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On (s, t)-relaxed strong edge-coloring of graphs

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Abstract In this paper, we introduce a new relaxation of strong edge-coloring. Let G be a graph. For two nonnegative integers s and t, an (s, t)-relaxed strong k-edge-coloring is an assignment of k colors to the edges of G, such that for any edge e, there are at most s edges adjacent to e and t edges which are distance two apart from e assigned the same color as e. The (s, t)-relaxed strong chromatic index, denoted by $\chi'_{(s,t)}(G)$, is the minimum number k of an (s, t)-relaxed strong k-edge-coloring admitted by G. This paper studies the (s, t)-relaxed strong edge-coloring of graphs, especially trees. For a tree T, the tight upper bounds for $\chi'_{(s,0)}(T)$ and $\chi'_{(0,t)}(T)$ are given. And the (1, 1)-relaxed strong chromatic index of an infinite regular tree is determined. Further results on $\chi'_{(1,0)}(T)$ are also presented.

Keywords Strong edge-coloring · Strong chromatic index · (s, t)-relaxed strong edge-coloring · (s, t)-relaxed strong chromatic index · Tree · Infinite Δ-regular tree

Mathematics Subject Classification 05C15

1 Introduction

In this paper, we consider undirected and simple graphs only, and we use standard notations in graph theory (cf. Bondy and Murty (2008)). Let G be a graph. Two edges e_1 and e_2 of G are adjacent (at distance one) if they meet at a common vertex; and

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two edges e_1 and e_2 are distance two apart if they are nonadjacent but adjacent to a common edge in G. The *degree* of an edge e (resp. a vertex u), denoted by $d(e)$ (resp. $d(u)$), is the number of edges (resp. vertices) adjacent to e (resp. u) in G. The maximum edge degree (resp. vertex degree) of a graph G is denoted by $\Delta_e(G)$ (resp. $\Delta(G)$), or simply Δ_e (resp. Δ) if there is no risk of confusion. For an edge e (resp. a vertex u), let $N(e)$ (resp. $NE(u)$) be the set of edges adjacent to e (resp. incident with u). Let a and b be two nonnegative integers with $a \leq b$. By $[a, b]$ we denote the set of integers $a, a+1, \ldots, b$.

A *strong k-edge-coloring* of G is an assignment of k colors to the edges of G in such a way that any two edges within distance two apart are assigned different colors. The *strong chromatic index*, denoted by $s\chi'(G)$, is the minimum number k of a strong k-edge-coloring admitted by G.

Strong edge-coloring was first studied by Fouquet and Jolivet (1983, 1984) for cubic planar graphs. The following conjecture was posed by Erdős and Nešetřil (1989):

Conjecture 1.1 *If G is a graph with maximum degree* Δ*, then*

$$s\chi'(G) \leq \begin{cases} \dfrac{5\Delta^2}{4} - \dfrac{\Delta}{2} + \dfrac{1}{4}, & \text{if } \Delta \text{ is odd;} \\ \dfrac{5\Delta^2}{4}, & \text{if } \Delta \text{ is even.} \end{cases}$$

Since then the bounds for strong chromatic index have been studied extensively. For graphs with $\Delta = 3$, Conjecture 1.1 was confirmed by Andersen (1992) and by Horák et al. (1993) independently. For $\Delta = 4$, Conjecture 1.1 says that $s\chi'(G) \leq 20$. At present, $s\chi'(G) \leq 22$ was the best result proved by Cranston (2006). We refer the readers for surveys (Horák 1990; Mahdian 2000; Molloy and Reed 1997; Wu and Lin 2008) about the strong chromatic index.

An application of strong edge-coloring is a channel assignment problem in wireless radio networks (Barrett et al. 2006). The graph (refer as interference graph) describes the network of transmitters. Colors of edges correspond to channels assigned to transmitters. To avoid interference, "close" transmitters (corresponding to the edges within distance two apart) are required to receive distinct channels. The main aim is to minimize the span of channels assigned to transmitters.

However, problems may be arisen if the channel resource is limited (or equivalently the channel span is restricted). That is to say, it is possible that with given channel span, one cannot construct a good strong edge-coloring. In this case, some kind of relaxation is necessary in assigning channels to transmitters. Gyárfás and Hubenko (2006) gave a kind of relaxation which was called semistrong edge coloring. In this paper, we introduce another kind of relaxation called (s, t)-relaxed strong edge-coloring, where s and t are two nonnegative integers.

In a strong edge coloring of a graph G, two edges that are distance 1 apart must receive different colors, and two edges that are distance 2 apart must also receive different colors. We call the former constraint 'distance one condition' and the later one 'distance two condition'. In an (s, t)-relaxed strong edge-coloring, both distance one and distance two conditions are relaxed to some extension, respectively. We formally define this concept below.

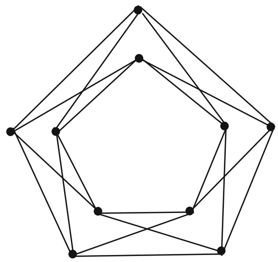

Fig. 1 The composition of C_5 and K_2^c

For any edge e of G, the edges adjacent to e are called *neighbors* of e, and the edges which are distance two apart from e are called 2-*neighbors* of e. Suppose f is an assignment of k colors to the edges of G. If, for any edge e of G, the number of neighbors of e with color $f(e)$ is at most s, and the number of 2-neighbors of e with color $f(e)$ is at most t, then f is called an (s, t)-*relaxed strong k-edge-coloring* of G. The (s, t)-*relaxed strong chromatic index* of G, denoted by $\chi'_{(s,t)}(G)$, is the minimum number k of an (s, t)-relaxed strong k-edge-coloring admitted by G.

Clearly, a $(0, 0)$-relaxed strong k-edge-coloring of G is a strong k-edge-coloring of G and so $\chi'_{(0,0)}(G) = s\chi'(G)$ for any graph G. In an (s, t)-relaxed strong edge-coloring of a graph G with maximum degree Δ, if $s = 0$ then, for any edge e, there are at most $2(\Delta - 1)$ edges that are distance two apart from e can receive the same color as e. Thus, if $s = 0$ and $t \geq 2(\Delta - 1)$ then $\chi'_{(s,t)}(G) = \chi'(G)$. An extremal case is that if $s \geq 2(\Delta - 1)$ and $t \geq 2(\Delta - 1)^2$, then $\chi'_{(s,t)}(G) = 1$.

It is very interesting to investigate how the relaxation on distance conditions causes the reduction of the strong chromatic index of a graph.

Given two graphs G and H, the *composition* of G and H, denoted by $G[H]$, has vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if $x = x'$ and $yy' \in E(H)$ or $xx' \in E(G)$. For a positive integer, let K_m^c denote the empty graph on m vertices. The composition of C_5 and K_2^c is illustrated as in Fig. 1.

In next section, by considering $(1, 0)$-relaxed and $(0, 1)$-relaxed strong chromatic indices of the graph $C_5[K_m^c]$, we illustrate the fact that even a little bit relaxation on distance conditions will cause a great reduction of strong chromatic index. In Sect. 3, for $1 \leq s \leq \Delta - 1$, it is proved that $\chi'_{(s,0)}(T) \leq \left\lceil \frac{\Delta}{s+1} \right\rceil + \left\lceil \frac{\Delta-1}{s+1} \right\rceil + 1$; furthermore, if $(\Delta - 1) \equiv 0 \pmod{s + 1}$ then $\chi'_{(s,0)}(T) \leq \left\lceil \frac{\Delta}{s+1} \right\rceil + \left\lceil \frac{\Delta-1}{s+1} \right\rceil$. This upper bound is proved to be tight. Section 4 investigates $(0, t)$-relaxed strong chromatic index of trees. It is proved that $\chi'_{(0,t)}(T) \leq 2\Delta - t$ for $1 \leq t \leq \Delta - 1$, $\chi'_{(0,t)}(T) \leq \Delta + 1$ for $\Delta \leq t \leq 2\Delta - 3$, and $\chi'_{(0,t)}(T) \leq \Delta$ for $t \geq 2\Delta - 2$. From these results, we see that, for trees, the relaxation on distance one condition causes much reduction of the strong chromatic index than the relaxation on distance two condition does. In the following section, we show that the $(1, 1)$-relaxed strong chromatic index of the infinite regular tree is equal to its $(1, 0)$-relaxed strong chromatic index. The final section proves that $\left\lceil \frac{\sigma(T)}{2} \right\rceil \leq \chi'_{(1,0)}(T) \leq \left\lceil \frac{\sigma(T)}{2} \right\rceil + 1$, where $\sigma(T)$ is defined in Sect. 3.

2 (1, 0)-relaxed and (0, 1)-relaxed strong chromatic index of the graph $C_5[K_m^c]$

Since any two edges of $C_5[K_m^c]$ are at most distance two apart, its strong chromatic index is equal to $\frac{5\Delta^2}{4} = 5m^2$, the number of its edges, attaining the upper bound in Conjecture 1.1.

Let $x_0, x_1, \ldots, x_4$ be the 5-cycle C_5. And let the vertices of K_m^c be $y_0, y_1, \ldots, y_{m-1}$. For $i = 0, 1, \ldots, 4$, let $V_i = \{(x_i, y_j) \mid j = 0, 1, \ldots, m-1\}$. Denote by H_i the subgraph of $C_5[K_m^c]$ induced by the vertex set $V_i \cup V_{i+1}$, where '+' is taken modulo 5. Then each H_i is a balanced complete bipartite graph on $2m$ vertices. It is clear that the edge set of $C_5[K_m^c]$ is $\bigcup_{i=0}^{4} E(H_i)$.

Theorem 2.1 *Let m be any positive integer. Then* $\chi'_{(1,0)}(C_5[K_m^c]) = \chi'_{(0,1)}(C_5[K_m^c]) = \lceil \frac{5}{2}m^2 \rceil$.

Proof Since any two edges of $C_5[K_m^c]$ are at distance at most 2, in a (1, 0)-relaxed (resp. (0, 1)-relaxed) strong edge coloring of $C_5[K_m^c]$, each color can be assigned to at most two different edges. It follows that $\chi'_{(1,0)}(C_5[K_m^c]) \geq \lceil \frac{5}{2}m^2 \rceil$ (resp. $\chi'_{(0,1)}(C_5[K_m^c]) \geq \lceil \frac{5}{2}m^2 \rceil$). We next show that $\lceil \frac{5}{2}m^2 \rceil$ colors are enough for a (1, 0)-relaxed (resp. (0, 1)-relaxed) strong edge coloring of $C_5[K_m^c]$.

First we suppose m is even. Then each H_i is m-regular and thus has a Euler tour. Let the edges along the Euler tour being $e_1, e_2, \ldots, e_{m^2}$. For $h = 1, 2, \ldots, \frac{m^2}{2}$, assign each pair of edges e_{2h-1} and e_{2h} a same color. In this way, one can get a (1, 0)-relaxed strong edge coloring of $C_5[K_m^c]$ using $\frac{5}{2}m^2$ colors. See Fig. 2 for an illustration. For $h = 0, 1, \ldots, \frac{m^2}{4} - 1$, assign the edge e_{4h+1} (resp. e_{4h+2}) with the same color as e_{4h+3} (resp. e_{4h+4}). In this way, one can get a (0, 1)-relaxed strong edge coloring of $C_5[K_m^c]$ using $\frac{5}{2}m^2$ colors. See Fig. 3 for an illustration. Thus the theorem holds if m is even.

Now suppose m is odd. For $i = 0, 1, \ldots, m-1$, let $P_i = (x_0, y_i)(x_1, y_{i+1}) \ldots (x_4, y_{i+4})$ and $C = \bigcup_{i=0}^{m-1} P_i \bigcup \{(x_4, y_{i+4})(x_0, y_{i+1}) | i = 0, 1, \ldots, m-1\}$, where the subscripts of the second coordinates are taken modulo m. It is straightforward to check that C is a Hamilton cycle of $C_5[K_m^c]$. And it is clear that, for $i = 0, 1, \ldots, 4$, $H_i \setminus E(C)$ has a Euler tour with $m(m-1)$ edges. Now, as in the previous paragraph, note that C is of odd length and $H_i \setminus E(C)$ has even number of edges, we can construct a (1, 0)-relaxed strong edge coloring of $C_5[K_m^c]$ using $\lceil \frac{5}{2}m^2 \rceil$ colors.

We next construct a (0, 1)-relaxed strong edge coloring of $C_5[K_m^c]$ using $\lceil \frac{5}{2}m^2 \rceil$ colors. For $i = 0, 1, \ldots, 4$, let $M_i = \{(x_i, y_j)(x_{i+1}, y_j) \mid j = 0, 1, \ldots, m-1\}$.

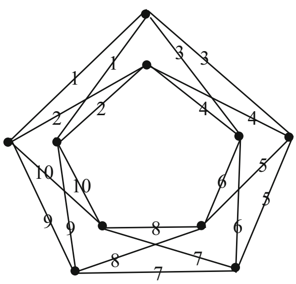

Fig. 2 A (1, 0)-relaxed strong 10-edge coloring of $C_5[K_2^c]$

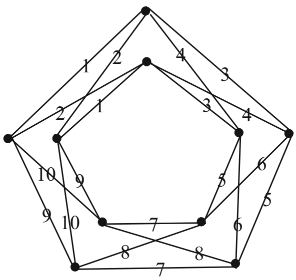

Fig. 3 A (0, 1)-relaxed strong 10-edge coloring of $C_5[K_2^c]$

Then $M_0 \cup M_2$, $M_1 \cup M_3$ and M_4 are three disjoint matchings of $C_5[K_m^c]$. And $\lceil \frac{5}{2}m \rceil$ colors are enough for the $5m$ edges in these matchings. For each $i \in \{0, 1, \ldots, 4\}$, $j, k \in \{0, 1, \ldots, m-1\}$ with $j \neq k$, assign the edge $(x_i, y_j)(x_{i+1}y_k)$ the same color as the edge $(x_i, y_k)(x_{i+1}y_j)$. In this way, we get a (0, 1)-relaxed strong edge coloring of $C_5[K_m^c]$ using $\lceil \frac{5}{2}m^2 \rceil$ colors. Thus the theorem also holds if m is odd. □

From the above theorem, for the graph $C_5[K_m^c]$, a unit relaxation on distance condition reduces the strong chromatic index by a half. In the corresponding channel assignment problem, this relaxation scheme will greatly save the channel span. Thus we think that this idea of relaxation will play an important role in the studies of strong edge colorings of graphs.

We next turn to relaxed strong edge colorings of trees, giving further examples how the relaxation on distance conditions causes the reduction of strong chromatic indices of graphs.

3 Some notations about trees

In a tree T, a vertex of degree 1 is called a *leaf* and a vertex of maximum degree is called a *major vertex*. For two vertices x and y in a tree T, the *distance between x and y*, denoted by $d(x, y)$, is the length of the unique (x, y)-path in T.

For a tree T, one can choose a vertex r as the *root* of T. A tree T with root r is called a r-tree, and is denoted by $T(r)$. Let $T(r)$ be a r-tree. For an edge uv of $T(r)$, if $d(u, r) = d(v, r) - 1$, then u is the *father* of v and v is a *son* of u. For any edge xy in $T(r)$, define $d(xy, r) = \min\{d(x, r), d(y, r)\}$. If $d(xy, r) = k$, then we call xy a *kth-generation edge descended from r*. For two edges e_1 and e_2 with distance one (resp. two) of $T(r)$, if e_1 is a kth-generation edge and e_2 is a $(k+1)th$ (resp. $(k+2)th$)-generation edge, then we say that e_1 is the *father* (resp. *grandfather*) of e_2 and e_2 is a *child* (resp. *grandchild*) of e_1. Let $F(e)$ and $GF(e)$ denote the father and the grandfather of e, respectively. And let $Ch(e)$ denote the set of children of e and $GCh(e)$ the set of grandchildren of e. Two edges incident with the root r, or with the same father are called *brothers*. If e_1 and e_2 have the same grandfather but different father, then we say e_1 and e_2 are *cousins*, and $F(e_1)$ is called an *uncle* of e_2, e_2 is called a *nephew* of $F(e_1)$.

Let $T(r)$ be a tree with root r. The *height* of $T(r)$ is defined as $R = \max\limits_{u \in V(T(r))} \{i | d(u, r) = i\}$. Let $V_i = \{u | d(u, r) = i\}$ and the vertices in V_i are said to be in the *ith-level* of $T(r)$. For $0 \leq i \leq R-1$, let E_i^r denote the set of *ith*-generation edges descended from r. Thus E_0^r is the set of edges incident with the root r. And E_1^r is the set of edges that are at distance 1 from r.

Let Δ (≥ 2) be a positive integer. A Δ-*regular tree* is a tree with maximum degree Δ and all vertices of degree less than Δ are leaves. Let T be a Δ-regular tree. If there is a vertex u of T such that the distance from all leaves to u are the same, then T is called a *complete* Δ-*regular tree*. The vertex u is called the *root* of T. A complete Δ-regular tree with height 1 is called a *star* and is denoted by $K_{1,\Delta}$. Let $T_\infty(\Delta)$ be an infinite tree with each vertex having the same degree Δ, which is called *infinite* Δ-*regular tree*.

Let uv be an edge of G, define $\sigma(uv) = d(u) + d(v) - 1$ and $\sigma(G) = \max_{uv \in E(G)} \{\sigma(uv)\}$. An edge uv in G is said to be σ-*tight* if $\sigma(uv) = \sigma(G)$. It is easy to see that $\sigma(G)$ is a lower bound for strong chromatic index $s\chi'(G)$, that is, $s\chi'(G) \geq \sigma(G)$ for any graph G. In Fandree et al. (1990) determined the strong chromatic index of trees.

Theorem 3.1 Fandree et al. (1990) *For any tree* T, $s\chi'(T) = \sigma(T)$.

In the following sections, we consider (s,t)-relaxed strong chromatic indices of trees. If the maximum degree of a tree is 2 then it is a path. Let P_n be a path with n vertices. Suppose $n \geq 6$. Then it is not difficult to prove that $\chi'_{(0,0)}(P_n) = \chi'_{(0,1)}(P_n) = 3$, $\chi'_{(1,0)}(P_n) = \chi'_{(1,1)}(P_n) = \chi'_{(1,2)}(P_n) = \chi'_{(0,2)}(P_n) = 2$, and $\chi'_{(2,2)}(P_n) = 1$. Thus we shall always assume $\Delta \geq 3$ in the following sections.

4 $(s, 0)$-relaxed strong chromatic index of trees

In this section, we study the $(s,0)$-relaxed strong chromatic index of tree with maximum degree at least 3 for all integers $s \geq 1$. In an $(s,0)$-relaxed strong edge-coloring, edges at distance two must receive different colors. And so, for any edge uv, if the color of uv appears more than once on edges in $NE(u)$ (resp. $NE(v)$), then it will not appear on edges in $NE(v) \setminus \{uv\}$ (resp. $NE(u) \setminus \{uv\}$). It follows that in an $(s,0)$-relaxed strong edge-coloring of a tree, each edge has at most $\min\{s, \Delta-1\}$ neighbors assigned the same color as itself. This implies that, for any tree T with maximum degree Δ, $\chi'_{(s,0)}(T) = \chi'_{(\Delta-1,0)}(T)$ for $s \geq \Delta - 1$. Thus we assume $1 \leq s \leq \Delta - 1$ in this section.

Let $\Delta(\geq 3)$ and s be two positive integers with $1 \leq s \leq \Delta - 1$. Let $k_1 = \left\lceil \frac{\Delta}{s+1} \right\rceil$ and $k_2 = \left\lceil \frac{\Delta-1}{s+1} \right\rceil$. Choose any vertex r as the root of $T_\infty(\Delta)$. The following algorithm will produce an $(s,0)$-relaxed strong edge coloring of $T_\infty(\Delta)$ using at most $k_1 + k_2 + 1$ colors.

Lemma 4.1 *Let* $\Delta(\geq 3)$ *and* s *be two positive integers with* $1 \leq s \leq \Delta - 1$. *If* $\Delta - 1$ *is divisible by* $s+1$, *then Algorithm 1 gives an* $(s,0)$-*relaxed strong edge-coloring of* $T_\infty(\Delta)$ *using* $k_1 + k_2$ *colors; otherwise, Algorithm 1 gives an* $(s,0)$-*relaxed strong edge-coloring of* $T_\infty(\Delta)$ *using* $k_1 + k_2 + 1$ *colors.*

Algorithm 1 Construct an $(s, 0)$-relaxed strong edge-coloring f for $T_\infty(\Delta)$ with $1 \le s \le \Delta - 1$

1: Assign integers $1, 2, \ldots, k_1$ $(= \lceil \frac{\Delta}{s+1} \rceil)$ to the edges in $NE(r)$ such that each integer is used at most $s + 1$ times;
2: **for** each $i := 1$ to ∞ **do**
3: **for** each vertex $v \in V_i$ **do**
4: Color all $\Delta - 1$ children of the edge $F(v)v$ with the least k_2 $(= \lceil \frac{\Delta-1}{s+1} \rceil)$ positive integers different from those in $f(NE(F(v)))$, such that each integer is used at most $s + 1$ times;
5: **end for**
6: **end for**

Proof It is clear that the coloring f produced by Algorithm 1 is an $(s, 0)$-relaxed strong edge-coloring of $T_\infty(\Delta)$. We next count the number of colors used by f.

Suppose uv is an edge of $T_\infty(\Delta)$ with $u = F(v)$. According to the algorithm, the Δ edges incident with the vertex u are assigned k_1 distinct colors if u is the root and $k_2 + 1$ colors otherwise, and the $\Delta - 1$ children of uv receive k_2 distinct colors. Since $k_1 \le k_2 + 1$, f uses at most $2k_2 + 1$ distinct colors. Recall that $k_1 = \lceil \frac{\Delta}{s+1} \rceil$ and $k_2 = \lceil \frac{\Delta-1}{s+1} \rceil$, we have $k_1 = k_2 + 1$ if $\Delta - 1$ is divisible by $s + 1$ and $k_1 = k_2$ otherwise. Therefore, if $\Delta - 1$ is divisible by $s + 1$, then f uses at most $2k_2 + 1 = k_1 + k_2$ colors; otherwise, f uses at most $2k_2 + 1 = k_1 + k_2 + 1$ colors. The lemma holds. □

Lemma 4.2 *Let* $\Delta(\ge 3)$ *and* s *be two positive integers with* $1 \le s \le \Delta - 1$. *Suppose* f *is an* $(s, 0)$*-relaxed strong edge-coloring of a complete* Δ*-regular tree with height at least 3. Then* f *uses at least* $k_1 + k_2$ *colors. Furthermore, if* $\Delta - 1$ *is not divisible by* $s + 1$, *then* f *uses at least* $k_1 + k_2 + 1$ *colors.*

Proof Let u be the root of T and v a child of u. If the color $f(uv)$ appears more than once on the Δ edges incident with u, then it can not be assigned to any children of uv. Since any two edges at distance two can not receive the same color, it is obvious that f uses at least $\lceil \frac{|NE(u)|}{s+1} \rceil + \lceil \frac{|Ch(uv)|}{s+1} \rceil = k_1 + k_2$ colors. Otherwise f uses at least $\lceil \frac{|NE(u)|-1}{s+1} \rceil + \lceil \frac{|NE(v)|}{s+1} \rceil = k_2 + k_1$ colors.

Now suppose $\Delta - 1$ is not divisible by $s + 1$. If f uses less than $k_1 + k_2 + 1$ colors, then it uses exactly $k_1 + k_2$ colors. If all edges incident with the root u receive different colors, then f uses at least $\Delta - 1 + \lceil \frac{\Delta}{s+1} \rceil = \Delta - 1 + k_1 > k_1 + k_2$ colors. Thus we assume that there are two edges, say uv and uw, receive the same color. Then $|f(NE(u))| = k_1$ and the $\Delta - 1$ children of uv receive exactly k_2 colors that are different from $f(uv)$. This implies that $|f(NE(v))| = k_2 + 1$. Since $\Delta \ge 3$, we have $k_2 = \lceil \frac{\Delta-1}{s+1} \rceil < \Delta - 1$. It follows that there are two children of uv, say vx and vy, having $f(vx) = f(vy)$. Now the number of colors left for the $\Delta - 1$ children of vx is $k_1 - 1$. Since $\Delta - 1$ is not divisible by $s + 1$, we have $k_1 - 1 = \lceil \frac{\Delta}{s+1} \rceil - 1 < \lceil \frac{\Delta-1}{s+1} \rceil$. Thus the children of vx cannot be colored properly. This is a contradiction and so f uses at least $k_1 + k_2 + 1$ colors. □

Combining Lemmas 4.1 and 4.2, we get the following results.

Theorem 4.3 *Let* $\Delta(\geq 3)$ *and* s *be two positive integers with* $1 \leq s \leq \Delta - 1$. *Then for any tree* T *with maximum degree* Δ,

$$\chi'_{(s,0)}(T) \leq \begin{cases} \left\lceil \frac{\Delta}{s+1} \right\rceil + \left\lceil \frac{\Delta-1}{s+1} \right\rceil, & \text{if } (\Delta - 1) \equiv 0 \pmod{s+1}; \\ \left\lceil \frac{\Delta}{s+1} \right\rceil + \left\lceil \frac{\Delta-1}{s+1} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

The upper bound in this theorem is sharp. It is attained by complete Δ-regular trees with height at least 3.

Corollary 4.4 *Let* T *be a complete* Δ*-regular tree with* $\Delta(\geq 3)$ *and let* s *be an integer with* $1 \leq s \leq \Delta - 1$. *If the height of* T *is at least 3, then*

$$\chi'_{(s,0)}(T) = \begin{cases} \left\lceil \frac{\Delta}{s+1} \right\rceil + \left\lceil \frac{\Delta-1}{s+1} \right\rceil, & \text{if } (\Delta - 1) \equiv 0 \pmod{s+1}; \\ \left\lceil \frac{\Delta}{s+1} \right\rceil + \left\lceil \frac{\Delta-1}{s+1} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

5 $(0, t)$-relaxed strong chromatic index of trees

In this section, we consider the $(0, t)$-relaxed strong chromatic index of trees. In a $(0, t)$-relaxed strong edge-coloring, adjacent edges must receive distinct colors and for each edge e there are at most t edges distance two apart from e that receive the same color as e. We distinguish the problem into three cases: $1 \leq t \leq \Delta-1$, $\Delta \leq t \leq 2\Delta-3$ and $t \geq 2\Delta - 2$.

The following algorithm (Algorithm 2) applies for the case $1 \leq t \leq \Delta - 1$. Choose any vertex r as the root of $T_\infty(\Delta)$. Starting from coloring edges incident with r, Algorithm 2 will produce a $(0, t)$-relaxed strong $(2\Delta - t)$-edge-coloring, denoted by f, for $T_\infty(\Delta)$. For an illustration, a partial $(0, 3)$-relaxed strong 7-edge-coloring for $T_\infty(5)$ is presented in Fig. 4.

Lemma 5.1 *For* $1 \leq t \leq \Delta - 1$, *Algorithm* 2 *gives a* $(0, t)$*-relaxed strong* $(2\Delta - t)$*-edge-coloring for* $T_\infty(\Delta)$.

Proof It is easy to see that the color assignment f in Algorithm 2 uses $2\Delta - t$ colors. So, we only need to prove that f is a $(0, t)$-relaxed strong edge-coloring. From the coloring procedure, it is clear that adjacent edges always receive different colors. And the following three observations are easy to see.

Observation A For any edge e, none of its children and grandchildren receives the same color as e.

Observation B For any edge e not incident with the root r, at most one of its uncles receives the same colors as e.

Observation C For any edge e, there are exactly $t - 1$ nephews of e having the same color as e.

Algorithm 2 Construct a $(0, t)$-relaxed strong edge-coloring f for $T_\infty(\Delta)$ with $1 \le t \le \Delta - 1$

Let the Δ edges incident with the root r be $ru_0, ru_1, \ldots, ru_{\Delta-1}$;
for each $i := 0$ to $\Delta - 1$ **do**
  Color ru_i with $i + 1$ (i.e., $f(ru_i) = i + 1$);
  Color the $\Delta - 1$ children of ru_i with the $\Delta - 1$ colors in $\{f(ru_{i+1}), f(ru_{i+2}), \ldots, f(ru_{i+t-1})\} \cup [\Delta + 1, 2\Delta - t]$ (the indices in subscripts are taken module Δ), such that all children of ru_i receive distinct colors;
end for
for each $j := 0$ to ∞ **do**
  for each $e \in E_j^r$ **do**
    Let the $\Delta - 1$ children of e be $e_0, e_1, \ldots, e_{\Delta-2}$;
    for each $i := 0$ to $\Delta - 2$ **do**
      Color the $\Delta - 1$ children of e_i with the $\Delta - 1$ colors in $([1, 2\Delta - t] \setminus \{f(e), f(e_0), f(e_1) \ldots, f(e_{\Delta-2})\}) \cup \{f(e_{i+1}), f(e_{i+2}), \ldots, f(e_{i+t-1})\}$ (the indices in subscript are taken module $\Delta - 1$), such that all children of e_i receive distinct colors;
    end for
  end for
end for

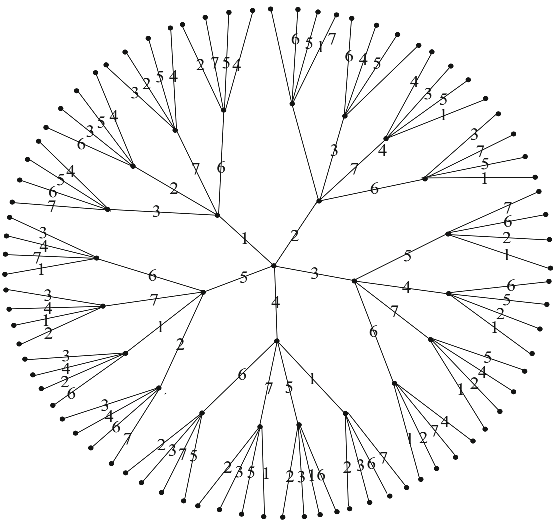

Fig. 4 A partial $(0, 3)$-relaxed strong 7-edge-coloring of $T_\infty(5)$

Combining observations A, B and C, for any edge e, there are at most t of its 2-neighbors having the same colors as e. Therefore, the coloring f produced by Algorithm 2 is a $(0, t)$-relaxed strong edge-coloring of $T_\infty(\Delta)$ using $2\Delta - t$ colors and the lemma holds. □

Lemma 5.2 *Let $\Delta(\ge 3)$ and t be two integers with $1 \le t \le \Delta - 1$. If T is a complete Δ-regular tree with height at least 3, then $\chi'_{(0,t)}(T) = 2\Delta - t$.*

Proof By Lemma 5.1, we have $\chi'_{(0,t)}(T) \le 2\Delta - t$. To prove the lemma, it suffices to prove that $\chi'_{(0,t)}(T) \ge 2\Delta - t$. Suppose to the contrary that $\chi'_{(0,t)}(T) \le 2\Delta - t - 1$. Let f be a $(0, t)$-relaxed strong edge-coloring of T using $2\Delta - t - 1$ colors. Let r be the root of T. Then $|E_0^r| = \Delta$ and $|E_1^r| = \Delta(\Delta - 1)$. Since f is a $(0, t)$-relaxed strong edge-coloring, edges in E_0^r must receive different colors. Assume $f(E_0^r) = [1, \Delta]$.

For each $i \in [1, 2\Delta - t - 1]$, let $L_i = \{e | e \in E_1^r \text{ and } f(e) = i\}$ and l_i be the cardinality of L_i. Since adjacent edges must receive different colors and each edge e can have at most t 2-neighbors colored with the same color as e, it is easy to see that $l_i \leq \Delta$ for $i \in [\Delta + 1, 2\Delta - t - 1]$ and $l_i \leq t$ for $i \in [1, \Delta]$. It follows that $\sum_{i=1}^{2\Delta-t-1} l_i \leq \Delta(\Delta - t - 1) + \Delta t = \Delta^2 - \Delta = |E_1^r|$. This implies that $l_i = \Delta$ for each $i \in [\Delta + 1, 2\Delta - t - 1]$ and $l_i = t$ for each $i \in [1, \Delta]$. Let u be a neighbor of r such that $f(ru) = 1$. Then since $l_i = \Delta$ for $i \in [\Delta + 1, 2\Delta - t - 1]$, each color in $[\Delta + 1, 2\Delta - t - 1]$ appears exactly once in $Ch(ru)$. The other t edges in $Ch(ru)$ receive colors from $[2, \Delta]$. Let these t colors be $c_1, c_2, \ldots, c_t$.

Let $GCh(ru)$ be the set of grandchildren of ru. Then $|GCh(ru)| = (\Delta - 1)^2$. For $i \in [1, 2\Delta - t - 1]$, let $L'_i = \{e | e \in GCh(ru) \text{ and } f(e) = i\}$ and l'_i be the cardinality of L'_i. Since f is a $(0, t)$-relaxed strong edge-coloring, it is not hard to prove the following four properties of f: (1) $l'_1 = 0$ (since $l_1 = t$); (2) $l'_i \leq t - 1$ for each $i \in \{c_1, c_2, \ldots, c_t\}$; (3) $l'_i \leq \Delta - 1$ for each $i \in [2, \Delta] \setminus \{c_1, c_2, \ldots, c_t\}$; (4) $l'_i \leq t$ for each $i \in [\Delta + 1, 2\Delta - t - 1]$. Therefore,

$$\begin{aligned}\sum_{i=1}^{2\Delta-t-1} l'_i &\leq t(t-1) + (\Delta - t - 1)(\Delta - 1) + (\Delta - t - 1) \cdot t \\ &= (\Delta - 1)^2 - t < (\Delta - 1)^2 = |GCh(ru)|.\end{aligned}$$

This is a contradiction. So $\chi'_{(0,t)}(T) \geq 2\Delta - t$ and the lemma follows. □

Lemma 5.3 *Let $\Delta(\geq 3)$ and t be two integers. Suppose T is a complete Δ-regular tree with height at least 3. Then $\chi'_{(0,t)}(T) = \Delta$ if and only if $t \geq 2\Delta - 2$.*

Proof It is obvious that $\chi'_{(0,t)}(T) \geq \Delta$ for each integer $t \geq 0$. Suppose $t \geq 2\Delta - 2$. Since T is a subgraph of $T_\infty(\Delta)$, to prove $\chi'_{(0,t)}(T) \leq \Delta$, it suffices to construct a $(0, t)$-relaxed strong Δ-edge-coloring f of $T_\infty(\Delta)$. Choose a root r for $T_\infty(\Delta)$ and color the Δ edges incident with r by the Δ colors $1, 2, \ldots, \Delta$. Let R be the height of T. Then, for $j = 0, 1, \ldots, R - 1$, for each edge e in E_j^r, color the $\Delta - 1$ children of e with the $\Delta - 1$ colors in $[1, \Delta] \setminus \{f(e)\}$. It is straightforward to check that for each edge e, the color $f(e)$ appears $2\Delta - 2$ times on the edges that are distance two apart from e. Thus f is a $(0, t)$-relaxed strong Δ-edge-coloring of $T_\infty(\Delta)$ for $t \geq 2\Delta - 2$. It follows that $\chi'_{(0,t)}(T) = \chi'_{(0,t)}(T_\infty(\Delta)) = \Delta$ for $t \geq 2\Delta - 2$.

On the other hand, let f be any $(0, t)$-relaxed strong Δ-edge-coloring of T. Let r be the root of T and u a child of r. It is not difficult to prove that at least $2\Delta - 2$ of 2-neighbors of ru are assigned the color $f(ru)$. It follows that if $t < 2\Delta - 2$, then $\chi'_{(0,t)}(T) > \Delta$. The lemma holds. □

Suppose T is a tree with maximum degree Δ. By Lemma 5.1, $\chi'_{(0,\Delta-1)}(T) \leq \chi'_{(0,\Delta-1)}(T_\infty(\Delta)) = 2\Delta - (\Delta - 1) = \Delta + 1$. Thus $\chi'_{(0,t)}(T) \leq \chi'_{(0,\Delta-1)}(T) \leq \Delta + 1$ for $\Delta \leq t \leq 2\Delta - 3$. We have the following theorem.

Theorem 5.4 *Let $\Delta(\geq 3)$ and $t(\geq 1)$ be two integers. If T is a tree with maximum degree Δ, then*

$$\chi'_{(0,t)}(T) \leq \begin{cases} 2\Delta - t, & \text{if } 1 \leq t \leq \Delta - 1; \\ \Delta + 1, & \text{if } \Delta \leq t \leq 2\Delta - 3; \\ \Delta, & \text{if } t \geq 2\Delta - 2. \end{cases}$$

The upper bound in this theorem is sharp. It is attained by complete Δ-regular trees with height at least 3.

Corollary 5.5 *Let $\Delta(\geq 3)$ and $t(\geq 1)$ be two integers. If T is a complete Δ-regular tree with height at least 3, then*

$$\chi'_{(0,t)}(T) = \begin{cases} 2\Delta - t, & \text{if } 1 \leq t \leq \Delta - 1; \\ \Delta + 1, & \text{if } \Delta \leq t \leq 2\Delta - 3; \\ \Delta, & \text{if } t \geq 2\Delta - 2. \end{cases}$$

Suppose T is a complete Δ-regular tree with height at least 3. By Corollary 5.5, $\chi'_{(0,1)}(T) = \chi'_{(0,0)}(T) = s\chi'(T) = 2\Delta - 1$. While, by Corollary 4.4, $\chi'_{(1,0)}(T) = \Delta$ if Δ is odd and $\chi'_{(1,0)}(T) = \Delta + 1$ if Δ is even, which is much less than $\chi'_{(0,1)}(T)$. In general, for $1 \leq s \leq \Delta - 1$, $\chi'_{(s,0)}(T)$ is close to $\lceil \frac{2\Delta}{s+1} \rceil$, while $\chi'_{(0,t)}(T)$ is equal to $2\Delta - s$. Thus, for such trees, the relaxation on distance one condition causes much reduction of the strong chromatic index than that on distance two condition. Thus, maybe, the upper bound for $\chi'_{(s,t)}(T)$ in the following theorem is not bad.

Theorem 5.6 *Let T be a tree with maximum degree Δ. Suppose s and t are two integers with $1 \leq s \leq \Delta - 1$ and $1 \leq t \leq 2(\Delta - 1)^2$. Then*

$$\left\lceil \frac{\Delta}{s+1} \right\rceil \leq \chi'_{(s,t)}(T) \leq \chi'_{(s,0)}(T) \leq \left\lceil \frac{\Delta}{s+1} \right\rceil + \left\lceil \frac{\Delta - 1}{s+1} \right\rceil + 1.$$

In the next section, we show that, sometimes, the upper bound in this theorem is sharp by proving that $\chi'_{(1,1)}(T_\infty(\Delta)) = \chi'_{(1,0)}(T_\infty(\Delta))$.

6 (1, 1)-relaxed strong chromatic index of $T_\infty(\Delta)$

We determine the (1, 1)-relaxed strong chromatic index of $T_\infty(\Delta)$ in this section.

Lemma 6.1 *Let $\Delta(\geq 3)$ be an odd integer. Then $\chi'_{(1,1)}(T_\infty(\Delta)) \geq \Delta$.*

Proof Suppose $\chi'_{(1,1)}(T_\infty(\Delta)) \leq \Delta - 1$. Let f be (1, 1)-relaxed strong $(\Delta-1)$-edge-coloring of $T_\infty(\Delta)$. Choose any vertex r as the root of $T_\infty(\Delta)$. Let $k = |f(NE(r))|$. Since Δ is odd and each color can appear at most twice in $NE(r)$, it is obvious that $\frac{\Delta+1}{2} \leq k \leq \Delta - 1$. By the distance condition, for the edges in $NE(r) \cup E_1^r$, the colors in $f(NE(r))$ can appear at most three times, and the other colors can appear at most

2Δ times. Then the number of edges in $NE(r) \cup E_1^r$ which can get colors is at most $N = 3k + (\Delta - 1 - k) \cdot 2\Delta$. Since $\frac{\Delta+1}{2} \leq k \leq \Delta - 1$ and $\Delta \geq 3$, we have

$$N = 3k + 2\Delta^2 - 2\Delta - 2\Delta k = \Delta^2 + (\Delta^2 - 2\Delta - (2\Delta - 3)k)$$
$$\leq \Delta^2 + \left(\Delta^2 - 2\Delta - (2\Delta - 3) \cdot \frac{\Delta + 1}{2}\right) = \Delta^2 + \left(\frac{3}{2} - \frac{3}{2}\Delta\right) < \Delta^2$$

Note that $|NE(r) \cup E_1^r| = \Delta + \Delta(\Delta - 1) = \Delta^2$. It is a contradiction and the lemma holds. □

Lemma 6.2 *Let* $\Delta(\geq 4)$ *be an even integer. Then* $\chi'_{(1,1)}(T_\infty(\Delta)) \geq \Delta + 1$.

Proof Suppose $\chi'_{(1,1)}(T_\infty(\Delta)) \leq \Delta$. Let f be (1, 1)-relaxed strong Δ-edge-coloring of $T_\infty(\Delta)$. Then for any vertex of $T_\infty(\Delta)$, we have the following result:

Claim Suppose u is an vertex in $T_\infty(\Delta)$. If $\Delta \geq 8$, then $|f(NE(u))| = \frac{\Delta}{2}$; if $4 \leq \Delta \leq 6$, then $\frac{\Delta}{2} \leq |f(NE(u))| \leq \frac{\Delta}{2} + 1$.

Proof Choose vertex u as the root of $T_\infty(\Delta)$. Assume $|f(NE(u))| = k$. Since Δ is even and each color can appear at most twice in $NE(u)$, it is obvious that $\frac{\Delta}{2} \leq k \leq \Delta$. With the similar argument as Lemma 6.1, the number of edges in $NE(u) \cup E_1^u$ which can get colors is at most $N = 3k + (\Delta - k) \cdot 2\Delta$. Suppose that $\Delta \geq 8$. If $k \geq \frac{\Delta}{2} + 1$, then

$$N = 3k + (\Delta - k) \cdot 2\Delta = 2\Delta^2 + (3 - 2\Delta)k$$
$$\leq 2\Delta^2 + (3 - 2\Delta) \cdot \left(\frac{\Delta}{2} + 1\right) = \Delta^2 + \left(3 - \frac{\Delta}{2}\right) \leq \Delta^2 - 1 < \Delta^2$$

It contradicts the fact $|NE(u) \cup E_1^u| = \Delta^2$. Thus, $k < \frac{\Delta}{2} + 1$, which implies that $|f(NE(u))| = \frac{\Delta}{2}$.

With the same argument, for $4 \leq \Delta \leq 6$, we can prove $|f(NE(u))| < \frac{\Delta}{2} + 2$. It implies that $\frac{\Delta}{2} \leq |f(NE(u))| \leq \frac{\Delta}{2} + 1$. Thus, the claim holds. □

Next, we distinguish the proof into three cases:

Case 1 $\Delta \geq 8$.

In this case, by the above claim, we know that $|f(NE(u))| = \frac{\Delta}{2}$ for any vertex u in $T_\infty(\Delta)$. This implies that any color in $f(NE(u))$ is used exactly twice on the edges in $NE(u)$. Suppose uv is any edge in $T_\infty(\Delta)$ and $f(uv) = a$. Then there exists an edge, say uu_1 (resp. vv_1), in $NE(u) \setminus \{uv\}$ (resp. $NE(v) \setminus \{uv\}$) satisfying $f(uu_1) = a$ (resp. $f(vv_1) = a$). So, we have uu_1 and vv_1 are both neighbors of uv and $f(uv) = f(uu_1) = f(vv_1)$. It is a contradiction since f is a (1, 1)-relaxed strong edge-coloring.

Case 2 $\Delta = 6$.

In this case, by the above claim, we know that $|f(NE(u))| = 3$ or 4 for any vertex u in $T_\infty(6)$. Suppose for some vertex u, $|f(NE(u))| = 4$. Choose u as the root of

$T_\infty(6)$. Without loss of generality, let the colors appearing on edges in $NE(u)$ be 1, 2, 3 and 4, where colors 1 and 2 appear twice and colors 3 and 4 appear once. Let $L_i = \{e \in E_1^u | f(e) = i\}$ and l_i be the cardinality of L_i. Since f is a (1, 1)-relaxed strong 6-edge-coloring, we have $l_i = 0$ for $i = 1, 2$, $l_i \leq 2$ for $i = 3, 4$, and $l_i \leq 12$ for $i = 5, 6$. Thus, $\sum_{i=1}^{6} l_i \leq 28 < 30 = |E_1^u|$. This is a contradiction. Therefore, for any vertex $u \in V(T_\infty(6))$, it satisfies $|f(NE(u))| = 3$. With the same argument as in Case 1, we can get a contradiction.

Case 3 $\Delta = 4$.

In this case, by the above claim, we know that $|f(NE(u))| = 2$ or 3 for any vertex u in $T_\infty(4)$. Suppose for some vertex u, $|f(NE(u))| = 3$. Choose u as the root of $T_\infty(4)$. Without loss of generality, let the colors appearing on edges in $NE(u)$ be 1, 2 and 3, where color 1 appears twice and colors 2 and 3 appear once. Let $L_i = \{e \in E_1^u | f(e) = i\}$ and l_i be the cardinality of L_i. Since f is a (1, 1)-relaxed strong 4-edge-coloring and $|E_1^u| = 12$, we have $l_1 = 0$, $l_2 = l_3 = 2$ and $l_4 = 8$. Then, we can assume $f(uv) = 1$, $f(vv_1) = 2$, $f(vv_2) = f(vv_3) = 4$. Next, we consider the edges in $GCh(uv)$. Let $L'_i = \{e \in GCh(uv) | f(e) = i\}$ and l'_i be the cardinality of L'_i. By the distance conditions, we have $l'_1 \leq 1$, $l'_2 \leq 1$, $l'_3 \leq 6$ and $l'_4 = 0$. Thus, $\sum_{i=1}^{4} l'_i \leq 8 < 9 = |GCh(uv)|$. This is a contradiction. Therefore, for any vertex $u \in V(T_\infty(4))$, it satisfies $|f(NE(u))| = 2$. With the similar argument as in Case 1, we can get a contradiction.

By the above cases, for any even integer $\Delta(\geq 4)$, $\chi'_{(1,1)}(T_\infty(\Delta)) \geq \Delta + 1$, and the lemma holds. □

Since a (1, 0)-relaxed strong edge-coloring is also a (1, 1)-relaxed strong edge-coloring, we have $\chi'_{(1,1)}(T_\infty(\Delta)) \leq \chi'_{(1,0)}(T_\infty(\Delta))$. Combining Theorem 4.3, Lemmas 6.1 and 6.2, we have the following result:

Theorem 6.3 *Let* $\Delta(\geq 3)$ *be an integer. Then*

$$\chi'_{(1,1)}(T_\infty(\Delta)) = \chi'_{(1,0)}(T_\infty(\Delta)) = \begin{cases} \Delta, & \text{if } \Delta \text{ is odd;} \\ \Delta + 1, & \text{if } \Delta \text{ is even.} \end{cases}$$

Question Are there any pairs of (s, t) such that $\chi'_{(s,t)}(T_\infty(\Delta)) < \chi'_{(s,0)}(T_\infty(\Delta))$? Could we characterize all pairs of (s, t) such that $\chi'_{(s,t)}(T_\infty(\Delta)) = \chi'_{(s,0)}(T_\infty(\Delta))$?

7 (1, 0)-relaxed strong edge-coloring for a tree

In this section, we consider the (1, 0)-relaxed strong edge-coloring for a tree T. For any edge $uv \in E(T)$, since $s = 1$ and $t = 0$, a color can be assigned to at most two edges in $NE(u) \cup NE(v)$. This implies that $\chi'_{(1,0)}(T) \geq \left\lceil \frac{\sigma(T)}{2} \right\rceil$. Next, we give a coloring algorithm and obtain the upper bound for $\chi'_{(1,0)}(T)$ in terms of $\sigma(T)$. For a function f of color assignment and an edge set E', let $f(E')$ be the set of colors

Algorithm 3 Construct a (1,0)-relaxed strong edge-coloring f for a r-tree

1: Assign integers $1, 2, \ldots, \left\lceil \frac{d(r)}{2} \right\rceil$ to the edges in $NE(r)$, such that each integer is used at most twice;
2: **for** each $i := 1$ to $R - 1$ **do**
3: **for** each vertex $v \in V_i$ **do**
4: **if** v is not a leaf and all children of the edge $F(v)v$ are uncolored;
5: **then** color all children of the edge $F(v)v$ with the least $\left\lceil \frac{d(v)-1}{2} \right\rceil$ positive integers different from those in $f(NE(F(v)))$, such that each integer is used at most twice;
6: **end for**
7: **end for**

appearing on edges in E'. Let T be a tree with root r. The following algorithm will construct an edge coloring of T with at most $(\left\lceil \frac{\sigma(T)}{2} \right\rceil + 1)$ colors.

Lemma 7.1 *For any tree T, Algorithm 3 gives a* $(1, 0)$*-relaxed strong* $(\left\lceil \frac{\sigma(T)}{2} \right\rceil + 1)$*-edge-coloring.*

Proof It is not hard to check that Algorithm 3 produces a $(1, 0)$-relaxed strong edge-coloring of T. To prove the theorem, it suffices to show that Algorithm 3 uses at most $\left\lceil \frac{\sigma(T)}{2} \right\rceil + 1$ colors.

Suppose uv is an edge of T with $u = F(v)$. At the stage when the children of uv (i.e., $Ch(uv)$) are colored, the number of colors needed for $Ch(uv)$ is $\left\lceil \frac{d(v)-1}{2} \right\rceil$. According to Algorithm 3, the edges in $NE(u)$ have used at most $\left\lceil \frac{d(u)-1}{2} \right\rceil + 1$ colors. Since

$$\left\lceil \frac{d(u)-1}{2} \right\rceil + 1 + \left\lceil \frac{d(v)-1}{2} \right\rceil \leq \left\lceil \frac{d(u)+d(v)-1}{2} \right\rceil + 1 \leq \left\lceil \frac{\sigma(T)}{2} \right\rceil + 1 \quad (1)$$

we conclude that f uses at most $\left\lceil \frac{\sigma(T)}{2} \right\rceil + 1$ colors. The lemma holds. □

Since $\left\lceil \frac{\sigma(T)}{2} \right\rceil$ is a lower bound for $\chi'_{(1,0)}(T)$, together with Lemma 7.1, we have the following theorem.

Theorem 7.2 *For any tree T,*

$$\left\lceil \frac{\sigma(T)}{2} \right\rceil \leq \chi'_{(1,0)}(T) \leq \left\lceil \frac{\sigma(T)}{2} \right\rceil + 1$$

The lower and upper bounds in this theorem are attainable. We next present some trees T with $\chi'_{(1,0)}(T)$ equal to the lower bound (resp. upper bound).

Let T be a tree and let uv be an edge of T. In the proof of Lemma 7.1, if $\sigma(T)$ is odd and uv is not σ-tight, then the inequality (1) is specified as

$$\left\lceil \frac{d(u)-1}{2} \right\rceil + 1 + \left\lceil \frac{d(v)-1}{2} \right\rceil \leq \left\lceil \frac{d(u)+d(v)-1}{2} \right\rceil + 1 < \left\lceil \frac{\sigma(T)}{2} \right\rceil + 1 \quad (2)$$

If uv is a σ-tight but $d(u)$ and $d(v)$ are both odd (this implies that $\sigma(T)$ is odd), then the inequality (1) is specified as

$$\left\lceil \frac{d(u)-1}{2} \right\rceil + 1 + \left\lceil \frac{d(v)-1}{2} \right\rceil = \frac{d(u)+d(v)}{2} = \frac{\sigma(T)+1}{2} = \left\lceil \frac{\sigma(T)}{2} \right\rceil \quad (3)$$

Thus, if every σ-tight edge uv of T has the property that both $d(u)$ and $d(v)$ are odd, then, by (2) and (3), the coloring function f produced by Algorithm 3 uses at most $\left\lceil \frac{\sigma(T)}{2} \right\rceil$ colors. Thus we have the following theorem.

Theorem 7.3 *Let T be a tree. If $\sigma(T)$ is odd and for any σ-tight edge, the degrees of its two endpoints are both odd, then $\chi'_{(1,0)}(T) = \left\lceil \frac{\sigma(T)}{2} \right\rceil = \frac{\sigma(T)+1}{2}$.*

Theorem 7.4 *Let T be a tree with $\sigma(T)$ being odd. Suppose uv is a σ-tight edge such that both $d(u)$ and $d(v)$ are even. If there are two σ-tight edges except uv incident with u and two σ-tight edges except uv incident with v, then $\chi'_{(1,0)}(T) = \frac{\sigma(T)+1}{2} + 1$.*

Proof Let u_1 and u_2 (resp. v_1 and v_2) be two vertices other than v (resp. u) that are adjacent to u (resp. v) such that $uu_1,\ uu_2$ (resp. $vv_1,\ vv_2$) are σ-tight. Then $d(u) = d(v_1)$. As $\sigma(T)$ is odd, the four vertices u_1, u_2, v_1, v_2 are all of even degree. Suppose the theorem is not true. Then $\chi'_{(1,0)}(T) = \frac{\sigma(T)+1}{2}$. Let f be a $(1, 0)$-relaxed strong $\frac{\sigma(T)+1}{2}$-edge-coloring. Since both $d(u)$ and $d(v)$ are even, it is clear that $f(uv)$ appears twice on edges in $NE(u)$ or twice on edges in $NE(v)$. Without loss of generality, assume $f(uv)$ appears twice on edges in $NE(u)$. It follows that $|f(NE(v))| = \frac{d(v)}{2} + 1$. This implies that at least one of $f(vv_1)$ and $f(vv_2)$ appears twice on edges in $NE(v)$. Without loss of generality, assume that $f(vv_1)$ appears twice on edges in $NE(v)$. But then there are $\frac{d(u)+d(v)}{2} - (\frac{d(v)}{2} + 1) = \frac{d(u)-2}{2} = \frac{d(v_1)-2}{2}$ colors left for $d(v_1) - 1$ edges in $NE(v_1) \setminus \{vv_1\}$. The number of colors for these edges are not enough. This is a contradiction. Thus the theorem holds. □

In the following, we consider the case that $\sigma(T)$ is even. By Theorem 7.2, if $\sigma(T)$ is even, then $\frac{\sigma(T)}{2} \leq \chi'_{(1,0)}(T) \leq \frac{\sigma(T)}{2} + 1$.

Lemma 7.5 *Let T be a tree with $\sigma(T)$ being even. Suppose f is a $(1, 0)$-relaxed strong $\frac{\sigma(T)}{2}$-edge-coloring and xy is a σ-tight edge of T. If $d(x)$ is even, then $f(xy) = f(xy')$ for some neighbor $y'(\neq y)$ of x.*

Proof Since $\sigma(T)$ is even, xy is σ-tight and $d(x)$ is even, $d(y)$ is odd. On the contrary, suppose that there doesn't exist any neighbor y' of x satisfying $f(xy) = f(xy')$. Then the edges in $NE(x)$ receive at least $\left\lceil \frac{d(x)-1}{2} \right\rceil + 1 = \frac{d(x)}{2} + 1$ distinct colors. Since the color $f(xy)$ can be used to another edge incident with y, the edges in $NE(y)$ receive at least $\left\lceil \frac{d(y)-2}{2} \right\rceil = \frac{d(y)-1}{2}$ colors that are different from the colors used on the edges in $NE(x)$. It follows that the number of colors used by f is at least $\frac{d(x)}{2} + 1 + \frac{d(y)-1}{2} = \frac{\sigma(T)}{2} + 1$. This is a contradiction and the lemma follows. □

Theorem 7.6 *Let T be a tree with $\sigma(T)$ being even. If there exists one of the following structures in T, then $\chi'_{(1,0)}(T) = \frac{\sigma(T)}{2}+1$: (1) There are two adjacent σ-tight edges uv and vw satisfied that $d(v)$ is odd. (2) For some edge uv such that $\sigma(uv) = \sigma(T) - 1$ and $d(u)$ is even. There are four neighbors uu_1, uu_2, vv_1 and vv_2 of uv such that $\sigma(uu_i) = \sigma(vv_i) = \sigma(T) - 1$ for $i = 1, 2$.*

Proof (1) On the contrary, suppose that $\chi'_{(1,0)}(T) = \frac{\sigma(T)}{2}$ and let f be a $(1, 0)$-relaxed strong edge-coloring of T with $\frac{\sigma(T)}{2}$ colors. Since uv and vw are σ-tight and they meet at a common vertex, we have $d(u) = d(w)$. Moreover, both $d(u)$ and $d(w)$ are even as $\sigma(T)$ is even and $d(v)$ is odd. By Lemma 7.5, $f(uv) = f(uv')$ for some neighbor v' ($\neq v$) of u, and $f(vw) = f(v''w)$ for some neighbor v'' ($\neq v$) of w. It follows that the number of colors used by f is at least

$$\frac{d(u)}{2} + \left\lceil \frac{d(v)-2}{2} \right\rceil + 1 = \frac{d(u)}{2} + \frac{d(v)-1}{2} + 1 = \frac{\sigma(T)}{2} + 1.$$

It is a contradiction and so (1) holds.

(2) Let T' be the subgraph of T induced by the edges incident with the vertices u, v, u_1, u_2, v_1, v_2. It follows that $\sigma(T') = \sigma(T) - 1$ and $\sigma(T')$ is odd. Since $d(u)$ is even and uu_i, vv_i are all σ-tight edges of T', by Theorem 7.4, we have $\chi'_{(1,0)}(T') = \frac{\sigma(T')+1}{2} + 1$. Then

$$\chi'_{(1,0)}(T) \geq \chi'_{(1,0)}(T') = \frac{\sigma(T')+1}{2} + 1 = \frac{\sigma(T)}{2} + 1.$$

Thus $\chi'_{(1,0)}(T) = \frac{\sigma(T)}{2} + 1$. □

According to Theorem 7.2, for any tree T, $\chi'_{(1,0)}(T)$ is either $\left\lceil \frac{\sigma(T)}{2} \right\rceil$ or $\left\lceil \frac{\sigma(T)}{2} \right\rceil + 1$. It is natural to ask whether the following problem is in class P.

Problem 7.7 $(1, 0)$-relaxed strong edge-coloring
Instance A tree T with $\sigma(T) = k$.
Question Is $\chi'_{(1,0)}(T) = \left\lceil \frac{k}{2} \right\rceil$?

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