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A new graph parameter and a construction of larger graph without increasing radio k-chromatic number

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Abstract For a positive integer $k \geq 2$, the radio k-coloring problem is an assignment L of non-negative integers (colors) to the vertices of a finite simple graph G satisfying the condition $|L(u) - L(v)| \geq k + 1 - d(u, v)$, for any two distinct vertices u, v of G and $d(u, v)$ being distance between u, v. The span of L is the largest integer assigned by L, while 0 is taken as the smallest color. An rc_k-coloring on G is a radio k-coloring on G of minimum span which is referred as the radio k-chromatic number of G and denoted by $rc_k(G)$. An integer h, $0 < h < rc_k(G)$, is a hole in a rc_k-coloring on G if h is not assigned by it. In this paper, we construct a larger graph from a graph of a certain class by using a combinatorial property associated with $(k - 1)$ consecutive holes in any rc_k-coloring of a graph. Exploiting the same property, we introduce a new graph parameter, referred as $(k - 1)$-hole index of G and denoted by $\rho_k(G)$. We also explore several properties of $\rho_k(G)$ including its upper bound and relation with the path covering number of the complement G^c.

Keywords Radio k-coloring · Hole · Maximum degree · Domination number · Path covering number

Mathematics Subject Classification 05C78 · 05C15

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1 Introduction

In wireless communication networks, the frequency assignment problem (FAP) is a well-known problem which deals with the assignment of frequencies (non-negative integers) to transmitters in an optimal manner such that the potential risk of interference, due to proximity of two transmitters having same or nearly same frequencies, is averted. The radio k-coloring problem, introduced by Chartrand et al. (2005) is a graph-theoretic variant of FAP where the vertices represent the transmitters and the edges are assigned according to the proximity of the transmitters. For any positive integer $k \geq 2$, a radio k-coloring L of a finite simple graph G is a mapping $L : V \rightarrow \mathbb{N} \cup \{0\}$ such that for any two vertices u, v in G,

$$|L(u) - L(v)| \geq k + 1 - d(u, v) \tag{1}$$

The span of a radio k-coloring L is defined to be $(max_{v \in V} L(v) - min_{v \in V} L(v))$ and is denoted by *span(L)*. The *radio k-chromatic number* of G, denoted by $rc_k(G)$, is defined to be $min_L\{span(L) : L \text{ is a radio } k\text{-coloring of } G\}$. Without loss of generality, we shall assume $min_{v \in V} L(v) = 0$ for any radio k-coloring L on G. Any radio k-coloring L on G with span $rc_k(G)$ is referred as $rc_k(G)$-coloring or simply rc_k-coloring (when the underlying graph is fixed). An integer $h,\ 0 < h < rc_k(G)$, is said to be a *hole* in a rc_k-coloring L on G if it is not assigned as a color to any vertex of G by L.

So far the radio k-coloring problem has been mostly studied for $k = diam(G)$, $k = diam(G) - 1$, $k = diam(G) - 2$, $k = 3$, $k = 2$. For $k = 2$, the problem becomes the $L(2, 1)$-coloring problem introduced by Griggs and Yeh (1992). Also note that $rc_2(G)$ is denoted as $\lambda_{2,1}(G)$ and rc_2-coloring of G is referred as $\lambda_{2,1}$-coloring of G. Any graph G which admits a $\lambda_{2,1}$-coloring without any hole is said to be full colorable.

The $L(2, 1)$-coloring problem has received extensive attention. Readers may find an extensive survey in Calamoneri (2011). Several interesting combinatorial properties of holes of a $\lambda_{2,1}$-coloring as well as existence of full colorable graphs have been studied in Adams et al. (2007), Fishburn and Roberts (2006) and Georges and Mauro (2005). On the other hand, for $k \neq 2$, the radio k-coloring problem has been studied for very few limited families of graphs including paths, trees and cycles (Chartrand et al. 2000; Khennoufa and Togni 2005; Li et al. 2010; Liu 2008), powers of paths and cycles (Liu and Xie 2004, 2009; Saha and Panigrahi 2012), toroidal grids (Saha and Panigrahi 2013) etc. Also lower bound for radio k-chromatic number has been studied in Saha and Panigrahi (2015). But no significant attempt has been made so far to understand the behaviour of holes in any rc_k-coloring, up to the best of our knowledge. Besides, constructing a larger graph G' containing a given graph G with $rc_k(G') = rc_k(G)$ is also an interesting combinatorial problem. Given a rc_k-coloring of G, this problem becomes even more intriguing if a rc_k-coloring of G' is required to maintain the previous assignment of colors to the vertices of G.

2 Perspective and our contribution

Due to rapid growth in the use of wireless communication services and the corresponding scarcity and the high cost of radio spectrum bandwidth, an important task is to construct an extended network (connected graph) from a given network (graph) such that (i) the frequency (colors) allocation scheme to the old stations (vertices) remains same and (ii) the unused frequencies for the old system can be assigned to the new stations. To address this issue, we study an interesting combinatorial property of consecutive holes of an rc_k-coloring of a graph. Based on the fact that no rc_k-coloring on G contains k consecutive holes (Sarkar and Adhikari 2015), we explore the structural properties of graphs whose every rc_k-coloring has $(k-1)$ consecutive holes and show that if such a graph G belongs to a certain class then G is an induced subgraph of a graph G^* in the same class such that $rc_k(G^*) = rc_k(G)$ and at least one rc_k-coloring on G^* contains at most $(k-2)$ consecutive holes. While describing an rc_k-coloring of G^* with such desired feature, we use the colors of an rc_k-coloring of G as well as some of its holes. As the holes in an rc_k-coloring are unused frequencies in between the allotted spectrum, this enables us to think of an expanded network whose frequency assignment is done by already used as well as some unused frequencies of the smaller network.

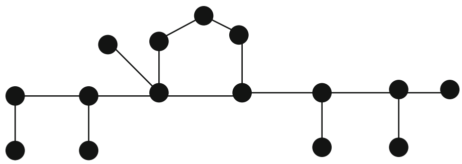
We refer the minimum number of occurrence of $(k-1)$ consecutive holes in any rc_k-coloring on G as the $(k-1)$-hole index of G and denote by $\rho_k(G)$ or simply ρ_k if there is no confusion regarding the graph. We show $\Delta(G)$, the maximum degree of a vertex in G, is a general upper bound of $\rho_k(G)$ and study the structure of G when $\rho_k(G) = \Delta(G)$ and $\rho_k(G) = \Delta(G) - 1$ for any $k \geq 2$. We also give another class such that for any graph G with n vertices in the class, if $rc_k(G) \geq (n-1)(k-1)$ then $\rho_k(G) = c(G^c) - 1$ and if $c(G^c) \geq 2$ then $\rho_{k_1}(G) = \rho_{k_2}(G)$, for $k_1,\ k_2 \geq 2$, where $c(G^c)$ is the path covering number of the complement G^c i.e., minimum number of vertex disjoint paths required to exhaust all vertices of G^c. These findings eventually extend the works of Georges and Mauro (2005) on hole index, i.e., the minimum number of holes in any rc_2-coloring, of any graph G and its similar relations with $\Delta(G)$ and $c(G^c)$. We further extend the notion of the island sequence (Georges and Mauro 2005), which is a non-decreasing sequence of cardinalities of maximal sets of consecutive integers used by an rc_2-coloring on G with minimum holes, in a general perspective of radio k-coloring for any $k \geq 2$ and prove some interesting results.

3 Preliminaries

Throughout this paper, unless otherwise stated, graphs are taken as finite and simple with at least two vertices and we assume $k \geq 2$. The vertex set of a graph G is denoted by $V(G)$. For any rc_k-coloring L on G, let $L_i = \{v \in V(G) | L(v) = i\}$ and $l_i = |L_i|$.

For definitions of disjoint union and Cartesian product of graphs (denoted by $G + H$ and $G \square H$ respectively, for two graphs G and H), Hamiltonian path, domination number (denoted by $\gamma(G)$), radius and centre of G (denoted by $rad\ G$ and $C(G)$ respectively) the reader is referred to West (2001). The disjoint union of l copies of G is denoted by lG. A set $S \subset V(G)$ is said to be a perfect dominating set if every vertex in $V(G) \setminus S$ is adjacent to exactly one vertex in S (see Haynes et al. 1998). The minimum cardinality of a perfect dominating set of G is the perfect domination number of G, denoted by $\gamma_p(G)$. Clearly $\gamma(G) \leq \gamma_p(G)$. The complete graph, cycle and path with n vertices are denoted as K_n, C_n and P_n respectively.

Fig. 1 Illustrating an example of $\mathcal{H}_k$, for $2 \leq k \leq 7$



We now define a new class of graphs for which we will show later that given any graph G in that class, either $\rho_k(G) = 0$ or G is an induced subgraph of G' in the same class such that $rc_k(G') = rc_k(G)$ and $\rho_k(G') = 0$ (see Theorem 4). Let $\mathcal{H}_k$ be the class of simple finite graphs such that $G \in \mathcal{H}_k$ if and only if for every edge e in G, there exists a vertex $\omega_e \in V(G)$ which is at a distance $\lfloor \frac{k}{2} \rfloor$ from one vertex incident to e and at a distance $\lceil \frac{k}{2} \rceil + 1 - k(mod\ 2)$ from the other vertex incident to e.

Remark 1 Note that $\mathcal{H}_k \subseteq \mathcal{H}_{k-1}$ for $k \geq 3$.

Explanation: Let $G \in \mathcal{H}_k$. Also let $e = (x,\ y)$ be any edge in G.

Case I Let $k = 2r$ and $\omega_e \in V(G)$ be such that $d(x,\ \omega_e) = r$ and $d(y,\ \omega_e) = r+1$. Let ω be the vertex immediately before w_e on a shortest path from x to ω_e. Then $d(x,\ \omega) = r - 1$ and $d(y,\ \omega) = r$. Since e is any edge in G, $G \in \mathcal{H}_{2r-1} = \mathcal{H}_{k-1}$.

Case II Let $k = 2r + 1$ and $\omega_e \in V(G)$ be such that $d(x,\ \omega_e) = r$ and $d(y,\ \omega_e) = r + 1$. Since e is any edge in G, $G \in \mathcal{H}_{2r} = \mathcal{H}_{k-1}$.

Examples of $\mathcal{H}_k$:

(i) $C_n \in \mathcal{H}_k$, where $n \geq 4$ and $2 \leq k \leq n - 1 - n\ (mod\ 2)$.
(ii) $C_n \Box P_m \in \mathcal{H}_k$, where $n \geq 4$, $m \geq 2$ and $2 \leq k \leq n - 1 - n\ (mod\ 2) + m - m\ (mod\ 2)$.
(iii) Let T be a tree. Then $T \in \mathcal{H}_k$, where $2 \leq k \leq 2.rad\ T + (-1)^{|C(T)|-1}$ and $rad\ T = min_{u \in V(T)}\{max_{v \in V(T)} d(u,\ v)\}$. Also $|C(T)|$ is the number of vertices u of T such that $max_{v \in V(T)} d(u,\ v)$ is minimum.
(iv) The graph in Figure 1 is an example of $\mathcal{H}_k$, for $2 \leq k \leq 7$.

Also let $\mathcal{G}_1$ be the class of triangle-free graphs and $\mathcal{G}_2$ be the family of graphs with a Hamiltonian path in every component of their complements. A complete bipartite graph, the Cayley graph $G = Cay(\mathbb{Z}_n, S)$, where $S = \mathbb{Z}_n \setminus \{\overline{0}, \overline{m}, -\overline{m}\}$ such that $gcd(m, n) > 1$, are some examples of $\mathcal{G}_2$. For these two classes, the following results will help us later to get some interesting properties of $\rho_k(G)$ (see Theorem 9 and Corollary 3).

Theorem 1 (Sarkar and Adhikari 2015) *Let $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ be a graph with n vertices. Then*

(i) $rc_k(G) \leq (n-1)(k-1)$ *if and only if* G^c *has a Hamiltonian path.*
(ii) $rc_k(G) = n(k-1) + r - k$ *if and only if* $c(G^c) = r$, *when* $r \geq 2$.

Beside this, the following lower bound for domination number of graph will be used in the proof of Theorem 7.

Theorem 2 (Haynes et al. 1998; Walikar et al. 1979) *For any graph G of order n,* $\gamma(G) \geq \lceil \frac{n}{\Delta(G)+1} \rceil$.

4 Preparatory definitions and results

We start with the following definitions.

Definition 1 A $(k-1)$-hole in a rc_k-coloring L on a graph G is a sequence of $(k-1)$ consecutive holes in L. If L has $(k-1)$ consecutive holes $i+1, i+2, \ldots, i+k-1$, then the corresponding $(k-1)$-hole is denoted by $\{i+1, i+2, \ldots, i+k-1\}$.

Remark 2 The minimum number of $(k-1)$-holes in any rc_k-coloring on G is the $(k-1)$-hole index of G. We refer the collection of all rc_k-colorings on G with $\rho_k(G)$ number of $(k-1)$-holes as $\Lambda_{\rho_k}(G)$.

Definition 2 The minimum span of a radio k-coloring on G with at most $(k-2)$ consecutive holes is defined as max-$(k-2)$-hole span of G and denoted by $\mu_k(G)$.

Remark 3 It is easy to observe that $rc_k(G) \le \mu_k(G)$, for any graph G. Clearly if $rc_k(G) = \mu_k(G)$, then $\rho_k(G) = 0$.

Definition 3 Let $L \in \Lambda_{\rho_k}(G)$. An Ω-set of L is a non-empty set A of non-negative integers, assigned by L, such that $s \in A$ only if $0 \le |s - s'| < k$, for some $s' \in A$. A maximal Ω-set of $L \in \Lambda_{\rho_k}(G)$ is a k-island or simply an island of L. The minimum and maximum element of an island I are said to be the left and right coast of I respectively, denoted by $lc(I)$ and $rc(I)$ accordingly. The left and right coasts of I are together called the coastal elements of I.

We now prove some results which will be useful in proving our main results.

Lemma 1 *For $k \ge 2$, let L be an rc_k-coloring of a graph G with a $(k-1)$-hole $\{i+1,\ i+2, \ldots,\ i+k-1\}$, where $i \ge 0$. Then there are two vertices $u \in L_i$ and $v \in L_{i+k}$ such that u and v are adjacent in G.*

Proof If possible, let no vertex of L_i be adjacent to any vertex in L_{i+k} in G. Define a new radio k-coloring $\hat{L}$ given by

$$\hat{L}(u) = \begin{cases} L(u), & \text{if } L(u) \le i, \\ L(u) - 1, & \text{if } L(u) \ge i + k. \end{cases}$$

Since no vertex of $\hat{L}_i$ is adjacent to a vertex of $\hat{L}_{i+k-1}$ in G and L is a proper radio k-coloring of G, $\hat{L}$ is a proper radio k-coloring of G with span $rc_k(G) - 1$, a contradiction. This completes the proof. □

In the next result, we consider G satisfies $\rho_k(G) \ge 1$.

Lemma 2 *Let G be a graph with $\rho_k(G) \ge 1$ and $L \in \Lambda_{\rho_k}(G)$. If $\{i+1,\ i+2, \ldots,\ i+k-1\}$ is a $(k-1)$-hole in L, then $l_i = l_{i+k}$ and the subgraph of G induced by $L_i \cup L_{i+k}$ is $l_i K_2$.*

Proof **Case I** Let $l_{i+k} \geq 2$. If possible, let $x \in L_{i+k}$ be such that x is not adjacent to any vertex in L_i. Define $\hat{L}$ by

$$\hat{L}(u) = \begin{cases} L(u), & \text{if } u \neq x, \\ i + k - 1, & \text{if } u = x. \end{cases}$$

Hence $\hat{L}$ is an rc_k-coloring with fewer $(k-1)$-holes, leading to a contradiction.
Case II Let $l_{i+k} = 1$ and $L_{i+k} = \{x\}$. If possible, let x be not adjacent to any vertex in L_i. Define $\hat{L}$ by

$$\hat{L}(u) = \begin{cases} L(u), & \text{if } L(u) \leq i, \\ L(u) - 1, & \text{if } L(u) \geq i + k. \end{cases}$$

Clearly, $\hat{L}$ is a radio k-coloring with span $rc_k(G) - 1$, a contradiction.

The above two cases suggest that each vertex of L_{i+k} is adjacent to some vertex in L_i. Also no two vertices in L_{i+k} can be adjacent to the same vertex in L_i. Hence $l_i \geq l_{i+k}$.

Now $l_{i+k} \geq 1$. If $l_i = 1$, we are done. Let $l_i \geq 2$. Then by a similar argument as before we can show each vertex $u \in L_i$ is adjacent to a unique vertex in L_{i+k}. Therefore $l_{i+k} \geq l_i$. Hence we have $l_i = l_{i+k}$.

Since no two vertices in L_i can be adjacent to the same vertex in L_{i+k} in G and vice versa, the subgraph of G induced by $L_i \cup L_{i+k}$ is $l_i K_2$.

This completes the proof. □

4.1 A recoloring from a coloring in $\Lambda_{\rho_k}(G)$ and an equivalence relation

Let $L \in \Lambda_{\rho_k}(G)$ and $I_0, I_1, \ldots, I_{\rho_k}$ be the islands of L such that $lc(I_{j+1}) = rc(I_j) + k$, $0 \leq j \leq \rho_k - 1$. For any i, $0 \leq i \leq \rho_k$, and m, $0 \leq m \leq (\rho_k - i)$, we define a new coloring $\hat{L}$ on G given by

$$\hat{L}(u) = \begin{cases} L(u), & \text{if } L(u) \notin \bigcup_{s=i}^{i+m} I_s, \\ rc(I_{i+m}) - t, & \text{if } L(u) \in \bigcup_{s=i}^{i+m} I_s \text{ and } L(u) = lc(I_i) + t,\ t \geq 0. \end{cases}$$

Clearly $\hat{L} \in \Lambda_{\rho_k}(G)$ and the islands of $\hat{L}$ are $I'_0, I'_1, \ldots, I'_{\rho_k}$ such that $I'_j = I_j$, for $j \notin \{i, i+1, \ldots, i+m\}$ and $rc(I'_{i+m-t}) = rc(I_{i+m}) - (lc(I_{i+t}) - lc(I_i))$, $lc(I'_{i+m-t}) = rc(I_{i+m}) - (rc(I_{i+t}) - lc(I_i))$, for $0 \leq t \leq m$. We refer this recoloring $\hat{L}$ as an α-recoloring of L.

Define a relation η on $\Lambda_{\rho_k}(G)$ given by $L_1 \eta L_2$ if and only if L_2 is obtained from L_1 by applying a finite number of suitable α-recolorings described above. Then η is an equivalence relation. Hence every radio k-coloring in $\Lambda_{\rho_k}(G)$ is η-related to some $L \in \Lambda_{\rho_k}(G)$ which has islands $I_0, I_1, \ldots, I_{\rho_k}$ with $lc(I_{j+1}) = rc(I_j) + k$, $0 \leq j \leq \rho_k - 1$ such that $|I_0| \leq |I_1| \leq \cdots \leq |I_{\rho_k}|$. Thus we assume, without loss of generality, that the islands $I_0, I_1, \ldots, I_{\rho_k}$ with $lc(I_{j+1}) = rc(I_j) + k$, $0 \leq j \leq \rho_k - 1$, of any $L \in \Lambda_{\rho_k}(G)$ satisfy the inequalities $|I_0| \leq |I_1| \leq \cdots \leq |I_{\rho_k}|$ and we refer the finite sequence $(|I_0|, |I_1|, \ldots, |I_{\rho_k}|)$ as island sequence of L.

Corollary 1 *Let G be a graph with $\rho_k(G) \geq 1$ and I, J be two distinct islands of $L \in \Lambda_{\rho_k}(G)$ with x and y as two coastal elements of I and J respectively. Then $l_x = l_y$ and the subgraph of G induced by $L_x \cup L_y$ is $l_x K_2$.*

Proof Considering some suitable α-recolorings, we can assume, without loss of generality, that $x = rc(I)$, $y = lc(J)$ and $y = x + k$. The proof follows from Lemma 2. □

Lemma 3 *Let G be a graph with $\rho_k(G) \geq 1$ and $\rho_k(G) = \Delta(G) - r$. Also let $L \in \Lambda_{\rho_k}(G)$ which has islands $I_0, I_1, \ldots, I_{\rho_k}$ with $lc(I_{j+1}) = rc(I_j)+k$, $0 \leq j \leq \rho_k - 1$ such that $|I_0| \leq |I_1| \leq \cdots \leq |I_{\rho_k}|$. Then $|I_j| = 1$, where $0 \leq j \leq \Delta(G) - 2r$.*

Proof If possible, let $|I_{\Delta(G)-2r}| \geq 2$. Then for every j, $\Delta(G) - 2r \leq j \leq \Delta(G) - r$, $lc(I_j) \neq rc(I_j)$ and these islands have total $2(r + 1)$ number of coastal elements. Let u be a vertex in G such that $L(u) = lc(I_0) = 0$. Then by Corollary 1, u is adjacent to at least one vertex from each of I_j, $1 \leq j \leq \Delta(G) - 2r - 1$ and at least two vertices from each of I_j, $\Delta(G) - 2r \leq j \leq \Delta(G) - r$ in G. Hence $d(u) \geq (\Delta(G) - 2r - 1) + 2(r + 1) = \Delta(G) + 1$, a contradiction. □

5 Main results

5.1 Larger graph with the same radio k-chromatic number

In this subsection, we obtain larger graph with reduced $(k - 1)$-hole index while keeping the radio k-chromatic number unchanged.

Theorem 3 *Let $G \in \mathcal{H}_k$ with $\rho_k(G) \geq 1$. Then there is a graph $G^* \in \mathcal{H}_k$ such that*

(i) *G is an induced subgraph of G^*;*
(ii) *$rc_k(G^*) = rc_k(G)$;*
(iii) *$\rho_k(G^*) = \rho_k(G) - 1$.*

Proof Let L be any rc_k-coloring on G. Since $\rho_k(G) \geq 1$, L has at least one $(k - 1)$-hole. Let $\{i+1, i+2, \ldots, i+k-1\}$ be a $(k-1)$-hole in L where i is an non-negative integer. By Lemma 1, there are two vertices $x \in L_i$ and $y \in L_{i+k}$ so that $e = (x, y)$ is an edge in G.
Case I Let ω_e be the vertex in G such that $d(x, \omega_e) = \lfloor \frac{k}{2} \rfloor$ and $d(y, \omega_e) = \lceil \frac{k}{2} \rceil + 1 - k(mod\ 2)$. We construct a new graph G^* by adding a new vertex ω^* and a new edge (ω^*, ω_e) to G. Note that $G^* \in \mathcal{H}_k$, since $G \in \mathcal{H}_k$ and $x = \omega_{e'}$, where $e' = (\omega_e, \omega^*)$. Also G is an induced subgraph of G^*. Define a new coloring L^* on G^* given by

$$L^*(u) = \begin{cases} L(u), & \text{if } u \in V(G) \\ i + \lceil \frac{k}{2} \rceil, & \text{if } u = \omega^* \end{cases}$$

We shall show that L^* is a proper radio k-coloring on G^*.
Now since $i+1,\ i+2,\ldots,\ i+k-1$ are consecutive holes in L, either

$$L^*(\omega_e) \geq L^*(y) + k + 1 - d(\omega_e,\ y) = i + k + \lfloor\frac{k}{2}\rfloor + k(mod\ 2) \tag{2}$$

or

$$L^*(\omega_e) \leq L^*(x) - (k + 1 - d(\omega_e,\ x)) = i - \lceil\frac{k}{2}\rceil - 1 \tag{3}$$

From Eq. (2), $L^*(\omega_e) - L^*(\omega^*) \geq k$ and from Eq. (3), $L^*(\omega^*) - L^*(\omega_e) > k$. Thus

$$|L^*(\omega^*) - L^*(\omega_e)| \geq k \tag{4}$$

Again

$$|L^*(\omega^*) - L^*(x)| = k + 1 - d(\omega^*,\ x) \tag{5}$$

and

$$|L^*(\omega^*) - L^*(y)| \geq k + 1 - d(\omega^*,\ y) \tag{6}$$

Let $u \in V(G^*)\backslash\{x,\ y,\ \omega_e,\ \omega^*\}$ be in the same connected component of G^* containing ω^*.

Case I(a) Let $d(x,\ u) = k - r_1$ and $d(y,\ u) = k - r_2$, where $0 \leq r_1,\ r_2 \leq k - 1$.

Now, $d(x,\ u) = k - r_1 \Rightarrow |L^*(u) - L^*(x)| \geq r_1 + 1$. This implies either $L^*(u) \leq i - r_1 - 1$ or $L^*(u) \geq i + r_1 + 1$, i.e., either $L^*(u) \leq i - r_1 - 1$ or $L^*(u) \geq i + k$, since $0 \leq r_1 \leq k - 1$ and $i+1,\ i+2,\ldots,\ i+k-1$ are holes in L.

Again, $d(y,\ u) = k - r_2 \Rightarrow |L^*(u) - L^*(y)| \geq r_2 + 1$. This implies either $L^*(u) \geq i + k + r_2 + 1$ or $L^*(u) \leq i + k - r_2 - 1$, i.e., either $L^*(u) \geq i + k + r_2 + 1$ or $L^*(u) \leq i$ since $0 \leq r_2 \leq k - 1$ and $i+1,\ i+2,\ldots,\ i+k-1$ are holes in L.

As $d(x,\ u) = k - r_1$ and $d(y,\ u) = k - r_2$, $0 \leq r_1,\ r_2 \leq k - 1$, so considering the above two situations, we get either $L^*(u) \leq i - r_1 - 1$ or $L^*(u) \geq i + k + r_2 + 1$.

Now let $L^*(u) \leq i - r_1 - 1$. Then $L^*(\omega^*) - L^*(u) \geq \lceil\frac{k}{2}\rceil + r_1 + 1$.

Also, $d(\omega_e,\ u) \geq d(x,\ u) - d(\omega_e,\ x) = \lceil\frac{k}{2}\rceil - r_1$. So, $d(\omega^*,\ u) \geq \lceil\frac{k}{2}\rceil + 1 - r_1$ implying $k + 1 - d(\omega^*,\ u) \leq \lfloor\frac{k}{2}\rfloor + r_1$. Hence we have $L^*(\omega^*) - L^*(u) > k + 1 - d(\omega^*,\ u)$. Therefore $|L^*(\omega^*) - L^*(u)| > k + 1 - d(\omega^*,\ u)$, if $L^*(u) \leq i - r_1 - 1$.

Consider the other way, i.e., let $L^*(u) \geq i + k + r_2 + 1$. Then $L^*(u) - L^*(\omega^*) \geq \lfloor\frac{k}{2}\rfloor + r_2 + 1$.

But $d(\omega_e,\ u) \geq d(y,\ u) - d(\omega_e,\ y) \Rightarrow d(\omega^*,\ u) \geq \lceil\frac{k}{2}\rceil - r_2 \Rightarrow k + 1 - d(\omega^*,\ u) \leq \lfloor\frac{k}{2}\rfloor + r_2 + 1$. Hence we have $L^*(u) - L^*(\omega^*) \geq k + 1 - d(\omega^*,\ u)$, i.e., $|L^*(u) - L^*(\omega^*)| \geq k + 1 - d(\omega^*,\ u)$, if $L^*(u) \geq i + k + r_2 + 1$.

Therefore

$$|L^*(\omega^*) - L^*(u)| \geq k + 1 - d(\omega^*,\ u), \tag{7}$$

whenever $u \in V(G^*) \setminus \{x,\ y,\ \omega_e,\ \omega^*\}$ with $d(x,\ u) = k - r_1$ and $d(y,\ u) = k - r_2$, $0 \leq r_1,\ r_2 \leq k - 1$.

Case I(b) Let $d(x,\ u) = k$ and $d(y,\ u) = k+1$. Then $L^*(u) \leq i-1$ or $L^*(u) \geq i+k$. Also, $d(\omega_e,\ u) \geq d(x,\ u) - d(\omega_e,\ x) = \lceil\frac{k}{2}\rceil$. So, $k + 1 - d(\omega^*,\ u) \leq \lfloor\frac{k}{2}\rfloor \leq |L^*(\omega^*) - L^*(u)|$.

Since x and y are adjacent in G, $|d(x,\ u) - d(y,\ u)| \leq 1$. Hence *Case I.(b)*, together with equation (7) of *Case I.(a)*, implies

$$|L^*(\omega^*) - L^*(u)| \geq k + 1 - d(\omega^*,\ u), \tag{8}$$

whenever $u \in V(G^*) \setminus \{x,\ y,\ \omega_e,\ \omega^*\}$ with $d(x,\ u) = k - r, 0 \leq r \leq k - 1$.

Now the only remaining possibility of *Case I* is the following.

Case I(c) Let $d(x,\ u) = k + r$, $r \geq 1$. Then $d(y,\ u) \geq k + r - 1 \geq k$. Therefore $|L^*(\omega^*) - L^*(u)| \geq \lfloor \frac{k}{2} \rfloor$.

Now, $d(\omega_e,\ u) \geq d(x,\ u) - d(\omega_e,\ x) = \lceil \frac{k}{2} \rceil + r$. So, $k + 1 - d(\omega^*,\ u) \leq \lfloor \frac{k}{2} \rfloor - r < |L^*(\omega^*) - L^*(u)|$, as $r \geq 1$. Thus

$$|L^*(\omega^*) - L^*(u)| > k + 1 - d(\omega^*,\ u), \tag{9}$$

whenever $u \in V(G^*) \setminus \{x,\ y,\ \omega_e,\ \omega^*\}$ with $d(x,\ u) = k + r, r \geq 1$.

The equations (4), (5), (6), (8) and (9) together suggest that L^* is a proper radio k-coloring of G^*.

Case II Let ω_e be a vertex in G such that $d(y,\ \omega_e) = \lfloor \frac{k}{2} \rfloor$ and $d(x,\ \omega_e) = \lceil \frac{k}{2} \rceil + 1 - k(mod\ 2)$. Similar to **Case I**, we construct a new graph G^* by adding a new vertex ω^* and a new edge $(\omega^*,\ \omega_e)$ to G. Note that $G^* \in \mathcal{H}_k$, since $G \in \mathcal{H}_k$ and $y = \omega_{e'}$, where $e' = (\omega_e,\ \omega^*)$. Also G is an induced subgraph of G^*. Define a new coloring $\hat{L}$ on G^* by

$$\hat{L}(u) = \begin{cases} L(u), & \text{if } u \in V(G) \\ i + \lfloor \frac{k}{2} \rfloor, & \text{if } u = \omega^* \end{cases}$$

Similar argument, as in the former case, shows $\hat{L}$ is a proper radio k-coloring of G^*.

As in both cases, G is an induced subgraph of G^* and G^* admits a radio k-coloring of span $rc_k(G)$, so $rc_k(G^*) = rc_k(G)$.

Since L is an arbitrary rc_k-coloring on G, applying the above argument for any $L \in \Lambda_{\rho_k}(G)$, we get $\rho_k(G^*) \leq \rho_k(G) - 1$. Again since G is an induced subgraph of G^* which has exactly one vertex more than G, $\rho_k(G^*) \geq \rho_k(G) - 1$. Hence $\rho_k(G^*) = \rho_k(G) - 1$. This completes the proof. □

Theorem 4 *Let $G \in \mathcal{H}_k$ be a graph with $\rho_k(G) \geq 1$. Then there exists a graph $G^* \in \mathcal{H}_k$, such that*

(i) *G is an induced subgraph of G^*;*
(ii) *$rc_k(G^*) = rc_k(G)$;*
(iii) *$\rho_k(G^*) = 0$.*

Proof By repeated use of Theorem 3, the proof follows immediately. □

Recall that $\mu_k(G)$ is the minimum span of any radio k-coloring on G with at most $(k - 2)$ consecutive holes. We now prove the following result.

Theorem 5 *Let n be a positive integer such that $n = rc_k(G)$, for some $G \in \mathcal{H}_k$. Then there exists a graph $G^* \in \mathcal{H}_k$, containing G as its induced subgraph, such that $rc_k(G^*) = n$ and $\mu_k(G^*) = rc_k(G^*)$.*

Proof If $\rho_k(G) = 0$ then $G^* = G$, otherwise the proof follows from Theorem 4. □

We now prove an interesting result for $L(2, 1)$-coloring. Recall that for any graph G, $rc_2(G) = \lambda_{2,1}(G)$ and G is full colorable if $\rho_2(G) = 0$.

Corollary 2 *Let n be a positive integer such that $n = \lambda_{2,1}(G)$, for some $G \in \mathcal{H}_2$. Then there exists a graph $G^* \in \mathcal{H}_2$, such that*

(i) *G is an induced subgraph of G^*;*
(ii) *$\lambda_{2,1}(G^*) = n$;*
(iii) *G^* is full colorable.*

Proof If $\rho_2(G) = 0$ then $G^* = G$, otherwise the proof follows from Theorem 4. □

5.2 A general upper bound of $\rho_k(G)$ and related results

We begin with an upper bound of $\rho_k(G)$.

Theorem 6 *Let G be a graph with $\rho_k(G) \geq 1$. Then $\rho_k(G) \leq \Delta(G)$.*

Proof Let $L \in \Lambda_{\rho_k}(G)$ which has islands $I_0, I_1, \ldots, I_{\rho_k}$. Let x be a coastal element of I_0 and $u \in L_x$. Then u is adjacent to some $v_y \in L_y$ in G, for each coastal element y of I_j, $1 \leq j \leq \rho_k(G)$ by Corollary 1. Hence $d(u) \geq \rho_k(G)$. Therefore $\rho_k(G) \leq \Delta(G)$. □

We now explore the structure of G when $\rho_k(G)$ attains its upper bound.

Theorem 7 *Let G be a graph of order n with $\rho_k(G) \geq 1$ and $\rho_k(G) = \Delta(G) = \Delta$. Then*

(i) *G is a Δ-regular graph;*
(ii) *$rc_k(G) = k\Delta$;*
(iii) *$n \equiv 0 \ (mod\ \Delta + 1)$;*
(iv) *$\gamma(G) = \gamma_p(G) = \frac{n}{\Delta+1}$;*
(v) *If $n \neq \Delta + 1$, then $\mu_k(G) = rc_k(G) + 1$.*

Proof Let $L \in \Lambda_{\rho_k}(G)$ which has islands $I_0, I_1, \ldots, I_{\rho_k}$ with $lc(I_{j+1}) = rc(I_j)+k$, $0 \leq j \leq \rho_k - 1$ such that $|I_0| \leq |I_1| \leq \ldots \leq |I_{\rho_k}|$. Then L has $\rho_k + 1$ number of islands.

If possible, let I be an island of L such that $lc(I) \neq rc(I)$. Then by Corollary 1, for any vertex v with $L(v)$ as a coastal element of any other island J of L, $d(v) \geq \rho_k + 1 = \Delta + 1$, a contradiction. Hence for every island I of L, $lc(I) = rc(I)$ i.e., the islands are singleton sets. Thus, for every vertex v of G, $L(v)$ is a coastal element of some island of L and so degree of each vertex is same, by Corollary 1. Hence G is Δ-regular.

As the islands are singleton sets, so $I_j = \{kj\}$, $0 \le j \le \Delta$. Hence $rc_k(G) = k\Delta$.

Again, by Corollary 1, $l_{ki} = l_{kj} = l$ (say), $0 \le i < j \le \Delta$. Hence $n = l(\Delta + 1)$, i.e., $n \equiv 0 \pmod{\Delta + 1}$.

Since the subgraph induced by $L_{ki} \cup L_{kj}$ is lK_2, for any $i,\ j$ with $0 \le i < j \le \Delta$ by Corollary 1 and $I_i = \{ki\}$, $0 \le i \le \Delta$, L_{ki} is a dominating set as well as a perfect dominating set of G, for every i, $0 \le i \le \Delta$. Hence $\gamma(G) \le \frac{n}{\Delta+1}$. Using Theorem 2, we get $\gamma(G) = \frac{n}{\Delta+1}$. Hence $\gamma_p(G) = \frac{n}{\Delta+1}$.

Let $n \ne \Delta + 1$. Then $l \ge 2$. Since $lc(I_j) = rc(I_j)$, for every $0 \le j \le \Delta$, using Corollary 1, we get a path $P:\ v_0,\ v_1, \ldots,\ v_\Delta$ in G such that $L(v_j) \in I_j = \{kj\}$, $0 \le j \le \Delta$. Define a new coloring $\hat{L}$ on G by

$$\hat{L}(u) = \begin{cases} L(u), & \text{if } u \ne v_j,\ 0 \le j \le \Delta, \\ L(u) + 1, & \text{if } u = v_j,\ 0 \le j \le \Delta. \end{cases}$$

Note that $\hat{L}$ is a radio k-coloring of G without $(k-1)$ consecutive holes. Therefore $\mu_k(G) \le span(\hat{L}) = rc_k(G) + 1$. But since $\rho_k(G) > 0$, $\mu_k(G) > rc_k(G)$. Hence $\mu_k(G) = rc_k(G) + 1$. □

There may be graph G for which $\rho_k(G) < \Delta(G)$ with $rc_k(G) = k\Delta(G)$ for some $k \ge 2$. For example, consider $G = K_n + K_1, n \ge 2$. Here $\rho_k(G) = n-2 = \Delta(G)-1$ and $rc_k(G) = k(n-1) = k\Delta(G)$, $k \ge 2$. Again let $G_1 = K_2 + 2K_1$ and $G_2 = C_4$. Then $rc_2(G_1) = 2 = 2\Delta(G_1)$ and $\rho_2(G_1) = 0 = \Delta(G_1) - 1$. Also, $rc_2(G_2) = 4 = 2\Delta(G_2)$ and $\rho_2(G_2) = 1 = \Delta(G_2) - 1$.

In this perspective, the following theorem gives an insight into the structure of a graph G when $\rho_k(G) = \Delta(G) - 1$ and $rc_k(G) = k\Delta(G)$.

Theorem 8 *Let G be a graph with $\Delta(G) \ge 2$ and $\rho_k(G) = \Delta(G) - 1$. If $rc_k(G) = k\Delta(G)$, then $G = H + K_1$, where $\rho_k(H) = \Delta(H)$ and $rc_k(H) = k\Delta(H)$.*

Proof Let $rc_k(G) = k\Delta(G) = k\Delta$ and $L \in \Lambda_{\rho_k}(G)$ which has islands $I_0, I_1, \ldots, I_{\rho_k}$ with $lc(I_{j+1}) = rc(I_j) + k$, $0 \le j \le \rho_k - 1$, such that $|I_0| \le |I_1| \le \ldots \le |I_{\rho_k}|$. Since $\rho_k(G) = \Delta(G) - 1$, by Lemma 3, $|I_j| = 1$ and so $I_j = \{kj\}$ for $0 \le j \le \Delta - 2$. Therefore $lc(I_{\Delta-1}) = k\Delta - k$. Since $\rho_k(G) = \Delta - 1$ and $rc_k(G) = k\Delta$, $rc(I_{\Delta-1}) = k\Delta$ and $|I_{\Delta-1}| \ge 3$. Hence $lc(I_{\Delta-1}) \ne rc(I_{\Delta-1})$.

Now using Corollary 1, for every vertex u with $L(u) \in I_j$, for $0 \le j \le \Delta - 2$, we have $d(u) = \Delta$ and hence u is not adjacent to any vertex whose color is a non-coastal element of $I_{\Delta-1}$. Let v be a vertex such that $k\Delta - k < L(v) < k\Delta$. Then v is isolated in G. Therefore any vertex, which is colored with an element of $I_{\Delta-1}$ other than the coastal elements, can be suitably recolored to obtain an rc_k-coloring of G with fewer number of $(k-1)$-holes than $\rho_k(G)$, a contradiction, unless $I_{\Delta-1}$ has only one element, say x, other than its coastal elements and $l_x = 1$. Hence $G = H + K_1$, where $\rho_k(H) = \rho_k(G) + 1 = \Delta(H)$ and $rc_k(H) = rc_k(G) = k\Delta(G) = k\Delta(H)$. □

5.3 Relation among $(k-1)$-hole index, path covering number and island sequence

The following two results deal with $(k-1)$-hole index and path covering number. The last result of this subsection explores an interesting property of island sequence.

Theorem 9 *For any graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, $\rho_k(G) = c(G^c) - 1$, if $rc_k(G) \geq (n-1)(k-1)$, where n is the number of vertices of G.*

Proof Let $r = c(G^c)$.

Case I

Let $rc_k(G) = (n-1)(k-1)$. Then by Theorem 1, G^c has a Hamiltonian path, say, $P: x_0, x_1, \ldots, x_{(n-1)}$. Define a radio k-coloring L on G by $L(x_i) = i(k-1)$, $0 \leq i \leq n-1$. Then L is an rc_k-coloring on G and $\rho_k(G) = 0 = c(G^c) - 1$.

Case II

Let $rc_k(G) > (n-1)(k-1)$. Then by Theorem 1, $r \geq 2$ and $rc_k(G) = n(k-1)+r-k$. Let $\mathcal{P} = \{P^{(1)}, P^{(2)}, \ldots, P^{(r)}\}$ be a minimum path covering of G^c and let the j-th vertex, $1 \leq j \leq p_i$, of the $P^{(i)}$, $1 \leq i \leq r$, be x^i_j, where p_i is the number of vertices of $P^{(i)}$. We define a radio k-coloring L of G as $L(x^i_j) = (\Sigma_{t=1}^{i-1} p_t - i + j)(k-1) + (i-1)k$. Then $span(L) = n(k-1) + r - k$, i.e., L is a rc_k-coloring on G. Also L has $(r-1)$ number of $(k-1)$-holes. Hence $\rho_k(G) \leq r - 1 = c(G^c) - 1$.

Let $\hat{L} \in \Lambda_{\rho_k}(G)$ and $I_0, I_1, \ldots, I_{\rho_k(G)}$ be the islands of $\hat{L}$. Then the vertices in the set $A_i = \{u : \hat{L}(u) \in I_i\}$ form a path $Q^{(i)}$ (say) in G^c, for $0 \leq i \leq \rho_k(G)$. Thus $\{Q^{(0)}, Q^{(1)}, \ldots, Q^{(\rho_k(G))}\}$ is a path covering of G^c. Hence $c(G^c) \leq \rho_k(G) + 1$.

Therefore $\rho_k(G) = c(G^c) - 1$. This completes the proof. □

Corollary 3 *For any graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, if $c(G^c) \geq 2$ then $\rho_{k_1}(G) = \rho_{k_2}(G)$, for any $k_1, k_2 \geq 2$.*

Proof Proof follows directly from Theorems 1 and 9. □

Corollary 4 *For any graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, if $c(G^c) \geq 2$, then there exists a finite sequence of positive integers which is admitted as an island sequence by some $L \in \Lambda_{\rho_k}(G)$, for every $k \geq 2$.*

Proof Let $r = c(G^c) \geq 2$. Also let $\mathcal{P} = \{P^{(1)}, P^{(2)}, \ldots, P^{(r)}\}$ be a minimum path covering of G^c and p_i be the number of vertices of $P^{(i)}$. Now by Theorem 1, for any $k \geq 2$, $rc_k(G) = n(k-1) + r - k$. Without loss of generality, we assume $p_1 \leq p_2 \leq \cdots \leq p_r$. Let the j-th vertex, $1 \leq j \leq p_i$, of $P^{(i)}$, $1 \leq i \leq r$, be x^i_j. We define a radio k-coloring L of G as $L(x^i_j) = (\Sigma_{t=1}^{i-1} p_t - i + j)(k-1) + (i-1)k$. Then L is a rc_k-coloring on G with $(r-1) = \rho_k(G)$ number of $(k-1)$-holes, by Theorem 9. So $L \in \Lambda_{\rho_k}(G)$. Now the colors of vertices in each path $P^{(i)}$ together form an island of L and each vertex receives distinct color. Therefore the island sequence of L is $(p_1, p_2, \ldots, p_r)$. As $k \geq 2$ is arbitrary, we are done. □

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