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The adjacent vertex distinguishing total chromatic numbers of planar graphs with $\Delta = 10$

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Abstract A (proper) total-k-coloring of a graph G is a mapping $\phi : V(G) \cup E(G) \mapsto \{1, 2, \ldots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $C(v)$ denote the set of the color of a vertex v and the colors of all incident edges of v. A total-k-adjacent vertex distinguishing-coloring of G is a total-k-coloring of G such that for each edge $uv \in E(G)$, $C(u) \neq C(v)$. We denote the smallest value k in such a coloring of G by $\chi''_a(G)$. It is known that $\chi''_a(G) \leq \Delta(G) + 3$ for any planar graph with $\Delta(G) \geq 11$. In this paper, we show that if G is a planar graph with $\Delta(G) \geq 10$, then $\chi''_a(G) \leq \Delta(G) + 3$. Our approach is based on Combinatorial Nullstellensatz and the discharging method.

Keywords Adjacent vertex distinguishing total coloring · Planar graph · Maximum degree

1 Introduction

Let G be a simple, undirected graph. Denote the vertex set, edge set, maximum degree and minimum degree of G by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ), respectively. The terminology and notation used but undefined in this paper can be found in Bondy and Murty (1976).

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A (*proper*) *total-k-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. A graph G is *total-k-colorable* if it admits a total-k-coloring. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G is total-k-colorable.

Given a *total-k-coloring* ϕ of G, let $C_\phi(v)$ denote the set of the color of v and the colors of the edges incident with v. If $C_\phi(u)$ is different from $C_\phi(v)$ for each edge uv, then this total-k-coloring is called *adjacent vertex distinguishing*, or *total-k-avd-coloring* for short. The smallest k is called the *adjacent vertex distinguishing total chromatic number*, denoted by $\chi''_a(G)$.

Let $\chi(G)$ and $\chi'(G)$ denote the vertex chromatic number and the edge chromatic number of G respectively. Then we have the following relation:

Proposition 1 *For any graph G, $\chi''_a(G) \le \chi(G) + \chi'(G)$.*

Suppose that G is a planar graph. Then $\chi(G) \le 4$ by the Four-Color Theorem (Appel and Haken 1977; Appel et al. 1977) and $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$ by Vizing (1964). So $\chi''_a(G) \le \Delta(G) + 5$. Particularly, since $\chi'(G) = \Delta(G)$ when $\Delta(G) \ge 7$ by Sanders and Zhao (2001), $\chi''_a(G) \le \Delta(G) + 4$. Zhang et al. proposed the following conjecture in Zhang et al. (2005):

Conjecture 1 (Zhang et al. 2005) *For any graph G with at least two vertices, $\chi''_a(G) \le \Delta(G) + 3$.*

Coker and Johannson (2012) used a probabilistic method to establish an upper bound $\Delta(G) + c$ for $\chi''_a(G)$, where $c > 0$ is a constant. Later, Huang et al. (2012) proved that $\chi''_a(G) \le 2\Delta(G)$ for any graph G with maximum degree $\Delta(G) \ge 3$. Conjecture 1 was confirmed for graphs with maximum degree at most three by Chen (2008) and independently by Wang (2007). Wang and Wang proved that this conjecture holds for outerplanar graphs (Wang and Wang 2010) and K_4-minor free graphs (Wang and Wang 2009). Huang and Wang proved that $\chi''_a(G) \le \Delta(G) + 2$ for planar graphs with maximum degree at least 14 in Wang and Huang (2014), and they also proved the following result:

Theorem 1 (Huang and Wang 2012) *Let G be a planar graph with maximum degree $\Delta(G) \ge 11$. Then $\chi''_a(G) \le \Delta(G) + 3$.*

In this paper, we prove the following result, which improves the bound in Huang and Wang (2012).

Theorem 2 *Let G be a planar graph with maximum degree $\Delta(G) \ge 10$. Then $\chi''_a(G) \le \Delta(G) + 3$.*

Recently the adjacent vertex distinguishing total coloring by sums has been considered. For a total-k-coloring ϕ of G, let $m_\phi(v)$ denote the total sum of colors of the edges incident with v and the color of v. If $m_\phi(u) \ne m_\phi(v)$ for each edge uv, then this total-k-coloring is called a *total-k-neighbor sum distinguishing-coloring*. The smallest number k is called the *neighbor sum distinguishing total chromatic number*. For this coloring, see Cheng et al. (2015), Ding et al. (2014), Dong and Wang (2014), Li et al. (2015), Li et al. (2013), Pilśniak and Woźniak (2015).

2 Notations and preliminaries

For a given planar graph G, a vertex of degree k (at least k, at most k) is called a *k-vertex* (*k^+-vertex*, *k^--vertex*). A face of degree k (at least k, at most k) is called a *k-face* (*k^+-face*, *k^--face*). Denote the set of faces of G by $F(G)$. For $x \in V(G) \cup F(G)$, let $d_G(x)$ denote the degree of x in G. For a vertex $v \in V(G)$, we use $N_k^G(v)$ to denote the set of k-vertices adjacent to v in G, and let $n_k^G(v) = |N_k^G(v)|$. Similarly, we define $n_{k^+}^G(v)$ and $n_{k^-}^G(v)$. If there is no confusion in the context, we usually write $n_k^G(x)$, $n_{k^+}^G(x)$ and $n_{k^-}^G(x)$ as $n_k(x)$, $n_{k^+}(x)$ and $n_{k^-}(x)$ respectively.

Suppose that ϕ is a total-k-avd-coloring of a planar graph G and $v \in V$. Recall $C_\phi(v)$ is the set of the color of v and the colors of the edges incident with v and $m_\phi(v)$ is the total sum of colors in $C_\phi(v)$. Obviously, for two adjacent vertices u and v, if $m_\phi(u) \neq m_\phi(v)$, then $C_\phi(u) \neq C_\phi(v)$. We call two adjacent vertices u and v *conflict* on ϕ if $C_\phi(u) = C_\phi(v)$. Let $D_\phi(v)$ denote the union of $C_\phi(v)$ and the colors of vertices adjacent to v. Now we state the Combinatorial Nullstellensatz.

Lemma 1 (Alon (1999), Combinatorial Nullstellensatz) *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose the degree $deg(P)$ of P equals $\sum_{i=1}^n k_i$, where each k_i is a non-negative integer, and suppose the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in P is non-zero. Then if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$, $i = 1, \ldots, n$, there exist $s_1 \in S_1, \ldots, s_n \in S_n$ so that $P(s_1, \ldots, s_n) \neq 0$.*

3 Proof of the main theorem

From Theorem 1 we know that if G is a planar graph with $\Delta(G) \geq 11$, then $\chi_a''(G) \leq \Delta+3$, so we only need to consider $\Delta(G) = 10$. Let G be a counterexample of Theorem 2 such that $|V(G)| + |E(G)|$ is as small as possible. Obviously, G is connected.

Let e be any edge of G and $H = G - e$. If $\Delta(H) = \Delta(G) = 10$, then by the minimality of G, $\chi_a''(H) \leq 13$. If $\Delta(H) = \Delta(G) - 1 = 9$, then by Proposition 1, $\chi_a''(H) \leq \Delta(H) + 4 = 13$. Therefore, $\chi_a''(H) \leq 13$ for both cases.

Note that if $P(x_1, x_2, \ldots, x_m)$ is a polynomial with $\deg(P) = n$, $k_1, k_2, \ldots, k_m$ are non-negative integers with $\sum\limits_{i=1}^{m} k_i = n$ and $c_P(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m})$ is the coefficient of monomial $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ in P, then $\frac{\partial^n P}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_m^{k_m}} = c_P(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}) \prod_{i=1}^m k_i!$. In the following, we always use MATLAB to calculate the coefficients of specific monomials. The codes are listed in Appendix.

3.1 Unavoidable configurations

Claim 1 *There is no edge $uv \in E(G)$ such that $d(v) \leq 6$ and $d(u) \leq 5$.*

Proof Assume to the contrary that G contains an edge uv such that $d(v) = t \leq 6$ and $d(u) = s \leq 5$, and $s \leq t$. Let $H = G - uv$. Then there exists a total-13-avd-coloring ψ of H by the above analysis. Without loss of generality, we assume that

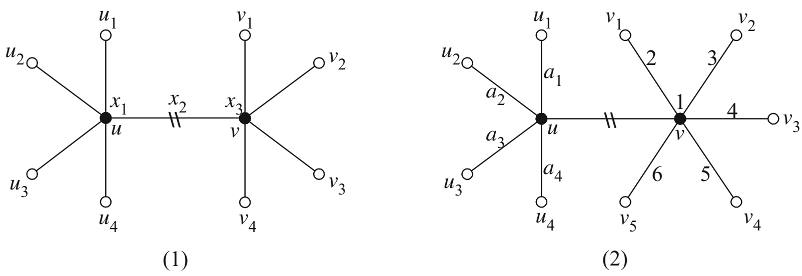

Fig. 1 Configurations in the proof of Claim 1

$C = \{1, 2, \ldots, 13\}$ is the set of all colors used in ψ. Let $u_1, u_2, \ldots, u_{s-1}$ be the neighbors of u other than v, and $v_1, v_2, \ldots, v_{t-1}$ be the neighbors of v other than u.
Case 1 $t \leq 5$. Without loss of generality, we may assume that $s = t = 5$ (We can get an easier proof for other cases). Erase the colors of u, v and denote this partial total-13-avd-coloring by ϕ'. Let $S_1 = C \setminus D_{\phi'}(u)$, $S_2 = C \setminus (C_{\phi'}(u) \cup C_{\phi'}(v))$ and $S_3 = C \setminus D_{\phi'}(v)$. Then $|S_i| \geq 5$ for $i = 1, 2, 3$. Now we extend ϕ' to G. We will color u, uv, v with the colors $s_i \in S_i$, $i = 1, 2, 3$ respectively (see Fig. 1(1)). Let ϕ denote the coloring after u, uv, v are colored. If $s_i - s_j \neq 0$ for $1 \leq i < j \leq 3$, then ϕ is a proper total coloring. If $m_\phi(u) \neq m_\phi(u_i)$, $m_\phi(v) \neq m_\phi(v_i)$ for $i = 1, 2, 3, 4$, and $m_\phi(u) \neq m_\phi(v)$, then ϕ is an adjacent vertex distinguishing coloring. Hence ϕ would be a total-13-avd-coloring if there exist $s_i \in S_i$, $i = 1, 2, 3$ such that $P(s_1, s_2, s_3) \neq 0$, where

$$P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + m_{\phi'}(u) - (x_3 + m_{\phi'}(v))) \prod_{i=1}^{4}(x_1 + x_2 + m_{\phi'}(u) - m_{\phi'}(u_i)) \prod_{i=1}^{4}(x_3 + x_2 + m_{\phi'}(v) - m_{\phi'}(v_i)).$$

By MATLAB, we obtain that $c_P(x_1^4 x_2^4 x_3^4) = c_Q(x_1^4 x_2^4 x_3^4) = 20 \neq 0$, where $Q(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)^2(x_2 - x_3)(x_1 + x_2)^4(x_2 + x_3)^4$. According to Lemma 1, since $\deg(P) = 12$ and $|S_i| \geq 5$, $i = 1, 2, 3$, there exist $s_i \in S_i$, $i = 1, 2, 3$ such that $P(s_1, s_2, s_3) \neq 0$. Coloring u, uv, v with s_1, s_2, s_3 respectively and then we obtain a total-13-avd-coloring of G, which is a contradiction.
Case 2 $t = 6$. Without loss of generality, we may assume that $s = 5$ and $t = 6$ (We can get an easier proof for other cases). Erase the color of u. We conclude that $d(u_i) \neq d(u)$ for any $i \in \{1, 2, 3, 4\}$ by Case 1. Suppose that $\psi(v) = 1$, $\psi(vv_i) = i + 1$ for $i \in \{1, 2, \ldots, 5\}$, and $\psi(uu_j) = a_j$ for $j \in \{1, 2, 3, 4\}$. Without loss of generality, assume $\{a_1, a_2, a_3, a_4\} \subseteq \{1, 2, \ldots, 10\}$ (see Fig. 1(2)). If there exists a color $x \in \{11, 12, 13\}$ such that coloring uv with x cannot result in conflicting vertices, then we color uv with the color x. Otherwise, without loss of generality, we can assume the conflicting vertices are v_1, v_2, v_3 respectively, which means that $C_\psi(v_i) = \{1, 2, \ldots, 6, i + 10\}$ for $i = 1, 2, 3$. Recolor v with a color $a \in \{7, 8, 9, 10\} \setminus \{\psi(v_4), \psi(v_5)\}$. Since now the possible conflicting vertices of v are v_4 and v_5, we can choose a color in $\{11, 12, 13\}$ to color uv such that v does not conflict with v_4 and v_5. Finally, we color u. Since

$d(u_i) \neq d(u), i = 1, 2, 3, 4$, u have at most 10 forbidden colors. Thus we can color u safely and then we obtain a total-13-avd-coloring of G, which is a contradiction. □

Suppose ϕ is a partial total-13-avd-coloring of G. We call ϕ to be a *nice* total-13-avd-coloring of G if only some 5^--vertices are not colored. Observe that every nice total-13-avd-coloring can be greedily extended to a total-13-avd-coloring of G since each 5^--vertex v has at most 10 forbidden colors by Claim 1.

Claim 2 *If v is a 7-vertex of G, then $n_{5^-}(v) \leq 2$. Moreover, if $n_{3^-}(v) \geq 1$, then $n_{5^-}(v) = 1$.*

Proof Let $v_1, v_2, \ldots, v_7$ be the neighbors of v with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_7)$.

(1) Suppose to the contrary that $n_{5^-}(v) \geq 3$. Then $d(v_1) \leq d(v_2) \leq d(v_3) \leq 5$. Let $H = G - vv_1 - vv_2 - vv_3$. Thus there exists a total-13-avd-coloring of H by the minimality of G. Erase the colors of v, v_1, v_2, v_3 and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ'. Let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2, 3$, and let $S_4 = C \setminus D_{\phi'}(v)$. Then $|S_i| \geq 5$ for $i = 1, 2, 3, 4$. We will color vv_i with the color $s_i \in S_i$ for $i = 1, 2, 3$ and color v with the color $s_4 \in S_4$. Let ϕ denote the partial coloring after vv_1, vv_2, vv_3 and v are colored. If $s_i - s_j \neq 0$ for $1 \leq i < j \leq 4$, then ϕ is a proper total coloring. If $m_\phi(v) \neq m_\phi(v_i)$, i.e., $\sum_{t=1}^{4} s_t + m_{\phi'}(v) - m_{\phi'}(v_i) \neq 0$ for $i = 4, 5, 6, 7$, then ϕ is an adjacent vertex distinguishing coloring. Hence ϕ would be a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2, 3, 4$ such that $P(s_1, s_2, s_3, s_4) \neq 0$, where

$$P(x_1, x_2, x_3, x_4) = \prod_{1 \leq i < j \leq 4} (x_i - x_j) \prod_{i=4}^{7} \left(\sum_{t=1}^{4} x_t + m_{\phi'}(v) - m_{\phi'}(v_i) \right).$$

By MATLAB, $c_P(x_1^4 x_2^3 x_3^2 x_4) = c_Q(x_1^4 x_2^3 x_3^2 x_4) = 1 \neq 0$, where $Q(x_1, x_2, x_3, x_4) = \prod_{1 \leq i < j \leq 4} (x_i - x_j)(\sum_{t=1}^{4} x_t)^4$. According to Lemma 1, since $\deg(P) = 10$ and $|S_i| \geq 5$ for $i = 1, 2, 3, 4$, there exist $s_i \in S_i, i = 1, 2, 3, 4$ such that $P(s_1, s_2, s_3, s_4) \neq 0$. Coloring vv_1, vv_2, vv_3, v with s_1, s_2, s_3, s_4 respectively and then we obtain a nice total-13-avd-coloring, which is a contradiction.

(2) Suppose $n_{5^-}(v) \geq 2$ when $n_{3^-}(v) \geq 1$. Then $d(v_1) \leq 3$ and $d(v_2) \leq 5$. Let $H = G - vv_1 - vv_2$. Then there exists a total-13-avd-coloring of H by the minimality of G. Erase the colors of v_1, v_2 and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ' and let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2$. Obviously, $|S_1| \geq 5$ and $|S_2| \geq 3$. Now we extend ϕ' to G and let ϕ denote the coloring after vv_1 and vv_2 are colored. Let s_1, s_2 correspond to the colors of vv_1, vv_2 respectively. Similar to (1), ϕ is a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2$ such that $P(s_1, s_2) \neq 0$, where

$$P(x_1, x_2) = (x_1 - x_2) \prod_{3 \leq i \leq 7} (x_1 + x_2 + m_{\phi'}(v) - m_{\phi'}(v_i)).$$

By MATLAB, $c_P(x_1^4 x_2^2) = c_Q(x_1^4 x_2^2) = 5$, where $Q(x_1, x_2) = (x_1 - x_2)(x_1 + x_2)^5$. According to Lemma 1, since $\deg(P) = 6$, $|S_1| \geq 5$ and $|S_2| \geq 3$, there exist

$s_i \in S_i, i = 1, 2$ such that $P(s_1, s_2) \neq 0$. Coloring vv_1, vv_2 with s_1, s_2 respectively and then we obtain a nice total-13-avd-coloring, which is a contradiction. □

Claim 3 *Suppose v is an 8-vertex of G. If $n_{4^-}(v) \geq 1$, then $n_{5^-}(v) \leq 3$. Moreover, if $n_{3^-}(v) \geq 2$, then $n_{5^-}(v) = 2$.*

Proof Let $v_1, v_2, \ldots, v_8$ be the neighbors of v with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_8)$.

(1) Suppose to the contrary that $n_{5^-}(v) \geq 4$ when $n_{4^-}(v) \geq 1$. Then $d(v_1) \leq 4$ and $d(v_2) \leq d(v_3) \leq d(v_4) \leq 5$. Let $H = G - vv_1 - vv_2 - vv_3 - vv_4$. Then there exists a total-13-avd-coloring of H by the minimality of G. Erase the colors of v_1, v_2, v_3 and v_4 and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ' and let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2, 3, 4$. Obviously, $|S_1| \geq 5$ and $|S_i| \geq 4$ for $i = 2, 3, 4$. Now we extend ϕ' to G and let ϕ denote the coloring after vv_1, vv_2, vv_3 and vv_4 are colored. Let s_1, s_2, s_3, s_4 correspond to the colors of vv_1, vv_2, vv_3, vv_4 respectively. Similar to Claim 2(1), ϕ is a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2, 3, 4$ such that $P(s_1, s_2, s_3, s_4) \neq 0$, where

$$P(x_1, x_2, x_3, x_4) = \prod_{1 \leq i < j \leq 4} (x_i - x_j) \prod_{5 \leq i \leq 8} \left(\sum_{t=1}^{4} x_t + m_{\phi'}(v) - m_{\phi'}(v_i) \right).$$

By MATLAB, $c_P(x_1^4 x_2^3 x_3^2 x_4) = c_Q(x_1^4 x_2^3 x_3^2 x_4) = 1$, where $Q(x_1, x_2, x_3, x_4) = \prod_{1 \leq i < j \leq 4} (x_i - x_j)(\sum_{t=1}^{4} x_t)^4$. According to Lemma 1, since $\deg(P) = 10$, $|S_1| \geq 5$ and $|S_i| \geq 4$ for $i = 2, 3, 4$, there exist $s_i \in S_i, i = 1, 2, 3, 4$ such that $P(s_1, s_2, s_3, s_4) \neq 0$. Coloring vv_1, vv_2, vv_3, vv_4 with s_1, s_2, s_3, s_4 respectively and then we obtain a nice total-13-avd-coloring, which is a contradiction.

(2) Suppose to the contrary that $n_{5^-}(v) \geq 3$ when $n_{3^-}(v) \geq 2$. Then $d(v_1) \leq d(v_2) \leq 3$ and $d(v_3) \leq 5$. Let $H = G - vv_1 - vv_2 - vv_3$. Then there exists a total-13-avd-coloring of H by the minimality of G. Erase the colors of v_1, v_2 and v_3 and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ' and let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2, 3$. Then $|S_1| \geq 5$, $|S_2| \geq 5$ and $|S_3| \geq 3$. Now we extend ϕ' to G and let ϕ denote the coloring after vv_1, vv_2 and vv_3 are colored. Let s_1, s_2, s_3 correspond to the colors of vv_1, vv_2, vv_3 respectively. Similar to Claim 2(1), ϕ is a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2, 3$ such that $P(s_1, s_2, s_3) \neq 0$, where

$$P(x_1, x_2, x_3) = \prod_{1 \leq i < j \leq 3} (x_i - x_j) \prod_{4 \leq i \leq 8} \left(\sum_{t=1}^{3} x_t + m_{\phi'}(v) - m_{\phi'}(v_i) \right).$$

By MATLAB, $c_P(x_1^4 x_2^3 x_3) = c_Q(x_1^4 x_2^3 x_3) = 5$, where $Q(x_1, x_2, x_3) = \prod_{1 \leq i < j \leq 3} (x_i - x_j)(\sum_{t=1}^{3} x_t)^5$. According to Lemma 1, since $\deg(P) = 8$, $|S_1| \geq 5$, $|S_2| \geq 5$ and $|S_3| \geq 3$, there exist $s_i \in S_i, i = 1, 2, 3$ such that $P(s_1, s_2, s_3) \neq 0$. Coloring vv_1, vv_2, vv_3 with s_1, s_2, s_3 respectively and then we obtain a nice total-13-avd-coloring, a contradiction. □

Claim 4 *Suppose v is a 9-vertex of G. If $n_{4^-}(v) \geq 1$ then $n_{5^-}(v) \leq 6$. Moreover, if $n_{3^-}(v) \geq 1$ and $n_{4^-}(v) \geq 2$, then $n_{5^-}(v) \leq 3$.*

Proof Let $v_1, v_2, \dots, v_9$ be the neighbors of v with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_9)$.

(1) Suppose to the contrary that $n_{5^-}(v) \geq 7$ when $n_{4^-}(v) \geq 1$. Then $d(v_1) \leq 4$ and $d(v_2) \leq d(v_3) \leq \cdots \leq d(v_7) \leq 5$. Let $H = G - vv_1 - vv_2 - \cdots - vv_7$. Thus there is a total-13-avd-coloring of H by the minimality of G. Erase the colors of $v, v_1, v_2, \dots, v_7$ and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ'. Let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2, \dots, 7$ and let $S_8 = C \setminus D_{\phi'}(v)$. Then $|S_1| \geq 8$, $|S_i| \geq 7$ for $i = 2, 3, \dots, 7$ and $|S_8| \geq 9$. Now we extend ϕ' to G and let ϕ denote the coloring after $vv_1, vv_2, \dots, vv_7$ and v are colored. Let $s_1, s_2, \dots, s_7$ and s_8 correspond to the colors of $vv_1, vv_2, \dots, vv_7$ and v respectively. Similar to Claim 2(1), ϕ is a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2, \dots, 8$ such that $P(s_1, s_2, \dots, s_8) \neq 0$, where

$$P(x_1, x_2, \dots, x_8) = \prod_{1 \leq i < j \leq 8} (x_i - x_j) \prod_{8 \leq i \leq 9} \left(\sum_{t=1}^{8} x_t + m_{\phi'}(v) - m_{\phi'}(v_i) \right).$$

By MATLAB, $c_P(x_1^7 x_2^5 x_3^4 x_4^3 x_5^2 x_6 x_8^8) = c_Q(x_1^7 x_2^5 x_3^4 x_4^3 x_5^2 x_6 x_8^8) = -1$, where $Q(x_1, x_2, \dots, x_8) = \prod_{1 \leq i < j \leq 8} (x_i - x_j) (\sum_{t=1}^{8} x_t)^2$. According to Lemma 1, since $\deg(P) = 30$, $|S_1| \geq 8$, $|S_i| \geq 7$ for $i = 2, 3, \dots, 7$ and $|S_8| \geq 9$, there exist $s_i \in S_i, i = 1, 2, \dots, 8$ such that $P(s_1, s_2, \dots, s_8) \neq 0$. Coloring $vv_1, vv_2 \dots, vv_7$ and v with $s_1, s_2, \dots, s_7$ and s_8 respectively and then we obtain a nice total-13-avd-coloring, a contradiction.

(2) Suppose to the contrary that $n_{5^-}(v) \geq 4$ when $n_{3^-}(v) \geq 1$ and $n_{4^-}(v) \geq 2$. Then $d(v_1) \leq 3$, $d(v_2) \leq 4$ and $d(v_3) \leq d(v_4) \leq 5$. Let $H = G - vv_1 - vv_2 - vv_3 - vv_4$. Thus there is a total-13-avd-coloring of H by the minimality of G. Erase the colors of v, v_1, v_2, v_3, v_4 and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ'. Let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2, 3, 4$ and let $S_5 = C \setminus D_{\phi'}(v)$. Then $|S_1| \geq 6$, $|S_2| \geq 5$, $|S_3| \geq 4$, $|S_4| \geq 4$ and $|S_5| \geq 3$. Now we extend ϕ' to G and let ϕ denote the coloring after vv_1, vv_2, vv_3, vv_4 and v are colored. Let s_1, s_2, s_3, s_4 and s_5 correspond to the colors of vv_1, vv_2, vv_3, vv_4 and v respectively. Similar to Claim 2(1), ϕ is a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2, \dots, 5$ such that $P(s_1, s_2, \dots, s_5) \neq 0$, where

$$P(x_1, x_2, \dots, x_5) = \prod_{1 \leq i < j \leq 5} (x_i - x_j) \prod_{5 \leq i \leq 9} \left(\sum_{t=1}^{5} x_t + m_{\phi'}(v) - m_{\phi'}(v_i) \right).$$

By MATLAB, $c_P(x_1^5 x_2^4 x_3^3 x_4^2 x_5) = c_Q(x_1^5 x_2^4 x_3^3 x_4^2 x_5) = 1$, where $Q(x_1, x_2, \dots, x_5) = \prod_{1 \leq i < j \leq 5} (x_i - x_j) (\sum_{t=1}^{5} x_t)^5$. According to Lemma 1, since $\deg(P) = 15$, $|S_1| \geq 6$, $|S_2| \geq 5$, $|S_3| \geq 4$, $|S_4| \geq 4$ and $|S_5| \geq 3$, there exist $s_i \in S_i, i = 1, 2, \dots, 5$ such that $P(s_1, s_2, \dots, s_5) \neq 0$. Thus we obtain a nice total-13-avd-coloring, a contradiction.

□

Claim 5 *Suppose v is a 10-vertex of G and $n_{2^-}(v) \geq 1$. Then $n_{5^-}(v) \leq 7$. Moreover, if $n_{3^-}(v) \geq 2$ and $n_{4^-}(v) \geq 3$, then $n_{5^-}(v) \leq 4$.*

Proof Let $v_1, v_2, \ldots, v_{10}$ be the neighbors of v with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_{10})$.

(1) Suppose to the contrary that $n_{5^-}(v) \geq 8$ when $n_{2^-}(v) \geq 1$. Then $d(v_1) \leq 2$ and $d(v_2) \leq d(v_3) \leq \cdots \leq d(v_8) \leq 5$. Let $H = G - vv_1 - vv_2 - \cdots - vv_8$. Thus there is a total-13-avd-coloring of H by the minimality of G. Erase the colors of $v, v_1, v_2, \ldots, v_8$ and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ'. Let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2, \ldots, 8$ and let $S_9 = C \setminus D_{\phi'}(v)$. Then $|S_1| \geq 10$, $|S_i| \geq 7$ for $i = 2, 3, \ldots, 8$ and $|S_9| \geq 9$. Now we extend ϕ' to G and let ϕ denote the coloring after $vv_1, vv_2, \ldots, vv_8$ and v are colored. Let $s_1, s_2, \ldots, s_8$ and s_9 correspond to the colors of $vv_1, vv_2, \ldots, vv_8$ and v respectively. Similar to Claim 2(1), ϕ is a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2, \ldots, 9$ such that $P(s_1, s_2, \ldots, s_9) \neq 0$, where

$$P(x_1, x_2, \ldots, x_9) = \prod_{1 \leq i < j \leq 9} (x_i - x_j) \prod_{9 \leq i \leq 10} \left(\sum_{t=1}^{9} x_t + m_{\phi'}(v) - m_{\phi'}(v_i) \right).$$

Since $c_P(x_1^9 x_2^6 x_3^5 x_4^4 x_5^3 x_6^2 x_7 x_9^8) = c_Q(x_1^9 x_2^6 x_3^5 x_4^4 x_5^3 x_6^2 x_7 x_9^8) = -1$, where $Q(x_1, x_2, \ldots, x_9) = \prod_{1 \leq i < j \leq 9}(x_i - x_j)(\sum_{t=1}^{9} x_t)^2$. According to Lemma 1, since $\deg(P) = 38$, $|S_1| \geq 10$, $|S_i| \geq 7$ for $i = 2, 3, \ldots, 8$ and $|S_9| \geq 9$, there exist $s_i \in S_i, i = 1, 2, \ldots, 9$ such that $P(s_1, s_2, \ldots, s_9) \neq 0$. Thus we obtain a nice total-13-avd-coloring, a contradiction.

(2) Suppose to the contrary that $n_{5^-}(v) \geq 5$. Then $d(v_1) \leq 2, d(v_2) \leq 3, d(v_3) \leq 4$ and $d(v_4) \leq d(v_5) \leq 5$. Let $H = G - vv_1 - vv_2 - \cdots - vv_5$. Thus there is a total-13-avd-coloring of H by the minimality of G. Erase the colors of $v_1, v_2, \ldots, v_5$ and denote this partial total-13-avd-coloring by ϕ'. Let C denote the set of all colors used in ϕ' and let $S_i = C \setminus (C_{\phi'}(v) \cup C_{\phi'}(v_i))$ for $i = 1, 2, \ldots, 5$. Obviously, $|S_1| \geq 6$, $|S_2| \geq 5$, $|S_3| \geq 4$, $|S_4| \geq 3$ and $|S_5| \geq 3$. Now we extend ϕ' to G and let ϕ denote the coloring after $vv_1, vv_2, \ldots, vv_5$ are colored. Let $s_1, s_2, \ldots, s_5$ correspond to the colors of $vv_1, vv_2, \ldots, vv_5$ respectively. Similar to Claim 2(1), ϕ is a nice total-13-avd-coloring, if there exist $s_i \in S_i, i = 1, 2, \ldots, 5$ such that $P(s_1, s_2, \ldots, s_5) \neq 0$, where

$$P(x_1, x_2, \ldots, x_5) = \prod_{1 \leq i < j \leq 5} (x_i - x_j) \prod_{6 \leq i \leq 10} \left(\sum_{t=1}^{5} x_t + m_{\phi'}(v) - m_{\phi'}(v_i) \right).$$

By MATLAB, $c_P(x_1^5 x_2^4 x_3^3 x_4^2 x_5) = c_Q(x_1^5 x_2^4 x_3^3 x_4^2 x_5) = 1$, where $Q(x_1, x_2, \ldots, x_5) = \prod_{1 \leq i < j \leq 5}(x_i - x_j)(\sum_{t=1}^{5} x_t)^5$. According to Lemma 1, since $\deg(P) = 15$, $|S_1| \geq 6$, $|S_2| \geq 5$, $|S_3| \geq 4$, $|S_4| \geq 3$ and $|S_5| \geq 3$, there exist $s_i \in S_i, i = 1, 2, \ldots, 5$ such that $P(s_1, s_2, \ldots, s_5) \neq 0$. Thus we obtain a nice total-13-avd-coloring, a contradiction. □

Suppose uv is an edge of G. We call u is *strong neighbor* of v, if $d(v) = 10$, $d(u) = 3$ and uv is incident with at least two 3-cycles. We use $n_{str}^G(v)$ to denote the number of strong neighbors of v in G.

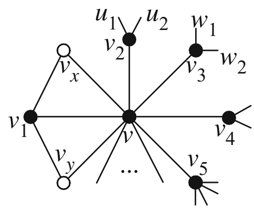

Fig. 2 Configurations in the proof of Claim 6

Claim 6 *Suppose v is a 10-vertex of G. If $n_{3^-}(v) \geq 4$ and $n^G_{str}(v) \neq 0$, then $n_{5^-}(v) = 4$.*

Proof Let $v_1, v_2, \ldots, v_{10}$ be the neighbors of v with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_{10})$.

Since $n_{3^-}(v) \geq 4$, if $n_{2^-}(v) \neq 0$, then $n_{5^-}(v) = 4$ by Claim 5. So we assume $n_{2^-}(v) = 0$. Suppose to the contrary that $n_{5^-}(v) \geq 5$. Then $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$ and $d(v_5) \leq 5$. Assume v_1 is a strong neighbor of v and v_x, v_y are common neighbors of v and v_1. Let u_1, u_2 be the neighbors of v_2 other than v and w_1, w_2 be the neighbors of v_3 other than v (see Fig. 2). Let $H = G - vv_1$. Thus there is a total-13-avd-coloring of H by the minimality of G. Erase the colors of $vv_2, vv_3, vv_4, vv_5, v_1, v_2, v_3, v_4, v_5$ and denote this partial total-13-avd-coloring by ϕ. Let C denote the set of all colors used in ϕ. Since $\phi(v_2u_1) \neq \phi(v_2u_2)$, without loss of generality, we assume $\phi(v_1v_x) \neq \phi(v_2u_1)$. Since $C \setminus (C_\phi(v) \cup \{\phi(v_2u_2)\})$ has at least six colors and then has at least six 5-element subsets, there exists at least one 5-element subset C', such that $C_\phi(v) \cup C'$ is different from any $C_\phi(v_j)$, $j = 6, 7, 8, 9, 10$. Now we will color $vv_1, \ldots, vv_5$ with C' properly to obtain a nice total-13-avd-coloring of G, which is a contradiction.

Case 1 $\phi(v_1v_y) \notin C'$. Since $d(v_5) \leq 5$, $|C_\phi(v_5)| \leq 4$, we can color vv_5 with a color $a_5 \in C' \setminus C_\phi(v_5)$. Since $d(v_i) \leq 3$ for $3 \leq i \leq 4$, $|C_\phi(v_i)| = 2$, we can color vv_4 with a color $a_4 \in (C' \setminus \{a_5\}) \setminus C_\phi(v_4)$ and color vv_3 with a color $a_3 \in (C' \setminus \{a_5, a_4\}) \setminus C_\phi(v_3)$. Notice that $\phi(v_1v_y) \notin C' \setminus \{a_3, a_4, a_5\}$, $\phi(v_2u_2) \notin C' \setminus \{a_3, a_4, a_5\}$ and $\phi(v_1v_x) \neq \phi(v_2u_1)$, therefore we can color vv_1 and vv_2 with a_1 and a_2 respectively such that $\{a_1, a_2\} = C' \setminus \{a_3, a_4, a_5\}$ and $a_1 \neq \phi(v_1v_x)$, $a_2 \neq \phi(v_2u_1)$.

Case 2 $\phi(v_1v_y) \in C'$. Notice that $\phi(v_1v_y) \notin C_\phi(v)$.

Case 2.1 $\phi(v_1v_y) \neq \phi(v_2u_1)$. We color vv_5, vv_4, vv_3 with a_5, a_4, a_3 successively such that $a_i \in (C' \setminus \{a_5, \ldots, a_{i+1}\}) \setminus C_\phi(v_i)$ for $3 \leq i \leq 5$, and if there exists an $i \in \{3, 4, 5\}$ such that $\phi(v_1v_y) \in (C' \setminus \{a_5, \ldots, a_{i+1}\}) \setminus C_\phi(v_i)$, then set $a_i = \phi(v_1v_y)$; If there exists an $i \in \{3, 4, 5\}$ such that $\phi(v_1v_y) \notin (C' \setminus \{a_5, \ldots, a_{i+1}\}) \setminus C_\phi(v_i)$ and $\phi(v_1v_x) \in (C' \setminus \{a_5, \ldots, a_{i+1}\}) \setminus C_\phi(v_i)$, then set $a_i = \phi(v_1v_x)$. If $\{\phi(v_1v_y), \phi(v_1v_x)\} \neq C' \setminus \{a_5, a_4, a_3\}$, say $\phi(v_1v_y) \notin C' \setminus \{a_5, a_4, a_3\}$, then similar to Case 1, we can color vv_2 and vv_1 with the colors in $C' \setminus \{a_5, a_4, a_3\}$ safely. So we only consider the case that $\{\phi(v_1v_y), \phi(v_1v_x)\} = C' \setminus \{a_5, a_4, a_3\}$. In this case, we have $\{\phi(v_1v_y), \phi(v_1v_x)\} \subseteq C_\phi(v_i)$ for every $i \in \{3, 4, 5\}$. Particularly, $C_\phi(v_3) = \{\phi(v_1v_y), \phi(v_1v_x)\}$. Assume $\phi(v_3w_1) = \phi(v_1v_x)$ and $\phi(v_3w_2) = \phi(v_1v_y)$. Now we erase the colors of vv_3, vv_4, vv_5.

Case 2.1.1 $\phi(v_1v_x) \neq \phi(vv_y)$. Firstly, we exchange the colors of v_1v_y and vv_y. Since $\phi(v_1v_y) \notin C_\phi(v)$, we obtain a partial total-13-avd-coloring, denoted by ϕ'. We can find at least one set $C'' \subseteq C\backslash(C_{\phi'}(v) \cup \{\phi'(v_2u_2)\})$ such that $|C''| = 5$ and coloring $vv_1, \ldots, vv_5$ with the colors in C'' (based on ϕ') will not lead to the conflicts of v with its neighbors. Observe $\phi'(v_3w_2) = \phi'(vv_y) \notin C''$. We color vv_i with a color $b_i \in (C'' \setminus \{b_5, \ldots, b_{i+1}\}) \setminus C_{\phi'}(v_i)$ for $4 \leq i \leq 5$, and then color vv_1 with a color $b_1 \in (C'' \setminus \{b_5, b_4\}) \setminus C_{\phi'}(v_1)$. Since $\phi'(v_3w_2) \notin C''$, $\phi'(v_2u_2) \notin C''$ and $\phi'(v_3w_1) = \phi'(v_1v_x) \neq \phi'(v_2u_1)$, therefore we can color vv_2 and vv_3 with b_2 and b_3 respectively such that $\{b_2, b_3\} = C''\backslash\{b_1, b_4, b_5\}$ and $b_2 \neq \phi'(v_2u_1)$, $b_3 \neq \phi'(v_3w_1)$.

Case 2.1.2 $\phi(v_1v_x) = \phi(vv_y)$. We exchange the colors of v_1v_y and vv_y, and the colors of v_1v_x and vv_x at the same time. Since $\phi(v_1v_y) \notin C_\phi(v)$ and $\phi(v_1v_x) \notin C_\phi(v) \setminus \{\phi(vv_y)\}$, we obtain a partial total-13-avd-coloring ϕ''. We can find at least one set $\hat{C} \subseteq C\backslash(C_{\phi''}(v) \cup \{\phi''(v_2u_2)\})$ such that $|\hat{C}| = 5$ and coloring $vv_1, \ldots, vv_5$ with the colors in $\hat{C}$ (based on ϕ'') will not lead to the conflicts of v with its neighbors. Now we color vv_i with a color $c_i \in (\hat{C} \setminus \{c_5, \ldots, c_{i+1}\}) \setminus C_{\phi''}(v_i)$ for $4 \leq i \leq 5$, and then color vv_1 with a color $c_1 \in (\hat{C} \setminus \{c_5, c_4\}) \setminus C_{\phi''}(v_1)$. Since $\phi''(v_2u_2) \notin \hat{C}$, we can color vv_2 with a color $c_2 \in (\hat{C} \setminus \{c_5, c_4, c_1\}) \setminus C_{\phi''}(v_2)$. Finally, we color vv_3. Since $\phi''(v_3w_2) = \phi''(vv_y) \notin \hat{C}$ and $\phi''(v_3w_1) = \phi''(vv_x) \notin \hat{C}$, we can find a color $c_3 \in (\hat{C} \setminus \{c_5, c_4, c_1, c_2\}) \setminus C_{\phi''}(v_3)$ to color vv_3.

Case 2.2 $\phi(v_1v_y) = \phi(v_2u_1)$. Since $\phi(v_1u_y) \notin C_\phi(v)$, we have $\phi(v_2u_1) \neq \phi(vv_x)$. If $\phi(v_1v_x) \neq \phi(vv_y)$, we only exchange the colors of v_1v_y and vv_y, and if $\phi(v_1v_x) = \phi(vv_y)$, we exchange the colors of v_1v_y and vv_y, and the colors of v_1v_x and vv_x at the same time. Denote the new partial total-13-avd-coloring by ψ. Then $\psi(v_1v_y) \neq \psi(v_2u_1)$, since $\phi(v_1v_x) \neq \phi(v_2u_1)$ and $\phi(vv_x) \neq \phi(v_2u_1)$, we claim that in both cases we have $\psi(v_1v_x) \neq \psi(v_2u_1)$. We can find at least one set $\tilde{C} \subseteq C\backslash(C_\psi(v) \cup \{\psi(v_2u_2)\})$ such that $|\tilde{C}| = 5$ and coloring $vv_1, \ldots, vv_5$ with the colors in $\tilde{C}$ (based on ψ) will not lead to the conflicts of v with its neighbors. If $\psi(v_1v_y) \notin \tilde{C}$, it becomes Case 1. Otherwise it becomes Case 2.1. □

3.2 Discharging process

We put all the 1-vertices and 2-vertices of G in V_1. Let $V_2 = V \setminus V_1$ and $H = G[V_2]$. For H, we have the following result:

Claim 7 *Let v be a vertex of H. Then the following properties hold:*

(1) $\delta(H) \geq 3$;
(2) *For any $k \in \{3, 4, 5\}$, $n_k^H(v) = n_k^G(v)$;*
(3) *There is no edge $uv \in E(H)$ such that $d_H(v) \leq 6$ and $d_H(u) \leq 5$.*

Proof Let v be a vertex of H. If $d_G(v) \leq 6$, then $n_{2^-}^G(v) = 0$ by Claim 1. If $d_G(v) = 7$, then $n_{2^-}^G(v) \leq 1$ by Claim 2. If $d_G(v) = 8$, then $n_{2^-}^G(v) \leq 2$ by Claim 3. If $d_G(v) = 9$, then $n_{2^-}^G(v) \leq 3$ by Claim 4. If $d_G(v) = 10$, then $n_{2^-}^G(v) \leq 4$ by Claim 5. So we conclude that if $d_G(v) \leq 5$, then $d_H(v) = d_G(v)$; If $d_G(v) \geq 6$, then $d_H(v) = d_G(v) - n_{2^-}^G(v) \geq 6$.

(1) Suppose to the contrary that there is a vertex $v \in V(H)$ such that $d_H(v) \leq 2$. Obviously, $d_G(v) \geq 3$. If $d_G(v) \leq 5$, then $d_H(v) = d_G(v) \geq 3$, if $d_G(v) \geq 6$, then $d_H(v) = d_G(v) - n^G_{2^-}(v) \geq 6$, which is a contradiction.

(2) Suppose u is a neighbor of v in H and $d_H(u) = k$, where $k \in \{3, 4, 5\}$. If $d_G(u) \neq k$, then we have $d_G(v) \geq 6$ by the above analysis, and then $d_H(v) \geq 6$, which is a contradiction. So $d_G(u) = k$ and $n^H_k(v) \leq n^G_k(v)$. On the other hand, if u is a neighbor of v in G and $d_G(u) = k$, where $k \in \{3, 4, 5\}$, then $d_H(u) = k$ by the above analysis, so $n^G_k(v) \leq n^H_k(v)$. Thus we conclude $n^H_k(v) = n^G_k(v)$.

(3) Suppose to the contrary that there is an edge $uv \in E(H)$ such that $d_H(v) \leq 6$ and $d_H(u) \leq 5$. Consider the degree of v in G. If $d_G(v) \leq 6$, then $d_G(u) \geq 6$ by Claim 1, so $d_H(u) \geq 6$, a contradiction. If $d_G(v) = 7$, then $n^G_{2^-}(v) = d_G(v) - d_H(v) \geq 1$. So $n^G_{5^-}(v) = n^G_{2^-}(v) = 1$ by Claim 2, that means v has no other neighbors with degree less than 6 in G. Thus $d_G(u) \geq 6$, so $d_H(u) \geq 6$, a contradiction. Similarly, we can also can obtain a contradiction when $d_G(v) = 8, 9, 10$ by Claims 3, 4 and 5. □

Due to Claim 7, we obtain the following observation:

Observation 1 *For any $f \in F(H)$, f is incident with at most $\lfloor \frac{d_H(f)}{2} \rfloor$ vertices of degree at most* 5.

Observation 1 can be easily deduced from Claim 7,

Let uv be an edge of H and $d(u) = k$, we call u a *bad k-neighbor* of v if the edge uv belongs to two 3-faces, and call u a *special k-neighbor* of v if the edge uv belongs to exactly one 3-face. We use $N^H_{kb}(v)$ and $N^H_{ks}(v)$ to denote the number of bad k-neighbors and number of special k-neighbors of v in H respectively, and let $n^H_{kb}(v) = |N^H_{kb}(v)|$ and $n^H_{ks}(v) = |N^H_{ks}(v)|$.

Observation 2 *Let v be a* 10*-vertex of H. Then*

(1) $n^H_{3s}(v) \leq 6$;

(2) $n^H_{3b}(v) + n^H_{4b}(v) + n^H_{5b}(v) \leq \frac{1}{2}(10 - n^H_{3s}(v))$.

Proof (1) Suppose that $n^H_{3s}(v) \geq 7$. Then the number of 3-faces incident with v and its special 3-neighbors is at least 7. So v must incident with three consecutive such 3-faces. According to the definition of special 3-neighbors, the second 3-face can not incident with special 3-neighbors of v, a contradiction.

(2) Since H is planar graph, we use $v_0, v_1, \ldots, v_9$ to denote the neighbors of v in H in clockwise order. For $0 \leq i \leq 9$, we call v_i and v_j *consecutive* if $j = i + 1$ modulo 10. Notice that the number of neighbors of v which are not special 3-neighbors is at most $10 - n^H_{3s}(v)$. Suppose that $n^H_{3b}(v) + n^H_{4b}(v) + n^H_{5b}(v) > \frac{1}{2}(10 - n^H_{3s}(v))$, then there exist two consecutive vertices u and w such that $\{u, w\} \subseteq N^H_{3b}(v) \cup N^H_{4b}(v) \cup N^H_{5b}(v)$. According to the definition of bad k-neighbors, u must be incident with w, which is contradict with Claim 7(3).

Using Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$ and the relation $\sum_{v \in V(H)} d_H(v) = \sum_{f \in F(H)} d_H(f) = 2|E(H)|$, we have

$$\sum_{v \in V(H)} (d_H(v) - 6) + \sum_{f \in F(H)} (2d_H(f) - 6) = -12.$$

That is

$$\sum_{v\in V(H)}(d_G(v)-n^G_{2^-}(v)-6)+\sum_{f\in F(H)}(2d_H(f)-6)=-12.$$

We use the discharging method. First, we give an initial charge function $w(v)=d_G(v)-d^G_{2^-}(v)-6$ for every $v\in V(H)$, and $w(f)=2d_H(f)-6$ for every $f\in F(H)$. Next, we design a discharging rule and redistribute weights accordingly. Let w' be the new charge after the discharging. We will show that $w'(x)\geq 0$ for all $x\in V(H)\cup F(H)$. This leads to the following obvious contradiction:

$$0\leq\sum_{x\in V(H)\cup F(H)}w'(x)=\sum_{x\in V(H)\cup F(H)}w(x)=-12<0,$$

hence demonstrates that no such a counterexample can exist.

The discharging rules are defined as follows:

R1 If v is a bad 3-neighbor of u in H, then u gives 1 to v.
R2 If v is a special 3-neighbor of u in H, then u gives $\frac{1}{2}$ to v.
R3 If v is a bad 4-neighbor of u in H, then u gives $\frac{1}{2}$ to v.
R4 If v is a bad 5-neighbor of u in H, then u gives $\frac{1}{5}$ to v.
R5 If $f\in F(H)$ is a 4-face, then f gives 1 to each incident 5^--vertex.
R6 If the degree of $f\in F(H)$ is at least 5, then f gives 2 to each incident 5^--vertex.

Now let us begin our analysis.

Let f be a face of H. Note that only vertices of degree at most 5 might receive weights from f. Suppose $d_H(f)=3$. Then no rule applies to f, so $w'(f)=0$. Suppose $d_H(f)=4$. Then there are at most two vertices of degree at most 5 on its boundary by Observation 1, so $w'(f)\geq 2\times 4-6-2=0$ by $R5$. Suppose $d_H(f)\geq 5$. Then $w'(f)\geq 2d_H(f)-6-2\lfloor\frac{d_H(f)}{2}\rfloor\geq 0$ by Observation 1 and $R6$.

Let v be a vertex of H. Obviously, $d_G(v)\geq d_H(v)\geq 3$.

Suppose $d_G(v)=3$. By Claim 1, $n^G_{5^-}(v)=0$ and $w(v)=-3$. Then we have $n^H_{5^-}(v)=0$ by Observation 1. So v gives no charge to its neighbors. We consider the faces incident with v in H. If v is incident with three 3-faces, then $w'(v)=-3+3=0$ by $R1$. If v is incident with exactly two 3-faces, then $w'(v)\geq -3+1+2\times\frac{1}{2}+1=0$ by $R1$, $R2$, $R5$ and $R6$. If v is incident with exactly one 3-face, then $w'(v)\geq -3+2\times\frac{1}{2}+2\times 1=0$ by $R2$, $R5$ and $R6$. Otherwise, v is incident with three 4^+-faces. So $w'(v)\geq -3+3\times 1=0$ by $R5$ and $R6$.

Suppose $d_G(v)=4$. By Claim 1, $n^G_{5^-}(v)=0$ and $w(v)=-2$. Then we have $n^H_{5^-}(v)=0$ by Observation 1. So v gives no charge to its neighbors. If all the faces incident with v are 3-faces, then $w'(v)\geq -2+4\times\frac{1}{2}=0$ by $R3$. If v is incident with exactly one 4^+-face, then $w'(v)\geq -2+2\times\frac{1}{2}+1=0$ by $R3$, $R5$ and $R6$. Otherwise, v receives at least $2\times 1=2$ from the incident 4^+-faces by $R5$ and $R6$, so $w'(v)\geq -2+2=0$.

Suppose $d_G(v)=5$. By Claim 1, $n^G_{5^-}(v)=0$ and $w(v)=-1$. Then we have $n^H_{5^-}(v)=0$ by Observation 1. So v gives no charge to its neighbors. If all the faces incident with v are 3-faces, then $w'(v) \geq -1 + 5 \times \frac{1}{5} = 0$ by $R4$. Otherwise, v receives at least 1 from the incident 4^+-faces by $R5$ and $R6$, so $w'(v) \geq -1 + 1 = 0$.

Suppose $d_G(v) = 6$. By Claim 1, $n^G_{5-}(v) = 0$ and $w(v) = 0$. Then we have $n^H_{5-}(v) = 0$ by Observation 1. So v gives no charge to its neighbors. Thus $w'(v) = w(v) = 0$.

Suppose $d_G(v) = 7$. Then by Claim 2, if $n^G_{3-}(v) \geq 1$, then $n^G_{5-}(v) = 1$, so $w'(v) \geq d_G(v) - 6 - n^G_{2-}(v) - n^H_3(v) - n^H_4(v) - n^H_5(v) = 1 - n^G_{5-}(v) = 0$ by Claim 7(2) and $R1 - R4$. Otherwise, $n^G_{3-}(v) = 0$ and $n^G_{5-}(v) \leq 2$, and then $n^G_{2-}(v) + n^H_3(v) = n^G_{3-}(v) = 0$ and $n^H_4(v) + n^H_5(v) = n^G_{5-}(v) - n^G_{3-}(v) \leq 2$ by Claim 7(2). So $w'(v) \geq 7 - 6 - \frac{1}{2} \times (n^H_4(v) + n^H_5(v)) \geq 0$ by Claim 7(2) and $R1$–$R4$.

Suppose $d_G(v) = 8$. According to Claim 3, Claim 7(2) and discharging rules, if $n^G_{3-}(v) \geq 2$, then $n^G_{5-}(v) = 2$, $w'(v) \geq 8 - 6 - n^G_{2-}(v) - n^H_3(v) - n^H_4(v) - n^H_5(v) = 2 - n^G_{5-}(v) = 0$; If $n^G_{3-}(v) = 1$, then $n^G_{5-}(v) \leq 3$, so $w'(v) \geq 8 - 6 - (n^G_{2-}(v) + n^H_3(v)) - \frac{1}{2}(n^H_4(v) + n^H_5(v)) = 2 - n^G_{3-}(v) - \frac{1}{2}(n^G_{5-}(v) - n^G_{3-}(v)) \geq 0$; If $n^G_{3-}(v) = 0$ and $n^G_4(v) \geq 1$, then $n^G_{5-}(v) \leq 3$, so $w'(v) \geq 8 - 6 - n^G_{3-}(v) - \frac{1}{2}(n^G_{5-}(v) - n^G_{3-}(v)) > 0$; Otherwise, $n^G_{4-}(v) = 0$ and $n^G_5(v) \leq 8$, so $w'(v) \geq 8 - 6 - (n^G_{2-}(v) + n^H_3(v) + n^H_4(v)) - \frac{1}{5}n^H_5(v) = 2 - n^G_{4-}(v) - \frac{1}{5}n^H_5(v) > 0$.

Suppose $d_G(v) = 9$. According to Claim 4, Claim 7(2) and discharging rules, if $n^G_{3-}(v) \neq 0$ and $n^G_{4-}(v) \geq 2$, then $n^G_{5-}(v) \leq 3$, so $w'(v) \geq 9 - 6 - 1 \times n^G_{5-}(v) \geq 0$; If $n^G_{3-}(v) = 1$ and $n^G_4(v) = 0$, then $n^G_{5-}(v) \leq 6$, so $w'(v) \geq 9 - 6 - (n^G_{2-}(v) + n^H_3(v)) - \frac{1}{2}n^H_4(v) - \frac{1}{5}n^H_5(v) = 3 - n^G_{3-}(v) - \frac{1}{2}n^G_4(v) - \frac{1}{5}(n^G_{5-}(v) - n^G_{4-}(v)) > 0$; If $n^G_{3-}(v) = 0$ and $n^G_4(v) \neq 0$, then $n^G_{5-}(v) \leq 6$, so $w'(v) \geq 9 - 6 - n^G_{3-}(v) - \frac{1}{2}(n^G_{5-}(v) - n^G_{3-}(v)) \geq 0$; Otherwise, $n^G_{4-}(v) = 0$ and $n^G_5(v) \leq 9$, so $w'(v) \geq 9 - 6 - n^G_{4-}(v) - \frac{1}{5}n^G_5(v) > 0$.

Suppose $d_G(v) = 10$. We consider two cases.

Case 1 $n^G_{2-}(v) \neq 0$. According to Claim 5, Claim 7(2) and discharging rules, if $n^G_{3-}(v) \geq 2$ and $n^G_{4-}(v) \geq 3$, then $n^G_{5-}(v) \leq 4$, so $w'(v) \geq 10 - 6 - 1 \times n^G_{5-}(v) \geq 0$; If $n^G_{3-}(v) = 2$ and $n^G_4(v) = 0$, then $n^G_{5-}(v) \leq 7$, so $w'(v) \geq 10 - 6 - n^G_{3-}(v) - \frac{1}{2}n^G_4(v) - \frac{1}{5}(n^G_{5-}(v) - n^G_{4-}(v)) > 0$; If $n^G_{3-}(v) = 1$, then $n^G_{5-}(v) \leq 7$, so $w'(v) \geq 10 - 6 - n^G_{3-}(v) - \frac{1}{2}(n^G_{5-}(v) - n^G_{3-}(v)) \geq 0$.

Case 2 $n^G_{2-}(v) = 0$. If $n^G_3(v) \leq 3$, then $n^H_3(v) \leq 3$ by Claim 7(2). According to Observation 2(2), $n^H_{3b}(v) + n^H_{4b}(v) + n^H_{5b}(v) \leq \frac{1}{2}(10 - n^H_{3s}(v)) \leq 5$. So $w'(v) \geq 10 - 6 - (n^H_{3b}(v) + \frac{1}{2}(n^H_{3s}(v) + n^H_{4b}(v) + n^H_{5b}(v))) \geq 4 - \frac{1}{2}(n^H_{3b}(v) + n^H_{3s}(v)) - \frac{1}{2}(n^H_{3b}(v) + n^H_{4b}(v) + n^H_{5b}(v)) \geq 4 - \frac{1}{2}n^H_3(v) - \frac{1}{2} \times 5 = \frac{1}{2}(3 - n^H_3(v)) \geq 0$ by $R1 - R4$; If $n^G_3(v) \geq 4$ and $n^G_{str}(v) \neq 0$, then $n^G_{5-}(v) = 4$ by Claim 6. So $w'(v) \geq 10 - 6 - (n^H_{3b}(v) + n^H_{3s}(v) + n^H_{4b}(v) + n^H_{5b}(v)) \geq 4 - n^H_{5-}(v) \geq 4 - n^G_{5-}(v) = 0$ by Claim 7(2) and $R1 - R4$; Otherwise, $n^G_3(v) \geq 4$ and $n^G_{str}(v) = 0$. Obviously, $n^H_{3b}(v) = 0$. Since $n^H_{3s}(v) \leq 6$ and $n^H_{4b}(v) + n^H_{5b}(v) \leq \frac{1}{2}(10 - n^H_{3s}(v))$ by Observation 2, $w'(v) \geq 10 - 6 - \frac{1}{2}n^H_{3s}(v) - \frac{1}{2}(n^H_{4b}(v) + n^H_{5b}(v)) \geq 4 - \frac{1}{2}n^H_{3s}(v) - \frac{1}{2} \times \frac{1}{2}(10 - n^H_{3s}(v)) \geq \frac{1}{4}(6 - n^H_{3s}(v)) \geq 0$ by $R2 - R4$.

This completes the whole proof of our theorem.

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Appendix

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%input
syms x1 x2 x3 x4 x5 x6 x7 x8 x9
%Claim1
Q1=(x1–x2)∗(x1–x3)^2∗(x2–x3)∗(x1+x2)^4∗(x2+x3)^4;
C1=diff(diff(diff(Q1,x1,4),x2,4),x3,4)/factorial(4)/factorial(4)/
   factorial(4)
%Claim2(1)
Q21=(x1–x2)∗(x1–x3)∗(x2–x3)∗(x1–x4)∗(x2–x4)∗(x3–x4)∗(x1+x2+x3+x4)^4;
C21=diff(diff(diff(diff(Q21,x1,4),x2,3),x3,2),x4,1)/factorial(4)/
    factorial(3)/2/1
%Claim2(2)
Q22=(x1–x2)∗(x1+x2)^5;
C22=diff(diff(Q22,x1,4),x2,2)/factorial(4)/2
%Claim3(1)
Q31=(x1–x2)∗(x1–x3)∗(x2–x3)∗(x1–x4)∗(x2–x4)∗(x3–x4)∗(x1+x2+x3+x4)^4;
C31=diff(diff(diff(diff(Q31,x1,4),x2,3),x3,2),x4,1)/factorial(4)/
    factorial(3)/2/1
%Claim3(2)
Q32=(x1–x2)∗(x2–x3)∗(x1–x3)∗(x1+x2+x3)^5;
C32=diff(diff(diff(Q32,x1,4),x2,3),x3,1)/factorial(4)/factorial(3)/1
%Claim4(1)
Q41=(x1–x2)∗(x1–x3)∗(x1–x4)∗(x1–x5)∗(x1–x6)∗(x1–x7)∗(x1–x8)∗(x2–x3)∗(x2–
    x4)∗(x2–x5)∗(x2–x6)∗(x2–x7)∗(x2–x8)∗(x3–x4)∗(x3–x5)∗(x3–x6)∗(x3–x7)∗
    (x3–x8)∗(x4–x5)∗(x4–x6)∗(x4–x7)∗(x4–x8)∗(x5–x6)∗(x5–x7)∗(x5–x8)∗
    (x6–x7)∗(x6–x8)∗(x7–x8)∗(x1+x2+x3+x4+x5+x6+x7+x8)^2;C41=diff(diff
    (diff(diff(diff(diff(diff(Q41,x1,7),x2,5),x3,4),x4,3),x5,2),x6,1),
    x8,8)/factorial(8)/factorial(7)/factorial(5)/factorial(4)/factorial
    (3)/2/1
%Claim4(2)
Q42=(x1–x2)∗(x1–x3)∗(x2–x3)∗(x1–x4)∗(x2–x4)∗(x3–x4)∗(x1–x5)∗(x2–x5)∗(x3–
    x5)∗(x4–x5)∗(x1+x2+x3+x4+x5)^5;C42=diff(diff(diff(diff(diff(Q42,x1,
    5),x2,4),x3,3),x4,2),x5,1)/factorial(5)/factorial(4)/factorial(3)/
    2/1
%Claim5(1)
Q51=(x1–x2)∗(x1–x3)∗(x1–x4)∗(x1–x5)∗(x1–x6)∗(x1–x7)∗(x1–x8)∗(x1–x9)∗(x2–
    x3)∗(x2–x4)∗(x2–x5)∗(x2–x6)∗(x2–x7)∗(x2–x8)∗(x2–x9)∗(x3–x4)∗(x3–x5)
    ∗(x3–x6)∗(x3–x7)∗(x3–x8)∗(x3–x9)∗(x4–x5)∗(x4–x6)∗(x4–x7)∗(x4–x8)∗
    (x4–x9)∗(x5–x6)∗(x5–x7)∗(x5–x8)∗(x5–x9)∗(x6–x7)∗(x6–x8)∗(x6–x9)∗(x7
    –x8)∗(x7–x9)∗(x8–x9)∗(x1+x2+x3+x4+x5+x6+x7+x8+x9)^2;
```

```
C51=diff(diff(diff(diff(diff(diff(diff(diff(Q51,x1,9),x2,6),x3,5),x4,4),
    x5,3),x6,2),x7,1),x9,8)/factorial(9)/factorial(8)/factorial(6)/
    factorial(5)/factorial(4)/factorial(3)/2/1
%Claim5(2)
Q52=(x1−x2)*(x1−x3)*(x2−x3)*(x1−x4)*(x2−x4)*(x3−x4)*(x1−x5)*(x2−x5)*(x3−
    x5)*(x4−x5)*(x1+x2+x3+x4+x5)^5;C52=diff(diff(diff(diff(diff(Q52,x1,
    5),x2,4),x3,3),x4,2),x5,1)/factorial(5)/factorial(4)/factorial(3)/
    2/1
%output
C1=20  C21=1  C22=5  C31=1  C32=5  C41=−1  C42=1  C51=−1  C52=1
```

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