



The strong chromatic index of graphs with edge weight eight

Lily Chen¹ · Shumei Chen¹ · Ren Zhao¹ · Xiangqian Zhou¹

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Abstract

The edge weight of a graph G is defined to be $\max\{d_G(u) + d_G(v) : uv \in E(G)\}$. The strong chromatic index of a graph is the minimum value of k such that the edge set of G can be partitioned into k induced matchings. In this article, we prove that the strong chromatic index of a graph with edge weight eight is at most 21.

Keywords Strong edge-coloring · Strong chromatic index · Edge weight

Mathematics Subject Classification 05C15

1 Introduction

All graphs in this article are simple and undirected. A *strong edge-coloring* of a graph G , first introduced by Fouquet and Jolivet (1983), is a proper edge-coloring of G such that any two edges joined by another one have distinct colors. Clearly, every coloring class is an induced matching of G . The minimum number of colors needed in a strong edge-coloring is called the *strong chromatic index*, denoted by $\chi'_s(G)$.

Let G be a graph and let Δ be the maximum degree of G . In 1989, Erdős and Nešetřil (1989) conjectured that $\chi'_s(G) \leq \frac{5}{4}\Delta^2$ if Δ is even, and $\chi'_s(G) \leq \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}$ if Δ is odd. They also found a class of graphs that can reach these upper bounds. Since then, there have been many progress on the conjecture, especially for various

✉ Lily Chen
lily60612@126.com

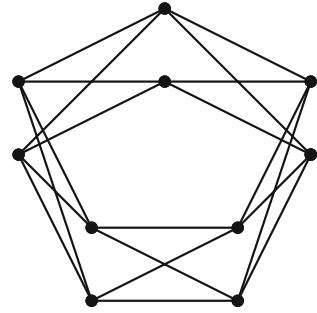
Shumei Chen
745778051@qq.com

Ren Zhao
ren-zhao@outlook.com

Xiangqian Zhou
xiangqian.zhou@wright.edu

¹ School of Mathematical Sciences, Huaqiao University, Quanzhou, China

Fig. 1 A graph with edge weight 8 and strong chromatic index 20



classes of sparse graphs, like planar graphs; k -degenerate graphs, bipartite graphs, graphs with small maximum average degree, and so on. The strong chromatic index on some special graphs have also been studied, such as unit distance graph (Debski 2019), and generalized Petersen graph (Yang and Wu 2018). For an overview of the strong edge-coloring, we refer the reader to the survey (Deng et al. 2019).

The *edge weight* of a graph G is defined to be $\max\{d_G(u) + d_G(v) \mid uv \in E(G)\}$. Note that, for every edge $uv \in E(G)$, the edges incident with u or v must receive different colors in a strong edge-coloring. So χ'_s is bounded below by the edge weight minus one. It would be interesting if one can find a relationship between the edge weight and an upper bound for χ'_s . The first such result was proved by Wu and Lin (2008) in 2008.

Theorem 1.1 (Wu and Lin (2008)) *If G is a connected graph with edge weight at most 5. Then either $\chi'_s(G) \leq 6$ or G is isomorphic to the graph obtained from C_5 by adding a new vertex connecting to a pair of nonadjacent vertices.*

Recently, Chen et al. (2020) proved the following theorem:

Theorem 1.2 (Chen et al. (2020)) *Let G be a graph.*

- (1) *if the edge weight of G is at most 6, then $\chi'_s(G) \leq 10$;*
- (2) *if the edge weight of G is at most 7, then $\chi'_s(G) \leq 15$.*

They gave a graph with edge weight 7 and strong chromatic index 13, which suggests that the bound of 15 may not be tight. In this article, we study graphs with edge weight 8 and prove the following result.

Theorem 1.3 *For any graph G with edge weight 8, $\chi'_s(G) \leq 21$.*

Note that the graph in Fig. 1 given by Erdős and Nešetřil (1989) has edge weight 8 and strong chromatic index 20. No known example of graphs with edge weight 8 would require 21 colors. We strongly believe that the following conjecture should be true.

Conjecture 1.4 *For any graph G with edge weight 8, $\chi'_s(G) \leq 20$.*

The proof of our result relies heavily on the following result of Huang et al. (2018).

Theorem 1.5 (Huang et al. (2018)) *If G is a graph with maximum degree 4, then $\chi'_s(G) \leq 21$.*

Note that every 4-regular graph has edge weight 8, so our result is a natural extension of Theorem 1.5. Likewise, Conjecture 1.4 is a non-trivial extension of Erdős and Nešetřil's Conjecture for $\Delta = 4$. While it might be true that Conjecture 1.4 would follow from Erdős and Nešetřil's Conjecture, the proof used in this article does not provide a direct link between these two conjectures, as we require the use of 21 colors in a number of important steps.

2 Proof of Theorem 1.3

We will use k -vertex (resp. k^- -vertex, k^+ -vertex) to denote a vertex of degree k (resp. at most k ; at least k). For $v \in V(G)$, we use $N(v)$ to denote the set of neighbors of v . For two edges e and e' , we say that e sees e' if they are incident with a common vertex or adjacent to a common edge. Apparently, such two edges must be assigned different colors in any strong edge-colorings of G .

From now on, we assume that G is a minimal counterexample to Theorem 1.3, that is, G has edge weight 8, $\chi'_s(G) \geq 22$, and $|V(G)|$ as small as possible. A *partial strong edge-coloring* of G is a proper edge-coloring of a subgraph G' of G with at most 21 colors such that any two edges that can see each other in G receive distinct colors. We will show that a certain partial strong edge-coloring of G can be extended to a strong edge-coloring of G . We denote by $C(e)$ the set of colors that are available for $e \in E(G) - E(G')$ and we let $c(e) = |C(e)|$. By a *good coloring*, we mean a strong edge-coloring using at most 21 colors.

Next we prove some basic properties of G .

Lemma 2.1 *The minimum degree of G is at least 3.*

Proof Assume that there exists a vertex $v \in E(G)$ with $d_G(v) \leq 2$. If $d_G(v) = 1$ and v' is the only neighbor of v , then the edge vv' can see at most 12 edges in $G - v$. So a good coloring of $G - v$ can be easily extended to a good coloring of G , a contradiction. If $d_G(v) = 2$ and v' and v'' are the two neighbors of v , then each of the edges vv' and vv'' can see at most 17 edges in $G - v$. So a good coloring of $G - v$ can again be easily extended to a good coloring of G , a contradiction. \square

It follows immediately that $d_G(v) \in \{3, 4, 5\}$ for any $v \in V(G)$.

Lemma 2.2 *A 3-vertex can not be adjacent to a 3-vertex.*

Proof Let v be a 3-vertex with neighbor set $\{v_1, v_2, v_3\}$. Assume without loss of generality that $d_G(v_1) = 3$. Then the edge vv_1 can see at most 18 edges of $G - v$, there are at least three colors available for the edge vv_1 , we color it with one of the colors. Now each of the edges vv_2 , and vv_3 can see at most 19 edges, and there are at least 2 colors for each of them, so a good coloring of $G - v$ can be extended to a good coloring of G , a contradiction.

The case in which one of v_2 and v_3 is a 3-vertex is done in a similar manner. \square

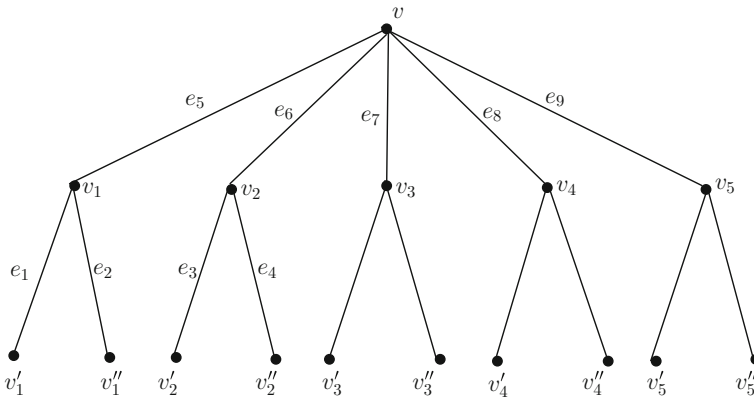


Fig. 2 A 5-vertex v and its neighbors

Lemma 2.3 *Every 3-vertex is adjacent to at least one 5-vertex.*

Proof Let v be a 3-vertex with neighbor set $\{v_1, v_2, v_3\}$. If none of them is a 5-vertex, then by Lemma 2.2, they must all be 4-vertices. Then each of the edges $vv_1, vv_2,$ and vv_3 can see at most 18 edges of $G - v$, therefore, there are at least 3 colors available for $vv_1, vv_2,$ and vv_3 . So a good coloring of $G - v$ can be extended to a good coloring of G . \square

By Theorem 1.5, we may assume that G has at least one 5-vertex.

Let G' be a subgraph of G with $|V(G')| < |V(G)|$. Suppose that G has a strong partial edge-coloring ϕ on G' . In order to extend ϕ to a good coloring of G , we construct a bipartite graph $B = B[X, Y]$ with bipartition (X, Y) , where a vertex $x \in X$ corresponds to an edge x in $E(G) - E(G')$, $Y = [21]$ is the set of all 21 colors, and a vertex $x \in X$ is adjacent to a vertex $y \in Y$ if and only if the color y is available for the edge x . By the construction of B , if B has a matching M that covers X , then we color each edge of $E(G) - E(G')$ with the color that is adjacent to it in M , hence the coloring ϕ of G' can be extended to a good coloring of G . The edges in X will be ordered by e_0, e_1, \dots, e_k . The degree sequence of X in B , denoted by $d_B(X)$, is the sequence (d_0, d_1, \dots, d_k) where $d_i = d_B(e_i)$ for $0 \leq i \leq k$. That is, d_i is the number of available colors for the edge e_i . Let $S = (s_0, s_1, \dots, s_k)$ and $S' = (s'_0, s'_1, \dots, s'_k)$ be two sequences of the same length. We say that S dominates S' if for each $i \in \{0, 1, \dots, k\}$, $s_i \geq s'_i$. We need the well-known Hall's Theorem (Hall 1935) in our proofs.

Theorem 2.4 *A bipartite graph $B = B[X, Y]$ has a matching that covers every vertex of X if and only if $|N_B(S)| \geq |S|$ for all $S \subseteq X$, where $N_B(S)$ is the set of all neighbors of vertices in S .*

Let v be a 5-vertex of G and let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$. Note that each of $v_i, 1 \leq i \leq 5$ is a 3-vertex. For each $1 \leq i \leq 5$, we use v'_i and v''_i to denote the neighbors of v_i other than v . We shall label the edges incident to $v, v_1,$ or v_2 as shown in Fig. 2.

Lemma 2.5 For any $\{i, j\} \subset \{1, 2, 3, 4, 5\}$, we have that $v_i v_j \notin E(G)$.

Proof Suppose otherwise. By symmetry, we may assume that $v_1 v_2 \in E(G)$. Let $G' = G - v$. Then each of the edges vv_1 and vv_2 can see at most 13 edges of G' ; and each of the edges $vv_3, vv_4,$ and vv_5 can see at most 17 edges of G' ; we can first color vv_3, vv_4, vv_5 , then color vv_1 and vv_2 to extend a good coloring of G' to a good coloring of G ; a contradiction. \square

Lemma 2.6 For any $\{i, j\} \subset \{1, 2, 3, 4, 5\}$, $N(v_i) \cap N(v_j) = \{v\}$.

Proof By symmetry, we only need show that v_1 and v_2 have no common neighbor other than v . It is easy to see that we have the following three cases.

- (1) $v'_1 = v'_2$ and $v''_1 = v''_2$.

In this case, we consider the subgraph $G - v$. Then each of vv_1 and vv_2 can see at most 16 edges of $G - v$; each of $vv_3, vv_4,$ and vv_5 can see at most 18 edges of $G - v$. We can first color $vv_3, vv_4,$ and vv_5 , then color vv_1 and vv_2 to extend a good coloring of $G - v$ to a good coloring of G ; a contradiction.

- (2) $v'_1 = v'_2$ and $v''_1 v''_2 \in E(G)$.

In this case, by Lemma 2.2, we have that $d_G(v''_1) \neq 3, d_G(v''_2) \neq 3$. By Lemma 2.1, $d(v) \in \{3, 4, 5\}$ for any $v \in V(G)$. If $d_G(v''_1) = 5$, since $v''_1 v''_2 \in E(G), d_G(v''_2) = 3$, a contradiction. Therefore, $d_G(v''_1) = 4$. Similarly, $d_G(v''_2) = 4$. We again consider the subgraph $G - v$. Then each of vv_1 and vv_2 can see at most 16 edges of $G - v$; each of $vv_3, vv_4,$ and vv_5 can see at most 18 edges of $G - v$. We can first color $vv_3, vv_4,$ and vv_5 , then color vv_1 and vv_2 to extend a good coloring of $G - v$ to a good coloring of G ; a contradiction.

- (3) $v'_1 = v'_2$ and the edges $v_1 v''_1$ and $v_2 v''_2$ do not see each other.

In this case, we will choose $G' = G - \{v, v_1, v_2\}$ and construct the bipartite graph $B[X, Y]$, where $X = E(G) - E(G') = \{e_1, e_2, \dots, e_9\}$ and $Y = [21]$. Then by calculating the least number of available colors for each edge in X , we have the degree sequence $d_B(X)$ dominates the sequence $(8, 6, 8, 6, 8, 8, 7, 7, 7)$. For any $S \subseteq X$ with $|S| \leq 8$, we have $|N_B(S)| \geq |S|$. Now if $|N_B(X)| \geq 9$, then by Hall's theorem, there exists a matching that covers X , which implies a good coloring of G' can be extended to a good coloring of G . So we may assume that $|N_B(X)| = 8$. If $C(e_2) \cap C(e_4) = \emptyset, |C(e_2) \cup C(e_4)| \geq 12$, then $|N_B(X)| \geq 12 \geq |X| = 9$, contrary to our assumption. So $C(e_2) \cap C(e_4) \neq \emptyset$. Since e_2 and e_4 do not see each other, we may color them with one color. Therefore the degree sequence $d_B(X - \{e_2, e_4\})$ dominates $(7, 7, 7, 7, 6, 6, 6)$. So for any $S \subseteq (X - \{e_2, e_4\})$, we have that $|N_B(S)| \geq |S|$, by Hall's theorem, there exists a matching that covers $X - \{e_2, e_4\}$, thus a good coloring of G' can be extended to a good coloring of G ; a contradiction.

Therefore v_1 and v_2 have no common neighbor other than v , that is, $N(v_1) \cap N(v_2) = \{v\}$. \square

Lemma 2.7 Neither $v_1 v'_1$ nor $v_1 v''_1$ can see the edges $v_2 v'_2$ or $v_2 v''_2$.

Proof Suppose otherwise. By symmetry, we may assume that the edge $v_1 v'_1$ can see the edge $v_2 v'_2$, that is, $v'_1 v'_2 \in E(G)$. By Lemma 2.6, $v'_1, v''_1 \notin \{v'_2, v''_2\}$. Since G has

no adjacent 3-vertices, both v'_1 and v'_2 are 4-vertices. We consider the following two cases:

Case 1: the edge $v_1v'_1$ also sees the edge $v_2v''_2$.

In this case, we have that $v'_1v''_2 \in E(G)$ and $d_G(v''_2) = 4$. Consider the subgraph $G - v$. Then vv_1 sees at most 17 edges of $G - v$; vv_2 sees at most 16 edges of $G - v$; and each of the edges $vv_3, vv_4,$ and vv_5 sees at most 18 edges of $G - v$. So we may color the edges in $E(G) - E(G')$ in the following order $vv_3, vv_4, vv_5, vv_1, vv_2$. This way we can extend a good coloring of G' to a good coloring of G ; a contradiction.

Case 2: the edge $v_1v'_1$ does not see the edge $v_2v''_2$.

In this case, we will choose $G' = G - \{v, v_1, v_2\}$ and construct the bipartite graph $B[X, Y]$, where $X = E(G) - E(G') = \{e_1, e_2, \dots, e_9\}$ and $Y = [21]$. Then the degree sequence $d_B(X)$ dominates $(6, 6, 6, 6, 8, 8, 7, 7, 7)$. Note that, for any proper subset S of X , we have that $|N_B(S)| \geq |S|$. So by Hall's Theorem, if $|N_B(X)| \geq 9$, then there exists a matching that covers X , and thus, a good coloring of G' can be extended to a good coloring of G ; a contradiction. So we may assume that $|N_B(X)| = 8$. Since the edge $e_1 = v_1v'_1$ does not see the edge $e_4 = v_2v''_2$, we may color e_1 and e_4 with the same color, now the degree sequence for the remaining edges dominates $(5, 5, 7, 7, 6, 6, 6)$. Again by Hall's theorem, a good coloring of G' can be extended to a good coloring of G , a contradiction. \square

Now we are ready to complete the proof for Theorem 1.3. Let $G' = G - \{v, v_1, v_2\}$ and we construct the bipartite graph $B[X, Y]$, where $X = E(G) - E(G') = \{e_1, e_2, \dots, e_9\}$ and $Y = [21]$. Then the degree sequence $d_B(X)$ dominates $(5, 5, 5, 5, 7, 7, 7, 7, 7)$. Note that for any $S \subseteq X$ with $|S| \leq 7$, we have that $|N_B(S)| \geq |S|$. So we may assume that there exists an $S_0 \subseteq X$ with $|S_0| = 8$ or 9 such that $|N_B(S_0)| < |S_0|$.

First assume that $|S_0| = 9$, that is, $S_0 = X$. Then $7 \leq |N_B(X)| \leq 8$. Note that $|C(e_2) \cup C(e_4)| \leq |N_B(X)| \leq 8$. Since $c(e_2), c(e_4) \geq 5$, $|C(e_2) \cap C(e_4)| \geq 2$. Similarly, $|C(e_1) \cap C(e_3)| \geq 2$. By Lemma 2.7, we may color e_1 and e_3 with the same color, and color e_2 and e_4 with the same color. Now the degree sequence for the remaining edges dominates $(5, 5, 5, 5, 5)$. By Hall's Theorem, there exists a matching covering these edges. So a good coloring of G' can be extended to a good coloring of G , a contradiction.

Next assume that $|S_0| = 8$ and $|N_B(S_0)| = 7$. Note that, if $\{e_1, e_2, e_3, e_4\} \subset S_0$, then we may again color e_1 and e_3 with the same color, and color e_2 and e_4 with the same color; thus extending a good coloring of G' to G , a contradiction. Therefore, S_0 contains exactly three edges of $\{e_1, e_2, e_3, e_4\}$. By symmetry, assume that $S_0 = X - e_1$. We may further assume that $|C(e_1) \cap C(e_3)| \leq 1$; as otherwise we may repeat the same coloring procedure to obtain a coloring of G . Therefore, $|C(e_1) \cup C(e_3)| \geq 9$, and hence, $C(e_1) - N_B(S_0) \neq \emptyset$. Now we choose a color from $C(e_1) - N_B(S_0)$ to color e_1 , and a color from $C(e_2) \cap C(e_4)$ to color the edges e_2 and e_4 . Now the degree sequence for the remaining edges dominates $(4, 6, 6, 6, 6, 6)$. By Hall's Theorem, a good coloring of G' can be extended to a good coloring of G ; a contradiction.

This completes the proof for Theorem 1.3.

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