, $j \geq i+2$. □

The *girth* of a digraph D with a cycle, denoted by $g(D)$, is the length of its shortest cycle.

Lemma 4.3 *Let D be a digraph of order $n \geq 13$. If it has a cycle and*

$$d(i) \geq n-2 \text{ for } i \in \mathcal{V},$$

then $g(D) \leq 4$.

Proof If D has a loop, we are done. Hence we always assume D is loopless. To the contrary suppose $g(D) \geq 5$. Assume there is a cycle as follows.

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow g(D) \rightarrow 1.$$

From the assumption, there is an arc between 1 and one of 3, 4. For the both cases, we can get shorter cycles, a contradiction. Therefore, $g(D) \leq 4$. □

Now we solve the case $n = k + t_0 + 4$.

Theorem 4.4 *Let n, k, t be positive integers with $k \geq 6t + 1$ and let $t_0 = \lfloor \log_2 t \rfloor$. If $n = k + t_0 + 4$, then*

$$ex(n, \mathscr{F}_{k,t+1}) = \frac{n(n-1)}{2} - t_0 - 4. \tag{4.1}$$

Moreover,

(1) if $t \geq 2^{t_0-1} \times 3$, $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is a $(1, k, t_0 + 4)$-transitive tournament or $D \cong H$ with $H \in \{H_i(k, \alpha) : i \in \alpha \cup \{1, k+1\}\} \cup \{H_{i_0}(k, i, \alpha) : i \in \langle k-2 \rangle$ and $|V_{i+1}| = |V_{i+2}| = 1\} \cup \{H(k, \alpha \cup \{i+1\}) - (i+k+1, i+k+2) : i \in \alpha$ and $i + 1 \in \langle k \rangle \setminus (\{1\} \cup \alpha)\}$, where α is an arbitrary t_0-subset of $\langle k \rangle \setminus \{1\}$;

(2) if $t < 2^{t_0-1} \times 3$, $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is a $(1, k, t_0 + 4)$-transitive tournament or $D \cong H$ with $H \in \{H_i(k, \alpha) : i \in \{1, k+1\}\} \cup \{H_{i_0}(k, i, \alpha) : i \in \langle k-2 \rangle$ and $|V_{i+1}| = |V_{i+2}| = 1\}$, where α is an arbitrary t_0-subset of $\langle k \rangle \setminus \{1\}$.

Proof We follow the strategy as follows. First we prove (4.4). Then we characterise the extremal digraphs. We distinguish two cases according to the maximum size of the induced subgraphs of order $n - 1$. In the first case, there is an induced subgraph containing $ex(n-1, \mathscr{F}_{k,t+1})$ arcs. Hence, it is easy to get the structure of the subgraph, then by adding a vertex we get the extremal digraph. In the other case, there is no induced subgraph containing $ex(n-1, \mathscr{F}_{k,t+1})$ arcs. We characterise the structure by analysing the degrees of vertices.

Let $D = (\mathcal{V}, \mathcal{A})$ be an $\mathscr{F}_{k,t+1}$-free of order n. Combining Theorem 3.4 with Lemma 3.1 we obtain

$$a(D) \leq \frac{n(n-1)}{2} - t_0 - 3. \tag{4.2}$$

Suppose equality in (4.2) holds. By Lemma 3.1, there exists $i \in \mathcal{V}$ such that

$$a(D - i) \geq \frac{(n-1)(n-2)}{2} - t_0 - 2.$$

Applying Theorem 3.4 to $D - i$, it is an initial and terminal $(1, k+1, t_0+2)$-transitive tournament. Without loss of generality, we assume $\mathcal{V}(D - i) = H(k, \alpha)$, where $\alpha \subset \langle k \rangle \setminus \{1\}$ with $|\alpha| = t_0$.

It is clear that $d(i) = n - 2$. By Lemma 3.3, $i \nrightarrow i$. Then by Lemma 4.1, $V_{k+1} \nrightarrow i \nrightarrow V_1$. Hence, i has both predecessors and successors. Let s' be the largest integer such that there exists a vertex $x \in V_{s'}$ with $x \rightarrow i$ and let t' be the smallest integer such that there exists a vertex $y \in V_{t'}$ with $i \rightarrow y$. By Lemma 4.2 we get $t' \geq s' + 2$. Recalling that $d(i) = n - 2$ and $i \nrightarrow i$, we get $t' \leq s' + 2$. Hence $t' = s' + 2$ and D is a $(1, k + 1, t_0 + 3)$-transitive tournament. It is clear that D is not $\mathscr{F}_{k,t+1}$-free.

Therefore, equality in (4.2) does not hold. It follows that

$$a(D) \leq \frac{n(n-1)}{2} - t_0 - 4.$$

On the other hand, any $(1, k, t_0+4)$-transitive tournament is $\mathscr{F}_{k,t+1}$-free and it has the maximum number of arcs. Hence, (4.1) holds.

Suppose D is a $(1, k, t_0 + 4)$-transitive tournament or $D \cong H$ with $H \in \{H_1(k, \alpha), H_{k+1}(k, \alpha)\} \cup \{H_{i_0}(k, i, \alpha) : i \in \langle k-2\rangle \textit{ and } |V_{i+1}| = |V_{i+2}| = 1\}$, where α is an arbitrary subset of $\langle k\rangle \setminus \{1\}$ with $|\alpha| = t_0$. Then $a(D) = n(n-1)/2 - t_0 - 4$. Moreover, it is easy to check that D is $\mathscr{F}_{k,t+1}$-free. When $t \geq 2^{t_0-1} \times 3$, it is clear that for any t_0-subset of $\langle k\rangle$ and each $i \in \alpha$, $D \in H_i(k, \alpha)$ is $\mathscr{F}_{k,t+1}$-free and has $n(n-1)/2 - t_0 - 4$ arcs. For $D \in \{H(k, \alpha \cup \{i+1\}) - (i+k+1, i+k+2) : i \in \alpha \textit{ and } i+1 \in \langle k\rangle \setminus (\{1\} \cup \alpha)\}$, it is clear that D is $\mathscr{F}_{k,t+1}$-free and it has $n(n-1)/2 - t_0 - 4$ arcs. Therefore, $D \in Ex(n, \mathscr{F}_{n-4})$ and we get the sufficiency part.

Next we prove the necessity part. Let $D \in Ex(n, \mathscr{F}_{k,t+1})$. Note that all subgraphs of D are $\mathscr{F}_{k,t+1}$-free. By Theorem 3.4 we have

$$a(D-i) \leq \frac{(n-1)(n-2)}{2} - t_0 - 2 \text{ for all } i \in \mathcal{V}.$$

We distinguish two cases.

Case 1. There exists $i \in \mathcal{V}$ such that

$$a(D-i) = \frac{(n-1)(n-2)}{2} - t_0 - 2.$$

By Theorem 3.4(iii), $D - i$ is an initial and terminal $(1, k+1, t_0+2)$-transitive tournament. We may assume $D - i = H(k, \alpha)$ with α is a t_0-subset of $\langle k\rangle \setminus \{1\}$. It is clear that

$$d(i) = a(D) - a(D-i) = n - 3. \tag{4.3}$$

By Lemma 3.3, we have

$$i \nrightarrow i. \tag{4.4}$$

By Lemma 4.1, $V_{k+1} \nrightarrow i \nrightarrow V_1$. If $\mathcal{V} \nrightarrow i$, we have $i \rightarrow \mathcal{V}\setminus V_1$, which implies that $D \cong H_1(k, \alpha)$. Similarly, if $i \nrightarrow \mathcal{V}$, we have $\mathcal{V}\setminus V_{k+1} \rightarrow i$. It follows that $D \cong H_{k+1}(k, \alpha)$. Now assume i has both predecessors and successors. Define s' and t' as before. By (4.3) and (4.4), $t' \leq s' + 3$. By Lemma 4.2, $t' \geq s' + 2$.

Suppose $t' = s'+3$. It is clear that $s' \in \langle k-2\rangle$. Moreover, $|V_{s'+1}| = |V_{s'+2}| = 1$ as $d^+(i) + d^-(i) = n - 3$. It follows that $D = H_i(k, s', \alpha)$ with $|V_{s'+1}| = |V_{s'+2}| = 1$ and $s' \in \langle k-2\rangle$. Suppose $t' = s' + 2$. By (4.3) and (4.4), $1 \leq |V_{s'+1}| \leq 2$. If $|V_{s'+1}| = 1$, we can conclude that $|V_{s'}| + |V_{s'+2}| \geq 3$. Otherwise, there is a walk of length k as follows.

$$1 \rightarrow \cdots \rightarrow s' \rightarrow s'+1 \rightarrow s'+2 \rightarrow \cdots \rightarrow k+1.$$

For each $j \in \alpha$, in the above walk we can replace j by $k+1+j$ to obtain a new walk. We can also replace $s'+1$ by i since $s' \rightarrow i \rightarrow s'+2$. So there are 2^{t_0+1} walks of length

k from 1 to $k+1$, a contradiction. Hence, we get $|V_{s'}| \geq 2$ or $|V_{s'+2}| \geq 2$. Moreover, there is a vertex in $V_{s'} \cup V_{s'+2}$ not adjacent with i. Without loss of generality, we assume that $|V_{s'}| = 2$ and $s'+k+1 \nrightarrow i$, i.e., $D \cong H(k, \alpha_1) - (s'+k+1, s'+k+2)$ with $\alpha_1 = \alpha \cup \{s'+1\}$ and $|V_{s'}| = 2$. If $|V_{s'+1}| = 2$, D is isomorphic to $H_{s'+1}(k, \alpha)$ with $s'+1 \in \alpha \setminus \{1\}$. For the both cases, when $t < 2^{t_0-1} \times 3$, D is not $\mathscr{F}_{k,t+1}$-free, a contradiction; when $t \geq 2^{t_0-1} \times 3$, D is $\mathscr{F}_{k,t+1}$-free and has the maximum number of arcs.

Case 2. For all $i \in \mathcal{V}$, $a(D-i) \leq (n-1)(n-2)/2 - t_0 - 3$. Then

$$d(i) \geq n-2 \text{ for all } i \in \mathcal{V}. \tag{4.5}$$

By Lemma (Huang and Lyu 2020a, Lemma 2.3) we have

$$d(i) \leq n \text{ for all } i \in \mathcal{V}. \tag{4.6}$$

Suppose $d(i_0) = n$. If $i_0 \to i_0$, we get $d^+(i_0) + d^-(i_0) = k + t_0 + 5$. Without loss of generality, we assume $d^+(i_0) \geq 3t + 3$. By Lemma 3.2, i_0 is joined with every vertex by exactly one arc. Moreover, $D - i_0$ contains no cycles of length less than 5. Combining with Lemma 4.3, $D - i_0$ contains no cycles. It is well known (West 1996, Lemma 1.4.23) that if each vertex of a diagraph has a successor, then this digraph contains a cycle. It follows that there is a vertex i_1 with no successors in $\mathcal{V} \setminus \{i_0\}$. (4.5) indicates i_1 has at least $n-3$ predecessors in $\mathcal{V} \setminus \{i_0\}$. Moreover, $|N^+(i_0) \cap N^-(i_1)| \geq 3t - 1$. Then there are more than t walks of length k in the following form.

$$i_0 \to \cdots \to i_0 \to x \to i_1,$$

where x is any vertex in $N^+(i_0) \cap N^-(i_1)$. We get a contradiction. Thus $i_0 \nrightarrow i_0$. Now we assume there is $i_0' \in \mathcal{V} \setminus \{i_0\}$ such that $i_0 \to i_0' \to i_0$. Adopting the same technique as above, we can get a contradiction. Therefore, we obtain

$$d(i) \leq n-1 \text{ for all } i \in \mathcal{V}. \tag{4.7}$$

We assert that D is loopless. Otherwise, we let $i_0 \to i_0$. By (4.5), $d^+(i_0) + d^-(i_0) \geq k + t_0 + 3$. Without loss of generality, we let $d^+(i_0) \geq 3t + 2$. By Lemma 3.2, each successor of i_0 except itself cannot be contained in any 2-cycle or loop. Then each vertex in $N^+(i_0)$ is incident with more than $d^+(i_0) - 2$ arcs in $D[N^+(i_0)]$. By Lemma 4.3 and Lemma 3.2, $D[N^+(i_0)]$ has only one cycle: $i_0 \to i_0$. Then there is a vertex, say i_1, with no successors in $N^+(i_0)$. Moreover, $|N^+(i_0) \cap N^-(i_1)| \geq 3t$. Then there are more than t walks of length k in the following form.

$$i_0 \to \cdots \to i_0 \to x \to i_1,$$

where x is any vertex in $N^+(i_0) \cap N^-(i_1)$. We get a contradiction with D is $\mathscr{F}_{k,t+1}$-free. Hence, D has no loop. Similarly, D has no 2-cycles.

Let

$$V_1 = \{i \in \mathcal{V} : d(i) = n - 1\} \text{ and } V_2 = \{i \in \mathcal{V} : d(i) = n - 2\}.$$

Recalling (4.5) and (4.7), $\mathcal{V} = V_1 \cup V_2$. From (4.1) we have $|V_1| = k - t_0 - 4$ and $|V_2| = 2t_0+8$. It is clear $D[V_1]$ is a tournament. Suppose $D[V_1]$ has a cycle. By (Huang and Lyu 2020a, Lemma 2.2), $D[V_1]$ has a 3-cycle, say, $a \to b \to c \to a$. Without loss of generality, we assume $N^+_{D[V_1]}(a) \geq t + 2$. By Lemma 3.3, $D[N^+_{D[V_1]}(a)]$ contains a cycle. Moreover, $D[N^+_{D[V_1]}(a)]$ has a 3-cycle. By Lemma 3.2, $D[V_1]$ is not $\mathscr{F}_{k,t+1}$-free, a contradiction. Thus $D[V_1]$ is a transitive tournament. For simplicity, we let $k_1 = k - t_0 - 4$. Without loss of generality, let $V_1 = \langle k_1 \rangle$ and

$$D[V_1] = T_{k_1}. \tag{4.8}$$

Moreover, Lemma 3.3 implies the following claim.

Claim 1 *D contains no cycles.*

We partition V_2 into $W_1, W_2, \ldots, W_{t_0+4}$ such that each W_i consisting of two vertices of V_2 which are not adjacent, say x_i and y_i. Given $i \in V_2$, let $s_i \in \langle k_1 \rangle$ be the largest integer such that $s_i \to i$ and $t_i \in \langle k_1 \rangle$ be the smallest integer such that $i \to t_i$. Here if $i \nrightarrow \langle k_1 \rangle$ let $s_i = 0$ and if $\langle k_1 \rangle \nrightarrow i$ let $t_i = k_1 + 1$. For each $i \in V_2$, it is joined with each vertex of $\langle k_1 \rangle$ by exactly one arc. By Claim 1 we have

$$t_i = s_i + 1 \ \text{ for all } i \in V_2, \tag{4.9}$$

which implies

$$\{1, \ldots, s_i\} \to i \to \{s_i + 1, \ldots, k_1\}. \tag{4.10}$$

Moreover, for $i, j \in V_2$,

$$\text{if } i \to j, \text{ then } s_i \leq s_j. \tag{4.11}$$

Given any $\alpha \subset V_2$, there are $n(\alpha)$ distinct neighborhoods in $\langle k_1 \rangle$ for the vertices in α. We partition α into $\bigcup_{i=1}^{n(\alpha)} U_i(\alpha)$ such that for each pair $j, l, s_j = s_l$ if and only if $j, l \in U_i$ with $i \in n(\alpha)$. We abbreviate $U_i(\alpha)$ as U_i if no confusion arises. Define $s(U_i) = s_j$ with $j \in U_i$ for $i = 1, \ldots, n(\alpha)$. We may assume $s(U_1) < s(U_2) < \cdots < s(U_{n(\alpha)})$. α is said to be insertable if $D[U_i]$ contains a path of length $|U_i|-1$ for $i \in \{1, \ldots, n(\alpha)\}$. There is a path of length $k_1 - 1$ as follows.

$$p : 1 \to 2 \to \cdots \to k_1.$$

If $\alpha \subseteq V_2$ is insertable, we can obtain a new walk by inserting the vertices of α into p following the strategy below.

$$1 \to \cdots \to s(U_1) \xrightarrow{P_1} s(U_1)+1 \to \cdots \to s(U_{n(\alpha)}) \xrightarrow{P_{n(\alpha)}} s(U_{n(\alpha)})+1 \to \cdots \to k_1,$$

where P_i is the path made up of all vertices of U_i. Note that if $s(U_1) = 0$, P_1 precedes 1; and if $s(U_{n(\alpha)}) = k_1$, k_1 precedes $P_{n(\alpha)}$ and the new path ends at $P_{n(\alpha)}$. Hence, we could obtain a new path of length $k_1 - 1 + |\alpha|$ from inserting vertices of α into p. In the remainder of this proof, our discussion is based on this fact: every tournament has a path containing all its vertices (Harary and Moser 1966, Theorem 4).

Claim 2 $s_{x_i} = s_{y_i}$ for $i \in \{1, \ldots, t_0 + 4\}$.

Otherwise, there is $b \in \{1, \ldots, t_0 + 4\}$ such that $s_{x_b} < s_{y_b}$. Let

$$\alpha_1 = \{y_b\} \cup \{x_i : i \in \langle t_0 + 4\rangle\}.$$

It is clear that if $D[U_i]$ is a transitive tournament for $i = 1, \ldots, n(\alpha)$, then α_1 is insertable. Without loss of generality, we assume $x_b \in U_{j_1}$ and $y_d \in U_{j_2}$ with $j_1 < j_2$.

Suppose for any distinct pair $\{i, j\} \subseteq \langle t_0 + 4\rangle \setminus \{b\}$, $W_i \to W_j$ or $W_j \to W_i$. We obtain a walk w of length k by inserting the vertices of α_1 into p. Assume x is its initial vertex and y its terminal vertex. Without loss of generality, we assume $x \in W_c$ and $y \in W_d$ with b, c, d three distinct integers. Let $\beta = \langle t_0+4\rangle \setminus \{b, c, d\}$. For each $i \in \beta$, we have two choices, i.e., we can replace x_i with y_i in α_1 to obtain a new insertable set. Then there are 2^{t_0+1} walks of length k with initial vertex x and terminal vertex y, a contradiction.

Now assume without loss of generality there are $c, d \in \langle t_0 + 4\rangle \setminus \{b\}$ such that $x_c \to x_d \to y_c$. Let

$$\alpha = \{y_b, y_c\} \cup \{x_i : i \in \langle t_0 + 4\rangle\}.$$

If $s_{x_c} \neq s_{y_c}$, then $D[U_i]$ is a tournament for $i = 1, \ldots, n(\alpha)$. It follows that α is insertable, so there is a walk of length $k+1$ obtained by inserting α into p. We can obtain a new walk of length k by deleting any vertex in $\langle k_1\rangle \setminus \{1, k_1, s_{x_b}+1, \min(s_{x_c}, s_{y_c})+1\}$ and joining its predecessor to its successor. There are more than t walks of length k with the same endpoints, a contradiction. Now we assume $s_{x_c} = s_{y_c}$. Then there is $j \in \{1, 2, \ldots, n(\alpha)\}$ such that x_c, x_d, y_c are contained in U_j. By Claim 1, we can partition $U_j \setminus \{x_c, x_d, y_c\}$ into $N^+_{U_j}(y_c)$, $N^+_{U_j}(x_d) \cap N^-_{U_j}(y_c)$, $N^+_{U_j}(x_c) \cap N^-_{U_j}(x_d)$ and $N^-_{U_j}(x_c)$, where $N^+_V(x)$ ($N^-_V(x)$) denotes the set of successors (predecessors) of x in the vertex set V. The vertices of each set make up a path. Then there is a path containing all vertices in U_j. Moreover, α is insertable.

By inserting α into p, we can obtain a walk w_1 of length $k+1$ with the initial vertex x and the terminal vertex y. Then new walks of length k with the same endpoints x, y could be obtained from w_1 by deleting any vertex of $\{2, \ldots, k_1 - 1\} \setminus \{s_{x_b} + 1\}$ and joining its predecessor to its successor. Note $t \geq 2$. Then there are more than t walks of length k with the same endpoints, a contradiction. Hence, we get Claim 2.

Claim 3 *For distinct pair $i, j \in \langle t_0 + 4\rangle$, either $W_i \to W_j$ or $W_j \to W_i$.*

Recalling (4.11), it is sufficient to consider the case $s_{x_i} = s_{x_j}$. Suppose there are $c, d \in \langle t_0 + 4\rangle$ such that $x_c \to x_d \to y_c$. If $y_d \to x_c$, we let

$$\alpha = \{y_c, y_d\} \cup \{x_i : i \in \langle t_0 + 4\rangle\}.$$

Applying the same analysis as in the proof of Claim 2, α is insertable. Then we can obtain a walk of length $k + 1$ with endpoints x, y by inserting α into p, say, w_1. We obtain walks of length k with endpoints x, y from the above walk by deleting one of vertex of $\langle k_1\rangle \setminus \{1, k_1\}$ and joining its predecessor to its successor. Then there are more than t walks of length k with the same endpoints, a contradiction. If $x_d \to y_d$, using the same analysis we get a contradiction.

Now assume $x_c \to y_d \to y_c$. Without loss of generality, we assume $x_j \in W_j$ and $x_l \in W_l$ such that x_j has no predecessor in V_2 and x_l has no successor in V_2. We let

$$\alpha = W_c \cup \{x_j, x_l\} \cup \{z_i : i \in \langle t_0 + 4\rangle \setminus \{c, j, l\}\},$$

where $z_i \in W_i$ for $i \in \langle t_0 + 4\rangle$. It is clear that α is insertable.

By inserting α into w we obtain walks of length k with the same endpoints. For $i \in \langle t_0 + 4\rangle \setminus \{c, j, l\}$, we have two choices for z_i. Then there are 2^{t_0+1} walks of length k with the same endpoints, a contradiction. Therefore, we get Claim 3.

Combining with (4.8), (4.10), (4.11), Claim 2 and Claim 3, D is a $(1, k, t_0 + 4)$-transitive tournament. This completes the proof. □

5 Proof of theorem 2.2

In this section, we give the proof of Theorem 2.2.

Proof of theorem 2.2 To complete the proof, we follow the strategy as follows. First we prove (2.1). The sufficiency of the second part is clear. It is sufficient to prove the necessity. Next we show that Theorem 2.2 holds for $n = k + t_0 + 5$ (Claim 1). Taking this fact as the inductional base, we use induction to complete the proof.

We first use induction on n to prove (2.1). By Theorem 4.4 we know (2.1) holds for $n = k + 4 + t_0$. Assume (2.1) holds for $n = k + 5 + t_0, \ldots, sk + r$, where $0 \le r < k$ and $s > 0$ are integers. Now consider the case $n = sk + r + 1$. Let u, v be integers such that $v < k$ and $n = uk + v$. Then $u = s$, $v = r + 1$ when $r < k - 1$, and $u = s + 1$, $v = 0$ when $r = k - 1$.

Given any $\mathscr{F}_{k,t+1}$-free digraph D of order n. For any $i \in \mathcal{V}$, since the digraph $D - i$ is $\mathscr{F}_{k,t+1}$-free, by the induction hypothesis we have

$$\begin{aligned} a(D - i) &\le ex(n - 1, \mathscr{F}_k) \\ &= \binom{n-1}{2} - \binom{s}{2}k - sr \\ &= \frac{(n-1)(n-2)}{2} - \frac{(s-1)(n-1)}{2} - \frac{(s+1)r}{2}. \end{aligned} \tag{5.1}$$

Applying Lemma 3.1 we have

$$\begin{aligned} a(D) &\le \frac{n(n-1)}{2} - \frac{(s-1)(n-1)}{2} - \frac{(s+1)r}{2} - s \\ &= \frac{n(n-1)}{2} - \frac{(s-1)n}{2} - \frac{(s+1)(r+1)}{2} \\ &= \frac{n(n-1)}{2} - \frac{(u-1)n}{2} - \frac{(u+1)v}{2}. \end{aligned}$$

Hence,

$$ex(n, \mathscr{F}_k) \le \frac{n(n-1)}{2} - \frac{(u-1)n}{2} - \frac{(u+1)v}{2}.$$

On the other hand, if D is a (u, k, v)-transitive tournament, then D is $\mathscr{F}_{k,t+1}$-free since it has no walk of length k. Moreover, the size of D is

$$a(D) = \frac{n(n-1)}{2} - \frac{(u-1)n}{2} - \frac{(u+1)v}{2}.$$

Hence, (2.1) holds.

Now we prove the second part. It is sufficient to prove the necessity since the sufficiency is clear. Next we prove that Theorem 2.2 holds for $n = k + t_0 + 5$ □

Claim 1 *The second part of Theorem 2.2 holds when $n = k + t_0 + 5$.*

Let $D \in EX(k + t_0 + 5, \mathscr{F}_{k,t})$. By (2.1), we have

$$d(i) \ge n - 2 \textit{ for all } i \in \mathcal{V}. \tag{5.2}$$

By Lemma 3.1, there exists some i_0 such that

$$a(D - i_0) = \frac{(n-1)(n-2)}{2} - t_0 - 4.$$

According to Theorem 4.4 we distinguish four cases.

Case 1. $D - i_0 \cong H_i(k, \alpha)$, where α is a t_0-subset of $\langle k\rangle \setminus \{1\}$ and $i \in \alpha \cup \{1, k+1\}$. Without loss of generality, we let $D - i_0 = H_i(k, \alpha)$, where α is a t_0-subset of $\langle k\rangle \setminus \{1\}$. By (5.2) we get $i \leftrightarrow i_0$ and $i + k + 1 \leftrightarrow i_0$. By Lemma 3.2, D is not $\mathscr{F}_{k,t+1}$-free, a contradiction.

Case 2. $D - i_0 \cong H_u(k, i, \alpha)$ with α is a t_0-subset of $\langle k\rangle \setminus \{1\}$, $i \in \langle k - 2\rangle$ and $|V_{i+1}| = |V_{i+2}| = 1$. Without loss of generality, we assume $D - i_0 = H_u(k, i, \alpha)$. Since u is incident with $n - 4$ arcs in $D - i_0$. Combining with (5.2), we obtain

$$u \to i_0 \to u. \tag{5.3}$$

Since two 2-cycles cannot be joint, there is at most one arc between i_0 and 1. It follows that $d(1) \le n - 2$. Recalling (5.2), we obtain $d(1) = n - 2$, which implies

$D-1 \in EX(k+t_0+4, \mathscr{F}_{k,t+1})$. From Theorem 4.4 $D-1$ contains no cycle, which contradicts (5.3).

Case 3. $D-i_0 \cong H(k, \alpha \cup \{i+1\}) - (i+k+1, i+k+2)$, where α is a t_0-subset of $\langle k \rangle \setminus \{1\}$, $i \in \alpha$ *and* $i+1 \in \langle k \rangle \setminus (\{1\} \cup \alpha)$. Without loss of generality, we let $D-i_0 = H(k, \alpha \cup \{i+1\}) - (i+k+1, i+k+2)$. Applying the same arguments as in Case 2 we can get a contradiction.

Case 4. $D-i_0$ is a $(1, k, t_0+4)$-transitive tournament. Given $\alpha = \{j_1, j_2, \ldots, j_{t_0+4}\} \subset \langle k \rangle$, denote by

$$V_i = \{i, i+k\} \text{ for } i \in \alpha, \;\; V_i = \{i\} \text{ for } i \in \langle k \rangle \backslash \alpha.$$

Without loss of generality, we let D be a digraph with vertex set

$$\mathcal{V} = \bigcup_{i=1}^{k} V_i \cup \{i_0\}$$

such that there is an arc (x, y) in $D-i_0$ if and only if $x \in V_i$, $y \in V_j$ with $i < j$. It is clear that

$$d(i_0) = a(D) - a(D-i_0) = n-2. \tag{5.4}$$

Now we assert that

$$i_0 \nrightarrow i_0. \tag{5.5}$$

Otherwise, $i_0 \to i_0$. By (5.4), $d^+(i_0) + d^-(i_0) = k + t_0 + 4 \geq 6t + 5$. Without loss of generality, we assume i_0 has at least $3t$ successors in $\mathcal{V} \backslash (\{i_0\} \cup V_k)$. For each vertex $i \in N^+(i_0) \backslash (\{i_0\} \cup V_k)$, there is a walk of length k with the initial vertex i_0 and the terminal vertex k in the following form.

$$\cdots \to i_0 \to i_0 \to i \to k.$$

We get a contradiction. Thus, (5.5) holds.

Let $s' \in \langle k \rangle$ be the largest integer such that there is a vertex $x \in V_{s'}$ with $x \to i_0$ and let $t' \in \langle k \rangle$ be the smallest integer such that there is a vertex $y \in V_{t'}$ with $i_0 \to y$. Here we let $s' = 0$ if $\mathcal{V} \nrightarrow i_0$ and let $t' = k+1$ if $i_0 \nrightarrow \mathcal{V}$. Without loss of generality, we let $s' \to i_0$ and $i_0 \to t'$.

We assert that $t' = s' + 2$. By (5.4) and (5.5), we obtain $t' \leq s' + 2$. To the contrary suppose $t' \leq s' + 1$. If $s' = 0$, we get $t' = 1$, i.e., $i_0 \to 1$. Then there is a walk of length k as follows.

$$i_0 \to 1 \to 2 \to \cdots \to k.$$

For each $i \in \alpha \backslash \{1, k\}$, we can replace i with $i + k$ in the above walk to obtain a new one. Hence, there are at least 2^{t_0+2} walks from i_0 to k with length k, a contradiction. If $s' = k$, applying the same arguments as above, we get a contradiction.

Now we assume $s' \in \{1, \ldots, k-1\}$. If $t' = s' + 1$, there is a walk of length k as follows.

$$1 \to 2 \to \cdots \to s' \to i_0 \to s' + 1 \to \cdots \to k.$$

Recalling (5.4) and (5.5), either $V_{s'} \to i_0$ or $i_0 \to V_{t'}$. Without loss of generality, we assume $V_{s'} \to i_0$. For each $i \in \alpha \backslash \{1, s'+1, k\}$, we can replace i with $i+k$ in the walk above. Hence, there are at least 2^{t_0+1} walks from 1 to k with length k, a contradiction.

Now we assume $t' \le s'$. Then there is a walk of length $k+1$ as follows.

$$1 \to 2 \to \cdots \to s' \to i_0 \to t' \to s' + 1 \to \cdots \to k.$$

For each $i \in \langle k \rangle \setminus \{1, s', t', k\}$, we can obtain a walk from the above walk by deleting i and joining its predecessor to its successor. Since $k \ge 6t + 1$, we have $|\langle k \rangle \setminus \{1, s', t', k\}| \ge t + 1$. Then there are more than t walks of length k with the same endpoints, a contradiction.

Therefore, we obtain $t' = s' + 2$. It follows that $V_i \to i_0$ for $i \le s'$ and $i_0 \to V_i$ for $i \ge s' + 2$, where $s' \in \langle k-1 \rangle$. Moreover, $|V_{s'+1}| = 1$. Hence, we get D is a $(1, k, t_0 + 5)$-transitive tournament. We get Claim 1.

We use induction on n to prove the second part of Theorem 2.2. Assume for $n \le sk + r$ and $0 \le r < k$, $D \in EX(n, \mathscr{F}_{k,t+1})$ if and only if D is a (s, k, r)-transitive tournament. Now consider the case $n = sk + r + 1$. Let u, v be integers such that $v < k$ and $n = uk + v$. Then $u = s$, $v = r + 1$ when $t < k - 1$, and $u = s + 1$, $v = 0$ when $r = k - 1$.

Suppose $D \in Ex(u, k, v)$. Applying Lemma 3.1 to D, by (2.1) we know there is some $i_0 \in \mathcal{V}$ such that equality in (5.1) holds. By the induction hypothesis we may assume $D - i_0$ is an (s, k, r)-transitive tournament.

Let $\{j_1, \ldots, j_r\}$ be an arbitrary r-subset of $\{1, \ldots, k\}$. Denote by

$$V_i = \{i, k+i, \ldots, (s-1)k + i, sk + i\} \text{ for } i \in \{j_1, \ldots, j_r\}$$

and

$$V_i = \{i, k+i, \ldots, (s-1)k + i\} \text{ for } i \in \{1, \ldots, k\} \backslash \{j_1, \ldots, j_r\}.$$

Without loss of generality, we let D be a digraph with vertex set

$$\mathcal{V} = \bigcup_{i=1}^{k} V_i \cup \{i_0\}$$

such that there is an arc (x, y) in $D - i_0$ if and only if $x \in V_i$, $y \in V_j$ with $i < j$.

Applying the same arguments as in Case 4 of Claim 1, we obtain D is a (u, k, v)-transitive tournament. This completes the proof. □

Acknowledgements The author is grateful to two anonymous referees whose comments helped to improve the presentation of this paper.

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