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Research on single-machine scheduling with position-dependent weights and past-sequence-dependent delivery times

Ji-Bo Wang[1] · Bo Cui[1] · Ping Ji[2] · Wei-Wei Liu[3]

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Abstract
This article studies scheduling problems with past-sequence-dependent delivery times (denoted by psddt) on a single-machine, i.e., the delivery time of a job depends on its waiting time of processing. We prove that the total (discounted) weighted completion time minimization can be solved in $O(n \log n)$ time, where n is the number of jobs, and the weight is a position-dependent weight. For common (denoted by con) and slack (denoted by slk) due-date assignment and position-dependent weights (denoted by pdw), we prove that an objective cost minimization is solvable in $O(n \log n)$ time. The model (i.e., psddt and pdw) can also be extended to position-dependent (time-dependent) processing times.

Keywords Delivery time · Position-dependent weight · Single-machine · Scheduling

1 Introduction

In a production situation, the phenomenon of past-sequence-dependent delivery times (denoted by psddt) can be found in electronic industry (Koulamas and Kyparisis 2010). Koulamas and Kyparisis (2010) introduced psddt to scheduling problems according to which the delivery time of a job depends on its waiting time of processing. Using the three-field notation of Graham et al. (1979), they proved that single-machine problem $1|psddt|X$ can be solved in polynomial time, where $(X \in \{C_{\max}, \sum_{i=1}^{n} C_i, L_{\max}, T_{\max}, \sum_{i=1}^{n} U_i\}$, $C_{\max}$ is the makespan (i.e., maximal

✉ Wei-Wei Liu
liuww_2010@163.com

1 School of Science, Shenyang Aerospace University, Shenyang 110136, People's Republic of China

2 Department of Industrial and Systems Engineering, The Hong Kong Polytechnic University, Hong Kong, People's Republic of China

3 Department of Science, Shenyang Sport University, Shenyang 110102, People's Republic of China

completion time value, represents the time interval needed to finish all jobs), $\sum_{i=1}^{n} C_i$ is the total completion time (represents the work-in-process inventory cost), $L_{\max}$ ($T_{\max}$) is the maximum lateness (tardiness) (represents the penalty cost in the case where there are job delays over its due date), $\sum_{i=1}^{n} U_i$ is the number of tardy jobs (represents the cost penalty incurred by a tardy job does not depend on how late it is, but depends on the fact that it is late). Liu et al. (2012a) showed that the problem $1|psddt, r_i, prmp|\sum_{i=1}^{n} C_i$ is polynomial time solvable if the jobs can be interrupted (i.e., job preemption is allowed), where r_i is the ready time of job J_i, $prmp$ denotes job preemption is allowed. Liu et al. (2012a) also proved that the non-preemptive problem $1|psddt, r_i|\sum_{i=1}^{n} C_i$ is NP-hard, and proposed an approximation algorithm to solve the $1|psddt, r_i|\sum_{i=1}^{n} C_i$ problem. Liu et al. (2012b) studied the problem $1|psddt|\rho$, where $\rho \in \{\sum_{i=1}^{n} w_i C_i, \sum_{i=1}^{n} w_i(1-e^{-\gamma C_i}), TADC = \sum_{i=1}^{n}\sum_{j=1}^{i}|C_i - C_j|\}$, γ ($0 < \gamma < 1$) is the discount factor, they proved that these problems remain polynomially solvable.

Meeting due-dates has always been one of the most important objectives. There are two more commonly used due-date assignment methods below: The con due-date assignment method denotes that all jobs have a common due-date, i.e., the due-date of job J_i is $d_i = d_{opt}$, and d_{opt} is a decision variable; The slk due-date assignment method means that all jobs have a common flow allowance, i.e., $d_i = p_i + q_{opt}$, where p_i is the completion time of job J_i, and q_{opt} is a decision variable. Liu et al. (2012b) also considered the con due-date assignment problem $1|psddt, con|\sum_{i=1}^{n}(\alpha_1 E_i + \alpha_2 T_i + \alpha_3 d_{opt})$, they proved that the problem is polynomial time solvable, where $E_i = \max\{0, d_{opt} - C_i\}$ ($T_i = \max\{0, C_i - d_{opt}\}$) is the earliness (tardiness) of job $J_i, \alpha_1, \alpha_2, \alpha_3$ are given constant values, C_i is the completion time of job J_i.

Liu (2013) and Wu and Wang (2016) studied learning effects problems with psddt. For following objectives: $TADC$, $\sum_{h=1}^{m} C_{\max}^{h}$ (i.e., the sum of load on all machines, $C_{\max}^{h}$ is the makespan of machine M_h, $h = 1, 2, \ldots, m$), $\sum_{i=1}^{n} C_i$, under parallel-machine and a position-dependent learning effect, Liu (2013) proved that these problems (i.e., $Pm|psddt|X$, $X \in \{TADC, \sum_{h=1}^{m} C_{\max}^{h}, \sum_{i=1}^{n} C_i\}$) are polynomially time solvable. Under truncated sum-of-processing-times-based learning effect and single-machine, Wu and Wang (2016) showed that some minimizations (i.e., $C_{\max}$, $\sum_{i=1}^{n} C_i^{\eta}$ ($\eta > 0$), $\sum_{i=1}^{n} L_i$, $\sum_{i=1}^{n} w_i C_i$, $L_{\max}$) are polynomially time solvable. Liu et al. (2013) and Yin et al. (2013) tackled problems with psd delivery times and deterioration effects. Under the parallel machine setting, Liu et al. (2013) presented polynomial algorithms for following objectives: $TADC, \sum_{h=1}^{m} C_{\max}^{h}, \sum_{i=1}^{n} C_i$. Under the single-machine setting, Yin et al. [8] showed that the makespan (total completion time) minimization is polynomial time solvable. Yin et al. (2013) also proved that some special cases of $\sum_{i=1}^{n} w_i C_i$ and $L_{\max}$ minimizations remain polynomially solvable. Yang and Yang (2012), and Zhao and Tang (2014) investigated problems with psddt and position-dependent processing times.

On the other hand, Kahlbacher (1992), Brucker (2001), Liu et al. (2017), Liu and Jiang (2020), and Jiang et al. (2020) considered scheduling models with pdw weights, i.e., the weight is not related to the job but to the position in which the job is scheduled. If the con due-date d is a given constant, Kahlbacher (1992) showed that problem $1\,|d, pdw|\sum_{i=1}^{n} \nu_i |L_{S_{(i)}}|$ is NP-hard, where ν_i is the weight of ith position in a sequence (i.e., pdw), $L_{S_{(i)}} = C_{S_{(i)}} - d$ is the lateness of job $J_{S_{(i)}}$, $S_{(i)}$ denotes the job scheduled in ith position under sequence S. Brucker (2001), concentrated on the con method, he proved that the problem $1\left|pdw, con, d_{opt}\right|\sum_{i=1}^{n}\nu_i|L_{S_{(i)}}| + \nu_0 d_{opt}$ (where d_{opt} is a decision variable, $\nu_0 \geq 0$) is polynomial time solvable. Liu et al. (2017) addressed the slk method, they showed that $1\left|pdw, slk, q_{opt}\right|\sum_{i=1}^{n}\nu_i|L_{S_{(i)}}| + \nu_0 q_{opt}$ ($\nu_0 \geq 0$ is the weight of q_{opt}) can also be solved in polynomial time. Liu and Jiang (2020) dealed with resource allocation scheduling with learning effects. For single-machine setting, they showed that con and slk methods remain polynomial time solvable, respectively. Jiang et al. (2020) investigated problems with pdw. Under proportionate flowshop setting, they proved that con and slk methods remain polynomial time solvable, respectively.

In this article, we focus on scheduling under model with psddt and pdw weights. Motivated by the phenomena in practice such that the weights are position-dependent weights due to the importance of position in service production system, we introduce position-dependent weights into the psddt model. To the best of our knowledge, there are no results on scheduling with pdw weights and psddt in literature. The remainder of the article is organized as follows: Sect. 2 formulates the model. Section 3 considers two scheduling problems without due-date constraint. Section 4 studies two due-date assignment problems. Section 5 provides some extensions. Conclusions are presented in Sect. 6.

2 Model description

Consider a set $\breve{N} = \{J_1, J_2, \ldots, J_n\}$ of simultaneously available jobs to be processed by a single-machine. Let p_i, s_i, and q_i be the processing time, starting time, and psddt of job J_i, respectively, $i = 1, 2, \ldots, n$, and let $S_{(i)}$ be the job scheduled in the ith position under sequence S. As in Koulamas and Kyparisis (2010), we have

$$s_{S_{(i)}} = \sum_{l=1}^{i-1} p_{S_{(l)}},\ q_{S_{(1)}} = 0 \text{ and } q_{S_{(i)}} = b s_{S_{(i)}} = b\sum_{l=1}^{i-1} p_{S_{(l)}}, \tag{1}$$

where $\sum_{l=1}^{0} p_{S_{(l)}} := 0$, and $b \geq 0$ is a normalizing constant. Let C_i denote the completion time of job J_i and $C_{S_{(i)}}$ can be defined analogously, then

$$C_{S_{(i)}} = s_{S_{(i)}} + p_{S_{(i)}} + q_{S_{(i)}} = (1+b)\sum_{h=1}^{i-1} p_{S_{(h)}} + p_{S_{(i)}},\ i = 1, 2, \ldots, n. \tag{2}$$

3 Scheduling without due-date

The aim of this section is to study $1|psddt, pdw|\sum_{i=1}^{n}\varpi_i C_{S_{(i)}}$ and $1|psddt, pdw|\sum_{i=1}^{n}(1 - e^{-\gamma C_{S_{(i)}}})$, where $\sum_{i=1}^{n}\varpi_i C_{S_{(i)}}$ is total weighted completion time and

$\sum_{i=1}^{n} \varpi_i (1-e^{-\gamma C_{S_{(i)}}})$ is total discounted weighted completion time, ϖ_i is the weight of ith position in a sequence (i.e., pdw), γ $(0 < \gamma < 1)$ is the discount factor.

Theorem 1 *For* $1|psddt, pdw|\sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$*, an optimal sequence can be obtained in* $O(n \log n)$ *time, i.e., by smallest processing time (SPT) first rule.*

Proof Let $S = (\tilde{S}_1, J_i, J_j, \tilde{S}_2)$ and $S^{\perp} = (\tilde{S}_1, J_j, J_i, \tilde{S}_2)$ denote two sequences, where $p_i \le p_j$, $\tilde{S}_1$ and $\tilde{S}_2$ are partial sequences (there are $r-1$ jobs in $\tilde{S}_1$). Under S and $S^{\perp}$, we have

$$\sum_{i=1}^{n} \varpi_i C_{S_{(i)}} = \sum_{i=1}^{r-1} \varpi_i C_{S_{(i)}} + \varpi_r[(1+b)\sum_{l=1}^{r-1} p_{[l]} + p_i] + \varpi_{r+1}[(1+b)\sum_{l=1}^{r-1} p_{[l]} + (1+b)p_i + p_j] + \sum_{i=r+2}^{n} \varpi_i C_{S_{(i)}}. \quad (3)$$

$$\sum_{i=1}^{n} \varpi_i C_{S^{\perp}_{(i)}} = \sum_{i=1}^{r-1} \varpi_i C_{S^{\perp}_{(i)}} + \varpi_r[(1+b)\sum_{l=1}^{r-1} p_{[l]} + p_j] + \varpi_{r+1}[(1+b)\sum_{l=1}^{r-1} p_{[l]} + (1+b)p_j + p_i] + \sum_{i=r+2}^{n} \varpi_i C_{S^{\perp}_{(i)}}. \quad (4)$$

From (3) and (4), we have $\sum_{i=1}^{r-1} \varpi_i C_{S_{(i)}} = \sum_{i=1}^{r-1} \varpi_i C_{S^{\perp}_{(i)}}$, $\sum_{i=r+2}^{n} \varpi_i C_{S_{(i)}} = \sum_{i=r+2}^{n} \varpi_i C_{S^{\perp}_{(i)}}$,

$$\sum_{i=1}^{n} \varpi_i C_{S_{(i)}} - \sum_{i=1}^{n} \varpi_i C_{S^{\perp}_{(i)}} = (\varpi_r + b\varpi_{r+1})(p_i - p_j).$$

From $p_i \le p_j$, then $\sum_{i=1}^{n} \varpi_i C_{S_{(i)}} - \sum_{i=1}^{n} \varpi_i C_{S^{\perp}_{(i)}} \le 0$. □

Theorem 2 *For* $1|psddt, pdw|\sum_{i=1}^{n} \varpi_i (1-e^{-\gamma C_{S_{(i)}}})$*, an optimal sequence can be obtained in* $O(n \log n)$ *time, i.e., by the SPT rule.*

Proof It is same as Theorem 1, except that, if $p_i \le p_j$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \varpi_i (1-e^{-\gamma C_{S_{(i)}}}) - \sum_{i=1}^{n} \varpi_i (1-e^{-\gamma C_{S^{\perp}_{(i)}}}) \\
&= \varpi_r (1-e^{-\gamma[(1+b)\sum_{l=1}^{r-1} p_{[l]} + p_i]}) + \varpi_{r+1}(1-e^{-\gamma[(1+b)\sum_{l=1}^{r-1} p_{[l]} + (1+b)p_i + p_j]}) \\
&\quad -\varpi_r (1-e^{-\gamma[(1+b)\sum_{l=1}^{r-1} p_{[l]} + p_j]}) - \varpi_{r+1}(1-e^{-\gamma[(1+b)\sum_{l=1}^{r-1} p_{[l]} + (1+b)p_j + p_i]}) \\
&= \varpi_r e^{-\gamma(1+b)\sum_{l=1}^{r-1} p_{[l]}} \left(e^{-\gamma p_j} - e^{-\gamma p_i})\right) \\
&\quad + \varpi_{r+1} e^{-\gamma(1+b)\sum_{l=1}^{r-1} p_{[l]} + p_i + p_j} \left(e^{-\gamma b p_j} - e^{-\gamma b p_i})\right) \\
&\le 0.
\end{aligned}$$

□

4 Due-date assignment problem

4.1 Con due-date

For the con method, we have $d_i = d_{opt}$ for all jobs, the aim is to determine d_{opt} and a job sequence such that

$$\sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt} = \sum_{i=1}^{n} \varpi_i |C_{S_{(i)}} - d_{opt}| + \varpi_0 d_{opt}, \tag{5}$$

is minimized, where $\varpi_i > 0$ $(i = 0, 1, 2, \ldots, n)$ is the pdw weights. Brucker (2001) considered the problem $1\left|pdw, con, d_{opt}\right| \sum_{i=1}^{n} \nu_i |L_{S_{(i)}}| + \nu_0 d_{opt}$, i.e., he does not consider psddt. Obviously, for the problem $1\left|psddt, pdw, con, d_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$, there are no-idle time between the jobs and first job's starting time is 0 (see Lemma 7.1 in Brucker 2001).

Now, a dummy job J_0 is adopted such that its processing time is $p_0 = 0$, position-dependent weight is ϖ_0, starting time is 0, we have

$$\sum_{i=1}^{n} \varpi_i |C_{S_{(i)}} - d_{opt}| + \varpi_0 d_{opt} = \sum_{i=0}^{n} \varpi_i |C_{S_{(i)}} - d_{opt}|, \tag{6}$$

and an optimal schedule can be given by $S = (S_{(0)}, S_{(1)}, \ldots, S_{(n)})$, where $S(0) = 0$.

Lemma 1 *For a given sequence* $S = (S_{(0)}, S_{(1)}, \ldots, S_{(n)})$ *of the problem* $1\left|psddt, pdw, con, d_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$, $d_{opt} = C_{S_{(k)}} = \gamma \sum_{i=1}^{k-1} p_{S_{(i)}} + p_{S_{(k)}}$, *where* k *is a median for the sequence* $\varpi_0, \varpi_1, \ldots, \varpi_n$,

$$\sum_{i=0}^{k-1} \varpi_i \le \sum_{i=k}^{n} \varpi_i \text{ and } \sum_{i=0}^{k} \varpi_i \ge \sum_{i=k+1}^{n} \varpi_i. \tag{7}$$

Proof Similar to the proof of Lemma 7.2 in Brucker (2001). □

Lemma 2 *For the problem* $1\left|psddt, pdw, d_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$, *we have*

$$\sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt} = \sum_{i=1}^{n} \vartheta_i p_{S_{(i)}}, \tag{8}$$

where

$$\vartheta_i = \begin{cases} \sum_{v=0}^{i} b\varpi_v + \sum_{v=0}^{i-1} \varpi_v, & i = 1, 2, \ldots, k-1; \\ \sum_{v=0}^{k} \varpi_v + \sum_{v=k}^{n} b\varpi_v, & i = k; \\ \sum_{v=i}^{n} \varpi_v + \sum_{v=i+1}^{n} b\varpi_v, & i = k+1, k+2 \ldots, n. \end{cases} \tag{9}$$

Proof For $S = (S_{(0)}, S_{(1)}, \ldots, S_{(n)})$, from Lemma 1, $d_{opt} = C_{S_{(k)}}$,

$$\begin{aligned}
&\sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt} \\
&= \sum_{i=0}^{k} \varpi_i (C_{S_{(k)}} - C_{S_{(i)}}) + \sum_{i=k+1}^{n} \varpi_i (C_{S_{(i)}} - C_{S_{(k)}}) \\
&= \sum_{i=0}^{k} \varpi_i \Bigg(bp_{S_{(i)}} \\
&\quad + (1+b) \sum_{v=i+1}^{k-1} p_{S_{(v)}} + p_{S_{(k)}}\Bigg) + \sum_{i=k+1}^{n} \varpi_i \left(bp_{S_{(k)}} + (1+b) \sum_{v=k+1}^{i-1} p_{S_{(v)}} + p_{S_{(i)}}\right) \\
&= \sum_{v=1}^{k-1} p_{S_{(v)}} \left(\sum_{i=0}^{v} b\varpi_i + \sum_{i=0}^{v-1} \varpi_i\right) \\
&\quad + p_{S_{(k)}} \left(\sum_{i=0}^{k} \varpi_i + \sum_{i=k}^{n} b\varpi_i\right) + \sum_{v=k+1}^{n} p_{S_{(v)}} \left(\sum_{i=v}^{n} \varpi_i + \sum_{i=v+1}^{n} b\varpi_i\right) \\
&= \sum_{i=1}^{n} \vartheta_i p_{S_{(i)}},
\end{aligned}$$

where ϑ_i $(i = 1, 2, \ldots, n)$ are given by (9). □

4.2 Slack due-date assignment

For the slk method, we have $d_{S_{(i)}} = p_{S_{(i)}} + q_{opt}$, where q_{opt} is a decision variable. The problem is to determine q_{opt} and a job sequence such that

$$\sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt} = \sum_{i=1}^{n} \varpi_i |C_{S_{(i)}} - d_{S_{(i)}}| + \varpi_0 q_{opt} \tag{10}$$

is minimized. Liu et al. (2017) considered the problem $1 \left| pdw, slk, q_{opt} \right| \sum_{i=1}^{n} \nu_i |L_{S_{(i)}}| + \nu_0 q_{opt}$, i.e., they does not consider psddt. Obviously, for the problem $1 \left| psddt, pdw, slk, q_{opt} \right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$, there are no-idle time between the jobs and the first job's starting time is 0 (see Liu et al. 2017).

Similar to Sect. 4.1, a dummy job J_0 is adopted such that its processing time is $p_0 = 0$, position-dependent weight is ϖ_0, starting time is 0, then

$$\sum_{i=1}^{n} \varpi_i |C_{S_{(i)}} - d_{S_{(i)}}| + \varpi_0 q_{opt} = \sum_{i=0}^{n} \varpi_i |C_{S_{(i)}} - d_{S_{(i)}}|, \tag{11}$$

and an optimal schedule is $S = (S_{(0)}, S_{(1)}, \ldots, S_{(n)})$.

Lemma 3 *For a given sequence* $S = (S_{(0)}, S_{(1)}, \ldots, S_{(n)})$ *of the problem* $1\left|psddt, pdw, slk, q_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$, $q_{opt} = C_{S_{(l)}} = \gamma \sum_{i=1}^{l-1} p_{\rho(i)} + p_{\rho(l)}$, *where* l *is a median for the sequence* $\varpi_0, \varpi_1, \ldots, \varpi_n$,

$$\sum_{i=0}^{l} \varpi_i \le \sum_{i=l+1}^{n} \varpi_i \text{ and } \sum_{i=0}^{l+1} \varpi_i \ge \sum_{i=l+2}^{n} \varpi_i. \tag{12}$$

Proof Similar to the proof of Liu et al. (2017). □

Lemma 4 *For the problem* $1\left|psddt, pdw, slk, q_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$, *the optimal total cost can be written as:*

$$\sum_{i=1}^{n} \varpi_i |L_{\rho(i)}| + \varpi_0 q_{opt} = \sum_{i=1}^{n} \varpi_i |C_{S_{(i)}} - d_{S_{(i)}}| + \varpi_0 q_{opt} = \sum_{i=1}^{n} \vartheta_i p_{S_{(i)}}, \tag{13}$$

where

$$\vartheta_i = \begin{cases} \sum_{v=0}^{i+1} b\varpi_v + \sum_{v=0}^{i} \varpi_v, & i = 1, 2, \ldots, l-1; \\ \sum_{v=0}^{l+1} \varpi_v + \sum_{v=l+1}^{n} b\varpi_v, & i = l; \\ \sum_{v=i+1}^{n} \varpi_v + \sum_{v=i+2}^{n} b\varpi_v, & i = l+1, l+2, \ldots, n-1; \\ 0, & i = n. \end{cases} \tag{14}$$

Proof For $S = (S_{(0)}, S_{(1)}, \ldots, S_{(n)})$, from Lemma 3, $q_{opt} = C_{S_{(l)}}$,

$$\begin{aligned} &\sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt} \\ &= \sum_{i=0}^{l+1} \varpi_i \left(C_{S(l)} - C_{S(i-1)}\right) + \sum_{i=l+2}^{n} \varpi_i \left(C_{S(i-1)} - C_{S(l)}\right) \\ &= \sum_{i=0}^{l+1} \varpi_i \left(b p_{S(i-1)} + (1+b) \sum_{v=i}^{l-1} p_{S(v)} + p_{S(l)}\right) \\ &\quad + \sum_{i=l+2}^{n} \varpi_i \left(b p_{S(l)} + (1+b) \sum_{v=l+1}^{i-2} p_{S(v)} + p_{S(i-1)}\right) \\ &= \sum_{v=1}^{l-1} p_{S(v)} \left(\sum_{i=0}^{v+1} b\varpi_i + \sum_{i=0}^{v} \varpi_i\right) + p_{S(l)} \left(\sum_{i=0}^{l+1} \varpi_i + \sum_{i=l+1}^{n} b\varpi_i\right) \\ &\quad + \sum_{v=l+1}^{n-1} p_{S(v)} \left(\sum_{i=v+1}^{n} \varpi_i + \sum_{i=v+2}^{n} b\varpi_i\right) \end{aligned}$$

$$= \sum_{i=1}^{n} \vartheta_i p_{S_{(i)}},$$

where ϑ_i $(i = 1, 2, \ldots, n)$ are given by (14). □

4.3 Optimal solution

The terms (8) and (13) can be minimized by sequencing the vectors ϑ_i and $p_{S_{(i)}}$ in opposite order (see Hardy et al. 1967) in $O(n \log n)$ time, therefore $1\left|psddt, pdw, con, d_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$ and $1\left|psddt, pdw, slk, q_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$ are solved by following algorithm:

Algorithm 1 *Step 1. By Lemma 1 (Lemma 3), calculate k (l).*
Step 2. By sequencing the vectors ϑ_i and p_i in opposite order (see (9) and (14)) to identify the optimal sequence.
Step 3. For con (slk) due-date assignment, set $d_{opt} = C_{S_{(k)}} = (1+b)\sum_{h=1}^{k-1} p_{S_{(h)}} + p_{S_{(k)}}$ $(q_{opt} = C_{S_{(l)}} = (1+b)\sum_{h=1}^{i-1} p_{S_{(h)}} + p_{S_{(l)}})$.

Theorem 3 *Algorithm 1 solves $1\left|psddt, pdw, con, d_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$ and $1|psddt, pdw, slk, q_{opt}| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$ in $O(n \log n)$ time, respectively.*

Proof Steps 1 and 3 require $O(n)$ time respectively, and Step 2 needs $O(n \log n)$ time, thus the complexity of Algorithm 1 is $O(n \log n)$. □

Note: If the con method d is a given constant, Kahlbacher (1992) proved that problem $1\,|d, pdw|\, \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}|$ is NP-hard, hence, $1\,|psddt, pdw, d|\, \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}|$ is also NP-hard. Hall and Posner (1991) showed that con due-date assignment problem $1\left|con, d_{opt}\right| \sum_{i=1}^{n} w_i |L_i|$ is NP-hard, where w_i is the job-dependent weight of J_i, hence, the job-dependent weights problem $1\left|psddt, con, d_{opt}\right| \sum_{i=1}^{n} w_i |L_i| + w_0 d_{opt}$ is also NP-hard, where w_0 is a given constant.

5 Extensions

5.1 Positional-dependent processing times

The positional-dependent processing times have been given by Mosheiov (2011), i.e., the processing time of job J_i, if scheduled in position r, is given by the function $f(i, r)$, (i.e., $p_i^A = f(i, r), i, r = 1, \ldots, n$). Biskup (1999) considered the model $f(i, r) = p_i r^a$, Mosheiov and Sidney (2003) considered the model $f(i, r) = p_i r^{a_i}$, Cheng et al. (2013) considered the model $f(i, r) = p_i \max\{r^a, \beta\}$, Wang et al. (2014) considered the model $f(i, r) = p_i \max\{r^{a_i}, \beta\}$, where $a \leq 0$ is the learning effect (Biskup 1999; Wang and Zhang 2015; Wang et al. 2020), $a_i \leq 0$ is the job-dependent learning effect, and β is a truncation parameter $(0 < \beta < 1)$ (Cheng et al. 2013; Wang et al. 2014, 2019; Lu et al. 2015).

Obviously, from (2), we have

$$\sum_{i=1}^{n} \varpi_i C_{S_{(i)}} = \sum_{i=1}^{n} \varpi_i \left((1+b) \sum_{h=1}^{i-1} p_{S_{(h)}}^{A} + p_{S_{(i)}}^{A} \right) = \sum_{i=1}^{n} \vartheta_i p_{S_{(i)}}^{A} = \sum_{i=1}^{n} \vartheta_i f(i,r), \tag{15}$$

where

$$\vartheta_i = \begin{cases} \varpi_1 + \sum_{v=2}^{n} (1+b)\varpi_v, \text{ for } i=1; \\ \varpi_2 + \sum_{v=3}^{n} (1+b)\varpi_v, \text{ for } i=2; \\ \ldots \\ \varpi_{n-1} + (1+b)\varpi_n, \quad \text{for } i=n-1; \\ \varpi_n, \quad \text{for } i=n; \end{cases} \tag{16}$$

Let

$$y_{i,r} = \begin{cases} 1, & \text{if } J_i \text{ is assigned to } r \text{ th position,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, we can formulate the sequence problem of $1|psddt, pdw, p_i^A = f(i,r)|\sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$ as following Assignment Problem:

$$\min \sum_{i=1}^{n} \sum_{r=1}^{n} \vartheta_r f(i,r) y_{i,r} \tag{17}$$

$$s.t. \sum_{i=1}^{n} y_{i,r} = 1; r = 1, 2, \ldots, n \tag{18}$$

$$\sum_{r=1}^{n} y_{i,r} = 1; i = 1, 2. \ldots, n \tag{19}$$

$$y_{i,r} = \{0, 1\} \tag{20}$$

where

$$\vartheta_r = \begin{cases} \varpi_1 + \sum_{v=2}^{n} (1+b)\varpi_v, \text{ for } r=1; \\ \varpi_2 + \sum_{v=3}^{n} (1+b)\varpi_v, \text{ for } r=2; \\ \ldots \\ \varpi_{n-1} + (1+b)\varpi_n, \quad \text{for } r=n-1; \\ \varpi_n, \quad \text{for } r=n. \end{cases} \tag{21}$$

Obviously, Lemmas 1, 2, 3 and 4 still hold when positional-dependent processing times are introduced, we have

$$\sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}/q_{opt} = \sum_{i=1}^{n} \vartheta_i p_{S_{(i)}}^A = \sum_{i=1}^{n} \vartheta_i f(i,r), \tag{22}$$

where, for the con and slk models, ϑ_i $(i = 1, 2, \ldots, n)$ are given by (9) and (14), respectively.

Similarly, we can formulate the sequence problem of $1|psddt, pdw, con/slk, d_{opt}/q_{opt}, p_i^A = f(i,r)| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}/q_{opt}$ as the assignment problem (17)–(20), where, for the con due-date assignment,

$$\vartheta_r = \begin{cases} \sum_{v=0}^{i} b\varpi_v + \sum_{v=0}^{i-1} \varpi_v, & i = 1, 2, \ldots, k-1; \\ \sum_{v=0}^{k} \varpi_v + \sum_{v=k}^{n} b\varpi_v, & i = k; \\ \sum_{v=i}^{n} \varpi_v + \sum_{v=i+1}^{n} b\varpi_v, & i = k+1, k+2 \ldots, n; \end{cases} \tag{23}$$

for the slk due-date assignment,

$$\vartheta_r = \begin{cases} \sum_{v=0}^{i+1} b\varpi_v + \sum_{v=0}^{i} \varpi_v, & i = 1, 2, \ldots, l-1; \\ \sum_{v=0}^{l+1} \varpi_v + \sum_{v=l+1}^{n} b\varpi_v, & i = l; \\ \sum_{v=i+1}^{n} \varpi_v + \sum_{v=i+2}^{n} b\varpi_v, & i = l+1, l+2, \ldots, n-1; \\ 0, & i = n. \end{cases} \tag{24}$$

Based on the above analysis, we have

Theorem 4 $1|psddt, pdw, p_i^A = f(i,r)| \sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$, $1|psddt, pdw, con, d_{opt}, p_i^A = f(i,r)| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$ *and* $1|psddt, pdw, slk, q_{opt}, p_i^A = f(i,r)| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$ *are solvable in* $O(n^3)$ *time, respectively.*

If $p_i^A = p_i \max\{r^a, \beta\}$, the term $\sum_{i=1}^{n} \vartheta_i f(i,r) = \sum_{i=1}^{n} \vartheta_i \max\{i^a, \beta\} p_{S_{(i)}}$ can be minimized by sequencing the vectors $\vartheta_i \max\{i^a, \beta\}$ and $p_{S_{(i)}}$ in opposite order (see Hardy et al. 1967), hence, we have

Theorem 5 $1|psddt, pdw, p_i^A = p_i \max\{r^a, \beta\}| \sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$, $1|psddt, pdw, con, d_{opt}, p_i^A = p_i \max\{r^a, \beta\}| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$ *and* $1|psddt, pdw, slk, q_{opt}, p_i^A = p_i \max\{r^a, \beta\}| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$ *are solvable in* $O(n \log n)$ *time, respectively.*

5.2 Time-dependent processing times

Time-dependent processing times (Gawiejnowicz 2008; Lu 2016) are adopted to scheduling model, i.e.,

$$p_i^A = p_i + cs_i, i = 1, \ldots, n, \tag{25}$$

where $c \geq 0$ (denotes common deterioration rate), s_i is starting time of J_i (Gawiejnowicz 2008; Lu 2016).

Obviously, for the problem $1|psddt, pdw, p_i^A = p_i + cs_i| \sum_{i=1}^n \varpi_i C_{S_{(i)}}$, we have

$$\begin{aligned} s_{[1]} &= 0, s_{[2]} = 0 + p_{[1]} + c \times 0 = p_{[1]}, \\ s_{[3]} &= s_{[2]} + p_{[2]} + c \times s_{[2]} = p_{[2]} + (1+c)p_{[1]}, \\ &\ldots, \\ s_{[i]} &= p_{[i-1]} + (1+c)p_{[i-2]} + \ldots + (1+c)^{i-2} p_{[1]}, \\ p_{[i]}^A &= p_{[i]} + cs_{[i]} = p_{[i]} + cp_{[i-1]} + c(1+c)p_{[i-2]} + \ldots + c(1+c)^{i-2} p_{[1]}. \end{aligned}$$

For the objective functions $\sum_{i=1}^n \varpi_i C_{S_{(i)}} = \sum_{i=1}^n \vartheta_i p_{S_{(i)}}^A$ and $\sum_{i=1}^n \varpi_i |L_{S_{(i)}}| + \omega_0 d_{opt}/q_{opt} = \sum_{i=1}^n \vartheta_i p_{S_{(i)}}^A$, from the above analysis, we have

$$\sum_{i=1}^n \vartheta_i p_{S_{(i)}}^A = \sum_{i=1}^n \psi_i p_{S_{(i)}}, \tag{26}$$

where

$$\begin{aligned} \psi_1 &= \vartheta_1 + c\vartheta_2 + c(1+c)\vartheta_3 + \ldots + c(1+c)^{n-2}\vartheta_n \\ \psi_2 &= \vartheta_2 + c\vartheta_3 + c(1+c)\vartheta_4 + \ldots + c(1+c)^{n-3}\vartheta_n \\ \psi_3 &= \vartheta_3 + c\vartheta_4 + c(1+c)\vartheta_5 + \ldots + c(1+c)^{n-4}\vartheta_n \\ &\ldots \\ \psi_{n-1} &= \vartheta_{n-1} + c\vartheta_n \\ \psi_n &= \vartheta_n, \end{aligned} \tag{27}$$

where, for $\sum_{i=1}^n \varpi_i C_{S_{(i)}}$, ϑ_i $(i = 1, 2, \ldots, n)$ are given by (16); for the con due-date model, ϑ_i $(i = 1, 2, \ldots, n)$ are given by (9); for the slk due-date model, ϑ_i $(i = 1, 2, \ldots, n)$ are given by (14).

Similarly, we have:

Theorem 6 *The problems* $1|psddt, pdw, p_i^A = p_i + cs_i| \sum_{i=1}^n \varpi_i C_{S_{(i)}}$, $1|psddt, pdw, con, d_{opt}, p_i^A = p_i + cs_i| \sum_{i=1}^n \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$ *and* $1|psddt, pdw, slk, q_{opt}, p_i^A = p_i + cs_i| \sum_{i=1}^n \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$ *can be solved in* $O(n \log n)$ *time, respectively.*

Table 1 Results of this paper

Problem	Complexity	
$1\|psddt, pdw\| \sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$	$O(n \log n)$	Theorem 1
$1\|psddt, pdw\| \sum_{i=1}^{n} \varpi_i (1 - e^{-\gamma C_{S_{(i)}}})$	$O(n \log n)$	Theorem 2
$1\left\|psddt, pdw, con, d_{opt}\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 d_{opt}$	$O(n \log n)$	Theorem 3
$1\left\|psddt, pdw, slk, q_{opt}\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 q_{opt}$	$O(n \log n)$	Theorem 3
$1\|psddt, pdw, p_i^A = f(i, r)\| \sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$	$O(n^3)$	Theorem 4
$1\left\|psddt, pdw, con, d_{opt}\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 d_{opt}$	$O(n^3)$	Theorem 4
$1\left\|psddt, pdw, slk, q_{opt}\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 q_{opt}$	$O(n^3)$	Theorem 4
$1\|psddt, pdw, p_i^A = p_i \max\{r^a, \beta\}\| \sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$	$O(n \log n)$	Theorem 5
$1\left\|psddt, pdw, con, d_{opt}, p_i^A = p_i \max\{r^a, \beta\}\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 d_{opt}$	$O(n \log n)$	Theorem 5
$1\left\|psddt, pdw, slk, q_{opt}, p_i^A = p_i \max\{r^a, \beta\}\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 q_{opt}$	$O(n \log n)$	Theorem 5
$1\|psddt, pdw, p_i^A = p_i + cs_i\| \sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$	$O(n \log n)$	Theorem 6
$1\left\|psddt, pdw, con, d_{opt}, p_i^A = p_i + cs_i\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 d_{opt}$	$O(n \log n)$	Theorem 6
$1\left\|psddt, pdw, slk, q_{opt}, p_i^A = p_i + cs_i\right\| \sum_{i=1}^{n} \varpi_i \|L_{S_{(i)}}\| + \varpi_0 q_{opt}$	$O(n \log n)$	Theorem 6

6 Conclusions

We addressed the problems with psddt and pdw (see Table 1). We proved that the problems $1|psddt, pdw| \sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$, $1|psddt, pdw| \sum_{i=1}^{n} \varpi_i (1 - e^{-\gamma C_{S_{(i)}}})$, $1\left|psddt, pdw, con, d_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}$, and $1\left|psddt, pdw, slk, q_{opt}\right| \sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 q_{opt}$ are solvable in $O(n \log n)$ time, respectively. For the position-dependent and time-dependent processing times extensions, we showed that the objective functions $\sum_{i=1}^{n} \varpi_i C_{S_{(i)}}$, and $\sum_{i=1}^{n} \varpi_i |L_{S_{(i)}}| + \varpi_0 d_{opt}/q_{opt}$ remain polynomially solvable respectively. Future research may consider the problem $1|psddt, pdw, B| \sum_{i=1}^{n} \varpi_i (1 - e^{-\gamma C_{S_{(i)}}})$, $B \in \{p_i^A = f(i, r), p_i^A = p_i + cs_i\}$, study multi-machine settings (such as flow shop and job shop scheduling), investigate resource allocation scheduling (Lu and Liu 2018; Li et al. 2018), or optimize other objective functions with deterioration and learning effects (see Wang et al. 2012a, b and Lu et al. (2016)).

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References

Biskup D (1999) Single-machine scheduling with learning considerations. Eur J Oper Res 115:173–178
Brucker P (2001) Scheduling algorithms, 3rd edn. Springer, Berlin
Cheng TCE, Wu C-C, Chen J-C, Wu W-H, Cheng S-R (2013) Two-machine flowshop scheduling with a truncated learning function to minimize the makespan. Int J Prod Econ 141:79–86

Gawiejnowicz S (2008) Time-dependent scheduling. Springer, Berlin
Graham RL, Lawler EL, Lenstra JK, Rinnooy Kan AHG (1979) Optimization and approximation in deterministic sequencing and scheduling: a survey. Ann Discret Math 5:287–326
Hall NG, Posner ME (1991) Earliness-tardiness scheduling problems, I: weighted deviation of completion times about a common due date. Oper Res 39:836–846
Hardy GH, Littlewood JE, Polya G (1967) Inequalities, 2nd edn. Cambridge University Press, Cambridge
Jiang C, Zou D, Bai D, Wang J-B (2020) Proportionate flowshop scheduling with position-dependent weights. Eng Optim 52(1):37–52
Kahlbacher H (1992) Termin-und Ablaufplanung-ein analytischer Zugang. Ph.D. thesis, University of Kaiserslautern
Koulamas C, Kyparisis GJ (2010) Single-machine scheduling problems with past-sequence-dependent delivery times. Int J Prod Econ 126:264–266
Li L, Yan P, Ji P, Wang J-B (2018) Scheduling jobs with simultaneous considerations of controllable processing times and learning effect. Neural Comput Appl 29:1155–1162
Liu M (2013) Parallel-machine scheduling with past-sequence-dependent delivery times and learning effect. Appl Math Model 37:9630–9633
Liu W, Jiang C (2020) Due-date assignment scheduling involving job-dependent learning effects and convex resource allocation. Eng Optim 52(1):74–89
Liu M, Zheng F, Chu C, Xu Y (2012a) New results on single-machine scheduling with past-sequence-dependent delivery times. Theor Comput Sci 438:55–61
Liu M, Zheng F, Chu C, Xu Y (2012b) Single-machine scheduling with past-sequence-dependent delivery times and release times. Inf Process Lett 112:835–838
Liu M, Wang S, Chu, (2013) Scheduling deteriorating jobs with past-sequence-dependent delivery times. Int J Prod Econ 144:418–421
Liu W, Hu X, Wang X-Y (2017) Single machine scheduling with slack due dates assignment. Eng Optim 49:709–717
Lu Y-Y (2016) Research on no-idle permutation flowshop scheduling with time-dependent learning effect and deteriorating jobs. Appl Math Model 40:3447–3450
Lu Y-Y, Liu J-Y (2018) A note on resource allocation scheduling with position-dependent workloads. Eng Optim 50:1810–1827
Lu Y-Y, Teng F, Feng Z-X (2015) Scheduling jobs with truncated exponential sum-of-logarithm-processing-times based and position-based learning effects. Asia Pac J Oper Res 32:1550026
Lu Y-Y, Jin J, Ji P, Wang J-B (2016) Resource-dependent scheduling with deteriorating jobs and learning effects on unrelated parallel machine. Neural Comput Appl 27:1993–2000
Mosheiov G (2011) Proportionate flowshops with general position-dependent processing times. Inf Process Lett 111:174–177
Mosheiov G, Sidney JB (2003) Scheduling with general job-dependent learning curves. Eur J Oper Res 147:665–670
Wang J-J, Zhang B-H (2015) Permutation flowshop problems with bi-criterion makespan and total completion time objective and position-weighted learning effects. Comput Oper Res 58:24–31
Wang J-B, Ji P, Cheng TCE, Wang D (2012a) Minimizing makespan in a two-machine flow shop with effects of deterioration and learning. Optim Lett 6:1393–1409
Wang J-B, Wang M-Z, Ji P (2012b) Single machine total completion time minimization scheduling with a time-dependent learning effect and deteriorating jobs. Int J Syst Sci 43:861–868
Wang X-R, Jin J, Wang J-B, Ji P (2014) Single machine scheduling with truncated job-dependent learning effect. Optim Lett 8:669–677
Wang J-B, Liu F, Wang J-J (2019) Research on m-machine flow shop scheduling with truncated learning effects. Int Trans Oper Res 26(3):1135–1151
Wang J-B, Gao M, Wang J-J, Liu L, He H (2020) Scheduling with a position-weighted learning effect and job release dates. Eng Optim 52(9):1475–1493
Wu Y-B, Wang J-J (2016) Single-machine scheduling with truncated sum-of-processing-times-based learning effect including proportional delivery times. Neural Comput Appl 27:937–943
Yang S-J, Yang D-L (2012) Single-machine scheduling problems with past-sequence-dependent delivery times and position-dependent processing times. J Oper Res Soc 63:1508–1515
Yin Y, Cheng TCE, Xu J, Cheng S-R, Wu C-C (2013) Single-machine scheduling with past-sequence-dependent delivery times and a linear deterioration. J Ind Manag Optim 9:323–339

Zhao C, Tang H (2014) Single machine scheduling problems with general position-dependent processing times and past-sequence-dependent delivery times. J Appl Math Comput 45:259–274

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