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On the edge metric dimension of convex polytopes and its related graphs

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Abstract
Let $G = (V, E)$ be a connected graph. The distance between the edge $e = uv \in E$ and the vertex $x \in V$ is given by $d(e, x) = \min\{d(u, x), d(v, x)\}$. A subset S_E of vertices is called an edge metric generator for G if for every two distinct edges $e_1, e_2 \in E$, there exists a vertex $x \in S_E$ such that $d(e_1, x) \neq d(e_2, x)$. An edge metric generator containing a minimum number of vertices is called an edge metric basis for G and the cardinality of an edge metric basis is called the edge metric dimension denoted by $\mu_E(G)$. In this paper, we study the edge metric dimension of some classes of plane graphs. It is shown that the edge metric dimension of convex polytope antiprism A_n, the web graph $\mathbb{W}_n$, and convex polytope $\mathbb{D}_n$ are bounded, while the prism related graph D_n^* has unbounded edge metric dimension.

Keywords Metric dimension · Edge metric dimension · Edge metric generator · Convex polytopes

1 Introduction

The metric dimension was first introduced independently by Slater (1975) and by Harary and Melter (1976), which has been widely investigated in a number of papers, see Cáceres et al. (2007), Hallaway et al. (2014), Chartrand et al. (2000), Sebő and Tannier (2004), Guo et al. (2012) and Chartrand and Zhang (2003) for more details. It was appeared in various areas including pharmaceutical chemistry (Chartrand et al. 2000), combinatorial optimization (Sebő and Tannier 2004), robot navigation (Khuller et al. 1996), and sonar (Slater 1975), etc. Let G be a finite, simple, and connected graph with the vertex set $V(G)$ and the edge set $E(G)$. Throughout the paper, when there is no scope for ambiguity, we write V and E instead of the vertex set $V(G)$ and the edge

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set $E(G)$, respectively. For every distinct vertices $u, v \in V$, the *distance* between the vertices u and v, denoted by $d(u, v)$, is the number of edges in a shortest path between them in G. A vertex $x \in V$ is said to *distinguish* a pair of vertices $u, v \in V$ if $d(u, x) \neq d(v, x)$. A set $S \subseteq V$ is a *metric generator* for G if every pair of vertices of G can be distinguished by some vertex in S. A *metric basis* of G is a metric generator of minimum cardinality. The cardinality of a metric basis, denoted by $\mu(G)$, is called the *metric dimension* of G. It was shown that computing the metric dimension of a graph is NP-hard (Khuller et al. 1996).

The edge metric dimension is a new variant of the metric dimension. It was introduced by Kelenc et al. (2018) and further investigated by Zubrilina (2018), Peterin and Yero (2018), Kratica et al. (2017) and Zhu et al. (2019). The *distance* between an edge $e = uv$ and a vertex x is defined as follows:

$$d(e, x) = \min\{d(u, x), d(v, x)\}.$$

A vertex $x \in V$ is said to *distinguish* two distinct edges $e_1, e_2 \in E$ if $d(e_1, x) \neq d(e_2, x)$. A set $S_E \subseteq V$ is an *edge metric generator* of a graph G if every two distinct edges are distinguished by some vertex of S_E. An edge metric generator with the smallest number of vertices is called an *edge metric basis* of G. The *edge metric dimension* of G, denoted by $\mu_E(G)$, is the cardinality of its edge metric basis. Kelenc et al. (2018) proved that computing the edge metric dimension of a graph is NP-hard.

For an ordered subset $S_E = \{x_1, x_2, \ldots, x_k\}$ of the vertex set V, the k-tuple $r(e|S_E) = (d(e, x_1), d(e, x_2), \ldots, d(e, x_k))$ is called the *edge metric representation* of an edge e with respect to S_E. In this sense, S_E is an edge metric generator for G if and only if for every pair of different edges e_1, e_2 of G, we have $r(e_1|S_E) \neq r(e_2|S_E)$.

An edge metric generator S_E is not necessarily a metric generator. In Kelenc et al. (2018), the authors proposed an realization question for edge metric dimension and metric dimension. Specifically, they stated that it is possible to find graphs for which the metric dimension equals the edge metric dimension, as well as other graphs G for which $\mu(G) < \mu_E(G)$ or $\mu(G) > \mu_E(G)$. In this paper, using four classes of plane graphs convex polytope antiprism A_n, the web graph $\mathbb{W}_n$, the prism related graph D_n^* and convex polytope $\mathbb{D}_n$, we further explore such situations by comparing the value of $\mu(G)$ and $\mu_E(G)$, where G denotes one of plane graphs A_n, $\mathbb{W}_n$, D_n^* and $\mathbb{D}_n$.

This paper is organized as follows. In Sect. 2, we recall some results related to the edge metric dimension of graphs. In Sect. 3, we study the edge metric dimension of antiprism A_n. In Sect. 4, the explicit expression for $\mu_E(\mathbb{W}_n)$ and $\mu_E(D_n^*)$ are obtained. The edge metric dimension convex polytope $\mathbb{D}_n$ is determined in Sect. 5. In the last section, we conclude the obtained results.

Throughout this paper, all vertex indices are taken to be module n.

2 Preliminaries

In this section, we recall some results on the edge metric dimension of graphs.

Let $G = (V, E)$ be a simple connected graph with the vertex set V and the edge set E. For a vertex v, let $N(v) = \{u \in V | uv \in E\}$ denote the *neighborhood* of the vertex v. $|N(v)|$ is called the *degree* of the vertex v, denoted by $deg_G(v)$. The *maximum degree* and the *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let $\mathbb{R}$ be a real number set and $\mathbb{Z}$ be the integer set. If $i, j \in \mathbb{R}$, we let $[\![i, j]\!] := \{x \in \mathbb{Z} | \; i \le x \le j\}$ stand for the discrete interval between i and j.

It is known that if G is a connected graph of order n, we have $1 \le \mu_E(G) \le n-1$.

Now, we recall the lower bounds of the edge metric dimension of a connected graph.

Proposition 1 (Kelenc et al. 2018) *If G is a connected graph and $\Delta(G)$ is the maximum degree of G, then we have $\mu_E(G) \ge \lceil log_2 \Delta(G) \rceil$.*

Proposition 2 (Kratica et al. 2017) *Let G be a connected graph and let $\delta(G)$ be the minimum degree of G. Then $\mu_E(G) \ge 1 + \lceil log_2 \delta(G) \rceil$.*

For a family $\mathscr{F}$ of connected graphs, we say that $\mathscr{F}$ has bounded edge metric dimension, if for every graph G of $\mathscr{F}$ there exists a constant $C > 0$ such that $\mu_E(G) \le C$; otherwise $\mathscr{F}$ has unbounded edge metric dimension.

If all graphs in $\mathscr{F}$ have the same edge metric dimension, then $\mathscr{F}$ is called a family with constant edge metric dimension. Paths P_n, cycles C_n for $n \ge 2$ and prism D_n are families of graphs with constant edge metric dimension.

We end this section with some useful facts about the metric dimension of antiprism A_n, the web graph $\mathbb{W}_n$, the prism related graph D_n^* and convex polytope $\mathbb{D}_n$.

Lemma 1 (Javaid et al. 2008) *Let A_n be an antiprism with $n \ge 3$. Then we have $\mu(A_n) = 3$.*

Lemma 2 (Imran et al. 2016) *For $n \ge 3$, let $\mathbb{W}_n$ be a web graph. Then we have*

$$\mu(\mathbb{W}_n) = \begin{cases} 2, & n \; is \; odd, \\ 3, & n \; is \; even. \end{cases}$$

Lemma 3 (Ali et al. 2012) *For $n \ge 6$, we have $\mu(D_n^*) = 3$.*

Lemma 4 (Imran et al. 2012) *Let $\mathbb{D}_n$ be the graph of convex polytope with $n \ge 3$. Then we have $\mu(\mathbb{D}_n) = 3$.*

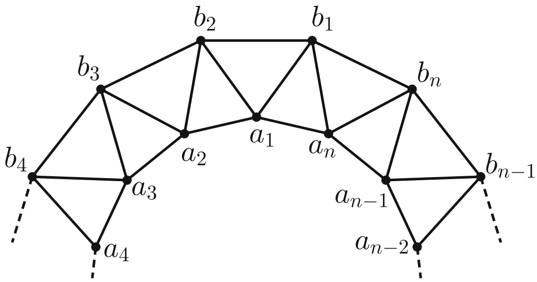
3 The graph of convex polytope antiprism A_n

In this section, we present the edge metric dimension of antiprism A_n with $n \ge 3$.

The antiprism A_n defined in Bača (1988) is a 4-regular graph which has $2n$ 3-sided faces, and a pair of n-sided faces respectively, see Fig. 1. It consists of an outer cycle $b_1, b_2, \ldots, b_n$, an inner cycle $a_1, a_2, \ldots, a_n$, and a set of $2n$ spokes $a_i b_i$ and $a_i b_{i+1}$. We have the vertex set $V(A_n) = \{a_i, b_i | 1 \le i \le n\}$, and the edge set $E(A_n) = \{a_i a_{i+1}, a_i b_i, a_i b_{i+1}, b_i b_{i+1} | 1 \le i \le n\}$.

Lemma 5 *For any edge metric generator S_E of A_n, S_E contains at least one vertex of outer cycle and one vertex of inner cycle, respectively.*

Fig. 1 Antiprism A_n



Proof Without loss of generality, assume $\{b_1, b_2, \ldots, b_n\} \bigcap S_E = \emptyset$. Since $S_E \neq \emptyset$, there exist some elements of inner cycle in S_E. In this case, we have $r(a_i b_i | S_E) = r(a_i b_{i+1} | S_E)$ for $1 \leq i \leq n$. It implies that S_E is not an edge metric generator. A contradiction. So there exists at least one vertex of outer cycle in S_E.

Similarly, we can show that there exists at least one vertex of inner cycle in S_E. □

Lemma 6 *If an edge metric generator S_E for A_n contains two vertices of one cycle, then S_E contains at least two vertices of the another cycle.*

Proof The result will be proved by showing that if an edge metric generator S_E for A_n contains $n-1$ vertices of one cycle, then S_E contains at least two vertices of the another cycle. Because of symmetry of the antiprism A_n, it is enough to show that if an edge metric generator S_E for A_n contains $n-1$ vertices of inner cycle, then S_E contains at least two vertices of the outer cycle. By Lemma 5, there exists at least one vertex of outer cycle in S_E. Assume that there is only one vertex, say b_j for $1 \leq j \leq n$, such that $b_j \in S_E$. Without loss of generality, we assume that $a_i \notin S_E$ and $a_k \in S_E$ $(1 \leq k \leq n, k \neq i)$. We divide the proof into two cases.

(1) If j or $j+n \in [\![i+1, i+\lceil \frac{n}{2} \rceil]\!]$, then we have $r(a_i b_{i+1} | S_E) = r(b_i b_{i+1} | S_E)$;
(2) If j or $j+n \in [\![i+\lceil \frac{n}{2} \rceil + 1, i+n]\!]$, then we have $r(a_i b_i | S_E) = r(b_i b_{i+1} | S_E)$.

Any of case above contradicts that S_E is an edge metric generator for A_n.
The lemma follows immediately from what we have proved. □

In the following, we give a lower bound for the edge metric dimension of antiprism A_n.

Corollary 1 *If A_n is an antiprism with $n \geq 3$, then $\mu_E(A_n) \geq 4$.*

Proof Immediate from Proposition 2 and Lemma 6. □

The metric dimension of antiprism A_n was investigated in Javaid et al. (2008), see Lemma 1. In the following, we determine the exact value of the edge metric dimension for antiprism A_n.

Theorem 1 *Let A_n be an antiprism with $n \geq 3$. Then we have*

$$\mu_E(A_n) = \begin{cases} 4, & n \text{ is even}, \\ 5, & \text{otherwise}. \end{cases}$$

Proof We divide our proof into two cases.

Case (I) n is even. Set $n = 2l$, where $l \in \mathbb{Z}$. Let $S_E = \{a_1, a_l, b_1, b_{l+1}\}$. To show that S_E is an edge metric generator for A_n, we give representations of any edge of $E(A_n)$ with respect to S_E. They are

$$r(a_i a_{i+1}|S_E) = \begin{cases} (i-1, l-i-1, i, l-i), & 1 \le i \le l-1, \\ (l-1, 0, l, 1), & i = l, \\ (2l-i, i-l, 2l-i, i-l), & l+1 \le i \le 2l-1, \\ (0, l-1, 1, l), & i = 2l. \end{cases}$$

$$r(a_i b_i|S_E) = \begin{cases} (i-1, l-i, i-1, l-i+1), & 1 \le i \le l, \\ (2l-i+1, i-l, 2l-i+1, i-l-1), & l+1 \le i \le 2l. \end{cases}$$

$$r(a_i b_{i+1}|S_E) = \begin{cases} (i-1, l-i, i, l-i), & 1 \le i \le l, \\ (2l-i+1, i-l, 2l-i, i-1), & l+1 \le i \le 2l. \end{cases}$$

$$r(b_i b_{i+1}|S_E) = \begin{cases} (1, l-1, 0, l-1), & i = 1, \\ (i-1, l-i, i-1, l-i), & 2 \le i \le l-1 (l \ge 3), \\ (l-1, 1, l-1, 0), & i = l, \\ (2l-i+1, i-l, 2l-i, i-l-1), & l+1 \le i \le 2l. \end{cases}$$

We note that there are no two edges having the same edge metric representations. So we have $\mu_E(A_n) \le 4$. Using the Corollary 1 we obtain $\mu_E(A_n) = 4$.

Case (II) n is odd. Set $n = 2l+1$, where $l \in \mathbb{Z}$. Let $S_E = \{a_1, a_{l+1}, a_{l+2}, b_1, b_{l+2}\}$. To show that S_E is an edge metric generator for A_n, we give representations of any edge of $E(A_n)$ with respect to S_E. They are

$$r(a_i a_{i+1}|S_E) = \begin{cases} (i-1, l-i, l-i+1, i, l-i+1), & 1 \le i \le l, \\ (l, 0, 0, l, 1), & i = l+1, \\ (2l-i+1, i-l-1, i-l-2, 2l-i+1, i-l-1), & l+2 \le i \le 2l, \\ (0, l, l-1, 1, l), & i = 2l+1. \end{cases}$$

$$r(a_i b_i|S_E) = \begin{cases} (0, l, l, 0, l), & i = 1, \\ (i-1, l-i+1, l-i+2, i-1, l-i+2), & 2 \le i \le l+1, \\ (2l-i+2, i-l-1, i-l-2, 2l-i+2, i-l-2), & l+2 \le i \le 2l+1. \end{cases}$$

$$r(a_i b_{i+1}|S_E) = \begin{cases} (0, l, l, 1, l), & i = 1, \\ (i-1, l-i+1, l-i+2, i, l-i+1), & 2 \le i \le l, \\ (l, 0, 1, l, 0), & i = l+1, \\ (2l-i+2, i-l-1, i-l-2, 2l-i+1, i-l-1), & l+2 \le i \le 2l+1. \end{cases}$$

$$r(b_i b_{i+1}|S_E) = \begin{cases} (1, l, l, 0, l), & i = 1, \\ (i-1, l-i+1, l-i+2, i-1, l-i+1,), & 2 \le i \le l (l \ge 2), \\ (l, 1, 1, l, 0), & i = l+1, \\ (l, 1, 1, l-1, 0), & i = l+2, \\ (2l-i+2, i-l-1, i-l-2, 2l-i+1, i-l-2), & l+3 \le i \le 2l+1. \end{cases}$$

Note that there are no two edges having same edge metric representation, which implies that $\mu_E(A_n) \le 5$. On the other hand, we show that $\mu_E(A_n) \ge 5$. Suppose on contrary that $\mu_E(A_n) = 4$. Then there are following possibilities to be discussed. Using Lemma 6, we know that an edge metric generator S_E for A_n contains at least two vertices of two cycles respectively.

Let a_1, a_i be two vertices on inner cycle and b_j, b_k be two vertices on outer cycle, where $2 \leq i \leq n$, $1 \leq j \neq k \leq n$. For $2 \leq i \leq l+1$, we only consider the following cases.

(1) If both of j, k aren't equal to $l+2$, then we have $r(a_{l+1}a_{l+2}|S_E) = r(a_{l+1}b_{l+2}|S_E)$.
(2) If one of the j, k equals $l+2$, we assume $j = l+2$. we only consider the following two cases.
   (i) If $k = 1$, then we have $r(a_{l+1}a_{l+2}|S_E) = r(a_{l+1}b_{l+1}|S_E)$;
   (ii) If $k \neq 1$, then we obtain $r(a_1a_n|S_E) = r(a_1b_1|S_E)$.

Thus, in every case we get a contradiction.

For $l+2 \leq i \leq n$, we can rename vertices and situation will be same as discussed above.

Hence, it follows from the above that there is no edge metric generator with four vertices for A_n implying that $\mu_E(A_n) = 5$ in this case. □

Remark 1 If A_n is an antiprism with $n \geq 3$, then $\mu(A_n) < \mu_E(A_n)$ by Lemma 1.

4 The prism related graphs

The prism D_n is a 3-regular graph which is obtained by the Cartesian product of a cycle C_n and the path P_2. It consists of an outer cycle $b_1, b_2, \ldots, b_n$, an inner cycle $a_1, a_2, \ldots, a_n$, and a set of n spokes a_ib_i. We have the vertex set $V(D_n) = \{a_i, b_i | 1 \leq i \leq n\}$, and the edge set $E(D_n) = \{a_ia_{i+1}, a_ib_i, b_ib_{i+1} | 1 \leq i \leq n\}$. The edge metric dimension of the prism D_n has been studied recently in Kratica et al. (2017). In this section, we extend this study to two classes prism related graphs which obtained from prism D_n by slight modifications. Furthermore, we consider how the edge metric dimension of prism D_n is affected by adding a single vertex.

4.1 The web graph $\mathbb{W}_n$

Koh et al. (1980) defined a web graph $\mathbb{W}_n$ (Fig. 2) as a stacked prism graph $P_3 \times C_n$ with the edges of the outer cycle removed. The web graph $\mathbb{W}_n$ also can be obtained from prism D_n by attaching a pendant edge b_ic_i at each vertex b_i of outer cycle of prism D_n. We have the vertex set $V(\mathbb{W}_n) = \{a_i, b_i, c_i | 1 \leq i \leq n\}$, and the edge set $E(\mathbb{W}_n) = \{a_ia_{i+1}, a_ib_i, b_ib_{i+1}, b_ic_i | 1 \leq i \leq n\}$. For convenience, we call vertices c_i where $1 \leq i \leq n$, the pendent vertices.

Theorem 2 *For the web graph $\mathbb{W}_n$ with $n \geq 3$, we have $\mu_E(\mathbb{W}_n) = 3$.*

Proof We consider the following two cases.

Case (I) n is even. Set $n = 2l$, where $l \in \mathbb{Z}$. Let $S_E = \{a_1, a_2, a_{l+1}\}$. To show that S_E is an edge metric generator for $\mathbb{W}_n$, we give representations of any edge of $E(\mathbb{W}_n)$

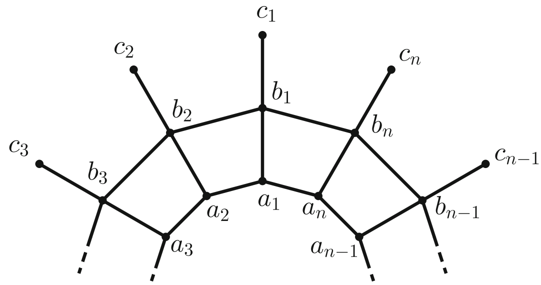


Fig. 2 The web graph $\mathbb{W}_n$

with respect to S_E. They are

$$r(a_i a_{i+1}|S_E) = \begin{cases} (0, 0, l-1), & i = 1, \\ (i-1, i-2, l-i), & 2 \le i \le l, \\ (l-1, l-1, 0), & i = l+1, \\ (2l-i, 2l-i+1, i-l-1), & l+2 \le i \le 2l. \end{cases}$$

$$r(a_i b_i|S_E) = \begin{cases} (0, 1, l), & i = 1, \\ (i-1, i-2, l-i+1), & 2 \le i \le l+1, \\ (2l-i+1, 2l-i+2, i-l-1), & l+2 \le i \le 2l. \end{cases}$$

$$r(b_i b_{i+1}|S_E) = \begin{cases} (1, 1, l), & i = 1, \\ (i, i-1, l-i+1), & 2 \le i \le l, \\ (l, l, 1), & i = l+1, \\ (2l-i+1, 2l-i+2, i-l), & l+2 \le i \le 2l. \end{cases}$$

$$r(b_i c_i|S_E) = \begin{cases} (1, 2, l+1), & i = 1, \\ (i, i-1, l-i+2), & 2 \le i \le l+1, \\ (2l-i+2, 2l-i+3, i-l), & l+2 \le i \le 2l. \end{cases}$$

Note that there are no two edges having same edge metric representation, which implies that $\mu_E(\mathbb{W}_n) \le 3$.

It remains to show that $\mu_E(\mathbb{W}_n) \ge 3$. The result will be proved by showing that there is no edge metric generator S_E with $|S_E| = 2$. Assume for a contradiction that $|S_E| = 2$, then there are the following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality, we assume that one vertex is a_1, and the other is a_i $(2 \le i \le l+1)$. For $2 \le i \le l$, we have $r(a_1a_n|S_E) = r(a_1b_1|S_E) = (0, i-1)$. And for $i = l+1$, we have $r(a_1a_2|S_E) = r(a_1a_n|S_E) = (0, l-1)$. A contradiction.

(2) Both vertices are in the outer cycle. Without loss of generality, we assume that one vertex is b_1, and the other is b_i $(2 \le i \le l+1)$. For $2 \le i \le l$, we have $r(b_1b_n|S_E) = r(b_1c_1|S_E) = (0, i-1)$. And for $i = l+1$, we have $r(b_1b_2|S_E) = r(b_1b_n|S_E) = (0, l-1)$. A contradiction.

(3) Both vertices are in the set of pendent vertices. Without loss of generality, we assume that one vertex is c_1, and the other is c_i $(2 \le i \le l+1)$. For $2 \le i \le l$, we have $r(a_1b_1|S_E) = r(b_1b_n|S_E) = (1, i)$. And for $i = l+1$, we have $r(a_2b_2|S_E) = r(a_nb_n|S_E) = (2, l)$. A contradiction.

(4) One vertex is in the inner cycle, and the other is in the outer cycle. Consider the vertex a_1, and the other is b_i $(1 \le i \le l+1)$. For $1 \le i \le l$, we have $r(b_1c_1|S_E) = r(b_1b_n|S_E) = (1, i-1)$. And for $i = l+1$, we have $r(a_2b_2|S_E) = r(a_nb_n|S_E) = (1, l-1)$. A contradiction.
(5) One vertex is in the inner cycle, and the other is in the set of pendent vertices. Consider the vertex a_1, and the other is c_i $(1 \le i \le l+1)$. For $i = 1$, we have $r(b_1b_2|S_E) = r(b_1b_n|S_E) = (1, 1)$. And for $2 \le i \le l+1$, we have $r(b_1b_2|S_E) = r(a_2b_2|S_E) = (1, i-1)$. A contradiction.
(6) One vertex is in the outer cycle, and the other is in the set of pendent vertices. Consider the vertex b_1, and the other is c_i $(1 \le i \le l+1)$. For $1 \le i \le l$, we have $r(a_1b_1|S_E) = r(b_1b_n|S_E) = (0, i)$. And for $i = l+1$, we have $r(a_2b_2|S_E) = r(a_nb_n|S_E) = (1, l)$. A contradiction.

So from above we conclude that there is no edge metric generator with two vertices for $\mathbb{W}_n$ implying that $\mu_E(\mathbb{W}_n) = 3$ in this case.

Case (II) n is odd.
Set $n = 2l + 1$, where $l \in \mathbb{Z}$. Let $S_E = \{a_1, a_2, a_{l+2}\}$. To show that S_E is an edge metric generator for $\mathbb{W}_n$, we give representations of any edge of $E(\mathbb{W}_n)$ with respect to S_E. They are

$$r(a_ia_{i+1}|S_E) = \begin{cases} (0, 0, l), & i = 1, \\ (i-1, i-2, l-i+1), & 2 \le i \le l+1, \\ (2l-i+1, 2l-i+2, i-l-2), & l+2 \le i \le 2l+1. \end{cases}$$

$$r(a_ib_i|S_E) = \begin{cases} (0, 1, l), & i = 1, \\ (i-1, i-2, l-i+2), & 2 \le i \le l+1, \\ (l, l, 0), & i = l+2, \\ (2l-i+2, 2l-i+3, i-l-2), & l+3 \le i \le 2l+1. \end{cases}$$

$$r(b_ib_{i+1}|S_E) = \begin{cases} (1, 1, l+1), & i = 1, \\ (i, i-1, l-i+2), & 2 \le i \le l+1, \\ (2l-i+2, 2l-i+3, i-l-1), & l+2 \le i \le 2l+1. \end{cases}$$

$$r(b_ic_i|S_E) = \begin{cases} (1, 2, l+1), & i = 1, \\ (i, i-1, l-i+3), & 2 \le i \le l+1, \\ (l+1, l+1, 1), & i = l+2, \\ (2l-i+3, 2l-i+4, i-l-1), & l+3 \le i \le 2l+1. \end{cases}$$

Again we see that there are no two edges having same edge metric representation, which implies that $\mu_E(\mathbb{W}_n) \le 3$.

By the similar arguments to Case (I), we have $\mu_E(\mathbb{W}_n) \ge 3$. Thus, we obtain $\mu_E(\mathbb{W}_n) = 3$. This completes the proof. □

Remark 2 Let $\mathbb{W}_n$ be a web graph with $n \ge 3$. If n is odd, then $\mu(\mathbb{W}_n) < \mu_E(\mathbb{W}_n)$; if n is even, then $\mu(\mathbb{W}_n) = \mu_E(\mathbb{W}_n)$ by Lemma 2,

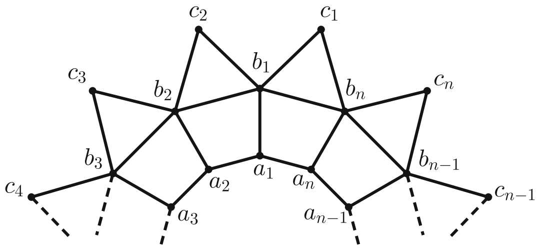


Fig. 3 The prism related graph D_n^*

4.2 The prism related graph D_n^*

The plane graph D_n^* (Fig. 3) defined in Ali et al. (2012) is also an extension of the prism D_n. It can be obtained from prism D_n by adding a new vertex c_i between the vertices b_{i-1} and b_i of the outer cycle with the vertex c_i joining to vertices b_{i-1} and b_i for $1 \le i \le n$, where $b_0 = b_n$. We have the vertex set $V(D_n^*) = \{a_i, b_i, c_i | 1 \le i \le n\}$, and the edge set $E(D_n^*) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+1}, b_i c_i, b_i c_{i+1} | 1 \le i \le n\}$.

Lemma 7 *Let $W = \{c_1, c_2, \dots, c_n\}$ be a subset of $V(D_n^*)$. For arbitrary edge metric generator S_E of D_n^*, S_E contains at least $\lceil \frac{n}{2} \rceil$ vertices of W.*

Proof Suppose that S_E contains at most $\lceil \frac{n}{2} \rceil - 1$ vertices of W for a contradiction. Without loss of generality, we assume vertices $c_i, c_{i+1} \notin S_E$. Then we have $r(b_i c_i | S_E) = r(b_i c_{i+1} | S_E)$, a contradiction. □

Remark 3 Let S_E be any edge metric basis for D_n^*. We note that S_E contains all *odd vertices* (vertex indices are odd) of W for odd n, while S_E contains either all *odd vertices* or *even vertices* (vertex indices are even) of W for even n.

In the next lemma, we give a lower bound for the edge metric dimension of D_n^*.

Lemma 8 *For $n \ge 5$, we have $\mu_E(D_n^*) \ge \lceil \frac{n}{2} \rceil + 1$.*

Proof We assume for a contradiction that the cardinality of subset S_E is equal to $\lceil \frac{n}{2} \rceil$ by Lemma 7. Using Remark 3, we take $S_E = \{c_i \in W |$ vertex indices i is odd $\}$ such that $|S_E| = \lceil \frac{n}{2} \rceil$. For even $i \in [n]$, we have $r(a_i b_i | S_E) = r(b_i c_i | S_E)$ and $r(a_{i-1} b_{i-1} | S_E) = r(b_{i-1} c_i | S_E)$, a contradiction. □

Theorem 3 *For the prism related graph D_n^* with $n \ge 3$, we have*

$$\mu_E(D_n^*) = \begin{cases} 4, & n = 3,\ 4, \\ \lceil \frac{n}{2} \rceil + 1, & otherwise. \end{cases}$$

Proof For $n = 3$ or $n = 4$, we have found the edge metric dimension by total enumeration and given edge metric bases of D_n^* in Table 1.

For $n \ge 5$, we consider the following four cases.

Let $S_E = \{a_1, c_1, c_3, c_5, \dots, c_{2l-1}\}$. We will show that S_E is an edge metric basis of D_n^* in Case (I) and (II), respectively.

Case (I) $n \equiv 0\ (mod\ 4)$.

Table 1 Edge metric bases of D_n^*

n	Basis	$\mu_E(D_n^*)$
3	$\{a_1, a_2, c_1, c_2\}$	4
4	$\{a_1, c_1, c_2, c_3\}$	4

In this case, we can write $n = 2l$, where $l \in \mathbb{Z}$. Let $S_1 = \{a_1, c_1, c_3, c_{l+1}, c_{2l-1}\}$. Next, we give representations of any edge of $E(D_n^*)$ with respect to S_1. They are

$$r(a_ia_{i+1}|S_1) = \begin{cases} (i-1, i+1, 2, l-i+1, i+3), & 1 \le i \le 2, \\ (i-1, i+1, i-1, l-i+1, i+3), & 3 \le i \le l-2, \\ (i-1, i+1, i-1, 2, 2l-i-1), & l-1 \le i \le l, \\ (2l-i, 2l-i+1, i-l, i-l+1, 2l-i-1), & l+1 \le i \le l+2, \\ (2l-i, 2l-i+1, 2l-i+3, i-l+1, 2l-i-1), & l+3 \le i \le 2l-3, \\ (2l-i, 2l-i+1, 2l-i+3, i-l+1, 2), & 2l-2 \le i \le 2l-1, \\ (0, 2, 3, l+1, 3), & i = 2l. \end{cases}$$

$$r(a_ib_i|S_1) = \begin{cases} (i-1, i, 3-i, l-i+1, i+2), & 1 \le i \le 2, \\ (i-1, i, i-2, l-i+1, i+2), & 3 \le i \le l-2, \\ (i-1, i, i-2, l-i+1, 2l-i-1), & l-1 \le i \le l, \\ (2l-i+1, 2l-i+1, i-2, i-l, 2l-i-1), & l+1 \le i \le l+2, \\ (2l-i+1, 2l-i+1, 2l-i+3, i-l, 2l-i-1), & l+3 \le i \le 2l-2, \\ (2l-i+1, 2l-i+1, 2l-i+3, i-l, i-2l+2), & 2l-1 \le i \le 2l. \end{cases}$$

$$r(b_ib_{i+1}|S_1) = \begin{cases} (i, i, 1, l-i, i+2), & 1 \le i \le 2, \\ (i, i, i-2, l-i, i+2), & 3 \le i \le l-2, \\ (i, i, i-2, 1, 2l-i-2), & l-1 \le i \le l, \\ (2l-i+1, 2l-i, i-2, i-l, 2l-i-2), & l+1 \le i \le l+2, \\ (2l-i+1, 2l-i, 2l-i+2, i-l, 2l-i-2), & l+3 \le i \le 2l-3, \\ (2l-i+1, 2l-i, 2l-i+2, i-l, 1), & 2l-2 \le i \le 2l-1, \\ (1, 1, 2, l, 2), & i = 2l. \end{cases}$$

$$r(b_ic_i|S_1) = \begin{cases} (1, 0, 2, l, 3), & i = 1, \\ (i, i, 3-i, l-i+1, i+2), & 2 \le i \le 3, \\ (i, i, i-2, l-i+1, i+2), & 4 \le i \le l-2, \\ (i, i, i-2, l-i+1, 2l-i-1), & l-1 \le i \le l, \\ (2l-i+2, 2l-i+1, i-2, 2i-2l-2, 2l-i-1), & l+1 \le i \le l+2, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l, 2l-i-1), & l+3 \le i \le 2l-1, \\ (2, 1, 3, l, 2), & i = 2l. \end{cases}$$

$$r(b_ic_{i+1}|S_1) = \begin{cases} (1, 1, 2, l, 3), & i = 1, \\ (i, i, i-2, l-i+1, i+2), & 2 \le i \le l-2, \\ (l-1, l-1, l-3, 2, l), & i = l-1, \\ (i, l, i-2, i-l, 2l-i-1), & l \le i \le l+1, \\ (l, l-1, i-2, 2, l-3), & i = l+2, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l, 2l-i-1), & l+3 \le i \le 2l-3, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l, i-2l+2), & 2l-2 \le i \le 2l-1, \\ (2, 0, 3, l, 2), & i = 2l. \end{cases}$$

When $1 \le i \le n$ and $i \ne 1, 2, 3, l, l+1, 2l-2, 2l-1, 2l$, we have $r(b_ic_i|S_1) = r(b_ic_{i+1}|S_1)$. In other cases, there are no two edges having same edge metric representation. For odd i, where $1 \le i \le n$ and $i \ne 1, 3, l+1, 2l-1$, we have

$r(b_{i-1}c_{i-1}|S_1\cup\{c_i\}) \neq r(b_{i-1}c_i|S_1\cup\{c_i\})$ and $r(b_ic_i|S_1\cup\{c_i\}) \neq r(b_ic_{i+1}|S_1\cup\{c_i\})$. It follows that $r(b_ic_i|S_E) \neq r(b_ic_{i+1}|S_E)$ for $1 \leq i \leq n$. Combining the above and Lemma 8, we obtain that S_E is an edge metric generator for D_n^*, which implies that $\mu_E(D_n^*) = \lceil \frac{n}{2} \rceil + 1$.

Case (II) $n \equiv 2$ $(mod\ 4)$.

In this case, we can write $n = 2l$, where $l \in \mathbb{Z}$. Let $S_1 = \{a_1, c_1, c_3, c_{l+2}\}$. Next, we give representations of any edge of $E(D_n^*)$ with respect to S_1. They are

$$r(a_ia_{i+1}|S_1) = \begin{cases} (i-1, i+1, 2, l-i+2), & 1 \leq i \leq 2, \\ (i-1, i+1, i-1, l-i+2), & 3 \leq i \leq l, \\ (l-1, l, l, 2), & i = l+1, \\ (2l-i, 2l-i+1, 2l-i+3, i-l), & l+2 \leq i \leq 2l-1, \\ (0, 2, 3, l), & i = 2l. \end{cases}$$

$$r(a_ib_i|S_1) = \begin{cases} (i-1, i, 3-i, l), & 1 \leq i \leq 2, \\ (i-1, i, i-2, l-i+2), & 3 \leq i \leq l, \\ (2l-i+1, 2l-i+1, i-2, 1), & l+1 \leq i \leq l+2, \\ (2l-i+1, 2l-i+1, 2l-i+3, i-l-1), & l+3 \leq i \leq 2l. \end{cases}$$

$$r(b_ib_{i+1}|S_1) = \begin{cases} (i, i, 1, l-i+1), & 1 \leq i \leq 2, \\ (i, i, i-2, l-i+1), & 3 \leq i \leq l, \\ (l, l-1, l-1, 1), & i = l+1, \\ (2l-i+1, 2l-i, 2l-i+2, i-l-1), & l+2 \leq i \leq 2l-1, \\ (1, 1, 2, l-1), & i = 2l. \end{cases}$$

$$r(b_ic_i|S_1) = \begin{cases} (1, 0, 2, l), & i = 1, \\ (i, i, 3-i, l-i+2), & 2 \leq i \leq 3, \\ (i, i, i-2, l-i+2), & 4 \leq i \leq l, \\ (2l-i+2, 2l-i+1, i-2, l-i+2), & l+1 \leq i \leq l+2, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l-1), & l+3 \leq i \leq 2l. \end{cases}$$

$$r(b_ic_{i+1}|S_1) = \begin{cases} (1, 1, 2, l), & i = 1, \\ (i, i, i-2, l-i+2), & 2 \leq i \leq l, \\ (2l-i+2, 2l-i+1, i-2, i-l-1), & l+1 \leq i \leq l+2, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l-1), & l+3 \leq i \leq 2l-1, \\ (2, 0, 3, l-1), & i = 2l. \end{cases}$$

When $1 \leq i \leq n$ and $i \neq 1, 2, 3, l+1, l+2, 2l$, we have $r(b_ic_i|S_1) = r(b_ic_{i+1}|S_1)$. In other cases, there are no two edges having same edge metric representation. For odd i, where $1 \leq i \leq n$ and $i \neq 1, 3, l+2$, we have $r(b_{i-1}c_{i-1}|S_1 \cup \{c_i\}) \neq r(b_{i-1}c_i|S_1 \cup \{c_i\})$ and $r(b_ic_i|S_1 \cup \{c_i\}) \neq r(b_ic_{i+1}|S_1 \cup \{c_i\})$. It follows that $r(b_ic_i|S_E) \neq r(b_ic_{i+1}|S_E)$ for $1 \leq i \leq n$. Combining the above and Lemma 8, we see that S_E is an edge metric generator for D_n^*, which implies that $\mu_E(D_n^*) = \lceil \frac{n}{2} \rceil + 1$.

Let $S_E = \{a_1, c_1, c_3, c_5, \ldots, c_{2l-1}, c_{2l+1}\}$. We will show that S_E is an edge metric basis of D_n^* in Case (III) and (IV), respectively.

Case (III) $n \equiv 1$ $(mod\ 4)$.

In this case, we can write $n = 2l + 1$, where $l \in \mathbb{Z}$. Let $S_1 = \{a_1, c_1, c_3, c_{l+3}\}$. Next, we give representations of any edge of $E(D_n^*)$ with respect to S_1. They are

$$r(a_i a_{i+1}|S_1) = \begin{cases} (i-1, i+1, 2, l+1), & 1 \le i \le 2, \\ (i-1, i+1, i-1, l-i+3), & 3 \le i \le l, \\ (2l-i+1, 2l-i+2, i-l, 2), & l+1 \le i \le l+2, \\ (2l-i+1, 2l-i+2, 2l-i+4, i-l-1), & l+3 \le i \le 2l, \\ (0, 2, 3, l), & i = 2l+1. \end{cases}$$

$$r(a_i b_i|S_1) = \begin{cases} (i-1, i, 3-i, l+i-1), & 1 \le i \le 2, \\ (i-1, i, i-2, l-i+3), & 3 \le i \le l+1, \\ (l, l, l, 1), & i = l+2, \\ (2l-i+2, 2l-i+2, 2l-i+4, i-l-2), & l+3 \le i \le 2l+1. \end{cases}$$

$$r(b_i b_{i+1}|S_1) = \begin{cases} (i, i, 1, l), & 1 \le i \le 2, \\ (i, i, i-2, l-i+2), & 3 \le i \le l, \\ (2l-i+2, 2l-i+1, i-2, 1), & l+1 \le i \le l+2, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l-2), & l+3 \le i \le 2l, \\ (1, 1, 2, l-1), & i = 2l+1. \end{cases}$$

$$r(b_i c_i|S_1) = \begin{cases} (1, 0, 2, l), & i = 1, \\ (i, i, 3-i, l-i+3), & 2 \le i \le 3, \\ (i, i, i-2, l-i+3), & 4 \le i \le l+1, \\ (2l-i+3, 2l-i+2, i-2, l-i+3), & l+2 \le i \le l+3, \\ (2l-i+3, 2l-i+2, 2l-i+4, i-l-2), & l+4 \le i \le 2l+1. \end{cases}$$

$$r(b_i c_{i+1}|S_1) = \begin{cases} (1, 1, 2, l), & i = 1, \\ (i, i, i-2, l-i+3), & 2 \le i \le l+1, \\ (2l-i+2, 2l-i+1, i-2, i-l-2), & l+2 \le i \le l+3, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l-2), & l+4 \le i \le 2l, \\ (2, 0, 3, l-1), & i = 2l+1. \end{cases}$$

Again we see that when $1 \le i \le n$ and $i \ne 1, 2, 3, l+2, l+3, 2l+1$, we have $r(b_i c_i|S_1) = r(b_i c_{i+1}|S_1)$. In other cases, there are no two edges having same edge metric representation. For odd i, where $1 \le i \le n$ and $i \ne 1, 3, l+3$, we have $r(b_{i-1} c_{i-1}|S_1 \cup \{c_i\}) \ne r(b_{i-1} c_i|S_1 \cup \{c_i\})$ and $r(b_i c_i|S_1 \cup \{c_i\}) \ne r(b_i c_{i+1}|S_1 \cup \{c_i\})$. It follows that $r(b_i c_i|S_E) \ne r(b_i c_{i+1}|S_E)$ for $1 \le i \le n$. Combining the above and Lemma 8, we see that S_E is an edge metric generator for D_n^*, which implies that $\mu_E(D_n^*) = \lceil \frac{n}{2} \rceil + 1$.

Case (IV) $n \equiv 3$ $(mod\ 4)$.

In this case, we can write $n = 2l + 1$, where $l \in \mathbb{Z}$. Let $S_1 = \{a_1, c_1, c_3, c_{l+2}\}$. Next, we give representations of any edge of $E(D_n^*)$ with respect to S_1. They are

$$r(a_i a_{i+1}|S_1) = \begin{cases} (i-1, i+1, 2, l-i+2), & 1 \le i \le 2, \\ (i-1, i+1, i-1, l-i+2), & 3 \le i \le l, \\ (2l-i+1, 2l-i+2, i-1, 2), & l+1 \le i \le l+2, \\ (2l-i+1, 2l-i+2, 2l-i+4, i-l), & l+3 \le i \le 2l, \\ (0, 2, 3, l+1), & i = 2l+1. \end{cases}$$

$$r(a_ib_i|S_1) = \begin{cases} (i-1, i, 3-i, l-i+2), & 1 \le i \le 2, \\ (i-1, i, i-2, l-i+2), & 3 \le i \le l+1, \\ (l, l, l, 1), & i = l+2, \\ (2l-i+2, 2l-i+2, 2l-i+4, i-l-1), & l+3 \le i \le 2l+1. \end{cases}$$

$$r(b_ib_{i+1}|S_1) = \begin{cases} (i, i, 1, l-i+1), & 1 \le i \le 2, \\ (i, i, i-2, l-i+1), & 3 \le i \le l, \\ (2l-i+2, 2l-i+1, i-2, 1), & l+1 \le i \le l+2, \\ (2l-i+2, 2l-i+1, 2l-i+3, i-l-1), & l+3 \le i \le 2l, \\ (1, 1, 2, l), & i = 2l+1. \end{cases}$$

$$r(b_ic_i|S_1) = \begin{cases} (1, 0, 2, l+1), & i = 1, \\ (i, i, 3-i, l-i+2), & 2 \le i \le 3, \\ (i, i, i-2, l-i+2), & 4 \le i \le l+1, \\ (l+1, l, l, 0), & i = l+2, \\ (2l-i+3, 2l-i+2, 2l-i+4, i-l-1), & l+3 \le i \le 2l+1. \end{cases}$$

$$r(b_ic_{i+1}|S_1) = \begin{cases} (1, 1, 2, l+1), & i = 1, \\ (i, i, i-2, l-i+2), & 2 \le i \le l, \\ (l+1, 2l-i+2, i-2, i-l-1), & l+1 \le i \le l+2, \\ (2l-i+3, 2l-i+2, 2l-i+4, i-l-1), & l+3 \le i \le 2l, \\ (2, 0, 3, l), & i = 2l+1. \end{cases}$$

When $1 \le i \le n$ and $i \ne 1, 2, 3, l+1, l+2, 2l+1$, we have $r(b_ic_i|S_1) = r(b_ic_{i+1}|S_1)$. In other cases, there are no two edges having same edge metric representation. For odd i, where $1 \le i \le n$ and $i \ne 1, 3, l+2$, we have $r(b_{i-1}c_{i-1}|S_1 \cup \{c_i\}) \ne r(b_{i-1}c_i|S_1 \cup \{c_i\})$ and $r(b_ic_i|S_1 \cup \{c_i\}) \ne r(b_ic_{i+1}|S_1 \cup \{c_i\})$. It follows that $r(b_ic_i|S_E) \ne r(b_ic_{i+1}|S_E)$ for $1 \le i \le n$. Combining the above and Lemma 8, we see that S_E is an edge metric generator for D_n^*, which implies that $\mu_E(D_n^*) = \lceil \frac{n}{2} \rceil + 1$. □

Remark 4 For $n \ge 6$, we obtain $\mu(D_n^*) < \mu_E(D_n^*)$ by Lemma 3.

5 The graph of convex polytope $\mathbb{D}_n$

The graph of convex polytope $\mathbb{D}_n$ (Fig. 4) defined in Bača (1988) is the trivalent plane graph which consists of $2n$ 5-sided faces and a pair of n-sided faces, respectively. We have the vertex set $V(\mathbb{D}_n) = \{a_i, b_i, c_i, d_i | 1 \le i \le n\}$, and the edge set $E(\mathbb{D}_n) = \{a_ia_{i+1}, a_ic_i, c_id_i, c_{i+1}d_i, b_id_i, b_ib_{i+1} | 1 \le i \le n\}$.

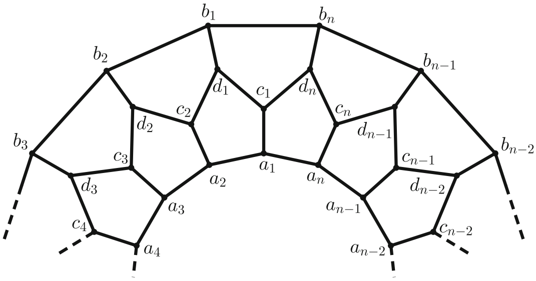
It has been proved in Imran et al. (2012) that the metric dimension of the convex polytope $\mathbb{D}_n$ is constant. In the following, we will prove that the edge metric dimension of $\mathbb{D}_n$ is the same as the metric dimension of $\mathbb{D}_n$.

Theorem 4 *Let the graph of convex polytope $\mathbb{D}_n$ be defined above. Then we have $\mu_E(\mathbb{D}_n) = 3$*

Proof We consider the following two cases.

Case (I) n is even.

Fig. 4 The graph of convex polytope $\mathbb{D}_n$



In this case, we set $n = 2l$, where $l \in \mathbb{Z}$. Let $S_E = \{a_1, b_1, c_l\}$. To show that S_E is an edge metric generator for $\mathbb{D}_n$, we give representations of any edge of $E(\mathbb{D}_n)$ with respect to S_E. They are

$$r(a_ia_{i+1}|S_E) = \begin{cases} (0,3,l-1), & i=1,\\ (i-1,i+1,l-i), & 2\le i\le l-1,\\ (l-1,l+1,1), & i=l,\\ (2l-i,2l-i+3,i-l+1), & l+1\le i\le 2l-1,\\ (0,3,l), & i=2l.\end{cases}$$

$$r(a_ic_i|S_E) = \begin{cases} (0,2,l), & i=1,\\ (i-1,i,l-i+1), & 2\le i\le l-1,\\ (l-1,l,0), & i=l,\\ (l,l+1,2), & i=l+1,\\ (2l-i+1,2l-i+3,i-l+1), & l+2\le i\le 2l.\end{cases}$$

$$r(c_id_i|S_E) = \begin{cases} (i,i,l-i+2), & 1\le i\le l-3,\\ (l-2,l-2,3), & i=l-2,\\ (l-1,l-1,1), & i=l-1,\\ (l,l,0), & i=l,\\ (l+1,l+1,2), & i=l+1,\\ (2l-i+2,2l-i+2,i-l+2), & l+2\le i\le 2l.\end{cases}$$

$$r(d_ic_{i+1}|S_E) = \begin{cases} (i+1,i,l-i+1), & 1\le i\le l-3,\\ (l-1,l-2,2), & i=l-2,\\ (l,l-1,0), & i=l-1,\\ (l+1,l,1), & i=l,\\ (l,l+1,3), & i=l+1,\\ (2l-i+1,2l-i+2,i-l+3), & l+2\le i\le 2l-1,\\ (l,2,l+1), & i=2l.\end{cases}$$

$$r(b_id_i|S_E) = \begin{cases} (i+1,i-1,l-i+1), & 1\le i\le l-2,\\ (l,l-2,1), & i=l-1,\\ (l+1,l-1,1), & i=l,\\ (2l-i+2,2l-i+1,i-l+2), & l+1\le i\le 2l-1\\ (2,1,l+1), & i=2l.\end{cases}$$

$$r(b_ib_{i+1}|S_E) = \begin{cases} (i+2, i-1, l-i), & 1 \le i \le l-2, \\ (l+1, l-2, 2), & i = l-1, \\ (l+1, l-1, 2), & i = l, \\ (2l-i+2, 2l-i, i-l+2), & l+1 \le i \le 2l-1 \\ (3, 0, l), & i = 2l. \end{cases}$$

Note that there are no two edges having same edge metric representation, which implies that $\mu_E(\mathbb{D}_n) \le 3$. It follows from Proposition 2 that $\mu_E(\mathbb{D}_n) = 3$ in this case.

Case (II) n is odd.

In this case, we set $n = 2l + 1$, where $l \in \mathbb{Z}$. Let $S_E = \{a_1, b_1, c_{l+1}\}$. To show that S_E is an edge metric generator for $\mathbb{D}_n$, we give representations of any edge of $E(\mathbb{D}_n)$ with respect to S_E. They are

$$r(a_ia_{i+1}|S_E) = \begin{cases} (0, 3, l), & i = 1, \\ (i-1, i+1, l-i+1), & 2 \le i \le l, \\ (l, l+2, 1), & i = l+1, \\ (2l-i+1, 2l-i+4, i-l), & l+2 \le i \le 2l+1. \end{cases}$$

$$r(a_ic_i|S_E) = \begin{cases} (0, 2, l+1), & i = 1, \\ (i-1, i, l-i+2), & 2 \le i \le l, \\ (l, l+1, 0), & i = l+1, \\ (2l-i+2, 2l-i+4, i-l), & l+2 \le i \le 2l+1. \end{cases}$$

$$r(c_id_i|S_E) = \begin{cases} (i, i, l-i+3), & 1 \le i \le l-2, \\ (l-1, l-1, 3), & i = l-1, \\ (l, l, 1), & i = l, \\ (l+1, l+1, 0), & i = l+1, \\ (l+1, l+1, 2), & i = l+2, \\ (2l-i+3, 2l-i+3, i-l+1), & l+3 \le i \le 2l+1. \end{cases}$$

$$r(d_ic_{i+1}|S_E) = \begin{cases} (i+1, i, l-i+2), & 1 \le i \le l-2, \\ (l, l-1, 2), & i = l-1, \\ (l+1, l, 0), & i = l, \\ (l+1, l+1, 1), & i = l+1, \\ (l, l+1, 3), & i = l+2, \\ (2l-i+2, 2l-i+3, i-l+2), & l+3 \le i \le 2l, \\ (l, 2, l+2), & i = 2l+1. \end{cases}$$

$$r(b_id_i|S_E) = \begin{cases} (i+1, i-1, l-i+2), & 1 \le i \le l-1, \\ (l+1, l-1, 1), & i = l, \\ (l+2, l, 1), & i = l+1, \\ (2l-i+3, 2l-i+2, i-l+1), & l+2 \le i \le 2l+1. \end{cases}$$

$$r(b_ib_{i+1}|S_E) = \begin{cases} (i+2, i-1, l-i+1), & 1 \le i \le l-1, \\ (l+2, l-1, 2), & i = l, \\ (2l-i+3, 2l-i+1, i-l+1), & l+1 \le i \le 2l \\ (3, 0, l+1), & i = 2l+1. \end{cases}$$

Again we see that there are no two edges having same edge metric representation, which implies that $\mu_E(\mathbb{D}_n) \leq 3$. It follows from Proposition 2 that $\mu_E(\mathbb{D}_n) = 3$ in this case. □

Remark 5 For the graph of convex polytope $\mathbb{D}_n$ with $n \geq 3$, then we have $\mu(\mathbb{D}_n) = \mu_E(\mathbb{D}_n)$ by Lemma 4.

6 Conclusion

In this paper, we have determined the exact value of the edge metric dimension of convex polytopes antiprism A_n, the web graph $\mathbb{W}_n$, the prism related graph D_n^* and convex polytope $\mathbb{D}_n$. We conclude that the edge metric dimension of web graph $\mathbb{W}_n$ and convex polytope $\mathbb{D}_n$ are constant, and antiprism A_n has bounded edge metric dimension while the prism related graph D_n^* has unbounded edge metric dimension.

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