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Equitable vertex arboricity of 5-degenerate graphs

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Abstract Wu et al. (Discret Math 313:2696–2701, 2013) conjectured that the vertex set of any simple graph G can be equitably partitioned into m subsets so that each subset induces a forest, where $\Delta(G)$ is the maximum degree of G and m is an integer with $m \geq \lceil \frac{\Delta(G)+1}{2} \rceil$. This conjecture is verified for 5-degenerate graphs in this paper.

Keywords Graph · Equitable coloring · Vertex arboricity · Equitable vertex arboricity · 5-Degenerate graph

1 Introduction

All graphs considered in this article are simple. Let G be a graph. We use $V(G)$, $E(G)$, $|G|$, $e(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, order, size, maximum degree and minimum degree of G, respectively. Denote by $G\backslash\{xy\}$ the graph obtained from G by deleting the edge xy but keeping two end vertices. For any vertex $v \in V(G)$, let $N_G(v)$ be the set of all neighbors of v in G. The degree of v, denoted by $d_G(v)$, is equal to $|N_G(v)|$. We use $d(v)$ instead of $d_G(v)$ if no confusion arises. For two disjoint subsets U and W of $V(G)$, the set of edges with one end in U and the other end in W is denoted by $E(U, W)$ and $e(U, W) = |E(U, W)|$.

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In particular, $e(v, W) = d_W(v)$ denotes the number of neighbors of v in W. Define $N_W(u), u \in U$ as the set of neighbors of u in W and $N_W(U) = \bigcup_{u\in U} N_W(u)$. We denote by $G[U]$ the subgraph of G induced by U.

We associate positive integers $1, 2, \ldots, k$ with colors and call f a *k-coloring* of G if f is a mapping from $V(G)$ to $\{1, 2, \ldots, k\}$. A k-coloring f of G is *equitable* if every color class has size $\lfloor \frac{|G|}{k} \rfloor$ or $\lceil \frac{|G|}{k} \rceil$. A coloring f is said to be *proper* if every two adjacent vertices receive different colors. The *equitable chromatic number* of a graph G, denoted by $\chi_=(G)$, is the minimum integer k such that G has a proper equitable k-coloring. Note that a graph G which is equitably k-colorable may not be equitably k'-colorable for $k' > k$. The smallest k such that G has proper equitable colorings for any number of colors greater than or equal to k is called the *equitable chromatic threshold* of G, and is denoted by $\chi_\equiv(G)$. Erdős (1964) conjectured that any graph with maximum degree $\Delta(G) \leq r$ has a proper equitable $(r + 1)$-coloring. The conjecture was proved by Hajnal and Szemerédi (1970). Recently, Kierstead and Kostochka (2008) gave a simpler proof of this theorem. There are two well-known conjectures regarding proper equitable colorings.

Conjecture 1 (Meyer 1973) *For any connected graph G, $\chi_=(G) \leq \Delta(G)$, with the exception that G is a complete graph or an odd cycle.*

Conjecture 2 (Chen et al. 1994) *For any connected graph G, $\chi_\equiv(G) \leq \Delta(G)$, with the exception that G is a complete graph, an odd cycle or a complete bipartite graph $K_{2m+1,2m+1}$.*

Note that Conjecture 2 is stronger than Conjecture 1 and has been verified for many classes of graphs, such as graphs with $\Delta(G) \leq 4$, see Chen et al. (1994), Chen and Yen (2012) and Kierstead and Kostochka (2012) or $\Delta(G) \geq \frac{|G|}{3} + 1$, see Chen et al. (1994), Chen and Yen (2012) and Yap and Zhang (1997), bipartite graphs (Lih and Wu 1996), outerplanar graphs (Yap and Zhang 1997), series-parallel graphs (Zhang and Wu 2011) and planar graphs with $\Delta(G) \geq 9$, see Zhang and Yap (1998) and Nakprasit (2012).

Here we consider a relaxed version of equitable coloring which is equitable tree-coloring. A k-coloring of a graph G is said to be a *k-tree coloring* if the induced subgraph $G[V_i]$ is a forest, where V_i denotes the set of vertices colored by i for each $i = 1, 2, \ldots, k$. The *vertex arboricity*, or *point arboricity* $va(G)$ of a graph G is the minimum integer k such that G admits a k-tree coloring. It is proved that $va(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph (Kronk and Mitchem 1975) and $va(G) \leq 3$ for every planar graph (Chartrand and Kronk 1969). The minimum integer k such that G has an equitable k-tree coloring is the *equitable vertex arboricity* of G, and is denoted by $va_{eq}(G)$. The *strong equitable vertex arboricity* of G, denoted by $va^*_{eq}(G)$, is the smallest m such that G has an equitable m'-tree coloring for every $m' \geq m$. It is easy to see that $va^*_{eq}(G) \geq va_{eq}(G)$. In Wu et al. (2013), Wu, Zhang and Li posed the following conjectures.

Conjecture 3 (Wu et al. 2013) $va^*_{eq}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ *for every graph G.*

Conjecture 4 (Wu et al. 2013) *There is a constant ζ such that $va^*_{eq}(G) \leq \zeta$ for every planar graph G.*

Conjecture 3 has been verified for graphs with $\Delta(G) \geq \frac{|V(G)|}{2}$ in Zhang and Wu (2014). In Wu et al. (2013), Wu, Zhang and Li proved that $va^*_{eq}(G) \leq 3$ for every planar graph with $g(G) \geq 5$ and $va^*_{eq}(G) \leq 2$ for every outerplanar graph and planar graph with $g(G) \geq 6$, where $g(G)$ denotes the girth of G. Zhang in 2015 proved that $va^*_{eq}(G) \leq 3$ for any planar graph such that all cycles of length at most 4 are independent and planar graph without 3-cycles and adjacent 4-cycles. Recently, Esperet, Lemoine and Maffray proved that $va^*_{eq}(G) \leq 4$ for every planar graph G (Esperet et al. 2015), which confirms Conjecture 4.

A *d-degenerate graph* is a graph G in which every subgraph has a vertex with degree at most d. In this paper, we confirm Conjecture 3 for 5-degenerate graphs. Clearly, since every planar graph has a vertex with degree at most 5, every planar graph is 5-degenerate. So the family of 5-degenerate graphs is a natural extension of the family of planar graphs. One strategy to tackle Conjecture 3 is to verify it for special classes of graphs. The family of planar graphs has been one of the candidates.

Theorem 5 *Let G be a 5-degenerate graph. Then $va^*_{eq}(G) \leq \lceil\frac{\Delta(G)+1}{2}\rceil$.*

2 Proof of Theorem 5

Proof □

Claim 1 *Let $m \geq 1$ be a fixed integer and $\mathbb{G}$ be the class of 5-degenerate graphs. Suppose that any graph G in $\mathbb{G}$ of order mt is equitably m-tree colorable for any integer $t \geq 1$. Then all graphs G in $\mathbb{G}$ are also equitably m-tree colorable.*

Proof We prove this Claim by induction on the order n of G. By the assumption, we may assume that $mt < n < m(t+1)$. If $m \leq 2$, then $n = m(t+1) - 1$. Now we consider the case that $m \geq 3$. Let $u \in V(G), d(u) = \delta(G) \leq 5$. By the induction hypothesis, $G - u$ has an equitable m-tree coloring ϕ. Let the color classes of ϕ be $V_1, V_2, \ldots, V_m$, where $|V_i| = t$ or $t+1$ for all $i \geq 1$. Since $d(u) \leq 5$, we assume each of $3, \ldots, m$ appears at most once in the colors of $N(u)$. If $|V_i| = t$ for some $i \geq 3$, then by adding u to V_i, we get an equitable m-tree coloring of G (having color classes $V_1, \ldots, V_{i-1}, V_i \bigcup \{u\}, V_{i+1}, \ldots, V_m$). Hence we can assume that $|V_i| = t+1$ for all $i \geq 3$ and we also have $n = m(t+1) - 1$. Let $G' = G \bigcup K_1$. Then G' is a graph of order $m(t+1)$ and $\delta(G') \leq 5$, thus by the assumption, G' is equitably m-tree colorable (and so is G). □

By Claim 1, we only need to show that $|G| = mt$, where $m \geq \lceil\frac{\Delta(G)+1}{2}\rceil$ and t are positive integers. We prove this theorem by induction on m. It is true for $m = 1$ trivially. Now we consider the case when $m \geq 2$. Let G be an edge-minimal 5-degenerate graph that is not equitably m-tree colorable with $|G| = mt$. Let $x \in V(G)$ such that $d(x) = \delta(G) = d \leq 5$. Let $x_1, x_2, \ldots, x_d \in N(x)$. By edge-minimality of G, the graph $G \backslash \{xx_1\}$ has an equitable m-tree coloring ϕ having color classes $V_1, V_2, \ldots, V_m$. Then there exists a cycle C of G passing through the edge xx_1 such that all vertices on C are colored with the same color, for otherwise ϕ is also an equitable m-tree coloring of G. Without loss of generality, we assume that $x, x_1, x_2 \in V_1$, $|N(x) \bigcap V_2| \leq 2$,

$|N(x) \bigcap V_i| \leq 1 (3 \leq i \leq m)$. Let $V_1' = V_1 \backslash (\{x\} \bigcup \{v \mid v \in V_1, N(v) \subseteq \bigcup_{i=2}^{m} V_i\})$. Then $|V_1'| \leq t-1$. Since $x_1, x_2 \in V_1'$, $V_1' \neq \emptyset$. We say a vertex $v \in A$ is movable to a set B if $G[B \bigcup \{v\}]$ contains no cycle, otherwise v is not movable to B, where $A, B \subseteq V(G)$ and $A \bigcap B = \emptyset$. So if v is not movable to B, then $d_B(v) \geq 2$ and there is a vertex $u \in N_B(v)$ such that $d_B(u) \geq 1$.

We first consider the case when $\delta(G) \leq 3$.

In this case, $|N(x) \bigcap V_i| \leq 1 (2 \leq i \leq m)$. If for each $v \in \bigcup_{i=2}^{m} V_i$, $d_{V_1'}(v) \geq 2$, then $e(\bigcup_{i=2}^{m} V_i, V_1') \geq 2(m-1)t$. However, this contradicts with $e(\bigcup_{i=2}^{m} V_i, V_1') \leq (\Delta - 1)(t-1)$ and $m \geq \lceil \frac{\Delta(G)+1}{2} \rceil$. Therefore, there exists $v \in V_i$ for some $i \in \{2, \ldots, m\}$ such that $d_{V_1'}(v) \leq 1$. Then we get an equitable m-tree coloring of G with color classes $(V_1 \backslash \{x\}) \bigcup \{v\}, V_2, \ldots, V_{i-1}, (V_i \backslash \{v\}) \bigcup \{x\}, V_{i+1}, \ldots, V_m$, a contradiction, too.

In the following, we deal with the case when $4 \leq \delta(G) \leq 5$.

Claim 2 *No vertex v in $\bigcup_{i=3}^{m} V_i$ is movable to V_1'.*

Proof Suppose that there is $v \in V_i (3 \leq i \leq m)$ which is movable to V_1'. Then we get an equitable m-tree coloring of G with color classes $(V_1 \backslash \{x\}) \bigcup \{v\}, V_2, \ldots, V_{i-1}, (V_i \backslash \{v\}) \bigcup \{x\}, V_{i+1}, \ldots, V_m$, a contradiction. □

Claim 3 *There exists a vertex $w \in V_2$ such that w is movable to V_1'.*

Proof By Claim 2, for each $v \in \bigcup_{i=3}^{m} V_i$, $d_{V_1'}(v) \geq 2$. Hence $e(V_1', \bigcup_{i=3}^{m} V_i) \geq 2(m-2)t$. Suppose that for any $w \in V_2$, w is not movable to V_1', then $d_{V_1'}(w) \geq 2$ and $e(V_1', \bigcup_{i=2}^{m} V_i) \geq 2(m-1)t$. This contradicts with $e(V_1', \bigcup_{i=2}^{m} V_i) \leq (\Delta - 1)(t-1)$ and $m \geq \lceil \frac{\Delta(G)+1}{2} \rceil$. □

Claim 4 *No vertex v in $\bigcup_{i=3}^{m} V_i$ is movable to V_2', where $V_2' = V_2 \backslash \{w\}$.*

Proof Suppose that there exists $v \in \bigcup_{i=3}^{m} V_i$, such that v is movable to V_2'. Then we get an equitable m-tree coloring of G with color classes $(V_1 \backslash \{x\}) \bigcup \{w\}, V_2' \bigcup \{v\}, V_3, \ldots, V_{i-1}, (V_i \backslash \{v\}) \bigcup \{x\}, V_{i+1}, \ldots, V_m$, a contradiction. □

Case 1 $m = 3$. Then $\Delta(G) \leq 5$.

We say a vertex $z \in V_2'$ is replaceable by a vertex $y \in V_3$ if $yz \in E(G)$ and y is movable to $V_2' \backslash \{z\}$.

Claim 5 *For every vertex $y \in V_3$, there is a vertex z in V_2' such that z is replaceable by y.*

Proof For any $y \in V_3$, we know that $2 \leq d_{V_1'}(y) \leq 3$ and $2 \leq d_{V_2'}(y) \leq 3$. If $d_{V_2'}(y) = 2$, any neighbor of y in V_2' is replaceable by y. So assume that $d_{V_2'}(y) = 3$ and $N_{V_2'}(y) = \{z_1, z_2, z_3\}$. If there is no path connecting z_i and z_j for some $i, j \in \{1, 2, 3\}, i \neq j$, then $\{z_1, z_2, z_3\} \backslash \{z_i, z_j\}$ is replaceable by y. Therefore, there are paths P_1, P_2, P_3 connecting z_1 and z_2, z_1 and z_3, z_2 and z_3, respectively. Because $G[V_2']$ is a forest, there is a common vertex $z \in V(P_1) \bigcap V(P_2) \bigcap V(P_3)$. Then z is replaceable by y. □

Claim 6 *If $z \in V_2'$ is replaceable by a vertex $y \in V_3$, then z is not movable to V_1'.*

Proof Suppose that $z \in V_2'$ is replaceable by vertex $y \in V_3$ and z is movable to V_1'. Then we get an equitable 3-tree coloring of G with color classes $(V_1 \backslash \{x\}) \bigcup \{z\}, (V_2 \backslash \{z\}) \bigcup \{y\}, (V_3 \backslash \{y\}) \bigcup \{x\}$, a contradiction. □

Since $|V_2'| \leq t - 1, |V_3| = t$, by Claim 5, there exists a vertex $z \in V_2'$ such that z is replaceable by at least two vertices in V_3. By Claim 6, $d_{V_1'}(z) \geq 2$ and $d_{V_2'}(z) \geq 1$. So $d_{V_3(z)} \leq 2$. If z is replaceable by at least three vertices in V_3, then at least two of them, say y_1, y_2 are not adjacent. Note that there are paths connecting z to y_1, y_2, so z has at most neighbor in $V_3 \backslash \{y_1, y_2\}$. Then we get an equitable 3-tree coloring with color classes $(V_1 \backslash \{x\}) \bigcup \{w\}, (V_2 \backslash \{z, w\}) \bigcup \{y_1, y_2\}, (V_3 \backslash \{y_1, y_2\}) \bigcup \{z, x\}$, a contradiction. Therefore, z is replaceable by exactly two vertices in V_3, say y_1, y_2. If $y_1 y_2 \notin E(G)$ or $G[(V_2' \backslash \{z\}) \bigcup \{y_1, y_2\}]$ is a forest, then we obtain an equitable 3-tree coloring of G as previously. So $y_1 y_2 \in E(G)$ and there exist $z_1, z_2 \in V_2'$ such that either (a) $y_1 z_1, y_2 z_2 \in E(G)$ and there is a path P connecting z_1 and z_2 or (b) $z_1 = z_2$. If (a) holds, since z is replaceable by y_1 and y_2, there are paths P_1, P_2 connecting z to z_1, z_2 in V_2', respectively. Since $G[V_2']$ is a forest, there is a vertex $z' \in V(P) \bigcap V(P_1) \bigcap V(P_2)$. If (b) holds, let $z_1 = z_2 = z'$. Since $d_{V_1'}(y_i) \geq 2$ and $y_1 y_2 \in E(G)$, hence $d_{V_2'}(y_i) = 2, i = 1, 2$. So in both cases, z' is replaceable by y_1 and y_2. If z' is replaceable by $y_3 \in V_3 \backslash \{y_1, y_2\}$, then $y_1 y_3 \notin E(G)$, we can get an equitable 3-tree coloring of G as previously, a contradiction. Hence z' is not replaceable by any vertex in $V_3 \backslash \{y_1, y_2\}$. This means that we can assign the vertices in V_2' such that each of them is replaceable by at most one vertex in V_3. However, this contradicts with $|V_2'| < |V_3|$.

Case 2 $m \geq 4$.

Claim 7 *For every vertex $v \in V_i, i \in \{3, \ldots, m\}$, if $d_{V_2'}(v) = 2$, then each vertex $u \in N_{V_2'}(v)$ is not movable to V_1' and $d_{V_2'}(u) \geq 1$.*

Proof Suppose that there exists $u \in N_{V_2'}(v)$ which is movable to V_1'. Then we get an equitable m-tree coloring with color classes $(V_1 \backslash \{x\}) \bigcup \{u\}, (V_2 \backslash \{u\}) \bigcup \{v\}, V_3, \ldots, V_{i-1}, (V_i \backslash \{v\}) \bigcup \{x\}, V_{i+1}, \ldots, V_m$, a contradiction. By Claim 4, v is not movable to V_2', so $d_{V_2'}(u) \geq 1$. □

Define by $V_i' = \{v \in V_i \mid d_{V_2'}(v) = 2\}, i \in \{3, \ldots, m\}$. Let $V_k' = \begin{cases} V_{m-1}', & \text{if } |V_{m-1}'| \geq |V_m'|; \\ V_m', & \text{otherwise.} \end{cases}$ We assume without loss of generality $V_k' = V_m'$.

Claim 8 $|V'_m| \geq 3$. *Furthermore, there exists a vertex* $z \in N_{V'_2}(V'_m)$ *such that* $d_{V'_m}(z) \geq 3$.

Proof Let $V'_{21} = \{v \in V'_2 \mid v \in N_{V'_2}(\bigcup_{i=3}^{m} V_i)$ and v has at least one neighbor in $V'_2\}$ and $V''_{21} = V'_{21} \bigcup \{v \in V'_2 \mid$ there is a path connecting v to some vertex $u \in V'_{21}$ in $V'_2\}$.

Suppose that $|V'_m| \leq 2$. Then by the former arguments and Claim 7

$$\begin{aligned}\Delta(G)|V''_{21}| &\geq \sum_{v \in V''_{21}} d(v)\\ &\geq |V''_{21}| + 2(m-4)t + 2(|V'_{m-1}| + |V'_m|) + 3(t - |V'_{m-1}| + t - |V'_m|)\\ &+ 2\max\{|N_{V'_2}(V'_{m-1})|, |N_{V'_2}(V'_m)|\}\\ &\geq |V''_{21}| + (2m-2)t,\end{aligned}$$

which is a contradiction with $m \geq \lceil \frac{\Delta(G)+1}{2} \rceil$ and $|V''_{21}| < t$.

Suppose that for any vertex $z \in N_{V'_2}(V'_m)$ we have $d_{V'_m}(z) \leq 2$. Since for each vertex $v \in V'_m$, $e(v, V'_2) = 2$, then $|N_{V'_2}(V'_m)| \geq |V'_m|$. Therefore,

$$\begin{aligned}\Delta(G)|V''_{21}| &\geq \sum_{v \in V''_{21}} d(v)\\ &\geq |V''_{21}| + 2(m-4)t + 2(|V'_{m-1}| + |V'_m|) + 3(t - |V'_{m-1}| + t - |V'_m|)\\ &+ 2\max\{|N_{V'_2}(V'_{m-1})|, |N_{V'_2}(V'_m)|\}\\ &\geq |V''_{21}| + 2(m-4)t + 2(|V'_{m-1}| + |V'_m|) + 3(t - |V'_{m-1}| + t - |V'_m|) + 2|V'_m|\\ &\geq |V''_{21}| + (2m-2)t,\end{aligned}$$

which is a contradiction with $m \geq \lceil \frac{\Delta(G)+1}{2} \rceil$ and $|V''_{21}| < t$. □

By Claim 8, there exists $z \in V'_2$ with $d_{V'_m}(z) \geq 3$. Since $G[V'_m]$ is a forest, we assume that $y_1, y_2 \in N_{V'_m}(z)$ and $y_1 y_2 \notin E(G)$. Then $(V_1 \backslash \{x\}) \bigcup \{w\}$, $(V_2 \backslash \{z, w\}) \bigcup \{y_1, y_2\}$ is an equitable 2-tree coloring of the graph $G[(V_1 \backslash \{x\}) \bigcup \{w\} \bigcup (V_2 \backslash \{z, w\}) \bigcup \{y_1, y_2\}]$. Let $G_1 = G \backslash G[(V_1 \backslash \{x\}) \bigcup \{w\} \bigcup (V_2 \backslash \{z, w\}) \bigcup \{y_1, y_2\}]$. Then we can see that $\Delta(G_1) \leq \Delta(G) - 4 = 2(\frac{\Delta(G)+1}{2} - 2) - 1 \leq 2(m-2) - 1$. Then we have $m - 2 \geq \lceil \frac{\Delta(G_1)+1}{2} \rceil$. By induction hypothesis, G_1 has an equitable $(m-2)$-tree coloring. So we can get an equitable m-tree coloring of G, a contradiction.

We complete the proof of the theorem.

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